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ABSTRACT

This document contains the proceedings of the 25th annual Conference of the International Group for the Psychology of Mathematics Education (PME). It features plenary lectures, research forums, discussion groups, working sessions, short oral communications, and poster presentations. Papers in Volume 1 include: (1) "The P in PME: Progress and Problems in Mathematics Education" (Jan De Lange); (2) "Linking Home and School Mathematics" (Martin Hughes); (3) "Algorithmic and Meaningful Ways of Joining Together Representatives within the Same Mathematical Activity: An Experience with Graphing Calculators" (Rina Hershkowitz and Carolyn Kieran); (4) "Even College Students Cannot Calculate Fractions: Mathematics Goals and Students' Achievement in Japan" (Yoshinori Shimizu); and (5) "Learning from History To Solve Equations" (Barbara van Amerom). Papers in Volume 2 include: (1) "Negation in Mathematics: Obstacles Emerging from an Exploratory Study" (Samuele Antonini); (2) "From the Decimal Number as a Measure to the Decimal Number as a Mental Object" (Cinzia Bonotto); (3) "The Concept of Parameter in a Computer Algebra Environment" (Paul Drijvers); and (4) "Australian and US Preservice Teachers' Perceptions of the Gender Stereotyping of Mathematics" (Helen J. Forgasz). Papers in Volume 3 include: (1) "Research on Attitudes in Mathematics Education: A Discursive Perspective" (Uwe Gellert); (2) "Effective Teachers of Second Language Learners in Mathematics" (Lena L. Khisty); (3) "Argumentative Processes in Problem Solving Situations: The Mediation of Tools" (Catia Mogetta); and (4) "Observations on the Nature of Quiet Disengagement in the Mathematics Classroom" (Elena Nardi and Susan Steward). Papers in Volume 4 include: (1) "Mathematics Subject Knowledge Revisited" (Stephanie Prestage and Patricia Perks); (2) "Challenging a Purely Mathematical Perspective on Teachers' Competence" (Jeppe Skott); (3) "Algebraic Understanding and the Importance of Operation Sense" (Elizabeth A. Warren); and (4) "Learning To Teach Mathematics Differently: Reflection Matters" (Terry Wood). (AA)

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Marja van den Heuvel-Panhuizen, Editor

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OF THE

25th CONFERENCE

OF THE

INTERNATIONAL GROUP FOR THE 'SYCHOLOGY OF MATHEMATICS EDUCATION

PME
25@NL

Utrecht - The Netherlands
July 12 - 17, 2001

Editor:
Marja van den Heuvel-Panhuizen

Freudenthal Institute
Faculty of Mathematics and Computer Science
Utrecht University
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Volume 1

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Volume 1

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PREFACE

PME is back home. 24 years after PME1, Utrecht is once again the spot where the International Group for Psychology of Mathematics Education meets. In 1977, one year after the birth of PME at the 3rd ICME Congress in Karlsruhe, the 1st PME conference was held in Utrecht, organized by Freudenthal. His friend Fischbein, the founder president of PME, had asked him whether *IOWO* would be able and inclined to do this job. The proposed organizers had some hesitation because of the short time for preparation and the expected number of participants. But there was another reason for being somewhat reluctant. To a certain degree there was a difference in scientific culture. Freudenthal and his colleagues saw themselves rather as engineers than as researchers and they were fairly critical regarding the psychologists' and educationists' view on mathematics education. Notwithstanding these doubts—or maybe because of them—this first conference was held in Utrecht and brought 86 people together: mathematicians, mathematics didacticians, but also psychologists and educationists. Since then, year after year they have met and inspired each other.

For the Dutch, of course, 1985 was another milestone in the history of PME. In that year, the 9th PME was held in Noordwijkerhout, organized by *OW & OC* with Streefland as the conference chair. Compared to the first conference—where mathematics education was more or less the working field of a number of different disciplines—now the didactics and the design of mathematics education became more and more a discipline of its own. The focus was on theory of mathematics education. Streefland saw it as a break-through. And at least for the Dutch it was so. It was the time that the theory of Realistic Mathematics Education, which so far was a theory in action, became more and more explicit.

Realistic Mathematics Education is still work in progress. The present stage of the development is characterized by revision, detecting blind spots and getting a more balanced approach. Having meetings with diverse approaches and perspectives is the best way to avoid narrow-mindedness and to create possibilities for growth. As a consequence we at the *Freudenthal Institute* welcomed the idea to organize the PME conference in The Netherlands in 2001. A meeting with the PME community is a guarantee for a rich variety of mathematical research domains, educational levels of mathematics education and types of research, and, last but least, including different views on the learning and teaching of mathematics.

There is still little agreement on what is the best view. Scientific research seems to support quite different approaches. Therefore it is no wonder that, today, the scientific quality of research is questioned and issues of methodology are discussed. PME is the forum par excellence to ask these questions and to discuss these issues. In this respect it is interesting to see how up-to-date the words are that Freudenthal used in his opening address to PME1. He told the audience to

> *"[l]ook and listen with an open mind and have the courage to notice and to report events that most people would consider as too silly to be noticed and to be reported—there will be a minority who can appreciate them, and this minority will be right."*

I think that we should interpretate his words as a warning against blindly following trends and as a stimulance of researchers' own responsibility. May this conference contribute to this.

The papers in the four volumes of the proceedings are grouped according to types of presentations: Plenary Lectures, Plenary Panel, Research Forums, Discussion Groups, Working Sessions, Short Oral Communications, Posters, and Research Reports. The plenary addresses and the research forum papers appear according to the order of presentation. The papers of the group activities are sequenced according to their number code. For the other types of presentations, within each group the papers are sequenced alphabetically by the name of the first author, with the name(s) of the presenting author(s) underlined.

There are two cross-references to help readers identify papers of interest to them:

- by research domain, according to the first author (p. 1-liii)
- by author, in the list of authors (p. 1-425).

I wish to express my appreciation to all the people who took part in the production of these proceedings. First of all, I thank all the authors of the papers who contributed to the richness and the scientific merits of the proceedings. Furthermore, I extend my thanks to the reviewers and the members of the Program Committee for their respectives roles in working with the papers. After they finished their job it was hard work to get the four volumes ready for the publisher. Several people were involved in this, but I am particularly indebted to Marianne Moonen for her dedication and time devoted to the preparation of the proceedings.

Although the increase of numbers of papers brought us more work than was expected, we could not resist the temptation to go digital. So for the first time in PME history, the PME Proceedings are available on cd-rom. Thanks to the team that succeeded in getting this product finished in time.

This conference received support from several sources, without which we could not have organized it to meet PME standards. We are grateful to the University Utrecht—and in particular to its "365 YEAR KNOWLEDGE @ LA CARTE" program—and to the Royal Netherlands Academy of Arts and Sciences (KNAW) for their support and facilities.

Last, but not least, many thanks to the members of the Local Organizing Committee for their willingness to share with me the responsibilities involved in this enterprise.

Marja van den Heuvel-Panhuizen
Utrecht, July 2001

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INTRODUCTION

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976. Its past presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada) and Stephen Lerman (UK).

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics educators
- To further a deeper understanding into the psychological aspects of teaching and learning mathematics and the implications thereof.

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11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128(vol.1), ED411129(vol.2)
13	1989	Paris, France	ED411140(vol.1), ED411141(vol.2), ED411142(vol.3)
14	1990	Oaxtepex, Mexico	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)
15	1991	Assisi, Italy	ED413162(vol.1), ED413163(vol.2), ED413 164(vol.3)
16	1992	Durham, USA	ED383 538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon, Portugal	ED383 537
19	1995	Recife, Brazil	ED411134(vol.1), ED411135(vol.2), ED411136(vol.3)
20	1996	Valencia, Spain	Being processed
21	1997	Lahti,Finland	ED416082(vol.1), ED416083(vol.2), ED416084(vol.3), ED416085(vol.4)
22	1998	Stellenbosch, South Africa	ED427969(vol.1), ED427970(vol.2), ED427971(vol.3), ED427972(vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	Being processed

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, California (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montréal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montréal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132(vol.1), ED411 133(vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137(vol.1), ED411138(vol.2), ED411139(vol.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	California	ED372917
16	1994	Louisiana	ED383533(vol.1), ED383534(vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494(vol.1), ED420495(vol.2)
20	1998	Raleigh, North Carolina	ED430775(vol.1), ED430776(vol.2)
21	1999	Mexico	ED433998
22	2000	Tucson, Arizona	Being Processed

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THE REVIEW PROCESS OF PME25

Research Forum

Five Research Forums had been suggested by the International Committee and the Program Committee for PME25: (1) Potential and pitfalls of techology tools in learning mathematics, (2) Early algebra, (3) Comparative views of mathematics goals and achievements, (4) Designing, researching and implementing mathematical learning environments — the research group "Mathe 2000", (5) Realistic mathematics education: Leen Streefland's work continues. For each Research Forum the proposed structure, the contents, the contributors and their role were reviewed and agreed by the Program Committee.

Research Reports

The Program Committee received 272 Research Report proposals. Each proposal was sent for blind review to three reviewers. As a rule, proposals with at least two recommendations for acceptance were accepted. The reviews of proposals with only one recommendation for acceptance were carefully read by at least two members of the Program Committee. When necessary, the Program Committee members read the full proposal and formally reviewed it. Proposals with three recommendations for rejection were not considered for presentation as research reports. Altogether, 171 research report proposals were accepted. When appropriate, authors of proposals that were not accepted as Research Reports were invited to re-submit their work — some in the form of a Short Oral Communication and some as a Poster Presentation.

Short Oral Communications and Poster Presentations

The Program Committee received 104 Short Oral Communication proposals and 31 Poster Presentation proposals. Each proposal was reviewed by at least two Program Committee members. From these proposals 89 Short Oral Communications and 28 Poster Presentations were accepted. Some authors of Short Oral Communication proposals were invited to re-submit their work as a Poster Presentation. Altogether, 113 Short Oral Communications and 43 Poster Presentations were accepted.

Overview Results

	RR	SO	PP
proposals	**272**	**104**	**31**
rejected proposals	70	7	3
rejected but SO	24		
rejected but PP	7	8	
suggested re-submission from RR		24	7
suggested re- submission from SO			8
accepted proposals	171 (1)*	113 = 89 (5) + 24 (8)	43 = 28 + 7 (3) + 8
accepted but cancelled	1	13	3
total number of presentations	**170**	**100**	**40**

* Between parentheses is the number of cancellations

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The PME23 Program Committee thanks the following people for their help in the review process:

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THE P IN PME: PROGRESS AND PROBLEMS

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A problem of math education in the real world

'Between a hundred and two hundred million students does not get any kind of math education'

'The statistice show that the number of illiterate citizens is rising and that in the countries of the MENA region 50% of the women will be illiterate in 2000'.

Obstacles in solving this problem

The philosophical gap:
' The question is not how we can improve the quality of education, but how we can improve the quality of formal education'.

The parrot effect:
Developing countries should look at and copy the curricula and tests from western countries.

The political vs. content gap:
Prerequisites for improving education according to World Bank: Lower the decifits; Let the GNP grow; Look for changes in the structure.

The gap between research and practice:
'Bent Christiansen would not have been comfortable in an english or american math education department. Reading, thinking, reflecting and discussing new ideas was more appriciated than the production of scientific articles and starting a production line for Ph.D's.'

The problem of mathematics in relation to math education

What is mathematics and what does this mean for math education?

Mathematics has evolved from the science of number, into the science of number and shape, into the science of number, shape and change into, at this moment, the science of patterns.

Mathematics education does not seem to reflect this development: another gap.

Mathematics and math education are under heavy pressure from society: why is mathematics important as it does not seem very relevant to the needs of society.

Modern mathematics was the mathematicians mathematics. What we want: the students mathematics. The content of students mathematics depend on the perception of the discipline, the role mathematics plays in society, the broadness of definition, the relation with other subjects, the goals for math education. The way we teach mathematics depend on many things-one of them the resutlts from research.

The Problem: The science of math education

In 1978 Freudenthal wrote the book: 'Preface to a Science of Mathematics Education'. The function was: To accelerate the birth of a science of mathematical education, which is seriously impeded by the unfounded view that such already exists.

That was almost 25 years ago. Where are we now?

'There is no agreement among leaders in the field about goals of research, important questions, objects of study, methods of investigation, criteria for evaluation, significant rtesults, major theories, or usefullness of results… a field in disarray.'

'Results obtained by different research schools are very difficult to compare and researchers prefer to stay within a strictly homogeneous reference system'.

Math education as a science is under pressure: University students close schools of education; colleagues find most research papers uninteresting (they deal with marginal details) and teachers cannot synthesize results into useful forms'.

The problem: gap between research and practice

In 1980 Freudenthal stated as one of the main problems of mathematics education: 'How to design educational development as a strategy for change?'

'Failure is the only possible outcome for any approach in which researchers hand their results to curriculum developers who are than expected to apply them in their practices'

'Wouldn't it be helpful to let outstanding teachers co-determine the selction and granting of research projects?'

The problem: the results of the practice of math education

TIMSS showes clearly that the actual outcome of at least part of the results of math education in a large number of countries leaves much to desire.

TIMSS also identifies some of the causes of the relatively disappointing results.

PISA tries to identify the functionality of math education for 15 year olds-seen form the perspective of mathematical litteracy. And although most western countries do have excellent facilties: books, paper, pencils, computers etc. the mathematical litteracy leaves a lot to be desired.

We should take more responsibility. We should adjust our agenda's.

We have made progress: if it were just by identifying clearly the problems as just mentioned. But we have not used the combined talents of all of us interested in math education. We have tried too much to build a research discipline. We should try to develop better math education. A research disciplin will be the result.

LINKING HOME AND SCHOOL MATHEMATICS

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Abstract

In this paper I want to address what I believe to be a fundamental educational problem – namely, that there is an important difference between the kinds of learning in which children engage at school, and the kinds of learning in which they engage outside school, particularly at home. This problem is closely connected with the well-known difficulties that learners experience with the application of knowledge from one context to another, particularly in the area of mathematics. In this paper I will look at two main ways in which home and school learning might be brought together in the area of mathematics. First, I look at approaches which import 'authentic' out-of-school problems into the mathematics classroom, and secondly, at those which export school mathematics problems into the home. I conclude that neither approach is sufficient to break down the barriers between home and school learning, and that more radical solutions are needed.

Summary

There is growing evidence that there are fundamental differences between the kinds of learning in which children engage at school and the kinds of learning in which they engage outside school, particularly at home. School learning, for the most part, is shaped by the curriculum, regularly assessed, strictly timetabled, and focused around artificial problems. It usually takes place in large horizontal age groups, with adults as instructors, and provides few opportunities for children themselves to act as teachers. Home learning, in contrast, is shaped by interest and need, rarely assessed, usually spontaneous, and focused around authentic problems. It usually takes place in small vertical age groups, with adults as models or guides, and provides substantial opportunities for children themselves to act as teachers.

This distinction between home and school learning can usefully be related to other distinctions within different theoretical perspectives. We can, for example, see it as an instance of the well-known distinction between 'formal' and 'informal' learning (eg Coffield, 2000). We can relate it to the contrast that Donaldson (1978, 1990) makes between 'embedded' and 'disembedded' thinking. Or we can link it to the distinction between 'cognitive' and 'situated' theories of learning, widely discussed in recent issues of the AERA journal Educational Researcher, and to the metaphors of 'acquisition' and 'participation' underlying these two different theoretical approaches (eg Sfard, 1998).

Whatever theoretical perspective is adopted, I will argue that the difference between home and school learning is closely related to the widespread problem of application – namely, that knowledge acquired in one context is frequently not available or not used in another context. While this problem is widespread across many subjects, it seems to be particularly acute in mathematics. Here there is considerable evidence that mathematical knowledge acquired in school does not readily transfer to out-of-school contexts (eg Hughes, 1986). At the same time, there is also evidence that mathematics knowledge acquired out-of-school does not readily transfer to school type problems (eg Nunes, Schliemann and Carraher, 1993).

In the rest of my paper I will look at ways in which we might address this problem. I will argue that there are two main approaches to linking home and school mathematics, and illustrate these approaches, and the issues they raise, with examples and evidence from my own recent research.

One attempt to make connections between home and school mathematics is by importing examples of 'out-of-school' problems into the classroom. This of course is a well-known approach widely used by mathematics educators. However, while this approach can often increase students' motivation to engage with classroom mathematics, it does not necessarily help break down the barrier between school maths and out-of-school maths. In my paper I will illustrate this approach, and the issues it raises, with examples drawn from a recent project in which we worked closely with primary school teachers trying to develop their practice (Hughes, Desforges and Mitchell, 2000).

The other main attempt to make connections between home and school is by sending home aspects of school mathematics. In many countries, this is traditionally done through 'homework', where students are required to work at home on problems related to the school curriculum. While homework provides opportunities for students to make links between school learning and out-of-school learning, our own research suggests that this does not often happen (Hughes and Greenhough, 2001). Instead homework becomes simply a piece of school work which happens to be done at home.

In the UK, there has been much interest in recent years in initiatives that attempt to go beyond traditional homework and involve parents more directly in their children's learning. One particularly well known scheme is IMPACT Maths (eg Merttens and Leather, 1993), whereby mathematical puzzles and games are sent home for children to carry out with their parents. In a recent study we looked systematically at what happens when children engage in a typical IMPACT activity with their parents, and compared this with what happens when children carry out the same activity with their teachers. Our findings suggest there are important differences between the ways in which parents and teachers help children, and that these may serve to accentuate rather than reduce the differences between home and school.

In the final section of my paper I will argue that the approaches described above, while not without merit, are not by themselves adequate to break down the barriers between home and school learning. A more radical approach is needed, one which gives greater recognition to the mathematical practices already occurring in children's homes and attempt to link these more directly with the practices of school.

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EXPLANATION, JUSTIFICATION AND ARGUMENTATION IN MATHEMATICS CLASSROOMS

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Abstract: *Current interest in mathematics learning that focuses on understanding, mathematical reasoning and meaning-making underscores the need to develop ways of analyzing classrooms that foster these types of learning. In this paper, I show that the constructs of social and sociomathematical norms, which grew out of taking a symbolic interactionist perspective, and Toulmin's scheme of argumentation, as elaborated for mathematics education by Krummheuer, provide us with a means to analyze aspects of explanation, justification and argumentation in mathematics classrooms, including means through which they may be fostered. In addition, I use the example of current research in a university-level differential equations class to show how these notions can inform instruction in higher-level mathematics.*

In his plenary address to PME in 1994, John Mason said,

> I have long concluded that it is very hard to say anything new that has not been said more eloquently elsewhere.... I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self discovery and development in which what others have learned has to be re-experienced by each traveller, re-learned re-integrated and re-expressed in each generation. (p. 177)

He went on to say that "All you can do, if you really want to be truthful is to tell a story" (p. 177).

My purpose in this paper is to tell a story that intends to capture something of what I have experienced and learned about explanation, justification and argumentation from a variety of mathematics classrooms that I have studied. At the same time, I intend to explain why I am telling you this story—what significance this story might have for someone else. In this regard, I am following the approach that Streefland (1993) took when he said that by analyzing his experiences, in his case in instructional design, his goal was to take what was an *after*-image for him and make it possible for that to become a *pre*-image for someone else working in the same arena. Similarly, my intention is to specify various aspects of explanation, justification and argumentation that have emerged from my analyses of classrooms so it becomes possible for others to use these aspects to inform their future efforts in mathematics education research and practice.

Background

The research activity that my colleagues and I have been engaged in is classroom based and involves conducting classroom teaching experiments that range from six weeks to an entire school year or university term. It involves developing instructional sequences and approaches as well as investigating teaching and learning as it occurs in the classroom. In this type of research, called developmental or design research (Cobb, Stephan, McClain, & Gravemeijer, in press; Gravemeijer, 1994), the researchers conduct ongoing analyses of classroom activity and use the results to inform instructional planning and decision making. It also involves retrospective analyses that attempt to explain the nature of the learning that took place and to explicate significant aspects of the learning situation.

Through analyzing these teaching experiments, we[1] have come to understand the importance of taking into account the social aspects of learning, including social interaction (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990; Yackel & Rasmussen, in press; Yackel, Rasmussen, & King, 2001). We have developed an interpretive framework for analyzing classrooms and we have explicated theoretical constructs within that framework in terms of our experiential base (Cobb & Yackel, 1996; Yackel & Cobb, 1996). Two constructs that are particularly relevant to issues of explanation, justification and argumentation are social norms and sociomathematical norms. Our investigations have pointed to the importance of these two constructs in clarifying both the functions that explanation, justification and argumentation serve and the means by which they might be fostered in the classroom. Further, the constructs of social and sociomathematical norms have been useful in clarifying how we might think of explanation, justification and argumentation in mathematics classrooms where understanding and meaning-making are the focus of instruction. In addition, in previous work, I have documented that children as young as second grade engage in sophisticated forms of explanation and justification and that their understanding of explanation advances as the school year progresses. Finally, I have argued (Yackel, 1997) that we can use Toulmin's argumentation scheme, as elaborated for mathematics education by Krummheuer (1995), as a methodological tool to demonstrate how learning progresses in a classroom.

All of these ideas were originally developed as a result of research in elementary school classrooms. A legitimate question is whether or not any of these results, including such constructs as social and sociomathematical norms, are useful in analyzing higher-level (including university-level) mathematics instruction in which the constraints are (or are perceived to be) different from those in the elementary grades. This question is particularly relevant since the amount of research at higher levels of mathematics instruction remains relatively small.

In this paper, I first discuss symbolic interactionism as a theoretical framework. Next, I explain what I mean by explanation and justification and discuss normative aspects of mathematics classrooms relative to explanation and justification. Then I describe how Toulmin's argumentation scheme can be used as a methodological tool to document learning in the classroom. I then discuss how this approach to argumentation can be used to explicate the teacher's role in the classroom. Finally, I show how these ideas, which are after-images of research conducted in mathematics classrooms at the elementary-school level, are being used as pre-images to inform current classroom-based research in a university-level mathematics class at my university. In doing so, I demonstrate their utility at the upper levels of mathematics learning. At the same time, the crucial role of explanation, justification and argumentation in mathematics learning from the lower to the higher levels of mathematics becomes apparent. Thus, I answer in the affirmative a question that was raised by Duval (2000) in his plenary lecture to PME last year when he asked "[I]s there something similar in the process of mathematics learning at the first levels and at upper levels?"

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[1] In using *we* I am referring to various colleagues that I have worked with to conduct and analyze teaching experiments over the past decade and a half including Paul Cobb, Terry Wood, Koeno Gravemeijer, Diana Underwood Gregg, Chris Rasmussen, and Karen King.

Theoretical Framework

In previous attempts to investigate explanation, justification and argumentation in mathematics classrooms, we have found it useful to take a symbolic interactionist perspective. Of the various approaches to social interaction, we have taken symbolic interaction as a theoretical lens for two reasons. First, it is compatible with psychological constructivism, which forms the theoretical basis for our investigation of individual learning (Cobb & Bauersfeld, 1995). Second, as Voigt (1996) points out, the symbolic interactionist approach is particularly useful when studying students' learning in inquiry mathematics classrooms[2] because it emphasizes both the individual's sense-making processes and the social processes without giving primacy to either one. Thus, we do not attempt to deduce an individual's learning from social processes or vice versa. Instead, individuals are seen to develop personal meanings as they participate in the ongoing negotiation of classroom norms.

The theory of symbolic interactionism has its roots in the work of George Herbert Mead, John Dewey and others and has been developed extensively by Herbert Blumer (1969). One of its defining principles is the centrality given to the process of interpretation in interaction. To put it another way, the position taken by symbolic interactionism is that in interacting with one another, individuals have to take account of (interpret) what the other is doing or about to do. Each person's actions are formed, in part, as she changes, abandons, retains, or revises her plans based on the actions of others. In this sense, social interaction is a process that *forms* human conduct rather than simply a setting in which human conduct takes place. As Blumer (1969) stated, "One has to *fit* one's own line of activity in some manner to the actions of others. The actions of others have to be taken into account and cannot be merely an arena for the expression of what one is disposed to do or sets out to do" (p. 8). Blumer further clarified that the term *symbolic* interactionism refers to the fact that the interaction of interest involves interpretation of action. Attempts to genuinely communicate involve understanding the meanings of another's actions (Rommetveit, 1985), and so involve symbolic interaction.

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[2] I follow Richards (1991) in using the label *inquiry* mathematics classrooms to describe those classrooms in which students engage in genuine mathematical discussions with each other and with the teacher. See Yackel (2000) for a detailed discussion of the social and sociomathematical norms that characterize inquiry mathematics classrooms.

In addition to interpreting actions of others, individuals engaged in interaction attempt to indicate to others, through their actions, what their own intentions are. Thus, actions have meanings both for the person making them and for the person(s) to whom the action is directed. In this sense there is a joint action that arises by the articulation of the participating actors' activity. Blumer (1969) emphasized the collective nature of such joint action as follows:

> A joint action, while made up of diverse component acts that enter into its formation, is different from any one of them and from their mere aggregation. The joint action has a distinctive character in its own right, a character that lies in the articulation or linkage as apart from what may be articulated or linked. Thus, the joint action may be identified as such and may be spoken of and handled without having to break it down into the separate acts that comprise it. (p. 17)

Blumer further pointed out that it is important to recognize that "the joint action of the collective is an interlinkage of the separate acts of the participants" (p. 17). As such, it has to undergo a process of formation and, even though it may be well established as a form of social action, each instance of it has to be formed once again. Consequently, the meanings and interpretations that underlie joint action are continually subject to challenge. As a result, both individual actions and the joint (collective) action of a group can change over time. Furthermore, this view of joint action supports the position that social rules, norms and values are upheld by a process of social interaction and not the other way around.

A second defining principle of symbolic interactionism, in addition to the centrality of interpretation, is that meaning is seen as a social product. Blumer (1969) elaborated this point as follows:

> It [symbolic interactionism] does not regard meaning as emanating from the intrinsic makeup of the thing that has meaning, nor does it see meaning as arising through a coalescence of psychological elements in the person. Instead, it sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate to define the thing for the person. Thus, symbolic interactionism sees meaning as social products, as creations that are formed in and through the defining activities of people as they interact. (pp. 4,5)

This view of meaning has important implications for how we interpret the results of classroom discursive activity. Since meanings grow out of social interaction, each individual's personal meanings and understandings are formed in and through the process of interpreting that interaction. Nevertheless, normative understandings are constituted. It is as these normative understandings are constituted that students develop their own interpretations of them. What we mean by saying that these understandings are normative is that there is evidence from classroom dialogue and activity that students' interpretations are compatible. It is in this sense that we say that students' interpretations or meanings are "taken-as-shared" (Cobb, Wood, Yackel, & McNeal, 1992).

Explanation and Justification

In this paper my interest is in explanation and justification as social constructs rather than as individual constructs. In this case, they are considered to be aspects of the discourse that serve communicative functions and are interactively constituted by the teacher and the students. Explanation and justification are distinguished, in part, by the functions they serve. Students and the teacher give mathematical explanations to clarify aspects of their mathematical thinking that they think might not be readily apparent to others. They give mathematical justifications in response to challenges to apparent violations of normative mathematical activity (Cobb et al., 1992). For example, consider the task "How can you figure out 16 + 8 + 14?" If a child responds, "I took one from the 16 and added it to the 14 to get 15 and 15; then I added the 15 and 15 to get 30, and the other 8 to get 38," we would infer that she was explaining her solution to others. However, a challenge that "you first have to add the 16 and the 8 and then add 14 to that sum" is a request for a justification.

Classroom norms that relate to mathematical explanation and justification are both social and sociomathematical in nature. Norm is a sociological construct and refers to understandings or interpretations that become normative or taken-as-shared by the group. Thus, norm is not an individual but a collective notion. One way to describe norms, in our case, classroom norms, is to describe the expectations and obligations that are constituted in the classroom.

By analyzing data from our initial teaching experiments, we were able to identify a number of social norms that characterized classroom interactions. These include that students are expected to develop personally-meaningful solutions to problems, to explain and justify their thinking and solutions, to listen to and attempt to make sense of each other's interpretations of and solutions to problems, and to ask questions and raise challenges in situations of misunderstanding or disagreement. In saying that these social norms characterized the classroom interactions, I mean that these ways of acting and of interpreting the actions of others became taken-as-shared. In subsequent classroom teaching experiments we had the constitution of these norms as an explicit goal. It is evident that each of these norms relates specifically to explanation and justification when taken as social constructs, as described above.

We were also able to identify normative aspects of interactions that are specific to mathematics. These we called sociomathematical norms (Yackel & Cobb, 1996). Normative understandings of what counts as mathematically different, sophisticated, efficient and elegant are examples of sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm. The distinction between social norms and sociomathematical norms is subtle. For example, the understanding *that* students are expected to explain their solutions is a social norm, whereas the understanding of *what counts* as an acceptable mathematical explanation is a sociomathematical norm.

We might ask how notions such as what counts as an acceptable explanation come to have meaning for students. To answer this question, we return to the symbolic interactionist position on meaning. This position is that meaning arises through interaction. Accordingly, the meaning of acceptable mathematical explanation is not something that can be outlined in advance for students to "apply." Instead, it is formed in and through the interactions of the participants in the classroom. As with all normative understandings, both explicit and implicit negotiations contribute to developing these understandings.

Elsewhere, I have documented second-grade children's evolving understanding of mathematical explanation and justification (Yackel, 1992; Yackel & Cobb, 1996). That analysis showed that initially children had to learn that their explanations and justifications needed a mathematical, rather than a social, basis. In that instance, the teacher initiated explicit discussions with the children about the basis for their explanations. For example, early in the school year a pupil changed her answer when the teacher asked the class if they agreed with her. In the subsequent discussion she revealed that she had interpreted the teacher's question as indicating that she had made an error. The teacher then used this as a paradigm case to discuss his expectation that students' answers should be based on mathematical reasoning with their explanations reflecting that reasoning. As the school year progressed students' explanations increasingly took on the character of descriptions of actions on objects that were experientially real for them. This expectation was both constituted and sustained, in part, through challenges that the students and the teacher made to explanations that described procedures. For example, when adding quantities such as 13 and 12 to get 25, explanations such as, "One and 1 are 2 and 3 and 2 are 5," were challenged by remarks such as, "That's a ten and that's another ten, and that's 20. And the answer is 25." Finally, by the end of the school year, some students took explanations as explicit objects of reflection and made comments such as, "How can someone understand what you mean? They don't know what you're referring to." The students making these challenge were taking the explanations as entities in and of themselves and were commenting on their overall potential as acts of communication within the classroom discourse. As with social norms, in subsequent teaching experiments we had the constitution of specific sociomathematical norms as an explicit goal, for example, that explanations should describe actions on objects that are experientially real for the students.

Argumentation

As noted earlier, my interest is in mathematical explanation and justification as interactional accomplishments and not as logical arguments. The focus is on what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered mathematically valid. In this sense my interest is in what Toulmin (1969) calls substantial rather than logical argument.

Krummheuer's work on argumentation (1995) provides the background for the approach I take. In his study of the ethnology of argumentation, Krummheuer analyzes argumentation using Toulmin's scheme of conclusion, data, warrant, and backing. According to this scheme, the conclusion is a statement that is made as though it is certain. It is a claim. The support one might give for the conclusion is the data. Warrant refers to the rationale that might be given to explain why the data are considered to provide support for the conclusion. Backing provides further support for the warrant, that is, the backing indicates why the warrant should be accepted as having authority (Toulmin, 1969). Krummheuer uses the notions of conclusions, data, warrants, and backing to explicate how argumentation is an interactive constitution of the participants. For him, an argumentation in any given situation "contains several statements that are related to each other in a specific way and that by this take over certain functions for their interactional effectiveness" (Krummheuer, 1995, p. 247). Statements do not have a function apart from the interaction in which they are situated. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact.

I have demonstrated previously (Yackel, 1997) that this approach to argumentation is useful as a methodological tool for documenting the collective learning of a class because it provides a way to demonstrate changes that take place over time. Further it helps to clarify the relationship between the individual and the collective, that is, between the explanations and justifications that individual students give in specific instances and the classroom mathematical practices[3] that become taken-as-shared. As mathematical practices become taken-as-shared in the classroom, they are beyond justification and, hence, what is required as data, warrants and backing evolves. Similarly, the types of rationales that are given as data, warrants, and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community. For example, in one second-grade classroom, explanations of solution methods that involved using thinking strategies became more cryptic over time. For a problem such as $5 + 6 = \_\_$, students initially gave explanations such as, "I know that 5 and 5 is 10 and 6 is just one more; it's just one more on the 5, so the answer is 11; one more than 10." Later a typical explanation was "5 and 5 is 10 so its 11." In this case, when explanations of the second type are no longer questioned, we can assume that the warrant and backing provided in the earlier type of explanation are now taken-as-shared.[4] At the same time, it is through explanations of the earlier type that the taken-as-shared understandings develop.

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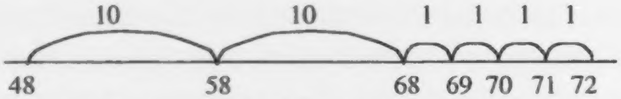
[3] The construct of *classroom mathematical practices* refers to the collective development of the classroom community. See Cobb (1998) for a discussion of this construct and for an illustrative example.

[4] It could be argued that students might not question this type of solution because they are disinterested or simply do not want to ask a question. We can rule out these possibilities in this case because of the classroom social norms that were

Finally, this approach to argumentation is useful for analyzing classroom dialogue to investigate the nature of the contributions made by various participants in the interaction. In particular, our analyses of second-grade classrooms show that the teacher frequently serves the role of calling for or herself making explicit data, warrants, and backing that may be only implicit in the explanations and justifications that students give (Yackel, 1997). Here again, I give an example from a second-grade class. In explaining how he figured out how much to add to 48 to get 72, Louis said, "It's 24. How I got it was I said, 48, 58, 68, 69, 70, 71, 72." The teacher was aware that approximately half of the children in the class did not yet know that the difference between 48 and 58 is 10. Therefore, the data that Louis gave to support his conclusion would not have explanatory relevance for them. A warrant would be needed. Without waiting for anyone to ask a question, she asked Louis to elaborate by saying, "You said 48, 58, how many is that?" Using the language of argumentation, we would say that the teacher was prompting Louis to give a warrant. After Louis responded "Ten," she continued, "And then you went to 68, and how many was that?" Louis replied, "Twenty." The teacher went on to say, "That was 20 in all and then you went by ones didn't you? You went 69, 70, 71, 72. ... So you went a ten and another ten and four ones and that's how you got 24." In this episode we see the teacher working together with Louis to develop the needed warrant. In addition, as she and Louis were developing the verbal warrant, the teacher drew the following diagram on the chalkboard.



This action of drawing the diagram linked Louis' solution to the empty number line,[5] a pedagogical notation that was being used regularly in the class at that time. In general, the overall intention was to use the empty number line as a way to notate and record students' thinking with the goal that it might later become a tool for students' reasoning. In this case, the numerals written below the line recorded the information that Louis gave as data while the arcs and numerals written above the line recorded the additional information supplied in the warrant. In this sense, the diagram took on the form of a record of an argumentation because it included a record of both the data and the warrant.

In this example, the teacher initiated both the diagram and the verbal elaboration of Louis' solution. Her actions accomplished several goals. One was to call attention to the argumentative support for the conclusion. Second, she contributed to the class' understanding of what is taken as argumentative support. Earlier, I noted that what

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operative. In this class students assumed the obligation of asking questions when they disagreed or did not understand. It is for this reason that we are able to interpret lack of questions as indicating taken-as-sharedness.

[5] The *empty* number line has no units marked on it. Therefore, unlike the (marked) number line that is often used in elementary school classrooms as a means for computing answers, the empty number line cannot be used for computing directly. Its power comes from the imagery that students use when they think about moving forward and backward from one number to another. Information about the distance between two numbers cannot be ascertained directly from the empty number line.

constitutes data, warrant, and backing is not predetermined but is negotiated by the participants as they interact. In this case, Louis' initial comments served as data for him. The teacher was aware that the only way these could serve as data for certain other students in the class was for additional information to be given, information that explained the relevance of Louis' comments. Finally, the teacher contributed to the goal that the empty number line might become a tool for reasoning by linking the data and warrant to the number line notation.

Explanation, Justification and Argumentation in a Differential Equations Class

In this section of the paper, I use the example of a differential equations class to demonstrate how the notions of explanation, justification and argumentation can be used to analyze the nature of the activity in a university-level mathematics class.

Data and Method

Over the past four years my colleague, Chris Rasmussen, has been investigating the teaching and learning of differential equations using his own classes as sites for investigation.[6, 7] During this time he has conducted classroom teaching experiments for all or part of the semester on three separate occasions. In each case, I have been an observer in the classroom and have participated in regular project meetings to discuss students' ways of reasoning, potential instructional activities and social aspects of the classroom.

Data from each teaching experiment consist of videotapes of each class session, including the small group work of two groups, field notes made by the observer(s) and the instructor, copies of students' work, and records of instructional activities and instructional decisions. Students' work includes in-class work, homework assignments, weekly electronic journal entries and reflective portfolios that students submitted twice in the semester. In addition, data include videotapes and artifacts from individual student interviews that were conducted with selected students at various times throughout each teaching experiment to gain initial information about their concepts related to functions and rates of change, to assess their understanding of key concepts of the course, and to inquire about their beliefs about mathematical activity and mathematics learning.

Social and Sociomathematical Norms Relating to Explanation and Justification

A specific goal in each teaching experiment was to constitute social and sociomathematical norms characteristic of inquiry instruction. Regarding explanation and justification, these include that students explain and justify their thinking, that they listen to and attempt to make sense of the explanations of others, and that explanations describe actions on objects that are experientially real for

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[6] Students in the differential equations classes at this university are typically engineering or mathematics majors. The typical class size is from 25 to 35 students.

[7] Rasmussen's work is supported by the National Science Foundation under grant number REC-9875388. I wish to thank him for permission to use episodes from his classes for this paper and to thank Jennifer Olszewski for making transcripts of selected episodes from videotapes.

them. Analysis of data from the first teaching experiment shows both *that* these norms were constituted and *how* the instructor initiated the negotiation of these norms (Yackel & Rasmussen, in press; Yackel, Rasmussen, & King, 2001).

The following example, taken from the second class session of the first teaching experiment, illustrates the constitution of social norms that foster explanation. The instructor began the class with a brief statement of the expectations he had for the students' mathematical activity. Then he orchestrated a whole class discussion of approximately twenty minutes in which he and the students discussed the rationale behind a differential equation they had used in the prior class session to indicate the rate of change of the recovered population in an infectious disease situation. The crucial aspect of this segment for purposes of this paper is the explicit attention the instructor gave throughout to the negotiation of social norms. Most of his comments were explicitly or implicitly directed toward expectations. For example, he said things such as:

> *Okay, can you explain to us then why it was 1/14 times I?*
> *What do the rest of the people think about that?*
> *Is that similar to what you were thinking?*
> *Anyone want to add to that explanation? Expand on it a little bit?*
> *So let's put that question out. So your question is ... Is that what I heard you say?*

In making these remarks, the instructor was attempting to influence the interpretations students made of how to engage in the discussion. From this perspective, it might seem that the teacher is the only one who contributes to the renegotiation of social norms. However, norms are interactively constituted as individuals participate in interaction. In this case, as the episode evolved, students contributed to the negotiation of the norms by increasingly acting in accordance with the expectations. As the discussion progressed, students not only responded to the instructor's questions, they initiated comments that showed that they were beginning to change their understanding of the classroom participation structure. For example, a few minutes into the discussion one student said, "I didn't quite understand what he [another student] said" and a few seconds later explained what he did understand and said, "What I don't understand, what I was asking about ..."

As the episode continued, several students asked questions, offered explanations, and asked for elaboration and clarification. In doing so, they too were contributing to the ongoing constitution of the social norms that students are expected to explain their thinking to others and ask questions and raises challenges when they do not understand. We find it encouraging that as early as the second class session understandings such as these were becoming normative. Further, analyses of classroom interactions throughout the semester provide evidence that these norms became well established. It became routine for students to explain their thinking, to ask questions and raise challenges, and to elaborate their explanations and justifications spontaneously, without prompting from the instructor. Furthermore, although we have not provided examples here, it became normative that explanations were about actions on (mathematical) objects that were experientially real for the students.[8] Thus, we would say that the class could be characterized as following an inquiry mathematics tradition. Classroom observations verify similar results in the subsequent teaching experiments.

Argumentation

When I observed Rasmussen's class in the spring of 2001 it was immediately apparent that his instructional approach had evolved to have an even stronger emphasis on justification and argumentation than previously observed. In addition, he had developed a unique approach to collaborative learning. Further, these two aspects were intimately intertwined.

In a typical collaborative learning situation the instructor poses a task or problem, students work in groups and after some time, often sufficient for most groups to complete the task, the class engages in a discussion of the solution methods students developed in their small groups. This is, in fact, the approach that Rasmussen took in the first two differential equations teaching experiments. The dramatic difference I noticed in this most recent class was that groups often worked for little more than two minutes before sharing their thinking. Whole class discussion might then continue for as much as 15 minutes before another short segment of two to four minutes of small group work took place. Typically this cycle was repeated four to five times in an 80-minute class period. Furthermore, the students' task during small group work was typically to "think about" some question or issue rather than to solve a specific problem. Because of the continual emphasis on reasoning, whole class discussions resulted in the *emergence* of key concepts, including slope fields, bifurcation diagrams and phase planes. In this sense, the instructional approach seems to have considerable potential for in-depth conceptual development that grows out of students' discursive activity.

The following example, which is taken from the second session of the 2001 class, is used to clarify and illustrate the approach and to explain how it contributed to the emphasis on argumentation. During the 80-minute class session, there were four small group segments of two to four minutes each, interspersed between five whole class discussions. The first 20 minutes of the class was devoted to continuing a discussion from the previous class about a predator-prey situation described by a pair of differential equations. Students did not yet have any analytic techniques for recovering the solutions to the differential equations but used informal and qualitative reasoning to "make sense" of the situation. Throughout the discussion the instructor repeatedly asked the students for the reasons for their claims. When students gave reasons, he did not evaluate their validity but instead asked other students what they thought and solicited other arguments. For example, he made remarks such as, "What do you think of his idea?" and "Okay, that's nice. Let me hear another argument for ..." From the point of view of norms, we would say that the instructor was initiating the expectations that students are to provide arguments to justify their claims, that they are to attempt to make sense of others' arguments, and that there might be more than one argument to support a given claim. From the point of view of argumentation, we would say that the discussion focused on data, warrants, and backing to support conclusions but not on the conclusions themselves.

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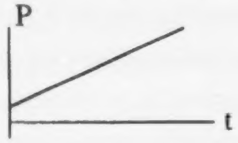
[8] I have limited the discussion to social norms for explanation. Similar analyses for sociomathematical norms have been conducted but space does not permit including them here. See Yackel and Rasmussen (in press) and Yackel, Rasmussen and King (2001) for a discussion of sociomathematical norms.

The instructor then posed a simplified version of the predator-prey problem that involved only one species, in this case fish in a lake, with the assumptions that there were unlimited resources, no competitors, and there was a continuous growth rate. With no additional information, except that the initial population was 10 units, students were instructed to "take just a few minutes, sketch for yourself and talk about it with the folks in your group what your graphs [for the number of fish over time] look like and the reasons for why you think the graphs look like that."

Most students quickly drew either a straight line graph with positive slope or an exponential graph, beginning on the vertical axis at a height of 10, and made remarks to their peers such as "I thought it looks something like an increasing exponential curve" and "They're unbounded so it's just gonna continue to grow." After less than two minutes the instructor called the class back together.

Instructor: All right. Let me ask you a question. ... Suppose you were in a group and that's the way that someone offered. (The instructor draws the graph shown below on the chalkboard.) What would you—would you agree or disagree and what reasons would you give for agreeing or disagreeing with that?



In the ensuing discussion several students offered reasons why they disagreed or agreed with a linear graph. Some of the reasons given were directly related to thinking about the scenario, for example, that fish lay eggs and typically the number of eggs is large. Other reasons were related more to thinking about how to interpret the assumption of continuous reproduction. It is important to note that the students did not have any equations or functional expressions to inform their reasoning. Instead, they were in the position of having only the verbal scenario as a basis for thinking about what issues might be relevant and how to take those issues into account in developing arguments for one graph or another. As a result several important issues related to logistic growth problems were brought up by students including that population growth is dependent on the existing population and that it is reasonable to think about a growth rate parameter for a particular species.

The instructor then posed the specific case for population growth described by the differential dP/dt = 0.2*P. He asked what appeared to be a simple question. "What are the units of the 0.2 in the equation? How do you think about what the two is?"

When one of the students he called on said that she did not know how to put it in words, the instructor asked the class to "Take a second here, talk with the person next to you and say, well how would I think about that? What is that two? What kind of quantity is that two and how do I think about what two is here?" Once again, after only about two minutes, in which students verbalized their ideas to their group partners, the instructor called the class together for a whole class discussion.

What became apparent from observing the class over many weeks and from preliminary analysis of classroom data is that these brief small-group segments resulted in extremely productive subsequent discussions. The nature of the small group task was not to solve a problem but to "think about" something and "develop reasons to support your thinking." Even though students had little time to explore their reasons with one another in depth during the short time allotted to small group discussion, these "interruptions" in the class discussion obligated students to pursue their own ideas momentarily. Furthermore, because of the social norms that were operative in the class, students accepted the obligation and did engage in thinking about the issues at hand and in sharing their thinking within their groups. As a result, the students had a basis for participating meaningfully in the subsequent class discussion. Further, since the discussion inevitably focused on their reasons, students were in the position to compare and contrast their reasons with those of others.

In the class session discussed here, several other important ideas emerged as a result of the small group "thinking" and the subsequent whole group discussions. These included that the placement of the P (population) axis is arbitrary. Shifts to the right or left indicate different starting times. A related idea that developed is that for autonomous differential equations solution functions are horizontal shifts of one another. Before the class session ended students were beginning to think about the relationship between slope marks that they might make on a coordinate graph (a foreshadow of the slope field) and the graph of the function of population over time. In each case, these notions emerged from the students' reasoning. However, it is important to note that the instructor took a very proactive role and had a number of semesters of prior experience on which to draw as he posed questions that he anticipated would bring forth various types of reasoning.

I have suggested that the emphasis in this class was on reasons, rather than on conclusions. Using Toulmin's language of argumentation, the emphasis was on data, warrants, and backing. Students were less engaged in solving problems than they were in reasoning or "thinking." Preliminary analysis of excerpts from class dialogue indicates that what became constituted as data, warrants, and backing was not fixed or predetermined by the content or by the instructor but was negotiated by the participants as they interacted. It is in this sense that we use the label "group thinking" to refer to the collective argumentation that develops as the students engage in reasoning interactively, both in their small groups and in the whole class setting. We have adopted this label from a student in the class who used it in his second journal. As the following quotation from his journal indicates, students became keenly aware of the powerful nature of discussions for their learning.

> One way of thinking about a particular problem from class that helped me enlarge my thinking is *group thinking* [emphasis added]. You are the first professor I had who actually tries to make us understand the material and not just spit out equations and answers to you. The *group thinking* lets me communicate and speak my mind. My reasoning to a particular problem might be different than someone else's but another person in the class might also think the same way I do.
>
> A specific problem I liked was the predator-prey problem. Everyone had a different idea about it, which made everyone have to think. The *group thinking* helps me sort ideas out. Also, *group thinking* helps me put in words what I am trying to say. *Group thinking* in a math class is new to me, but I like it so far.

I would argue that the reason *group thinking* is so powerful for students' learning is that it emphasizes what most mathematicians and mathematics educators consider to be the essence of mathematics—mathematical reasoning and argumentation.

Conclusion

When investigating teaching and learning in the mathematics classroom there are many aspects that one might focus on. In this paper, I have pointed to aspects that relate to explanation, justification and argumentation. I have argued that a symbolic interactionist perspective gives us a way to make sense of the social aspects of the classroom such as social and sociomathematical norms. I have argued further that Krummheuer's approach to argumentation which uses Toulmin's scheme in a collective sense provides us with a way to explain why an emphasis on explanation and justification in a mathematics classroom leads to mathematics learning that emphasizes reasoning.

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REVISITING THE GOALS AND THE NATURE OF MATHEMATICS FOR ALL IN THE CONTEXT OF A NATIONAL CURRICULUM

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Abstract

A movement of educational change has been developed in Portugal, aiming at giving the schools a larger autonomy in curricular decisions. In reconceiving the view about the curriculum, the concept of "competence" plays a central role while the process of innovation constitutes a major aspect. This movement is described and analysed, in particular by discussing the notion of competence and the characteristics of the process of curriculum development. A special focus is on the way in which mathematical competence for all may be interpreted and how it is related to developments in mathematics education. The analysis of obstacles emerging in a large-scale educational change may be relevant for discussion in an international context and offers some suggestions for future research and debate.

1. New trends in the Portuguese educational system

The evolution and recent developments of the Portuguese educational system in the last two decades, as well as the related perspectives on the curriculum, and on the mathematics curriculum in particular, raise some issues that might be of general interest for reflection and discussion.

1.1. The context: some contradictory aspects

In 1974, when we could finally restore the democracy after 48 years of a dictatorship, the extension of compulsory education up to 6 years of school – instead of the old and traditional primary school of 4 years – was a recent fact. This hardly reflected the reality in the poorest regions of the country. Illiteracy was very high among adult population. It was only in 1986 that a new general law for education fixed compulsory school attendance for all 6-15 year olds (that is, a "basic school" of nine years). In fact, we had to wait almost for the end of the twentieth century to see practically all our children and youngsters of those ages in our classrooms. At the same time, it was only after 1995 that pre-school education became a new reality, finally involving in 2000 the large majority of children of the 3-5 age level.

As a result of its rapid and original development, in particular due to the fact that its basis was partly designed during a sort of "revolutionary" period, this educational system has some characteristics that may be seen as contradictory.

On the one hand, we have a number of very "advanced" laws and regulations. In our system, there is a comprehensive and "inclusive" basic school of nine years for all, similar to the Scandinavian traditional organisation, and a secondary education of three years, with different branches (general, vocational, professional) all of them giving equal access to further studies. Since 1997 teachers of all school segments, from pre-school to university, must have the same academic degree at the beginning of their profession. Moreover, all teachers may use, from time to time, a sabbatical year or they can apply to a bursary in order to develop projects or to do a post-graduate course. In the second and third cycles of basic education (10-15 year olds), teachers work a maximum of 22 and a minimum of 14 class periods of 50 minutes per week, according to their age.

On the other hand, all this was conceived and implemented on a tradition of a centralised and rigid school system, which was affected by lack of resources in many schools – a problem that has taken us many years to overcome. It was only in the nineties that a new law to increase the administrative autonomy of schools was adopted. Dominant public views about the curriculum tend to see it as a set of disciplines with "programmes" indicating for each subject what (and how) must be "covered" each school year. This process is actually mediated by the power of the textbooks – even if those programmes are aligned with modern international trends in most disciplines. Although there are no national exams except at the end of the secondary school, assessment is dominated by traditional written tests. Although there is an official logic of evaluating students progress throughout each cycle, the old "principle" of deciding that the student must repeat the same school year if he/she does not succeed in two or three disciplines still guides the way how many people think and act, both outside and inside the school.

When focusing on mathematics education, there are also other contradictory aspects in the situation, just like in most European countries. However, Portugal probably stands in a clearer contrast. In Portugal, the community of mathematics education (teachers, teacher educators and researchers) has become quite strong in these last fifteen years. All these groups support and are involved in the Association of Teachers of Mathematics (APM), the biggest association of its kind, which was created in 1986 and has now about 6000 members. This number would be equivalent (proportionally to the population) to 24000 in Spain, more than 30000 in France or UK, and approximately 9000 in Holland! About 2000 participants attend the annual meeting organised by APM. Many Portuguese teachers and researchers began to participate in international congresses on mathematics education: in ICME-8, in 1996, they formed the third largest national representation among European countries, after Spain and UK.

At the same time, a considerable number of teachers are following post-graduate studies on education, in particular on various aspects of the didactics of mathematics, and some of them are participating in projects which involve both a strong component of curricular innovation and a research dimension. Since the creation of APM, and following the experience of the MAT789 project (Abrantes, 1993), the co-operation between teachers and researchers became an interesting feature of

Portuguese mathematics education – see, for example, Oliveira et al. (1997), Ponte et al. (1998) or Porfírio and Abrantes (1999). It is not surprising that curricular innovation and teachers professional development became two major areas of research.

However, this community is under a strong public pressure, as a consequence of the scores in the exams at the end of secondary school or the very low position of Portugal in the rankings of the international comparative studies. Like elsewhere, as Keitel and Kilpatrick (1998) have pointed out, these rankings are frequently used without any serious consideration about what they mean or do not mean. They are used, for example, as an argument to propose not only the return to a greater emphasis on training routine skills, but also "solutions" like more exams and more comparative studies! Many teachers seem to have mixed feelings. They have sympathy with the new ideas about curriculum development, but they are also moving in a culture of school still dominated, inside and outside the school, by the old ideas and values.

1.2. Recent developments in basic education

In the last four years, after one year of debate and preparation, the Ministry of Education initiated a movement towards a new curricular organisation. The starting point was the consideration that the traditional structure was not adequate to ensure significant learning experiences to all children and to avoid school failure and abandon.

For the "basic education", this movement started with a project, labelled as "flexible management of the curriculum". This project aimed at giving the schools a larger autonomy in the decisions about the various disciplines and connections among them, as well as about new interdisciplinary components – a "project area", an "oriented study area" and a "citizenship area". This autonomy relates to the teaching and learning process and refers both to the activities to be developed and the time and space dedicated to each component of the curriculum. Emphasis is put on the role of the teachers and their collective structures in school, namely – at 2nd and 3rd levels of basic education (10-15 year olds) – the class council, this is, the group of professionals who work directly with each group of students.

This movement was justified by the need to promote a new conception of the curriculum, both the intended curriculum and the implemented one. The former requires the educational authorities to express the curriculum in terms of "essential competences" and types of "educational experiences" that the school should consider for all pupils (in each cycle), in opposition to the usual programmes of content topics to be covered and corresponding methods (in each year). The latter challenges the teachers and the schools to assume a much larger responsibility in the search for the adequate decisions for the specific pupils they work with, taking into account their cultural and social environment, their educational needs and the human and material resources that exist or can be made available. In other words, under the guidance of a national curriculum expressed in general and broad terms, the process of curriculum implementation is seen like a project to be conceived and developed by the school, including more specific projects concerning each individual class.

From 1997 on, schools could participate in the so-called “flexible management of the curriculum” movement by presenting their own curricular projects, under a minimum number of general rules. This participation, on a completely free basis, began with 10 schools in 1997/98, increasing in the three following years up to 184 schools all over the country. These schools constituted a sort of informal network exchanging materials and points of view, and participating in local, regional or national meetings organised by the educational authorities.

1.3. The case of mathematics

Meanwhile, the Ministry of Education started to produce draft versions for discussion of the so-called “essential competences”. Some documents focussed on aspects crossing all school subjects, while others related specifically to the various disciplines. This activity has been developed by working groups with a strong participation of members of the associations of teachers, together with researchers and other professionals.

In the case of mathematics, the corresponding document states the ultimate end of mathematics in basic schools as follows:

> Mathematics is a part of the cultural patrimony of human kind and a way of thinking, which should be made accessible to all. Every child and youngster should have the opportunity
>
> ☐ to contact, at an adequate level, with the fundamental ideas and methods of mathematics and appreciate its value and nature;
>
> ☐ to develop the capacity of using mathematics to solve problems, reason and communicate, as well as the self-confidence to do it.

This document refers then to major aspects of “mathematical competence” for all in the following way:

> The mathematical competence that all pupils should develop through the basic education integrates attitudes, skills and knowledge, and includes:
>
> ☐ the disposition and capacity to think mathematically, this is, to explore problematic situations, search for patterns, formulate and test conjectures, make generalisations, think logically;
>
> ☐ the pleasure and self-confidence in developing intellectual activities involving mathematical reasoning and the conception that the validity of a statement is related to the consistence of the logical argumentation rather than to some external authority;
>
> ☐ the capacity to discuss with others and communicate mathematical thoughts through the use of both written and oral language adequate to the situation;
>
> ☐ the understanding of notions such as conjecture, theorem and proof, as well as the capacity to examine the consequences of the use of different definitions;
>
> ☐ the disposition to try to understand the structure of a problem and the capacity to develop problem solving processes, analyse errors and try alternative strategies;

☐ the capacity to decide about the plausibility of a result and to use, according to the situation, mental computational processes, written algorithms or technological devices;

☐ the tendency to "see" the abstract structure underlying a situation, from daily life, nature or art, involving either numerical or geometrical elements or both.

In a second part, the document elaborates on what this means, in terms of each of the main areas of the mathematics curriculum – Numbers and Operations, Geometry and Measurement, Statistics and Probability, Algebra and Functions – through all the basic education and in each of the three age level cycles. Finally, pointing out problem solving as a general guide-line, it states that all pupils in their mathematics classroom should be frequently involved in mathematical investigations, projects, practical tasks, discussions, reading and writing about mathematics, exploration of connections inside mathematics and relating it to other areas, as well as they should have various opportunities to use technology, manipulatives and games in relation to their mathematical activities.

2. Scope and meaning of the process of educational change

What is going on in Portuguese basic education is undoubtedly a change of paradigm. Other reforms in the past introduced interesting innovations, but all of them left untouched the power of central authorities in defining the curriculum, the usual way of testing and implementing it, the traditional separation between curriculum guidelines and school organisation, and the nature of teacher's role and professional activity. The current movement, for the first time in the history of our education, has to do with changes in all these aspects. Generally speaking, it could be described as a change from the "conventional" paradigm to a new one influenced by the "constructivist" and the "critical" paradigms – to use the terms of Galbraith (1993).

The evolution of the concept of curriculum in relation to school organisation and teacher development has been largely discussed in the literature – for example, Fullan and Hargreaves (1992), Fullan (1993), Goodson (1997) and many others. From a theoretical perspective, the current movement in Portugal – tending to view the curriculum as a project in the context of the school as a learning organisation – is far from being original. In practical and political terms, we can find some similarities with the reform developed in Spain about ten years ago. On the other hand, the formulation of the intended (national) curriculum in terms of "essential competences" is not original as well – there are other cases, for example, in Belgium or in Québec.

However, the recent evolution in Portugal deserves some exploration and discussion for three major reasons. Firstly, it is worth elaborating on the adopted concept of competence since the term "competence" is not used everywhere with the same meaning. Secondly, it seems promising to explore the way in which "competence" is interpreted in the specific context of the mathematics curriculum. Thirdly, it may be relevant to focus on the process of curriculum development and innovation, which is quite unusual at a national level.

2.1. The concept of "competence"

The shift from content topics and objectives to competences requires a clarification about the meaning of the term "competence". I do not intend to introduce a universally accepted definition, yet I would like to avoid ambiguity and misunderstanding.

Following Perrenoud (1997), it should be clear that, in spite of a possible confusion with a behaviouristic interpretation, the term "competence" does not indicate some kind of specific behaviour that "can be observed", neither does it refer to performance. In this author's view, competence is related to the process of activating resources (knowledge, skills, strategies) in a variety of contexts, namely problematic situations. Perrenoud quotes Chomsky (1977) to support the distinction between competence and performance, and the idea that competence is related to the capacity to improvise, but emphasising the fact that, in his view, competence develops as a result of learning and not spontaneously.

Short (1985) has shown that the concept of competence may be used (or misused) with several different meanings ranging from a connotation with behaviour and performance to an identification with a quality of a person or a state of being. In this last conception, the holistic nature of competence is emphasised. Knowledge is obviously involved, as well as the skill necessary to use it, but this use is an emancipatory action, based on reflection and implicating some degree of autonomy.

It may be interesting to note that there is a parallel evolution of the key concept used by the studies on literacy. Initially, "alphabetisation" was identified with school attendance; in a second phase, the important matter was the acquisition of the knowledge, whether or not the person had attended a given school level; finally, the focus of literacy moved from the acquisition to the use of the knowledge in concrete situations (Kirsch and Mosenthal, 1993). Since this is not necessarily limited to the direct application to routine situations, the (mathematical) literacy could be interpreted as the (mathematical) competence that all students should be helped to develop in the school.

In Portugal, the reform in the late eighties pointed out that educational goals went far beyond content knowledge, including skills and attitudes as well. The programmes stated three lists of general objectives to involve these three kinds of goals. After this, however, the programme for each school year indicated the content topics to be covered, together with "specific objectives" and methodological suggestions related to those topics. It is not surprising that a common interpretation of the intended curriculum tended to see skills (for example, deductive reasoning or problem solving strategies) and attitudes (for example, persistence or solidarity) as elements to be "added" to the content knowledge.

In the present movement, the concept of competence intends to emphasise the idea of integration of knowledge, skills and attitudes, where integration is the key idea. The choice of the expression "essential competences" is a deliberate attempt to distinguish what is being proposed from the "basic skills" or the "minimal objectives", which were common expressions in the official discourse some years ago. This distinction is a particularly important pedagogical and political issue in a country where education for all is a relatively recent principle and it is necessary to resist to systematic proposals to achieve this goal by creating hierarchies and inequalities among students.

2.2. Mathematical competence in a national curriculum for all

Resnick (1987) has consistently argued in favour of the idea that basic skills and higher order skills cannot be clearly separated. She also added the role of attitudes, namely by stating that the school should cultivate a broad disposition to higher order thinking. The integration of cognitive abilities and motivation is especially emphasised: "Motivation for learning will be empty if substantive cognitive abilities are not developed, and the cognitive abilities will remain unused if the disposition to thinking is not developed" (p. 50). Integration also played a central role in the definition of "mathematical power" as it was introduced by the NCTM (1989) Standards.

In the above list (section 1.3) of aspects of mathematical competence for all, the concern with the integration of knowledge, skills and attitudes is quite apparent. We should also add the clear concern with beliefs and conceptions about mathematics, which play an important role in students' learning process (Borasi, 1990; Schoenfeld, 1992). This aspect has been almost always absent in the curricular guidelines defined at an official and national level.

Another characteristic of the above listed aspects is the explicit attention to the nature of mathematics. As Bishop (1991) points out, it is not enough to teach (some) mathematics, it is indeed necessary to educate *about*, *through* and *with* mathematics. In this point, it should be emphasised that the mentioned aspects of mathematical competence cannot be seen in isolation from the educational experiences that all children should live in school, namely investigations and projects involving both mathematical ideas and their relations with different sorts of problems. Obviously, the idea is not to "enrich" the knowledge of facts and the training of procedures with some sort of rhetoric about the nature of mathematics as a science.

This concern with mathematical activity in relation to understanding the nature of mathematics is a central issue in several different approaches. Bishop (1991), in the search for "mathematical similarities", points out six activities that are "significant (...) for the development of mathematical ideas in any culture" (p. 23) – counting, locating, measuring, designing, playing and explaining. Goldenberg (1996) proposes "habits of mind" as organisers of the curriculum – for example, the tendency to describe relations and processes or the tendency to look for invariants. The NCTM (2000) states "process standards" to refer to "ways of acquiring and using content knowledge" (p. 29).

The present challenge we face is to help all children to develop their mathematical competence in a way that will avoid interpretations reinforcing the perspective of a curriculum of training procedures, skills and rules (for all) with the expectation that this kind of training will constitute (for some) a pre-requisite to future uses of mathematics. To do so, we have indeed to question the basis of the "technique-oriented" curriculum which has never been done except in some small-scale innovative projects. "A technique curriculum cannot educate (…) For the successful child it is at best a training, for the unsuccessful child it is a disaster" (Bishop, 1991, p.9). Perrenoud (1997) insists that the formulation of the curriculum in terms of competences should be strongly connected to the purpose of striving against school failure and taking into account all children, namely those with a cultural background not similar to that of the "traditional school".

If, at the level of the intended curriculum for all, our option is to reconceive the components of the mathematical competence, together with a larger variety of kinds of educational experiences, then a consequence is a reconsideration of the extension and complexity of topics included in the curriculum. For Bishop (1991), the curriculum should be relatively broad (in the variety of contexts offered) and elementary (in the mathematical content). Similarly, when discussing the problem of the construction of competences in the school, Perrenoud (1997) points out that, if our option is education rather than instruction, then it is necessary to reverse the tendency to include the teaching of more and more topics in the compulsory school curricula.

In the current movement in Portugal, it was announced that the number of content topics considered in the curriculum, in every discipline, would be reduced. This was not yet done for reasons that have to do with the process of curriculum development that I will comment on in the next section.

2.3. The process of curriculum development

The most original aspect of the recent development in Portuguese educational system is, probably, the fact that a curricular reform at a national level (indeed not even labelled as a "reform") is not following the RDD (standing for "research-development-dissemination") model.

The criticism on this strongly dominant model of curriculum development and implementation is far from being recent. Twenty years ago, Howson, Keitel and Kilpatrick (1981) have discussed the origins, assumptions, values and consequences on mathematics education pointing out emergent alternative perspectives. More recently, in the context of the so-called realistic mathematics education, Dutch researchers have developed this discussion into new and promising directions.

Gravemeijer (1994) explains that, in his approach, curriculum development is embedded in a holistic framework, taken from the concept of "educational development" as Freudenthal (1991) uses it. A central idea is that the process of curriculum innovation has to consider all the actions needed from the initial purpose to the actual change, incorporating teacher education, counselling, assessment and opinion shaping. Furthermore, unlike the RDD model, initial theory is much like a philosophy or a vision and it will evolve in the interaction between theoretical and empirical justifications.

The last reform in Portugal, in the late eighties, constituted an almost perfect example of the RDD model. Teams of invited experts prepared new programmes during two or three years; these programmes were implemented in a small number of "experimental schools" where motivated teachers worked together and prepared their own materials in the absence of textbooks; finally, after suffering slight corrections, the programmes were "generalised" to all schools. The "consumers" were introduced to the new finished "product", usually in the form of new textbooks. The result was not surprising: to solve the visible problems of low take-up, dilution and corruption of major ideas of the intended curriculum – to use the terms of Burkhardt (1989) – those responsible for the reform claimed that intensive teacher training programmes should then be developed.

The current movement, which as already mentioned was initiated four years ago, has a very different nature. Schools have been invited to participate by elaborating their own projects of curriculum development while, at a central level, different sorts of working documents are produced, namely draft versions of the "essential competences". These documents are discussed, criticised and modified in a process that takes into account the feedback of the schools and the contributions of universities and professional associations. As I have indicated earlier, a major principle is a large autonomy of schools, in relation both to the various disciplines and the new curricular areas dedicated to support students' projects and periods of personal and group study under teachers' guidance. Meanwhile, together with many formal and informal meetings organised by the Ministry and by the schools themselves, teachers participate in in-service initiatives, namely in the form of workshops and small projects – which are valued and credited for progression in the teachers' career just like traditional courses. The co-operative work among teachers inside the school has become, probably, the hallmark of the movement.

Throughout these last four years, the number of schools joining the movement increased significantly: 10, 33, 92, 184. An interesting result of the process was that teachers leading projects in the former schools began to be more and more invited by colleagues of other schools and meeting organisers to participate in conferences, debates and workshops, a kind of activity traditionally reserved for researchers and teacher educators.

In January 2001, a new law was adopted for curricular organisation. From now on, there are not compulsory uniform regulations about the exact amount of weekly time and the precise topics to be considered year by year in each discipline. Instead of that, schools are invited to make their own decisions about a number of relevant aspects, both at a school level and at a class level. Together with general guidelines focusing on the priority of experimental and practical teaching methods, a new version of the document stating the essential competences and educational experiences will constitute the main official reference. For each cycle, it is indicated the minimum time to be dedicated to each curricular component (group of disciplines or interdisciplinary area) and the maximum number of hours per day to compulsory activities. About 20% of the total time correspond to periods of work where there are not programmatic prescriptions at all – against the traditional 0%.

This new law includes a number of recommendations that emerged from the experience of the schools involved in the movement – which in any case may be adapted or modified. One of them is the recommendation to organise class activities in periods of 90 minutes, instead of the traditional 50 minutes. The main official arguments invoke better conditions to promote practical and investigative work in the classroom, the use of technology and other materials and the goal of reducing the number of different subject matters in each day.

It should be noted that, although we will enter into a new stage of the process of curriculum innovation, guidelines are far from being completely "ready". For example, the way in which programmes will evolve to constitute working materials for teachers is yet to be determined. The evolution of textbooks is another issue for debate. Refusing a top-down model for development, we do not have a new "system" to be generalised in a precise moment.

In the particular case of mathematics, the present stage of the debate suggests an evolution of the working documents mentioned earlier (section 1.3). The discussions about how we could characterise the mathematical competence for all indicated, for example, that it is necessary to make more explicit the uses of mathematics in relation to other areas and the "real world", and especially the role of mathematics in education for democratic citizenship. Generally speaking, a movement towards crossing the objectives indicated for the various curricular areas and referring them more clearly to common and central aims of the basic education will become a priority.

3. Obstacles and problems

Rather than examining the details, it may be relevant for the international community to discuss major issues and obstacles raised by a movement with the described characteristics. I will concentrate on some of them.

My first observation is that if creating an alternative to the RDD model for curriculum development is a difficult task even in small-scale projects, it becomes much harder in the context of an educational reform at a national level, especially if the tradition is that of a centralised system. The dominant conception of development claims for well-defined and "teacher-proof" curricula carefully tested before generalisation and high quality textbooks as key factors to improve teaching and learning. This seems to be a popular view shared by influential sectors of the scientific community and the society at large.

Public opinion is here a necessary, yet very complex, element to be considered. The strength of a movement based on the interaction between theoretical and practical developments, which is a gradual and long-term process in nature, seems to be at the same time its weakness. At a political level, it is not easy to respond to the accusation of delaying quick and clear answers. Guidelines appearing to be ill defined, as well as the absence of new programmes with exact and precise indications, become a factor of criticism. It is interesting to note that, in this context, the single proposal to give schools the possibility of organising classes in periods of 90 minutes is pointed out almost like a "revolution", provoking unusual public debates about education.

My second observation is that, even inside schools, together with the public pressure mentioned above, it is obviously difficult to deal with uncertainty. If the current movement constituted an opportunity for innovative teachers and school leaders to organise teaching and learning contexts more adequate to their students, for others it is a source of problems. There is a tendency to look for models in the initiatives of "more experienced" schools; however this becomes difficult when there are several different models and there is not an official one.

This tension between autonomy and security is amplified by the emergence of the rhetoric associated to the educational change. This is a common phenomenon in periods of reform, but it is particularly negative when change is a matter of process, not only of content. The tendency of emphasising the "pedagogically correct" and criticising all "deviations", characteristic of all sectors including some educational authorities and researchers, is in fact a force towards the adoption of uniform solutions and contradictory with the goal of a larger autonomy of schools and teachers.

My third observation is related to the concept of competence and, in the case of mathematics, the definition of mathematical competence. Doubts and criticism on the presented proposal showed that a broad concept is difficult to be widely accepted. Terms like *disposition* (to think mathematically), *pleasure* (in developing intellectual activities) or *tendency* (to look for the abstract structure) have been especially criticised with the argument that it is very difficult to make such things "operational". This seems to reflect the difficulty in getting the understanding or the acceptance of the idea that integration of cognitive and non-cognitive components is essential to the concept of competence.

Obviously, this is not a new problem, caused by the adoption of a terminology based on the concept of competence. Similar discussions tend to occur, regardless of the terms in which we base our definitions. Proposals aligned with using and applying mathematics in schools (de Lange, 1996), valuing mathematical investigations (Ernest, 1991) or adhering to the "rebirth" of project teaching (Bishop, 1995), which are consistent with the development of mathematical competence in a broad sense, are often accepted as complementary methods or a sort of applications but not necessarily as the essence of the curriculum. Clearly, the problem is the resistance to question and abandon the technique-oriented curriculum. A central aspect of this problem has to do with assessment and control, leading to my last observation.

Several authors have pointed out that conceptions and practices about assessment did not evolve to match developments in conceptions and practices about other curricular components. For example, Niss (1993) refers to "an increasing mismatch and tension between the state of mathematics education and current assessment practices" (p. 4). Assessment of the development of mathematical competence requires observation in different situations and confidence in the teacher's professional judgements, while the central role of standardised tests and exams may become a strong obstacle to flexibility, adequacy and diversity.

It is possible that, in the last years, a wider range of assessment modes and instruments – for example students' written productions – has begun to be increasingly accepted and used. However, reinforcing the dominance of tests and exams, the recent influence of the way in which international comparative studies tend to be interpreted and used has a powerful effect against educational change.

These studies could be relevant to provide information about important aspects of mathematical competence. However, presenting scores as indicators of curriculum achievement, and tending to view curriculum as unproblematic, context-free and culture-free (Keitel and Kilpatrick, 1998), the use of these studies, namely the emphasis on rankings, constitutes a serious obstacle to new conceptions and practices of curriculum development.

The problem is well known. Keitel and Kilpatrick (1998) show how, in the USA and in Germany, lower scores in the TIMSS test in comparison with Asian countries are used as an argument to urge teachers to return to a curriculum based on 'core knowledge' or to claim for funding to develop more sophisticated instruments for measuring students' performance. In the UK, the Secretary of State for Education and Employment says that "numeracy is an important life skill, but evidence shows that standards of school mathematics have not been high enough to enable us to compete internationally" (DfEE, 1998).

In Portugal, the situation is about the same, with the difference that scores were even lower. Porfírio and Abrantes (2000) have presented a paradigmatic example of popular notions of culture, school and mathematics, taken from a TV programme organised in the sequence of the publication of the international rankings. When a mathematics educator tried to introduce an example, the moderator immediately commented: "About mathematics we don't understand anything beyond the Pythagoras theorem". However, all evening, the moderator and two other opinion-makers criticised the school for the low scores of students in mathematics tests, while tried to proof the "lack of culture" of our youth by asking some questions about facts related to poetry, history and geography to some students present in the studio.

> These people showed that their level of mathematical ignorance was deeper than they even could realise. (...) They view mathematics as a 'building' and assume that they can only

> remember some pieces of that building. They don't have any idea about the nature of mathematical activity or about the way in which mathematical ideas are generated, develop and relate to other ideas. For them, however, this fact is not relevant in cultural terms. (Porfírio and Abrantes, 2000, p.278)

Presently, in Portugal, while the movement of curriculum innovation tries to emphasise flexibility and adequacy of teaching methods to students' characteristics and consideration of their social and cultural backgrounds, the "societal" values of competitiveness and standardisation – of guidelines, methods and "objective" results – tend to favour the reinforcement of a technique-oriented curriculum. The Ministry of Education is strongly criticised for not publicising rankings of schools based on students' scores in national tests. A popular argument is that everybody has the right to know what are the "best schools", the "best teachers" and the "best teaching methods". The need to compete with other countries is generally added as well, together with the argument of "globalisation". As Keitel (2000) observes, this concept is ambiguous; it has frequently a connotation opposite to the values of cross-country collaboration, interaction and co-operation at different levels.

4. Final remarks

In these last decades, much work has been done in the field of mathematics education. Generally the research focus is on the child and relates to learning processes or students' conceptions. In some cases there is concern about the evolution of these processes and conceptions in contexts of curriculum innovation, as well as about models of curriculum development. However, the remark of Burkhardt (1989) that the study of curriculum change on a large scale is neglected, "partly for practical reasons but mainly because of a lack of attention to system issues" (p. 9) seems to continue valid.

There is also a considerable work on teachers' professional development but, again, the emphasis is not generally in collective and social processes relating this development to the dynamics of curriculum change, except maybe in the context of innovative projects.

A major obstacle to develop promising approaches to curriculum innovation may be probably found in political and social issues, namely in "popular" conceptions about education and educational change. In particular, public conceptions about mathematics and mathematics learning in schools seem to play a central role in favouring the perpetuation of a technique-oriented curriculum. However, this influence is seldom studied. When discussing the factors that cause students' (and teachers') conceptions of mathematics, the literature has almost always pointed out the way in which mathematics is presented in school and the dominant use of scores on standardised tests as measures of academic success. There is no doubt about this strong and direct influence. But this is not a story of teachers and students only. What is the role of the society at large and how does it work? What about the role of the scientific community?

We can find a few references in the literature, revealing this concern. Borasi (1990) admits that "social stereotypes (…) may certainly play a role in shaping students' conceptions" (p. 177). As I have pointed out above, Freudenthal (1991) includes "opinion shaping" in the set of actions needed in a process of educational development. However, the recent evolution in Portuguese educational system strongly suggests that we need to know much more about these issues.

Finally, I would like to observe that the frequent shifts in my text between mathematics education and educational change at large reflect the impossibility of isolating the mathematics curriculum from the school curriculum as a whole – especially when the basic education is the context. Similarly, the shifts between a descriptive/analytic perspective and a normative one (Niss, 1996) result from the impossibility of ignoring personal values and options when the goals of mathematics education are concerned. These two aspects should not be neglected whenever research and debate focus on actual change at a national level.

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PATHWAYS IN MATHEMATICS TOWARDS EQUITY: A 25 YEAR JOURNEY

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ABSTRACT

A brief and selective overview of historical evidence of females' involvement in mathematics precedes a review of developments in research on gender and mathematics learning over the past 25 years. Evidence is presented that gender equity concerns have attracted considerable research attention by (mathematics) educators in many countries, and that over time the body of work on gender and mathematics education has increasingly reflected a greater diversity of inquiry methods used to examine and unpack critical factors. Research Reports presented at PME contain only limited evidence of these trends.

INTRODUCTION

Before turning to the main theme of this paper, developments in research on gender and mathematics learning since the founding of PME, it is appropriate to offer some brief, if inevitably selective, glimpses of females' involvement in mathematics in earlier times.

The first eminent woman mathematician

"It's true", wrote Woolfe (1996) in a popular novel, "that Hypatia, the ancient Greek mathematician, was the first eminent woman mathematician ... that anyone seems to know about" (p. 7). Readers are left in no doubt that her prowess in mathematics was both revered and feared.

> She had a huge following, and distinguished students came from Europe, Asia and Africa to hear her.... Cyril was fearful of her popularity and her religion, and incited a mob of fanatics, who dragged her to a church, murdered her with shells, and then burned her. This happened at the height of her fame, when she was 45. All her writings have been lost. (Woolfe, 1996, p. 7)

Hypatia's life, it appears, continues to fascinate those inside and outside the mathematics community. A brief biographical sketch, with reference to her mathematical excellence and her violent death, was recently included in the *Five things you didn't know about* regular column in a popular daily metropolitan newspaper (Hill-Douglas, 2001).

The Ladies Diary

A less spectacular, but more pervasive, example of females' involvement in mathematics is also worth mentioning. Some 300 years ago, in 1704 to be more precise, John Tipper launched the first almanack specifically aimed at women, under the title " The *Ladies Dairy*[1] or the *Women's Almanack*[2]". Considered to be the prototype of the popular eighteenth century ladies' pocket books and diaries, it was itself highly successful, with an unbroken publication run until 1840 when it combined with the *Gentleman's Dairy* and continued to be published under the latter name for another 30 years.

The contents of The *Ladies Dairy* make fascinating reading, with hints about choosing a life partner, optimistic messages about the status of women, and pithy advertisements:

> Never marry a vicious man in hopes of reclaiming him afterwards; for those who are habituated to any manner of debauching or vice, if you think to reclaim by fair means, or by foul, you will find yourself fatally mistaken. (*The Ladies Diary*, 1704, p. 5)[3]
>
> The method that God observed in the creation, plainly shows women to be the most excellent of created beings. Which method of proceeding was from the less to the more noble beings, namely from the mineral to the vegetable; from thence to the animal kingdom; all of which being finished, he made men, and last of all women, in whom all the creation was perfected and its beauty complete. God having made women, ended his work, having nothing else more excellent to create… (*The Ladies Diary*, 1705, p. 15)
>
> Artificial teeth, set in so firm, as to eat with them, and so exact, as not to be distinguished from natural; they are not to be taken out at night, as is by some falsely suggested, but may be worn years together …. (They) are an ornament to the mouth and greatly help the speech. (*The Ladies Diary*, 1979, p. 48)

It is not clear why Tipper included two mathematics problems in the *Ladies Diary* of 1707. As master at Bablake school and a mathematician of considerable ability, he was certainly qualified to do so. At the same time, "the law, which the first contributors imposed on themselves, of not only proposing, but also answering all questions in rhyme, was not favourable to the development of Mathematical genius" (Leybourn, 1817, p. viii). Nevertheless, the inclusion of mathematics problems in all subsequent issues explains my interest in the *Ladies Dairy*.

Tipper's formula proved successful. "My almanac sold this year beyond mine and the company of Stationers' expectations, so that of 4000 which they printed, they had not one left by New Year's tide" (Ellis, 1843, p. 314), he wrote in a letter to a friend. Some years later, the then editor of the *Diary* wrote:

> I believe that the *Diary* has the good fortune to fall into a multitude of hands which mathematical books seldom or never would … [T]he fair

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[1] Or *Ladies' Diary* in later issues

[2] I am indebted to Dr Teri Perl for her willingness to share her *Ladies Diary* materials with me

[3] The spelling in this and subsequent *Diary* extracts has been modernized

> sex may be encouraged to attempt mathematics and philosophical knowledge, they here see, that their sex have as clear judgements, a sprightly quick wit, a penetrating genius, and as discerning and sagacious faculties as ours, and to my knowledge do, and can, carry them thro' the most difficult problems. I have seen them solve, and am fully convinc'd, their works in the *Ladies Diary* are their own solutions and compositions….Foreigners would be amaz'd when I show them no less than 4 or 500 several letters from so many several women, with solutions geometrical, arithmetical, algebraical, astronomical and philosophical. (*The Ladies Diary*, Editorial, 1718)

Reference to the "discerning and sagacious faculties as ours" suggests that editor Beighton assumed that the *Diary* would be read by males. His words also foreshadow a theme to be mentioned later in the paper - the belief that for women to be the equal of males was high praise indeed.

The passage of time and the inadequacy of historical records make it difficult to quantify the full impact of the *Ladies Diary*. Yet, there is little doubt that the mathematical content of the *Ladies Diary* was taken seriously. In due course two substantial collections of "the useful and entertaining parts, both mathematical and poetical" of the *Diary* were published – the first by Charles Hutton (1775), professor of mathematics at the Royal Military Academy, and editor of the *Diary* from 1774 to 1818, the second by Thomas Leybourn (1817).

Problems, taken from the *Ladies Diary* of 1707, 1769, and 1814 respectively, and shown below, give some idea of the publication's mathematical content[4].

> If to my age there added be
> One half, one third, and three times three;
> Six score and ten the sum you'ld see,
> Pray find out what my age may be. (Question 2, *Ladies Diary*, 1707)
>
> Dear ladies, you with ease may find*
> A matchless hero's name,
> Who was beloved by mankind,
> And mounted up to fame:
> To serve his country boldly dar'd,
> Hot sulphur, smoke, and fire,
> And long campaigns' fatigue he shar'd,
> To conquer proud Monsieur.

* viz. From the equations

$$\begin{aligned} w + x + y + z &= 52 \\ wx + yz &= 360 \\ wz + xy &= 280 \\ wy + xz &= 315, \end{aligned}$$

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[4] The selection is biased by concerns about space constraints. I have selected shorter rather than longer problems.

> where w, x, y, and z denote the places of the letters in the alphabet composing the gentleman's name. (Question 597, proposed by the frequent contributor Mr Tho. Sadler, *Ladies Diary*, 1969)
>
> If a globe, of 1½ foot diameter, be put to float in common water; required the area of the section at the surface of the water, when the specific gravity of the globe, to that of water, is as 3 to 5? (Question 1269, proposed by Mr Joseph Williams, of Canterbury, *Ladies Diary*, 1814)

Although the overwhelming majority of problems were posed and answered by males, there is evidence of females' mathematical activities scattered throughout issues of the *Ladies Diary*. The address to "Dear ladies" in the second of the problems reproduced is suggestive. Included among the seven correct answers for the third problem printed in the *Diary* was one from a *Miss* Susannah Jackson from *Mile End.*

The decision by a succession of editors to award a prize for the first correct solution received to selected mathematical problems enables at least partial tracing of those who engaged in mathematical problem solving. Perusal of other records shows that the surnames of many of the early female contributors matched those of well known male mathematicians or scientists. Thus having a brother or husband knowledgeable about, and sympathetic to, mathematics and scientific pursuits appeared a distinct advantage for females interested in mathematics. Significantly, contemporary research findings have revealed the benefits of a nurturing environment, access to needed materials, and support from critical others as facilitating achievement in mathematics. Perl's (1979) assertion that the solutions contributed by women were confined mainly to arithmetic or algebra problems is also consistent with females' preferences in mathematics reported in some contemporary research.

Those familiar with the life of Mary Somerville, the Scottish mathematician, have further indirect evidence of the importance of publications like the *Ladies Diary*. Mary was born in 1780. Her family attached far greater importance to the education of their sons than of their daughter. For her it was deemed sufficient to be taught to read the Bible by her mother, although when she was ten she was sent to a fashionable boarding school for 12 months. From there she emerged "with a taste for reading, some notion of simple arithmetic, a smattering of grammar and French, poor handwriting and abominable spelling" (Patterson, 1974. p. 270). Some years later, quite fortuitously, she came across an algebra problem which aroused her curiosity. In Mary's own words:

> At the end of the magazine, I read what appeared to me to be simply an arithmetical question, but on turning the page I was surprised to see strange looking lines mixed with letters, chiefly Xs and Ys, and asked "What is that"? "Oh", said (my) friend, "it's a kind of arithmetic; they call it Algebra; but I can tell you nothing about it"... On going home I

> thought I would look if any of our books could tell me what was meant by Algebra. (Somerville, 1873, p. 54)

Instead of encouraging this thirst for knowledge, her father forbade her studying mathematics. “We must put a stop to this”, Mary recounted him saying, “or we shall have Mary in a straightjacket”. Such beliefs also proved persistent. Decades later an American physiologist argued that “a young woman might learn algebra, but [he added] when the limited sum of energy flowed to the overwrought brain, it harmed the natural growth of ovaries” (Tyack & Hansot, 1988, p. 37). Nevertheless, Mary persevered with her mathematical studies. Her most effective mentor was the Scotsman William Wallace, then editor of the *Gentlemen's Diary,* to which she sent a number of contributions. The close relationship between it and the *Ladies Diary* was fostered through cross references between the two publications. The eventual merging of the two publications has already been mentioned.

In Mary Somerville's case the influence of a popular magazine that also contained mathematics problems has been recorded for posterity. It is tempting to speculate that many other intelligent women were stimulated to achieve mathematical literacy through the mathematics section of the *Ladies Diary.* That copies of the *Diary* found their way into Australian libraries may reflect the priorities of some of that country's early settlers.

After this brief historical context, it is time to consider more recent trends, starting with the period approximating PME's creation.

MATHEMATICS AND GENDER

Who Cares?

In a recent article, Lubienski and Bowen (2000) reported the results of their attempt to identify major areas of mathematics education research activity, including “the attention given to various equity groups and topics by the mathematics education research community” (p, 627). Their data source comprised 48 major national and international educational research journals accessible through ERIC and likely to include at least some mathematics education-related research. Eventually 3,000 articles were counted and categorized over the period selected: 1982 to 1998. The accuracy of their results, Lubienski and Bowen readily admitted, was heavily dependent on the accuracy of the ERIC descriptors and their categorization of those descriptors. Nevertheless, their findings offer a useful, if rough, measure of research interest among mathematics educators in gender issues.

According to their search, approximately 20 % of the articles (623 out of the total 3,011) concerned with mathematics education contained an equity theme, i.e., they contained a focus on gender, ethnicity, class, or disability. The majority of these, 323, were concerned with gender. In other words, some 10 % of all the articles identified contained gender as a factor of interest. Frequency of such articles varied with journal type. For example, in journals broadly classified as general educational and psychological, 15.2 % and 14.1 % articles of the articles respectively contained a gender theme; for those grouped under US and international mathematics education journals the figures were 8.9 % and 7.7 % respectively. The thrust of these articles is the focus of the next section.

Identifying a "problem"[5]

During the 1970s, much research effort was directed at documenting gender differences in participation in mathematics courses and in performance on mathematical tasks and tests. A then timely "state of the art" summary read as follows:

> Are there sex differences in mathematics achievement? ... No significant differences between boys' and girls' mathematics achievement were found before boys and girls entered elementary school or during early elementary years. In upper elementary and early high school years significant differences were not always apparent. However, when significant differences did appear they were more apt to be in the boys' favor when higher-level cognitive tasks were being measured and in the girls' favor when lower-level cognitive tasks were being measured.... Is there "sexism" in mathematics education? If mathematics educators believe that there is a sex difference in learning mathematics (as was evidenced in the reviews cited) and have not attempted to help girls achieve at a similar level to boys, then this question must be answered in the affirmative. (Fennema, 1974, p. 137)

That concern about females' participation and performance in mathematics was not confined to the USA is evident from the excerpt below, taken from the report of the Victorian (Australia) Committee on Equal Opportunity in Schools:

> A large portion of mathematical ability resides in women and is potentially untapped. It has been a long-term aim of our educational system to develop individual talent, and the serious imbalance apparent in inculcating mathematics competence in men compared to women, demonstrates how far our achievement has fallen short of that ideal. (1977, p. 152)

Developments and explanations

The presentation below of the different phases in research on gender and mathematics education as sequential is simplistic and convenient rather than an accurate chronological representation. Trends described in earlier time spans have persisted in later research work; elements of those highlighted in the discussion of later years could be gleaned in earlier work.

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[5] The contents of this and following sections were shaped by collaborations and discussions with many colleagues. Of these, I wish to single out Elizabeth Fennema and Helen Forgasz as being particularly influential.

Early trends

Documenting current performance and participation differences, exploring likely contributing factors, and assessing the effectiveness of selected intervention strategies succinctly described the scope of research and scholarly activities with a focus on gender and mathematics in the mid 1970s and much of the 1980s. Gender differences in mathematics learning were typically assumed to be the consequences of inadequate educational opportunities, social barriers, or biased instructional methods and materials. It was further assumed that the removal of school and curriculum barriers, and if necessary the resocialisation of females, would prove to be fruitful paths for achieving gender equity. Male (white and western) norms of performance, standards, participation levels, and approach to work were generally accepted uncritically as optimum and to be attained by all students. When failing to reach these, females were considered deficient, or to use a theme from Kaiser and Rogers (1985), perceived as *a problem in mathematics*. They were to be encouraged and helped to *assimilate*. This notion, helping females attain achievements equal to those of males was consistent with the tenets of *liberal feminism.*

The 1980s

The *assimilationist* and *deficit* model approaches proved persistent throughout the 1980s and continued to guide many of the intervention initiatives aimed at achieving gender equity. Levels of males' and females' participation and performance in mathematics subjects continued to be reported in research studies, but now more frequently with an attempt at analysis - perhaps in conjunction with an examination of government policies. At the same time, different voices were beginning to be heard, undoubtedly influenced by work developed in the broader research community. The themes fuelled by Gilligan's (1982) *In a different voice*, and the feminist critiques of the sciences and of the Western notions of knowledge were particularly powerful. What other factors might be contributing to gender differences in mathematics education? Should we accept, uncritically, the way in which mathematics was being taught and valued? Should young women strive to become like young men or should the formers' goals, ambitions, and values instead be celebrated? What, crucially, might differentiate between a single-sex and co-educational school or class environment? Should we aim for uniformity or 'can different be equal'? In which setting might mathematics be taught and learnt more effectively by males and females? Should we accept only those conditions and approaches favoured by males? Such questions led to interventions which attempted to make the contents of mathematics less alienating to females. Rather than expect them to aim for male norms, attempts were made to use females' experiences and interests to shape the mathematics taught and methods of instruction. Females were to be perceived as *central to mathematics* and *mathematics* as being *reconstructed* (Kaiser & Rogers, 1985).

The assumptions of the "women as central to mathematics" phase were not without danger. Attempts to focus on women with exceptional and rare mathematical talents proved problematic. Some of these portrayals, it seemed, simply confirmed how difficult it was for an "ordinary" (female) student to become an "extraordinary" mathematician, what hardships needed to be endured, what challenges to be overcome, what prices to be paid? Programs which valued and nourished qualities and characteristics presumed to be exclusively female could be thought to imply, directly or indirectly, that these were innate to females and alienate those who did not possess them. This essentialism risked perpetuating traditional gender stereotypes rather than redressing gender inequities. Nevertheless, recognition that previously unchallenged assumptions, traditions, and cultural exclusivity needed to be examined and possibly redefined was overdue.

The more critical attempts to analyse and deconstruct explanations for gender differences in mathematics learning and the clearer recognition that different perspectives inevitably lead to differences in the ways in which the interventions aimed at challenging inequities were framed were noteworthy developments in the later years of the 1980s. The attempts to make females more *central to mathematics* and of exploring the *reconstruction of mathematics* soon accelerated and diversified. The assumptions of *liberal feminism* that discrimination and inequalities faced by females were the result of social practices and outdated laws were no longer deemed sufficient or necessary explanations. Instead, emphasis began to be placed on the pervasive power structures imposed by males for males. The acceptance of (white, western) male norms, the assumption that females aspire to these standards and modes of behaviours, and the presentation of a deficit model of womanhood in which girls and women are positioned as victims with deficit aims and desires were also challenged. Some researchers wished to settle for nothing less than making fundamental changes to society. Advocates of this approach, often classed as *radical feminists,* considered that the long term impact of traditional power relations between men and women more broadly, and in mathematics more specifically, could only be redressed through such means.

Further changes – beyond the 1980s

Attempts to explore the interaction between gender and other background variables - socio-economic status and cultural and ethnic affiliations, for example – have intensified in the past decade. The concerns of *social feminists* voiced in the community at large, that females from working class backgrounds are often particularly disadvantaged in the home, in the labour force, and in access to leisure pursuits, have also influenced research in mathematics education. The genuine efforts made to mirror as comprehensively as possible the complex web of factors - personal, situational, and social - which might shed light on issues of gender and mathematics are reflected in more complex research designs and in designs relying for their conception, execution, and data analysis on multiple research methods.

An interim summary

In brief, gender equity concerns have represented a significant item on the research agenda of (mathematics) educators in many countries - in highly technological societies as well as developing nations. International comparisons, formal and informal, have highlighted the roles of class and culture. For a given society, the status of mathematics in the lives of females is invariably linked to their status in that society. Male norms, and acceptance of difference without value judgments, have been more likely to be challenged in countries with active and long standing concerns about equity issues. Collectively, the body of work on gender and mathematics education reflects an increasing diversity in the inquiry methods used to examine and unpack critical factors. More radical feminist perspectives are being adopted, females are less frequently considered as a homogeneous group, and scholarly evaluations of interventions are becoming more prevalent. At the same time there is a clearer recognition of the extent to which the personal beliefs and theoretical orientation of the researchers undertaking the work influence inclusion and exclusion of variables and modes of data gathering. No longer is it simplistically assumed that the planning, execution, reporting, and interpretation of research are value free.

It is difficult to quantify the extent to which perceptions about gender and mathematics learning have changed. A recent research study (Leder & Forgasz, 2000) provides one measure. A sample of approximately 860 students in coeducational high schools in Victoria, Australia, completed a questionnaire aimed at tapping gender stereotypes about aspects related to the learning of mathematics. For each of 30 statements students were asked to indicate whether they believed (1) the statement to be definitely more likely to be true for boys than girls, (2) probably more likely to be true for boys than girls, (3) there was no difference between boys and girls, (4) probably more likely to be true for girls than boys, or (5) definitely more likely to be true for girls than boys. The data obtained from administration of that questionnaire were compared with findings reported in previous relevant research (see Table 1).

Table 1. Predictions based on previous research and findings from the study (Italics bold)

	ITEM	Pred	***Find***		ITEM	Pred	***Find***
1	Maths is their favourite subject	M	***F***	16	Distract others from maths work	M	***M***
2	Think it is important to understand the work	F	***F***	17	Get wrong answers in maths	F	***M***
3	Are asked more questions by the maths teacher	M	***M***	18	Find maths easy	M	***F***
4	Give up when they find a maths problem too difficult	F	***M***	19	Parents think it is important for them to study maths	M	***nd***
5	Have to work hard to do well	F	***M***	20	Need more help in maths	F	***M***
6	Enjoy mathematics	M	***F***	21	Tease boys if they are	M	***M***

ITEM	Pred	***Find***	ITEM	Pred	***Find***
			good at maths		
7 Care about doing well	M/F	***F***	22 Worry if they don't do well in maths	M/F	***F***
8 Think they did not work hard enough if don't do well	M	***F***	23 Are not good at maths	F	***M***
9 Parents would be disappointed if they don't do well	M	***F***	24 Like using computers to solve maths problems	M	***M***
10 Need maths to maximise employ opportunities	M	***M***	25 Teachers spend more time with them	M	***nd***
11 Like challenging maths problems	M	***nd***	26 Consider maths boring	F	***M***
12 Are encouraged to do well by the maths teacher	M	***nd***	27 Find maths difficult	F	***M***
13 Maths teacher thinks they will do well	M	***F***	28 Get on with their work in class	F	***F***
14 Think maths will be important in their adult life	M	***F***	29 Think maths is interesting	M	***F***
15 Expect to do well in maths	M	***F***	30 Tease girls if they are good at maths	M	***M***

There were only eight items, it can be seen from Table 1, for which the responses were consistent with previous findings. These items were largely related to the learning environment and to peers. For example, boys were still believed more likely to distract others from their work (Item 16) and to like using computers to solve problems (Item 24). Girls, there continued to be agreement, were more likely to get on with their work in class (item 28). In the past, boys were generally believed to have more natural ability for mathematics than girls, were considered to enjoy mathematics more, and to find it more interesting than did girls. Yet the more recent data revealed that, on average, students now consider boys more likely than girls to give up when they find a problem too challenging (Item 4), to find mathematics difficult (Items 27 & 18), and to need additional help (Item 20). Girls were considered more likely than boys to enjoy mathematics (Item 6) and find mathematics interesting (Item 29). Responses on so many items inconsistent with previous findings surely implies that changes have occurred over time in gendered perceptions related to mathematics education, that, in other words, the energy expended on documenting gender inequities and attempting to redress them have left their mark. It is perhaps worth adding that the gendered perceptions captured by the questionnaires are fully consistent with interview data gathered in recent studies involving students from elementary school to university (Forgasz & Leder, 2001; Landvogt, Leder, & Forgasz, 1998).

A focus on gender and PME activities

Comparisons of research attention on mathematics and gender between the wider mathematics education and PME communities reveal an ambiguous picture. On the one hand, females have figured quite prominently in PME activities. Four of the ten presidents to date have been female (though the first four presidents were males). From the outset, females have been active presenters of Research Reports: females were sole or co-authors of more than one-third of the Research Reports presented at the third PME conference, for example.

Yet those leafing through *PME Proceedings* will observe a more subdued emphasis on research concerned with gender and mathematics among the PME community than within the mathematics education research community at large. This may be a reflection of the beliefs expressed by participants at the earliest PME conferences that issues of gender differences were considered irrelevant in their own countries[6].

Inspection of *PME Proceedings* soon reveals an inconsistency in the listings of Research Reports, with *Proceedings* editors clustering them by category in some years, but not in others. When clustering occurred, there was considerable variation in the number of articles listed under each heading. For example, the Research Reports delivered by the 70 presenters at PME2, and included in the 1978 *Proceedings,* were grouped into five themes: *The Acquisition of Mathematical Concepts, The Learning of Generalisation and Proof, Interpersonal Aspects of Communication, The Nature of Mathematical Thinking,* and *Intuitive and Reflection Processes in Mathematics.* The 1982 *Proceedings* were divided into 12 categories, with the number of entries ranging from nine (*Concept Formation*) to one (*Discovery Learning* and *Neurophysiology*); the 1993 *PME Proceedings* contained16 different categories, with entries ranging from eleven (*Epistemology, Metacognition, and Social Construction* and *Problem Solving*) to one (*Probability, Statistics, and Combinatorics*). These examples show that having few *Research Reports* in a particular category was no barrier to that topic being highlighted on the Contents pages. Yet even in years in which a number of Research Reports contained the words "sex" or "gender" in the title, inclusion of gender as a category heading was rare, with the 1984 *Proceedings* in which two papers were listed under the heading *Girls and Mathematics* a notable exception. Presumably, then, the interest for research into gender and mathematics exhibited by some PME participants was not necessarily shared by the editor(s) of the *PME Proceedings*. Greater attention to the key words provided by Research Report authors to describe the content of their paper might provide a more equitable listing in future.

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[6] I am indebted to Alan Bishop for sharing with me his recollections about the early PME conferences.

As it stands, it is no easy task to trace how the topic of gender and mathematics has been explored in PME Research Reports.

PME and research on gender and mathematics

In the time and space available, it is impossible to give a comprehensive summary of all papers with a strong gender theme delivered at PME conferences. To summarise briefly and with no attempt at full coverage of Research Reports which included "gender" or "sex" in the title:

- Compared with other topics covered, gender issues have apparently been of limited interest to those presenting Research Reports at PME
- Sample details in the early papers rarely included the numbers of males and females involved in the research study, though this information was included in later reports
- Early papers in particular contained the "females are deficient" theme (e.g., Barboza, 1984[7])
- Others contained the "male norms are the standard measure for comparison" theme (e.g., Collis & Taplin, 1984; Leder, 1986; Mukuni, 1987; Kuyper & Otten, 1989, Visser 1988)
- The variety and complexity of the methodologies and instruments used for data collection increased with time, in line with the research reported in other settings and vehicles (e.g., Underwood, 1992)
- Early findings were increasingly revisited and previous assertions about the effect of gender on mathematics learning challenged (e.g., Forgasz, Leder & Gardner, 1996; Forgasz & Leder, 2000 – gender stereotyping of mathematics; George, 1999 and Gorgorió, 1992 – gender differences in visual representation; Pehkonen, 2000 – mathematical reasoning)

As indicated earlier, this list is most aptly described as indicative of the scope of research reported at PME, and does not aim to be exhaustive. Yet it prompts an inevitable question. To judge from the contents of the Research Reports included in Conference Proceedings, would those hoping to hear cutting edge research - whether experimental or theoretical, qualitative or quantitative – be more likely to be satisfied or disappointed by the fare at PME conferences? Where are the reports of research studies, detailed in other venues, in which more radical feminist perspectives are being adopted, females are less frequently considered as a homogeneous group, and fine grained rather than collective data are presented? Where are the reports of scholarly evaluations of large scale interventions? Or detailed case studies which focus on individual rather than group differences? Or reflective accounts of the impact of the personal beliefs and theoretical orientation of the researchers undertaking the research on design of the study, data gathering decisions, choice of instrumentation? From personal experience I know that these issues are of interest to members of PME and are discussed within venues such as Discussion and Project

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[7] Papers which appeared in *PME Proceedings* are not cited separately in the reference list

Groups - group activities which have scant permanent or written records. It does not take long to decide that the format adopted for PME written Research Reports, and let me add a format carefully and sensibly selected for many good reasons, favours the reporting of studies with certain data and research designs but discourages the reporting of others. Tracing the debate on gender and mathematics within the Proceedings of PME conferences has been an instructive and, for me, provocative exercise.

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PLENARY PANEL

Theme
25 years of PME: Past and Future Challenges

Coordinator
Catherine Sackur

Panelists
Allen Bell
Jorge Falcão
Andrea Peter-Koop
Fred Goffree

Plenary Panel

25 years of PME: Past and Future Challenges

Panelists: Alan Bell, University of Nottingham, Great Britain
Jorge Falcão, University of Pernambuco, Recife, Brazil
Andrea Peter-Koop, University of Muenster, Germany
Fred Goffree, Freudenthal Institute, Utrecht, The Netherlands
Coordinator: Catherine Sackur GECO-IREM, Nice, France

Introduction

In 1977, PME 1 took place in Utrecht; PME 25, in 2001, is now coming back to Utrecht.

After infancy, childhood and teenage, 25 is the age of adulthood.

What are the characteristics of adulthood? autonomy and responsibility.

Let's leave aside the dreams of childhood, forget the passions of teenage and face the responsibilities that we have towards the international community: that of presenting a coherent body of researches in mathematics education for the future.

In this panel, four experts will analyze the work which has already been done by the members of our community, will examine the way questions and ideas have developed and will try to give guidelines for future work..
Our four experts are: a psychologist (Jorge Falcão), a teacher (Andrea Peter-Koop), a teachers' educator (Fred Goffree) and a researcher in mathematics education (Alan Bell).

Alan Bell will start with a review of major trends throughout the 25 years of PME; comparing PME 2 and PME 23 he will make clear that some topics still need investigation.
The second expert will be Jorge Falcão, whose presentation will lead us to the discussion of the contribution of psychology to our field.
Andrea Peter-Koop will talk about the necessity of research for in-service and pre-service teachers. We'll discuss what kind of research is most appropriate for them.
The first part of the panel will end with Fred Goffree's talk. Fred Goffree has been working with student teachers in various institutions. He has experienced different techniques and methods, being close to Hans Freudenthal, here in Utrecht. He intents to question the links between theory and practice, and "developmental research".

After the four presentations, we will ask you to discuss some proposals for new trends in mathematics education's research.

I went through the very rich texts which have been proposed to us, and I would like now to share with you some of the questions which came to my mind.

- You will certainly notice that two of our experts are concerned by the great number of Research Reports presented in our PME meetings, and by the difficulty to have an integrated view of the actual research in our community. Does this mean that, in the future, we should change the ways in which PME members present their research, or should "we" devote some time in building synthesis? who could be this "we"?
- Do we need more psychology to understand mathematics learning and teaching? Should there be more psychology in the training of mathematics' teachers?
- Are the concepts of metaphors and competencies-in-action useful to describe mathematical concepts? If metaphors are useful for mathematics education, could they at the same time be didactic obstacles?
- About research and teacher education (inservice or preservice teachers), there are different questions: shall we take for granted that research should be part of the training of teachers? if the answer is yes, what is the role of research in teacher education? how much research should we devoted to teacher education?
- One can observe a change in the methodology of our work: large scale researches seem to be more or less abandoned in favor of classroom researches or observation of rather small groups of students. Could this phenomena be linked to a questioning about mathematics education: are teachers supposed to "teach skills" (the evaluation being then easily feasible through large scale studies), or is mathematics education "learning to act like a mathematician" (which study requires another methodology)?

No doubt that you will have many other questions to debate with the panelists. To the **P**, the **M**, the **E** of **PME** we will, of course, add the **R** of Research.

The answer to these questions should lead us to some sort of « agenda » for the next 25 years of PME.

May this panel contribute to bring PME to its full development. ***Rendez-Vous in Utrecht in 2026!***

25 Years of PME – and the next 25

Alan Bell

To review adequately 25 years of work in PME and make suggestions for the future is a daunting task. In the time available for preparation, and for presentation now, I can only highlight some trends which I perceive; I hope to provoke every one here to contribute and to make this a review more representative of the perceptions of all of us of this very substantial and growing work of ours. I decided to look at the first published PME proceedings - PME2, Osnabrueck 1978, and to compare them with those of a recent meeting, PME23 in Haifa, 1999. focusing on questions such as

- What new *knowledge* do we now have on the themes addressed in that early meeting?
- In what ways have the *methodologies* changed?

At PME2 there were 70 participants from 11 countries, and 26 talks were given –so they were all plenary. In the one 400-page volume, some of them occupy over 30 pages; the mean is about 15. The following were among the main themes addressed; these are the ones I will try to trace in later work.

- ***Students' understanding of particular mathematical concepts***, common intuitive preconceptions & misconceptions & normal development
- ***Students' understanding and learning of general strategies*** - of Mathematics and of Problem-solving, including mathematical thinking and the place of these aspects in the curriculum
- ***Teaching/Learning Interactions*** in the classroom
- ***The Learning Process*** and the role of ***Reflection***
- ***Dissemination and Implementation in School Practice***

Students' understanding of concepts

This theme has attracted by far the greatest volume of work in PME. In 1978, Vergnaud spoke about practical arithmetic problems, requiring identification and use of one of the four basic operations. Such problems embody a much greater variety of mathematical structures than the four operations themselves, when one takes into account the actual context, the order of events, and whether quantities are involved or pure numbers, and are typically solved by students in many different ways, relating to different symbolic and diagrammatic representations..
Problems with the initial quantity unknown are markedly more difficult than the others, requiring a reversal of thought and resistance to the conventional associations of cue-words for subtraction or addition.

Studies such as this, which analyse the space of problems in a given conceptual field, and observe students' approaches to them, identifying key difficulties and misconceptions, have formed a rich vein of study throughout the 25 years of PME. For example, there have been many studies focusing on difficulties with decimals, and, more recently, on probability. At PME23, Amir & Linchevski identified a 'representativeness' misconception which fails to distinguish sufficiently between the probability of a particular *order* of TTHHTH compared with TTTTTT, and the probability of these (unordered) combinations of T and H; and Ayres & Way show how children's predictions in sampling are influenced by the success or otherwise of their immediately prior prediction.

Also at PME23, Silver reported a study of students' generation of all the different nets of a cube, identifying the modes of classification they used, This year (PME25) there is a study of the stages young children pass through in coming to make up composite geometric shapes from basic pieces. These last two were descriptive studies in classroom learning settings, in contrast to the quoted earlier ones, which used a more rigorous comparison of responses to carefully designed questions. This tension between realism and rigour is visible in many recent studies. There is also the question Are these stages obvious, or is there anything surprising about the results?

- **Do we now have enough knowledge of student concepts and misconceptions in the major mathematical fields, or is more still needed?**
- **Is such work best incorporated in teaching studies?**

Some of this earlier work has led to the development of general teaching methodologies. Our own Diagnostic Teaching Project (Bell, 1993) used a 'conflict-discussion' method, in which a lesson started with a few critical problems (chosen in the light of the previous research), some soluble in the 'obvious' way, others intended to elicit a particular misconception and thus creating a cognitive conflict which would be resolved by inter-pupil discussion. In an equal-time comparison with a 'positive-only' method, which taught the correct concepts before posing the critical problems, the conflict method showed great superiority in retention over several months, in spite of the smaller (because more intensive) coverage of the material. Carpenter's Cognitively Guided Instruction project had a similar philosophy.

The Learning of General Processes

My 1978 paper considered both general problem-solving strategies and the more mathematics-specific processes of generalisation and proof. On the former I was able to quote a study reported by Scott (1977) which showed superior results in 16 year-old maths exams for students who, in primary school, had taken part in weekly scientific problem-solving sessions, in which they were faced with puzzling science situations (eg some objects floating & others sinking) and had to find explanations by formulating questions to ask of the teacher, who would only respond Yes or No. In view of the significance of this transfer of general strategy training to concept learning it is surprising that it has not been followed up.

More recent studies of students' work in open problem-situations include one in PME23 by English on problem posing and model generation and a Discussion Group in PME22 supported by a substantial publication by Pehkonen on open-ended problems and their use in the classroom. But these were descriptive, not experimental studies of transfer. What we have (in the paper by English) is more detailed report of the students' work on one of the tasks.

On the learning of general *mathematical* strategies, there has been ongoing work focused on the learning of proof. In my 1978 talk, using a number of simple generalisations, to be found or verified for truth, I distinguished levels of proof-explanations in students' responses. Most pupils up to age 15 relied on incomplete checks of a few cases, few gave explanations in terms of more basic principles, and none reached a higher stage of using explicitly stated starting points.

Three examples can be given from PME23 of work in this field. Boero identifies four 'processes generating conditionality' which link with proving activities. This could be the beginning of a study which might develop in a similar way to that on arithmetic problems discussed earlier. In another paper, Douek explores the explicit and implicit problem-solving and proof strategies which undergraduates use; she argues that the imposition of formal proof structures is harmful and that effective proof activity depends on intellectual qualities fully developed during ordinary, demanding argumentative activities other than proving.
In this field there are broad underlying questions about **transfer.**

- **How far does the acquisition of general strategies for problem-solving, generalising and proving improve the subsequent learning of particular mathematical concepts, skills and applications – or of learning in general?**

- **What aspects of mathematics are the most important for students' education – the knowledge of particular concepts, OR the experience of exploring problems like a mathematician?**

How do we attack these questions?

The social environment for learning

These questions can be seen as relating to the **depth** of learning, since it is in the quest for deeper understanding of mathematics that we are led to think about these general strategies. Another step along the same path raises questions about the social and personal environment in which learning takes place, and how far it is attuned to the ways of thinking and feeling of the learners. A relevant study in PME23 was that of Boaler, who compared the very different mathematical learning environments in two English schools. One was traditional and hard working; in the other, the students engaged in a variety of individual and group projects in an informal social environment. This group out-performed the other in the standard exams. We cannot be sure, of course, that all other aspects of the two situations were similar, which suggests a need for a more extended study.

Teaching/learning interactions

Bauersfeld's 1978 talk identified a number of characteristic question-and-answer sequences common in mathematics classrooms; one he called 'funnelling', in which the questions led the pupils steadily closer to an answer the teacher had in mind. Subsequent work has included the study of transcripts of such exchanges, more recently using videotape as well. Such activity is clearly especially valuable for groups of teachers, who can discuss their interpretations of what they see or read, and become aware of possibly unsuspected aspects of their own practice. In PME23, this field was the subject of a plenary lecture by Steinbring and of meetings of an ongoing project group and discussion group. My question here is

Do we have here mainly a methodology for gaining insights, or are there also significant new concepts for teachers to use?

Learning Processes and The Role of Reflection

Though this is widely acknowledged as a most important theme, it has received little explicit attention. It can be argued that reflection is an essential component of the notion of *advance organisers*, of the *institutionalisation* phase of the French didacticians' teaching/learning problematique, and of the cognitive conflict which figures in the teaching experiments mentioned above. But if the psychologists are right, it is a powerful enhancer of learning which should appear in every teaching episode. Two recent projects with this focus are the Australian Project for Enhancing Effective Learning (PEEL) and our own Pupils' Awareness of their Learning Processes. (for both, see Bell et al, PME21). The former work arose from the perception that, in the typical secondary classroom, students participated rather passively in an activity designed and controlled by teacher and school, and that learning could be improved if they became more aware of the purposes of the various activities and took part in decisions about what and how they studied. This achieved some success, but was hard to establish and maintain, as it proved to require the involvement of the teachers taking the classes in question for most of their subjects. In our work, pupils devised their own tests, taught other pupils, interviewed each other on recent work and took part in their own assessment. They also discussed the purposes of different lessons, such as those for practising skills, learning concepts, developing strategies for investigation. Questionnaire results showed improvements in these perceptions, but the circumstances of the project (insufficient length, in particular) did not enable us to evaluate changes in mathematics test performance.

- **Do we need more focused studies of the ways in which reflection can enhance learning?**

Dissemination and Implementation

The general problem of assimilation of new insights into the school system is a long-term one. Jack Easley addressed it in a talk at PME2. He had conducted an intensive study of the lack of adoption of well-founded innovations, particularly in science education, reporting that teachers reject the recommendations of experts as unworkable with the students they have, and much inservice activity fails to achieve a meeting of minds. Not so many studies of dissemination have been reported at PME, in spite of its great importance. Such work requires more substantial time and resources than are available to most of our members. However, there was a welcome contribution to PME23 on this theme, in a plenary lecture by Ruthven on the development and attempted dissemination of a calculator-aware curriculum in the UK, initially led by Hilary Shuard. He concluded that a successful innovation required a very thorough analysis of the entire content, progression and teaching methods of the curriculum, and its adoption as part of a coherent and committed programme of school development and ultimately of systemic reform, rather than the isolated responsibility of individual teachers. Such a programme the National Numeracy Initiative, has in fact been developed and prescribed by the UK government, with detailed lesson programmes and considerable inservice support. However, this has coincided with a politically inspired ban on calculator use until the last two years of primary education! Such are the hazards securing useful implementation of our work.

Another substantial study was conducted by Brown and her London team, on Effective Teachers of Numeracy. A substantial sample of Primary schools was used, and each teacher's success in improving pupils' performance on the standard tests over a year was correlated with various teacher characteristics. The most successful were those who had a well-connected relational understanding of the mathematics curriculum, often gained from in-service courses with this aim.

- **How can we find out more about the processes by which new knowledge and insights eventually influence the education of our students?**

Methodological Trends

In general, there has been a great expansion in the number of works presented, and many have been on a smaller scale than in 1978. Much detail has been filled in, especially in the field of students' understanding, and more work has been done in realistic classroom settings. At the same time, there has been perhaps less work integrating all these particulars into coherent bodies of knowledge. The Research Fora give more opportunities for extended exposition and critique, and this is a significant development; but they tend to adopt their own theories, and aim to display the power of the theory to provide a framework for constructing a teaching material, and explaining the results of its use. Aspects of the theory itself are not generally tested.

- **How can we encourage more critical tests of major theories?**
- **Do we need more integrative review studies?**

The trend in ***methodology*** I see is towards more observational studies of students' responses, often in classroom settings, to some innovatory curriculum element. The purpose of these is usually to demonstrate the superior effectiveness of the innovation, but there is often no formal comparison; one draws conclusions by reference to a mental comparison with what one would normally expect in the topic in question.

This raises the question of how to read and interpret such reports. We have moved a long way from the assumption that we are establishing hard scientific generalisations beyond reasonable doubt. So the way in which knowledge accumulates for each of us is by reading many reports, each very specific in terms of topic, method, underlying theoretical assumptions, and integrating them in much the same way as we do our general knowledge of the world. I used to think of this as being like reading good novels, which generally have some general insights to convey about the human condition, but clothed in a specific story which adds interest and also convinces us by showing how such things might happen. The analogy is not perfect, because our reports are not meant to contain a fictional element! But the fact remains that what is taken from it depends on the perceptions and preconceptions of the reader.

- **Is this an appropriate way of conducting and reporting research in our field?**

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Learning environment for mathematics in school: towards a research agenda in psychology of mathematics education

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1. *A little historical introduction...*

In February 16th 1630, a Dutch naval fleet arrived at the coast of Pernambuco, by this time a province of Brazil, owned by Portugal. The Dutch army took possession of Pernambuco and three other neighbor-provinces; in Pernambuco, the historical village of Olinda (by this time the capital of the province) was easily dominated, but Dutch military responsible soon realized that this village would be very hard to keep military, because of its topological irregular relief with many hills. Because of this, Dutch governor Waerdenburch decided to burn Olinda, and to move to a small village of fishermen at about 10 kilometers at the south.

Three aspects pleased the Dutch and made this specific village their final choice for establishing a fortified military nucleus: it was topologically regular, had a very good natural sea port and was situated at the estuary of two rivers, Capibaribe and Beberibe. Besides, the medium height of the village considering the sea level was negative in some regions, what made it very familiar to people coming from Holland, "the country stolen from the North Sea" (Gonsalves de Mello, 1978).

Count Maurício de Nassau is an important character in this historical scenario: he arrives at Pernambuco in 1637, as a general governor sent by Dutch Western Indians Company. Once arrived, he decides to build a real town, *his* town, in the village occupied by the Dutch after the burning of Olinda. This town took the name of "Mauritsstadt" or "Stadt Mauritia", the first Dutch name of the village of Recife, today capital of the state of Pernambuco. Among many important initiatives in economy and political administration, Count Maurício de Nassau was also interested in public education, since he believed that the Dutch domain over Brazilian lands could not be completely established without an educational effort addressed to Brazilian natives (and Portuguese descendents born in Brazil). Because of this point of view, Nassau founded the very first public school of Recife (by this time Mauritsstad), open to Dutch children (descendents of free Dutch citizens – "*vrijeluijden*"), Brazilian-Portuguese children, Brazilian natives (Brazilian indians – specially from the tribes Tapuia and Tupi) and even Brazilian-African black children, by this time slaves. A public day school was created, where Brazilian and Dutch boys could start learning from 5 years of age; in these schools they could learn Dutch language, religion, and handicraft works ("*hantwerken*"). During this first period of education, mathematics and science were not offered: Dutch authorities believed that a long period of preparation would be necessary before offering such school subjects; besides, they believed that only few students would be able to go ahead, facing mathematical and scientific learning. There are some historical evidences showing that mathematical and scientific knowledge were considered crucial to the Dutch military and political project for Mauritsstadt and the whole province. This approach allows important Brazilian historians to consider that Maurício de Nassau founded "(...) the most modern and original cultural center not only in Brazil but also in the hole XVIIth century America (Gonsalves de Mello, op. cit.). The Dutch, nevertheless, had no time to test their political educational and strategically propositions in Pernambuco: in 1654, a Brazilian-Portuguese army has successfully taken Pernambuco back to Portuguese domain.

2. *Learning environment for mathematics in school: from past to future*

Dutch conquerors seemed to be the very first group in the history of Pernambuco to demonstrate a clear preoccupation towards education as civil right to be largely offered to all citizens (including native indians and black slaves - and not a *privilege* to be restricted to very few aristocrats and their descendents). They highlighted some other important aspects: 1. Efforts in order to offer an adequate curriculum at school; 2. Dialogue between school/formal knowledge and outside-school knowledge; 3. Place of mathematics and science in school curriculum (*what* mathematics to offer, and *when*). Today, 347 years later, these aspects are far from being implemented, particularly when we think about schooling in the context of a poor state (Pernambuco) in a so-called "emergent country" (Brazil). Important psychological aspects are connected to these questions, being good issues to a research agenda in psychology of mathematics education. These aspects are mentioned and discussed in two main sections below. Discussing them in this Dutch PME, nowadays, is an amazing task for a psychologist from former Mauritsstadt, today Recife (Pernambuco), with far family roots in the Dutch community that tried the *Mauritsstad utopia* in these tropical and luxuriously beautiful lands.

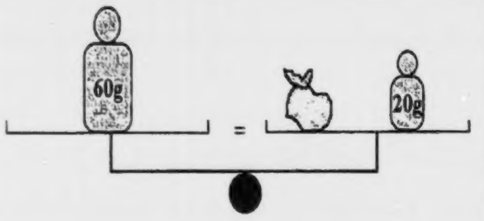
2.1. *Conceptualization in mathematics*

2.1.1. Mathematical concepts as *models*, instead of taxonomic categories.

- ***There are not "mathematical essences" (radical anti-Platonic perspective).***
- ***There are not exemplary-concrete cases for mathematical concepts: mathematical concepts are metaphorical models.***

According to G. Lakoff and R. Núñez, there is a strong "mythology" concerning conceptualization in mathematics, a kind of "romance of mathematics" (sic), according to which mathematics would be "abstract and disembodied" (yet it *is* certainly real), having "objective existence", providing "structure to the universe" and being "independent of and transcending the existence of human beings or any beings at all" (Lakoff & Núñez, 2000, pp. xv). We refuse this belief about mathematics, proposing, in accordance with these authors, that mathematics comes from *us* (human beings), instead of from heaven or the outer universe. Mathematics is definitely a human construction, not the way towards platonic-transcendent truths of the universe. By human construction we mean, quoting once more the authors above, not (...) a *mere historically contingent social construction*", but "(...) a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, *and our long social and cultural history*" (Lakoff & Núñez, op. cit., pp. 09; italics added). Two important conclusions emerge from this epistemological assumption: first, mathematical concepts are rooted in human bodily realities, as well as in socio-cultural activities; second, the learning of these concepts can not be limited to the experience of receiving axioms and developing (by direct imitation, perhaps) the "right" way of thinking (mathematical proof).

We agree with philosopher E. Cassirer propositions about conceptual development, specially when he stresses the role of *modeling* in concept formation and development (Cassirer, 1977). In fact, concepts in general are much more than descriptive, taxonomic categories based on common aspects, as proposed by many psychologists in the past (see, for example, the classic research work started by E. Heidbreder (1946), followed by Bruner, Goodnow & Austin (1956), and many other psychologists up to present days). This is especially true in the case of mathematical concepts; from a mathematical standpoint, mathematical concepts express *functions*; from a psychological perspective, they express *relations* issued from perceptual input and gathering. These relations are psychologically represented through symbolic tools like natural or formal language, in many degrees of combination. These combinations open the way to the didactically important psychological activity of *metaphorization*, as illustrated below:



60g = x + 20g then
60g − 20g = x + 20g − 20g (because of the metaphorical connection between the balance scale and the principle of equivalence in the treatment of algebraic equations).

40g = x (then, the unknown weight is 40g, which is the final step of the algebraic algorithm).

Figure 1: The metaphorical connection between balance scales and algebra (Reproduced from Da Rocha Falcão, 1995).

Symbolic representation is a key psychological aspect in the development of algebra and many other *conceptual fields* (Vergnaud, 1990) in mathematics because of two points: first, it is not a result or superstructure of operational structures, as proposed in the context of Piagetian theory (Piaget, 1970) but rather a constituent of concepts, with operational invariants and situational links that gives socially shared meaning to knowledge (Vygotsky, 1985); second, it opens to a particular individual a wide range of symbolic cultural tools that, as *cultural amplifiers* (Bruner, 1972), enables one to access new instances of conceptual construction. So, representations provide ***metaphors*** that can be useful as pedagogical tools; these metaphors help in amplifying pre-existing schemes, since they provide semantic links between structured knowledge and new pieces of information. In this process of enrichment of meaning, a quite important psychological sub-process is represented by the explicitation of *theorems-in-action* (Vergnaud, op. cit.), upon which are established many practical competencies exercised in daily life. The proposal of the two-pan balance scale represents an effort of offering a metaphor of algebraic equivalence between equations, based in the conservation of a pre-established functional equality between each side of an equation. The construction of meaning for the equivalence of equations (essential aspect for the comprehension of algebraic algorithms) is initially connected to the familiar idea of *equilibrium*, in the context of a culturally familiar artifact, the balance-scale. This idea of equilibrium is frequently poorly explicited, although people can make a competent use of a two-pan balance scale in order to sell or buy fish in Brazilian popular markets; nevertheless, equilibrium as theorem-in-action is based upon *explicitable* principles. As a metaphor, the balance-scale offers a context of cultural functioning where complex mathematical concepts (algebraic equivalence and algorithmic manipulation) can be initially rooted in competencies and theorems-in-action, enriching pre-existing schemes. This last aspect, concerning the mobilization of competencies-in-action for conceptualization in mathematics, led us to the next point of this brief theoretical navigation.

2.1.2. Concepts as schemes are not only limited to the language domain, but also cover *competencies-in-action*.

- ***Mathematical competence (as many other complex social competencies) cannot be restricted to the domain of language: language constitutes cognition, but cognition is larger than language.***

The theoretical position defended here tries to establish the relations between thought and language in the context of what we propose to call the *psychology of human gesture* (Vergnaud, 1998). The human gesture, in its functional richness, was once part of H. Wallon's theoretical contributions, who distinguished three levels of evolution development: the *automated* or *habitual* gesture, the *adaptative-motoric gesture* in interaction with the external ambient, and finally the *symbolic gesture*, connected to an objet not empirically present (Wallon & Lurçat, 1962). The gesture, as a theoretical subject in cognitive psychology, is directly connected to the notion of *scheme*, as defined by Gérard Vergnaud: it is a prototype of human cognitive activity, present in all periods of human development, from childhood to adult life (Vergnaud, op. cit.). As an organizing and organized structure based on schemes, the human gesture is characterized by the following aspects: a) a *goal*, eventually divided in secondary or sub-goals: a pre-linguistic 18 months old child can look for a small ball which has disappeared under a table, making bodily gestures with his/her head towards the direction he/she imagines the small ball is going to re-appear (the opposite side of the table under which the child saw the ball disappearing); b) *an ordered set of actions:* the same child in the example above will be able to organize his/her actions in order to achieve his/her main goal; c) *identification of material tools and empirical information*: in his/her efforts to recuperate the small ball, the same child will be able to take into account situational variations and constancy, like those concerning the speed of the ball, length of the table under which the ball moves, and so on; d) *calculations concerning the actions to be performed, information to be coordinated, controls to carry out:* a gesture action, when performed many times with the same goal, can be reformulated in the course of the action itself, in order to take into account a certain degree of situational variation. In other words, the child is able to modulate his/her gesture actions in the context of a certain (and necessarily limited) spectrum of variations.

The previous considerations allow us to refine the definition of human gesture in terms of an invariant organization of behavior for a limited class of situations. It must be said that this same definition can be used to schemes in general and *concepts* in particular. Among human schemes, it is important to mention the *competencies-in-action*, like those shown by handicraft workers, but also some competencies of very high-level (in terms of their *educational capital*) workers (researchers, engineers, specialized technicians, and so on). These so-called competencies-in-action have two major characteristics: firstly, they are *effective*, in the sense that they are cultural tools for daily (and culturally situated) life; secondly, these competencies are very hard to *express* by any symbolic means (natural language, graphic representations, mathematical models, and so on). Taking these competencies into account in mathematical conceptualization implies in connecting school activity to other socio-cultural contexts, as discussed in the next section.

2.2. Learning mathematics as a cultural activity: interest of negotiations between teacher and students.

- ***Didactic contract is a very effective variable in the process of teaching and learning of mathematics***
- ***Argumentation (seen as the process of changing of opinion in an interlocutory context) is a key process in the learning of mathematics, but it is not the only effective (pedagogically speaking) process.***

Mathematical teaching and learning, as a specific cultural activity, is organized by certain consensual rules, most of them not explicitly expressed. These rules express previously established goals (both behavioral and attitudinal), giving rise to a *didactic contract* that guides social activity in the classroom (Schubauer-Leoni & Perret-Clermont, 1997). Examples of such rules are, for example, "a mathematical problem has always one (and only one) answer", or "the teacher always knows the answers for mathematical problems". There is a clear interest in the use of these rules as *didactic variables* (Brousseau, 1998), in a research context, since many psychological schemes can be addressed and amplified in the context of classroom activity. It seems to be possible, as shown by Da Rocha Falcão and colleagues, to propose certain activities by previous contract, even if these activities are completely discrepant concerning usual school habits and rules (Da Rocha Falcão et. al., 2000). These researchers were able to propose symbolic manipulation of literal symbols, in the context of algebraic sentences, to very young school children; the classroom contract that allowed this activity is illustrated below. The students were invited to compare two plastic boxes containing different and unknown quantities of small balls, these deposits being covered with paper in order not to allow visual inspection of the quantity of balls inside them. Two students were invited to hold the deposits in front of the group; each of these two students was "marked" by an icon of a happy face (☺) for the one owning more balls, and a sad face (☹) for the other owning fewer balls (the icons were drawn in sheets of paper and put on the floor, in front of each student: see illustration on the left). It was then proposed to the students that we would represent the unknown quantity of balls by a *letter* (they suggested to adopt the first letter of the name of the student holding the plastic box). All the group of students was able to model a relationship using letters to represent the quantity of balls in each deposit: $G > J$ as well as the passage to the equality considering the difference (if $G > J$, then $G = J + D$ or $J = G - D$, representing the unknown difference by the letter D). When the protocols of the activity above were closely examined in terms of argumentative exchanges, we could realize the occurence of three kinds of discoursive actions, according to the analytical model proposed by S. Leitão (see Leitão, 2000): 1. *Pragmatic actions*, proposing situational condictions in order to allow argumentation; 2. *Argumentative actions*, i.e., speech turns adressing the negotiation of divergent points of view; 3. *Epistemic actions*, consisting of offering specific information about the topic in discussion. These three actions were not equally distributed in the didactic sequence proposed (Araújo et. al., 2001): there was an preponderance of epistemic actions, in the context of which the teacher offered information and tried to justify what he was proposing (allowing the students to ask questions and to express their doubts). These data show that even when we make special didactic efforts to propose a certain topic, there is not any guarantee that argumentative speech-actions will occur. On the other hand, even though we strongly believe in the role of argumentative actions in knowledge building, these specific speech actions are one among other possibilities in the context of discursive (and effective) pedagogical exchanges in the context of classroom.

3. Conclusions and final remarks

Research about conceptualization is seen here as central in a research agenda proposed by psychologists, in the interdisciplinary field of mathematics education. The psychological nature of mathematical concepts, the context of development and learning of them and the place of symbolical representation in concept building are important theoretical domains for psychological inquiry. For Dutch conquerors, in the XVIIth century, education in general, and scientific / mathematical education in particular, were crucial to build a modern society, based on free opinion, trade and religion. For us, psychological researchers in mathematical education from a "melting-pot" country like Brazil, these issues are crucial for us to survive inside mathematics education research community.

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FROM "TEACHER RESEARCHERS" TO "STUDENT TEACHER RESEARCHERS" – DIAGNOSTICALLY ENRICHED DIDACTICS

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Being in the "age of adulthood", our professional responsibility also requires that we focus our attention and efforts more closely on the dichotomy between classroom teachers and researchers. This seems to be of crucial importance since numerous studies report that teachers often view research as having little relevance to their practice or specific classrooms. Therefore, inservice activities designed by educational researchers based on (their) research results frequently fail to "achieve a meeting of minds" as Alan Bell has phrased it in his previous paper. Furthermore, looking at the historical issues of "teachers as researchers" Dawn Brown (1997) stresses the fact that classroom research has traditionally been viewed as the domain of researchers and university lecturers and NOT as that of classroom teachers. Despite the acknowledgeable efforts of individuals, I think this view is still shared on both sides by many teachers and researchers.

I do not want to deny the importance of research from an "external" perspective, i.e. with respect to the psychological aspects of learning environments in school mathematics that Jorge Falcão has outlined and which clearly require the expertise of psychologists in order to document and analyse respective phenomena. On the other hand, teacher inquiry does not only foster the development of their own professional skills but can offer insights, i.e. on patterns of student interaction, while constructing mathematical meaning, based on long-term data that is hardly accessible by temporary visitors to the classroom, or even during clinical interviews. Therefore, their findings inform mathematics education research and curriculum development.

In the following I will argue as to why I think that teachers as researchers and especially student teachers as researchers should be given more attention with respect to teacher education in the future. But before I elaborate on this notion and our experiences with this approach, I briefly want to "look back" and highlight the achievements and contributions of PME members that inspired and guided the work with student teachers.

Teachers as researchers: The work of PME members

During the past 25 years an increasing number of single research reports, short orals and posters focussing on the challenges faced and insights gained by teachers` research but also on its constraints have been presented and discussed and this panel presentation does not allow me to report all of them or even a selection. However, this work is well documented in the PME proceedings and therefore accessible to everybody interested in this facet of mathematics education research and development. Instead, for my "look back" at previous PME work I want to refer to the treatment of this topic over a period of time in a collaborative setting.

Between 1988 and 1996 a group of PME members continuously worked *on* and *in* the context of teachers as researchers. A discussion group initiated by Stephen Lerman and Rosalinde Scott-Hodgetts transformed into a working group in 1990 as interest grew and membership stabilised. Over the years, the participants of the working group explored the dialectical relationship between teaching and classroom research in the belief that mathematics teachers can and should carry out research in their classrooms. In 1997 their work resulted in the publication of a book titled „Developing practice: Teachers` inquiry and educational change" which was edited by Vicky Zack, Judy Mousley and Chris Breen. The 18 chapters of the various authors working in different cultural settings document quite clearly that the teacher as researcher approach has led to an impetus and encouragement for teachers to actively participate in educational inquiry, its increasing integration in inservice training programs, and the need for collaboration among teacher-researchers themselves and with university-based-researchers and administrators. These trends with respect to mathematics teacher professional development are also reflected in two other publications that have resulted from PME working groups focussing on teacher professional development in the last decade – however these books do not primarily focus on teacher-researchers:

- "Mathematics teacher education. Critical international perspectives" (Jaworski, Wood & Dawson 1999)
- "Working towards a common goal: Collaborative paths in mathematics teacher education" (Peter-Koop, Begg, Breen & Santos-Wagner, forthcoming)

In addition, the chapters in a book on teachers` inquiry (Zack et al.1997) highlight a number of important questions and desires which characterise and determine teacher research and which are also relevant for the notion of student teacher research.

The difference between teacher research and teacher experience

A frequently raised question is concerned with the difference between teacher research and experience, since (most) teachers carefully plan units and individual lessons, implement these plans, observe and reflect on results and consequently adapt their preparations and actions accordingly. In this context, Mousley (1997, 2) understands research in a similar way to algebraic expressions which have been generalised from particular situations:

> *I believe that research transcends the limitations of experience, in that what is learned from experience is abstracted and articulated to the extent that it can be applied in new areas (i.e. elsewhere in time or place or content etc.) ... Teacher inquiry also transcends experience in that it involves more than making space in our lives for alternative practices: it entails the construction of a new level of thought.*

Reflection seems to play a crucial part in the research process which clearly differs from reflective processes involved in experience. Kemnis (1985) stresses the political dimension of reflection in teacher research, pointing out that it goes beyond inward looking and arguing that it has meaning in relation to historically embedded contexts – thus it stimulates social change as well as personal change. Another difference between individual experience and research is concerned with the need for reporting the design, methodological approaches and outcomes of research. Student teachers who under the supervision of an university researcher engage in classroom research as part of their initial teacher training program, at the same time learn how to reflect and report their findings in an academic manner.

The chapters by Eileen Philipps and Vicki Zack clearly demonstrate the benefits of teacher inquiry for the teacher professionalisation process. Phillips (1997, 16) in this context uses the metaphor "to make the transition from searching to researching" while Zack (1997, 181) characterises researching from the inside as "generative and transformative". With respect to introducing student-teachers to classroom research that could mean that one can plant a long-term seed towards teachers' own research which can foster and sustain improvements in mathematics teaching and learning. However, both Philipp's and Zack's chapters also stress the enormous challenges and commitments that the engagement in teacher research puts on an already demanding profession – "one is speaking of two jobs" (Zack 1997, 187).

The need for different levels of support for teacher-researchers

Support and acknowledgement from university-based colleagues as well as school administrators are therefore crucial factors. Valero, Gómez and Perry (1997) in their chapter recognise that administrators need to be involved in school-based research in two ways – to contribute their expertise and to provide the necessary support for the teachers. However, Barbara Jaworski (1997, 178) furthermore raises an important question: "In what ways can the development of teaching through teacher research be a concept originating with and driven by teachers themselves?". Chris Breen (1997) obviously shares a similar concern with respect to teachers who want to focus on their classroom as part of postgraduate studies and stresses their vulnerability which leads them to engage in research projects which they originally would not have chosen themselves. Student teacher researchers therefore should be encouraged to develop their own research questions as they grow professionally. The experience of collaborative relationships with peers might help to assist with the development of further research questions and the implementation and reporting of this research.

Student teachers as researchers

The involvement of student teachers as 'teacher researchers' concurs with the idea of developing a community of practice (Lave & Wenger 1991) in which they can experience how scientific analyses can help enlighten them to those aspects of classroom practice, professional skills and knowledge that they personally perceive as important (Jungwirth et al. 2001). In the context of a pre-service education system, student teachers can benefit in a similar way to classroom teachers. The Austrian/ German mathematics education researchers Jungwirth, Steinbring, Voigt and

Wollring (2001) for example have shown this with respect to the 'interpretative classroom research' approach. They found that interpretative studies carried out by teacher researchers can help to reveal what is hidden in practice and what one has to understand in order to learn how to teach effectively and/or to change mathematics teaching. Respectively, Krainer (1998, 7) notes a current international trend in teacher education:

> *There are more and more international reports about involving (prospective or practising) mathematics teachers in research projects and integrating research components in teacher education courses where reflection and networking are important dimensions.*

The participation of student teachers[1] in qualitative mathematics education research projects has become increasingly popular in several German universities. The current trend towards the integration of student teachers in qualitative research projects from the perspective of the responsible university lecturers has the following reasons:

While the student teachers are usually concerned with one sub-question within the research project, they also become familiar with the global research interest, methodological considerations and the analysis of a substantial part of the data because they work in co-operation with fellow student teachers who are also involved in the study.

They have the opportunity to draw their own conclusions and consequences with respect to research results obtained from their own and their fellow students' involvement. Therefore they are not solely dependent on research findings described in the literature and/or lectures.

They become sensitive to empirical findings and the respective research designs. Furthermore, they frequently experience how difficult it is to translate a supposedly rather simple empirical question into an appropriate research design, how many specifications are comprised in such a design and how differentiatedly a respective finding has to be assessed.

It is also expected that the active involvement in a didactical research project will help them not only to develop a rather 'imperative' perspective with respect to mathematics education in the sense of learning how to teach (best) but also in addition a `diagnostical´ perspective. This means that the students should learn what to expect in certain teaching and learning environments in order to adjust their instruction and individual student support accordingly.

This approach which is referred to as 'diagnostically enriched didactics' corresponds with recent developments with respect to the improvement of mathematics teaching and learning at the school level. Reform-oriented curriculum documents which follow a constructivist framework describe desirable teaching and learning environments in the mathematics classroom as open towards a variety of different ways to solve a set goal (a specific task or problem etc.), supportive of the different individual approaches and strategies, and communicative with respect to the discussion of individual approaches and results in order to reflect on sensible and necessary standardisations in mathematics.

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[1] The student teachers usually use the results of their interpretative analyses as a basis for their *first teacher state exam* thesis which clearly has a scientific character and which together with written and oral exams concludes the university teacher based training program.

In order to be able to manage the variety of different approaches chosen and explored by the pupils, the teacher has to be able to assess how each of these approaches can contribute to and influence the joint discussion. Therefore he/she will need diagnostic abilities on the basis of mathematical subject knowledge as well as knowledge about the social-interactive dimension of mathematics learning. Furthermore, it is a highly desirable aim of teacher education programs to enable prospective teachers to realise and understand the 'universal in the special case' and to approach the special case in a diagnostic way. Such a diagnostic approach is required in the daily practice of classroom teachers. This explains why classroom oriented research (and especially projects that include student teachers as qualitative researchers) should not solely focus on 'typical' effects in the sense of frequent classroom occurrence but also reflect on significant however less frequent cases. These cases and not so much the rather common and widespread ones present crucial challenges for teachers.

Interpretative classroom research

Most of the current German mathematics education research projects that involve student teachers are based on the 'interpretative research paradigm'. This methodological approach was developed by Heinrich Bauersfeld and his colleagues (1988). Interpretative classroom research seeks to investigate typical structures by analysing single cases which are regarded as exemplary. Their focus is the 'universal in the special case' and the goal of the interpretation is to comprehensively perceive and understand the (inter)actions of the observed individuals.

The significance of the interpretative research paradigm is related to an international change from content-based and individual-psychological approaches towards inter-personal human relations in (mathematics) education in the past decade. Current theoretically based research contributions increasingly stress the social dimension of both mathematics (Davis & Hersh 1981) and mathematics learning (Steffe et al. 1996) for the development and extension of mathematical knowledge.

The data collection and interpretation phases of studies employing the interpretative classroom approach usually follow a strict procedure consisting of four stages:

1. video recordings
2. comprehensive transcriptions of the video recordings with respect to either the full document or selected segments of the recording that are relevant to a respective research question(s)
3. the *sequential interpretation* of the data by an 'interpretation team' of four or five individuals (student teachers, teachers, teacher researcher)

4. the *specific interpretation* of the results on the basis of relevant literature and research findings by an individual student teacher researcher.

Interim findings with respect to student teachers' learning processes

According to the experiences of the author and fellow German colleagues, the student teachers' perception of the observed 'classroom research reality' demonstrates several phases (Wollring 1994). Immediately after the recording the participating students seem to underrate the richness of the pupils' contributions while they tend to overrate their own moderation and instructional abilities. Quite frequently they express their initial disappointment with the quality of the data collection and question the suitability of the data sample for in-depth-analysis. During the first viewings of the video document however these perceptions start to change and the performance of the children involved in the study appears to be richer than first envisaged. This impression often continuously increases during the transcription process when suddenly very informative and highly differentiated perspectives on the observed classroom episode arise. The following interpretative analyses frequently lead to the identification of further 'deeper' i.e. more differentiated/specialised research questions.

The value of the participation of the student teachers as 'teacher-researchers' in a current study on third and fourth graders' co-operative problem solving strategies (Peter-Koop) as well as in Wollring's (1994) study on kindergarten and elementary school children's understanding of probability is reflected in the self-evaluation of their work. An evaluation questionnaire that was given to the student teachers after the completion of their sub-project addressed the following aspects:

- individual motivation for their involvement in this interpretative project
- their learning about the underlying mathematical topic
- their experiences with co-operative learning during group work
- their dealing with the technical requirements of the study
- their experiences with the preparation of transcripts and data interpretation
- the benefits and difficulties of peer co-operation during the interpretation process
- their reflection on pupils' learning and their individual teacher behaviour.

Conclusions

In summary, the analyses of the student teachers' retrospective responses to the evaluation questionnaire suggest that the benefits of the interpretative classroom approach with respect to teacher preparation can be seen on three different levels, which of course may partly overlap:

The student teachers learn about an important aspect of elementary mathematics. However, one can argue that other learning environments within teacher education courses also facilitate learning about mathematical 'content knowledge' as well as 'pedagogical content knowledge'. Their statements demonstrate that the student teachers learn to 'listen' to pupils with respect to their thinking and – in Peter-Koop's current study – their collaborative problem solving strategies.

All student teachers who participated to date indicated that they appreciate their involvement, because it provides them with opportunities for intense observation of children and children's learning. They deal with and reflect on real examples of pupils' behaviour, learning, interaction etc., which are believed to be more powerful than examples created by the lecturer (Jungwirth et al. 2001).

Finally, research designs that are based on the interpretative approach can enable the involved student teachers to learn about themselves as teachers. During the interpretation stages most student teachers used the opportunity to critically reflect on their individual classroom behaviour, interaction and instruction skills that became evident during their assistance of the pupils' group work. One student teacher's conclusion, which is representative for the majority of the replies, highlights the importance of active involvement and the opportunity for personal reflection of one's actions for the individual student teacher:

> *In my opinion interpretative analyses are an important addition to lectures and school practicals. But you have to conduct the teaching yourself, be responsible for the transcription and actively involved in the interpretation of your transcript. Only then you can find out how well you can relate to children, learn about your mistakes and how children react to you.*

Acknowledgement

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A consumer's point of view

Fred Goffree

1 What are we aiming for exactly?

There have been twenty-five PMEs since the forming of the study group in 1976 (Karlsruhe) and the first meeting in Utrecht. Were the initial expectations too high? Probably, although I can't produce any figures from the PME 1977 proceedings that would prove the validity of this assumption – they could not be found anywhere. What no one certainly would have expected was the tremendous growth of the PME community in those 25 years. The proceedings from Osnabrück 1978 (a very modest brochure) to Hiroshima 2000 (a four-part, wonderfully finished volume) all bear witness to that. As a comparative PME outsider, I had to make do with the proceedings. Arranged next to each other in the FI bookcase, they take up almost a metre of shelf space, so I needed seven-league boots to get through them. Fortunately, to provide an answer to the question 'what are we aiming for exactly?', I had already chosen a context, a domain, a position, a perspective and a viewing direction:

Context	Primary teacher education
Domain	Research in primary school mathematics
Position	Active consumer of research (processes and outcomes)
Perspective	Can the education of primary teachers gain more from PME research?
Viewing direction	Where theory and practice come together

To run ahead of my findings, I can say right away that there are indeed prospects, but only on condition that researchers opt more frequently for developmental research as defined by Hans Freudenthal and as conducted in the institute bearing his name. Besides that, other methods of reporting need to be learned, maybe using IT potential.

2 Background

It is obvious that the above did not appear just like that; as is the case with every researcher, my personal history did enter into it. Although researchers have the tendency to side-step personal issues in their work, I would like to say something on that score.

After ten years of designing maths lessons whilst working as a teacher educator with student teachers at a Teacher Training College in the Netherlands, round about 1968, I joined the CMLW, the predecessor of IOWO (Institute for the Development of Mathematics Education) that later became the Freudenthal Institute. The ten IOWO years were dominated by the Wiskobas (mathematics in primary schools) project, which modernised mathematics education in the Netherlands. The working methods of developmental research flourished in the Wiskobas team: idea, design, debate, try-out, reflection, discussion, revision, etc. Practice and theory converged in this developmental research and, in addition to that, new theories were framed. As Freudenthal used to say, 'like sparks flying off real work', or, as Leen Streefland later put it: 'The development of a researcher is the source of theory formation.'

I was one of the two members of the Wiskobas team who were mainly interested in the effects the developmental research activities had on teacher education and I learned to see educational research work from the teacher education perspective. Not only were practical results and theoretical conclusions of interest, so was the cycle of designing and research in the classroom. Piaget's method of clinical interviews with children was adopted and modified to Freudenthal's method, which had become known from publications as 'Walks with Bastiaan'. In those day, instead of teaching the 'class as a whole', student teachers conducted conversations with small groups of children about mathematical tasks or phenomena from daily life which they attempted to link to mathematics with the children.

After the IOWO had closed, on 1 January 1981, I wrote a 3-volume course book for student teachers, 'Mathematics & Didactics', a reflection on my 10 years of research and development work. The book included admirable designs, discussions of theory in the team, practical experience with it in class, comments made by pupils, to name a few. That way, students were invited to experience for themselves the Wiskobas routine – linking theory and practice, the recurring bottleneck in this corner of the scientific community, as a matter of course.

Midway through the 1990s, the work done by Lampert and Bell in Michigan (1998) offered a new stimulus to research and development work on teacher education in the Netherlands, resulting in the Mile project. Wil Oonk will be telling us more about this project in his presentation.

3 The (pre-service) education of primary mathematics teachers.

The first conference of a new European research community (in formation) was held in 1999, in Osnabrück: European Research in Mathematics Education. Working Group No. 3 was engaged in research work on mathematics teacher education. (Krainer, Goffree and Berger, 1999). There we distinguished between:

1. Research in the perspective of teacher education (the researcher does not have teacher education in mind).
2. Research in the context of teacher education (the researcher is a teacher educator, a teacher or a student teacher).
3. Research in teacher education (teacher education is itself the object of research).

Keeping my five special points of interest in mind, I glance through the list of proceedings of 24 PMEs.

Most hits fall under category 1. My attention was drawn to the paper by Bednar and Janvier (Bernadette), which they presented in Warwick (1979): 'The understanding of place value'. In my view, the points that receive special attention in this paper are of relevance to student teachers and the way the research is presented and written certainly has student teachers in mind. I would mention the following:

- The mathematical beginning: clarify the notion of place value (content knowledge, according to Shulman 1986).
- Next, a didactical follow-up: what is meant by 'understanding place value' (pedagogical content knowledge).
- Apply this knowledge to design a series of problems:
  - for diagnostic testing;
  - for learning units.
- Single out the main difficulties and strategies of the children.

What is it about this research that makes it so appropriate from the teacher education perspective?.It takes as its point of departure the student teacher's own level of proficiency (what is place value?). The topic (understanding) is then 'didacticised'. Following that, the 'theory' is applied to the design of teaching after which focused tests are done with children. The long and the short of it is that previously acquired (theoretical) knowledge is applied both for the benefit of practice and in the practical situation itself.

Whilst saying this, I could also cite Alan Bell, who also ventured into this field at the PME in Berkeley a year later: 'Designing teaching in the light of research on understanding'. So this is about theoretical knowledge that has been produced in research done by 'others', and can be regarded as a continuation of what Bednar and Janvier proposed.

In the 1997 proceedings my attention is also attracted by several papers related to the use made of 'clinical interviews' in the spirit of Piaget and Freudenthal ("Some reflections on the construction of the idea of number in 6-year-olds", "How does reflective thinking develop?", "A child's eye view of learning mathematics. Has Piaget found it?").

4 A philosophy of primary mathematics teacher education

The way in which I have picked out the best bits of PME research in the examples I have given above has a lot to do with my outlook on teacher education. As I mentioned before, I wrote a course (work)book for student teachers based on my experiences as member of the Wiskobas team (1971-1981). That was why in the early 1980s I was frequently on the look-out for useful material in research and development work. Material I then found suitable was material that illustrated, as it were, my vision of training, which itself was still in the making. Now, a good many years, development projects and investigations later, I am able to enumerate the building blocks that constitute my perception. Together, they form the underlying criteria for picking out the best bits, far more so than the 5 limitations I referred to previously. In connection with this, I would like to bring them to the fore to use them as an impetus towards the wishes I want to give to PME.

- As a school subject, mathematics provides excellent working material for prospective primary school teachers. (PML 1998)
- Where possible, practical situations form a starting point in teacher education.
- Elementary school practice is not solely represented by what crops up during fieldwork. Digitised teaching episodes (real teaching and real TV) make the study of practice possible. (Goffree & Oonk, 2001$^{a\ en\ b}$)
- Student teachers 'construct' practical knowledge through the investigation of real teaching practice. (Lampert & Ball 1998; Oonk 1997)
- 'Practical knowledge is what 'moves' the teacher. It is a way of narrative knowing'. (Gudmundsdottir 1995)
- Theory becomes integrated in practice (situations) by following Donald Schön's 'reflective conversations'. (Schön 1983)

The latter especially is of vital importance because teacher educators can demonstrate their expertise in the form of 'theory in action'.

I hope therefore that the work done by researchers will yield multiple practice situations complete with theoretical reflections by the researchers themselves. The researcher could, in his reflections, also include his own development and the onset of theory formulation.

The PME 1985 proceedings again caught my eye. It was one of the two meetings I attended in person. Richard Lesh presented his 'idea analysis', which had been worked out two years earlier in the book (Lesh and Landau 1983) that made a profound impression on me. How do mathematical concepts come about, from intuitive notions to formal mathematics, the influence of education and environment, and so on. Food for the reflective practitioners among the teacher educators, was my first thought.

5 Narrative knowing

Practical knowledge is therefore to a considerable extent made up of practical episodes. I had also collected a wide selection of episodes of my own when I wrote my Mathematics & Didactics. Those are written episodes about education in the classroom, with pupils and a teacher. (Among the favourites were John Holt, Herbert Ginsburg, Erlwänger). Later on, the filmed episodes from Mile were added, which the student teachers are actually required to use for making up their own episodes.

At the time, I opted for a very short episode that Alan Bishop showed me whilst we were working together in BACOMET on the chapter 'Classroom organisation and dynamics'. (Christiansen, Howson and Otte 1986). You can find it in the 1985 PME proceedings. I cite the episode and a possible reflective conversation. The background to this is the question to what extent PME research can contribute to realising such episodes + theories, which contain relevant practical knowledge (prospective) teachers have. (Harvey et al. 1982, p.28).

David

D	15's odd and q 1/2's even.
RH	15's odd and q 1/2's even? Is it?
D	Yes.
RH	Why is a 1/2 even?
D	Because, erm, 1/4's odd and 1/2 must be even.
RH	Why is 1/4 odd?
D	Because it is only 3.
RH	What is only 3?
	A 1/4.
RH	A 1/4 is only 3?
D	That's what I did in my division.

At this point another child joined in to explain to the teacher:

R	Yes, there's three parts in a quarter, like on a clock. It goes 5, 10, 15.
RH	Oh, I see.
R	There's only three parts in it.
RH	Oh, so you've got three lots of 5 minutes makes a quarter of an hour.
D	Yes. No. Yes, yes, yes.

A reflective conversation with the situation
Here, 'David' is the situation and in his reflective conversation, the educator can approach this from three different angles, thinking out aloud.

"Starting from the structure of pedagogical content knowledge:
It is possible to envisage 'models' between the intuitive notions of children and formal mathematics, which make connecting the two easier. These models are sometimes given in the book or provided by the teacher (just think of....), but sometimes the children think up models themselves. That is what David does, he links the dial of a clock to fractions. This is not really that extraordinary, several educationalists did the same thing with fraction circles, except that, in principle, they did not include the ready-made 12x5 minute structure. A lucky thing for David that

you can divide the number 12 (the dial) up in so many ways. Just think how many fractions are possible using it. (...)

But there is more than that. We can learn something from the teacher as well. He attempted to project himself into the role of pupil, on the basis of his own knowledge of fractions (educational content and teaching method). That didn't go very well. We can understand how and as a result of what he was set on the wrong track. But take care, he did not proceed on the assumption that David was talking nonsense. An important didactic principle: Do not presume that a pupil just says things off the top of his head. He has thought it through, although it might be completely wrong. (...)

Imagining the thought process pupils go through is vital in arithmetic and mathematics and indispensable in interactive lessons and discussing solutions afterwards."

6 As a conclusion

Here, I was adopting the teacher educator position and that of a consumer of educational research. I take an eclectic stance as I draw what I want from the rich source that can be used. My wish to create practice situations and to couple them to a (descriptive, explanatory or creative) theory so as to enable student teachers to acquire (construct) adequate knowledge of practice, calls for research to be done on the shop floor of mathematics teaching. The researcher, who, with regard to teacher education, is customer-oriented, allows none of the data to go to waste and he also keeps his own reflective notes and thoughts on his personal development during the research work. Video and audio recordings can be made (using information technology), anecdotes, remarks made by pupils, their written work, teachers' logbooks, tests and marks, and any other things that make mathematical classrooms a rich fund of learning for children and their future teachers.

My wish is that future PME research be conducted and reported in large part in the way outlined above, from the perspective of teacher education.

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MATHEMATICS EDUCATION IN THE NETHERLANDS

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"POLDER MATHEMATICS"

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MATHEMATICS EDUCATION IN THE NETHERLANDS

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Sheltered behind its dikes, the Netherlands more or less escaped the New-Math wave that swept the world in the 1960's. Inspired by Freudenthal, the Netherlands developed its own brand of mathematics education, currently known as realistic mathematics education (RME). His adagio of "mathematics as a human activity" was worked out by members of the Freudenthal Institute and its predecessors, IOWO, and OW&OC. New ideas have taken shape in prototypes of instructional sequences that are construed in developmental research (or design research). Teacher enhancement materials, tests, and background publications accompany these prototypical sequences. These materials form a source for textbook authors, teacher trainers, and school councillors and test developers. Mediated by this group, the new ideas have found their way to the instructional practice in schools.

In our presentation we will try to give an overview of what is going on in mathematics education in Dutch schools. We will use the innovation of mathematics education as pursued by the Freudenthal Institute as a background. In doing so, we will start by elaborating Freudenthal's idea of mathematics as a human activity. Subsequently we will describe the Dutch school system and the curricular innovations that lead to the current curriculum. Next we will look at the use of context problems, followed by examples of the reinvention approach in Dutch textbooks. Next, will look at the role of technology, and we will close with a discussion of the Dutch approach to geometry.

Mathematics as a human activity

Freudenthal was an outspoken opponent of the 'new mathematics' of the 1960s that took its starting point in a one-side structuralistic interpretation of the attainments of modern mathematics, especially set theory. Since the applicability of mathematics was also often problematic, he concluded that mathematics had to be taught in order to be useful. He observed that this could not be accomplished by simply teaching a 'useful mathematics'; that would inevitably result in a kind of mathematics that was useful only in a limited set of contexts. However, he also rejected the alternative: 'If this means teaching pure mathematics and afterwards showing how to apply it, I'm afraid we shall be no better off. I think this is just the wrong order' (Freudenthal 1968: 5). Instead, mathematics should be taught as mathematizing, and this view of the task of school mathematics was not only motivated by its importance for usefulness. For Freudenthal mathematics was first and foremost an activity. As a research mathematician, doing mathematics was more important to Freudenthal than mathematics as a ready-made product. In his view, the same should hold true for mathematics education: mathematics education was a process of doing mathematics that led to a result, mathematics-as-a-product. In traditional mathematics education the result of the mathematical activities of others was taken as a starting point for instruction. Freudenthal (1973) characterized this as an anti-didactical inversion. Things were upside down if one started by teaching the result of an activity rather than by teaching the activity itself.

History & school system

We will discuss some of the history of the many innovations in mathematics education in the Netherlands to offer some background for a sketch the mathematics curriculum in the Dutch schools. We will start out with the first projects of the IOWO, the Wiskobas project that aimed at primary school and Wiskivon that aimed at lower secondary. While changes in the primary-school curriculum were the result of a cumulative effect of a long-term process of gradual changes, the government usually mandated the changes in secondary education. In the former case, there was an indirect influence by the Wiskobas project and its successors. In the latter case, the government systematically gave our institute the responsibility of developing (prototypes of) new curricula—as was the case with the Hewet project and the Hawex project (both upper secondary), the project W12-16 (for the age group 12-16; together with the National Institute for Curriculum Development, SLO), and the recent Profi project (new curriculum for the 'exact' stream in upper secondary education). Current projects involve goals and learning route for primary school, special education, lower secondary and vocational training.

What is "realistic"?

In RME context problems play a role from the start onwards. Here they are defined as problems of which the problem situation is experientially real to the student. Under this definition, a pure mathematical problem can be a context problem too. Provided that the mathematics involved offers a context, that is to say, is experientially real for the student. In RME, the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by the students themselves.

Reinvention

Freudenthal proposed 'guided reinvention' as an alternative for the 'anti-didactical inversion'. Moreover, guided reinvention offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics. This principle is elaborated in many instructional sequences. As an example of the reinvention of an algorithm in primary school, we will discuss the reinvention of the long division. Further we will describe how reinvention plays out in some topics in algebra.

Text books
Most curricular changes in secondary school are based on government decisions. Usually the Freudenthal Institute would be asked to develop new curricula. The government then would mandate these new curricula, and textbook authors would use the prototypical materials that were developed to ground the new curriculum as the basis for the new textbook series. In primary school the influence was more indirect, here inspiring results of developmental research would find their way to textbooks via (journal) publications, conferences and personal contacts. In both cases, differences are to be expected between the original intent of the researchers and the actual textbooks, and their use.

Technology
The role of technology in mathematics education is growing that also holds for the Netherlands. We will show some examples of software that is used in primary and secondary education. This will include the use of web-based applets. In addition to this we will special attention to the use of graphic calculators. These are integrated in the Dutch curriculum; moreover, the use of graphic calculators is an integral part of the final exams.

Geometry
Geometry in the Dutch curriculum can be type-casted as "vision geometry". The informal experiential knowledge that students have is taken as a starting point for geometry instruction. We will present some examples of vision geometry from primary- and secondary-school textbooks. Further we will discuss how geometry is used in the new Profi-curriculum to foster the students' experience with proving.

RESEARCH FORUM 1

Theme

Potential and pitfalls of technology tools in learning mathematics

Coordinators

Carolyn Kieran & Rina Hershkowitz

Session 1
Presider: Ricardo Nemirovsky

- Introduction to the session by Carolyn Kieran and Rina Hershkowitz
- *"Algorithmic and meaningful ways of joining together representatives within the same mathematical activity: An experience with graphing calculators"* Rina Hershkowitz and Carolyn Kieran
- Reaction by Rosamund Sutherland
- *"A meta study on IC technologies in education. Towards a multidimensional framework to tackle their integration"* Jean-Baptiste Lagrange, Michèle Artigue, Colette Laborde, and Luc Trouche
- Reaction by Paul Drijvers

Session 2
Discussion leader: Ricardo Nemirovsky

- Continued reaction and proposing of discussion questions by Rosamund Sutherland and Paul Drijvers
- Audience discussion of these questions in small groups
- Questions and comments from the floor addressed to presenters or discussants
- Wrap-up by the presenters

POTENTIAL AND PITFALLS OF TECHNOLOGICAL TOOLS IN LEARNING MATHEMATICS: INTRODUCTORY REMARKS

Research Forum Coordinators: Carolyn Kieran and Rina Hershkowitz

For the past two decades, various computer and calculator technologies have been making their way into the teaching and learning of school mathematics. Much of the theoretical and empirical research that has been carried out with these new environments has tended to look for the value-added component provided by the technology. However, there are research works that show that this value-added component is not easily achieved and that, in certain contexts, the use of technology may block learning processes such as problem solving, justifying, and so on. The aim of this research forum is to present a balanced picture of the role being played by technology in the teaching of mathematics, while considering, in particular, the issue of the potential and pitfalls of technological tools in learning mathematics. Two main papers and two related commentaries discuss this issue from various perspectives and approaches.

In many respects, the two main papers are quite different. One is a meta-study drawn from a corpus of 662 research papers and is led by the goal "of building tools for understanding" the integration of technological tools in the classrooms. The other is a case study investigating a special situation of such integration in a particular classroom and is led by the goal of better understanding the cognitive and contextual aspects in such a situation of mathematical learning.

Nevertheless, as is expressed in the commentary papers, there are several threads that tie the contributions together. Both of the main papers are aware and even fascinated by the potential of the technological tools in learning mathematics but argue that this potential has to be investigated deeply and massively in order to be able to use them properly. Both differentiate between the potential of the tool itself and the mental structure that is built by the learner with the mediation of the tool. The meta-study paper emphasizes that "the instrumental dimension of the IC Technologies distinguishes a technological artefact and the instrument that a human being is able to build from this artefact." The case study paper follows the ways in which students produce and use, or try to use, technological artefacts (the representatives) in order to construct mathematical meaning while investigating a problem situation.

Finally, the authors of both papers and both commentaries believe that, without understanding the ways in which technological artefacts mediate the construction of the learner's mental structures, our considerations and decisions about technology-based mathematics learning might be ill-founded, and potentials might become pitfalls.

ALGORITHMIC AND MEANINGFUL WAYS OF JOINING TOGETHER REPRESENTATIVES WITHIN THE SAME MATHEMATICAL ACTIVITY: AN EXPERIENCE WITH GRAPHING CALCULATORS

Rina Hershkowitz — Carolyn Kieran
Weizmann Institute of Science — Université du Québec à Montréal

In designing mathematics learning with the mediation of computerized tools, one of the crucial questions to be considered is how much and in what way we would like the tool «to do the work» for the students. In problem situations where the solution is achieved mostly via graphical representatives, and algebraic models are mostly used as «keys» for obtaining graphical representatives on the screen, the algebraic representatives and their form seem minimized in importance, and students may tend to generate them from tables by mechanistic-algorithmic procedures. The above questions will be mainly demonstrated within the description and analysis of a case study involving a group of three 10th graders (about 16 years of age) working together to investigate and solve a problem situation on the topic of functions, having graphing calculators (TI-83 Plus) at their disposal. The role of contextual factors is highlighted by means of contrasts with the work of students on the same problem from another country.

Introduction

In designing as well as in studying a classroom learning activity in a computerized mathematics learning environment, one should consider contextual factors of various origins, like: (a) the mathematical content to be learned and its epistemological structure; (b) the learners, their mathematical knowledge, culture, and the history with which they started the researched activity; (c) the classroom culture and norms, the role of the teacher, the learning organization--in small groups or individually--etc.; and (d) the potential «contribution» of the computerized tool.

This study discusses questions related to the above factors and how they lead students to use a computerized tool and benefit from it mathematically or the opposite. The questions will be mainly demonstrated within the description and analysis of a case study involving a group of three 10th graders working together to investigate and solve a problem situation on the topic of functions, having graphing calculators (TI-83 Plus) at their disposal. The protocol analysis of the group work will reveal a dialectical problem-solving process that develops between two ways of making use of the computerized tool: a mechanistic-algorithmic one and another one that is led by students' search for meaning.

Contrasting the work of the group against the work of groups in a classroom from a different country will shed some light on the roles of the above contextual factors in students' making beneficial use of the computerized tool.

Some Theoretical Comments

During the past 20 years or so, the potential of computerized tools to develop and support *mathematization* by students working on problem situations has disclosed many positive aspects of their integration into school curricula. This can take the form of amplification and reorganization (Pea, 1985; Dörfler, 1993) and of experiencing new "mathematical realism" (Balacheff & Kaput, 1996). A common claim is that the computerized tools can play an advantageous role in assisting students to make connections between and within various representations of the same mathematical entity (e,g., Kaput, 1992). Teasley and Roschelle (1995) have documented instances of computer environments serving to disambiguate student thinking in the exploration of novel mathematical phenomena, by easy flexible transformations between representations. The power of a grapher to smoothly transform a function from its algebraic to its graphical representation, and the availability of the corresponding numerical data directly from the graph (by "walking on it"), make it possible to deal with problem situations involving complicated functions at an early stage of learning.

However, students have also been found to use technological tools in non-thinking and non-productive ways. Goldenberg (1988) has, for instance, warned of the ways in which students can misinterpret computer-based graphical representations of functions because they have not chosen an appropriate window; while Guin and Trouche (1998/1999) have argued that a surprising result produced by a graphing calculator does not necessarily induce a question on the part of students.

Looking closer at the interaction that learners have with a computerized tool in the classroom, one should take into account the epistemological power of the technology (Balacheff & Kaput, 1996), for example, the ways in which this power emerges from the multi-representational nature of the tool, the availability of different representations and different *representatives* of the same mathematical object (i.e., displays within the representations, Schwarz & Dreyfus, 1995), and the smooth transformation between them. Schwarz and Hershkowitz (in press) investigated how representatives can mediate the construction of meaning in mathematics learning. They claimed that from an epistemological point of view the relations between the mathematical entity and its representatives are inherently ambiguous. Representatives either may represent mainly the most prototypical examples of the mathematical entity or, because they are partial, may often be ambiguous in the sense that only some of the critical properties of the entity are displayed in the representative.

The ambiguity and the power of the computerized tool encourage the production of various representatives upon need, and also stimulate students' requirement for and ability with interweaving representatives together. In so doing, students may extract the invariant properties of the mathematical entity and thus overcome the ambiguity within the single representative. This process is part of the students' search for mathematical meaning. On the other hand, technological environments can induce students to reach «false representatives» (those that do not represent the critical properties of the mathematical entity at all) or to interweave representatives together in a non-meaningful, algorithmic fashion, as for example in the automatic extending of the numerical values of a spreadsheet column (Ponte & Carreira, 1992).

In this paper, we discuss two ways in which students interweave together the tool-based representatives: a mechanistic-algorithmic way (where students combine representatives in non-thinking, rote ways), and a meaningful way. Moreover, we tell the story of the use of two kinds of representatives in the process of problem solving: representatives that do not represent the properties of the mathematical objects involved at all, and representatives that do.

The Study

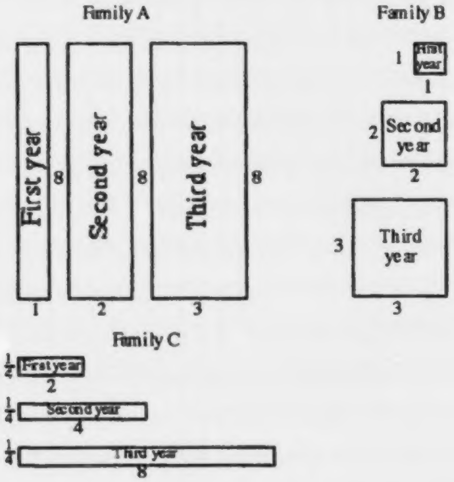
The case study involves three 10th graders from a Montreal high school working together to investigate and solve a mathematical problem situation, having graphing calculators (TI-83 Plus) at their disposal. The analysis of their activity is described in four rounds. On the whole, the investigative activity done by this group and their solution strategies are monitored by their need for a search for meaning. But, in Rounds 1 and 2, the process of "joining together tool-based representatives," which the students engage in, is mostly of the first kind, the mechanistic-algorithmic, which leads students towards some «false representatives.» Rounds 3 and 4 provide evidence of a different way of joining together representatives, one that is clearly characterized by a search for meaning. The progress of the group from Rounds 1 and 2 into Rounds 3 and 4 is led by critical thinking and is supported by the students' control of the "joining-together-representatives" functions, which are part of the strength of the graphing calculator.

The problem situation and some comments about its history

The following are the parts of the problem situation that are relevant to this paper.

Growing Rectangles

Each of the following three families of rectangles has its growth pattern:



In family A the width grows each year by one unit; the length remains constant at 8 units. In B the width and the length of the rectangle grow each year by one unit. In C the length doubles each year, and the width remains equal to 1/4.

Please investigate the problem in groups. At first try to generate various hypotheses concerning the following questions:

1. *Please compare the areas of the 3 families of rectangles over the years. What are their initial situations? Which family (or families) takes over the other families (or family) and when?*
2. *In which years will the area of each family exceed 1000 square units?*

Now check your hypotheses with mathematical tools (the help of the graphing calculator is recommended). Try to be as accurate as possible.

Please write a report as a common product of your group.

Try to describe your conjectures and what they were based on. What kinds of debates did your group have?

Try to describe the ways in which you solved the problem and in what ways you were using the graphing calculator

This activity was first tried in an introductory one-year-long course on functions with the mediation of graphical calculators (TI-81) in Grade 9 in Israel (for details see Hershkowitz & Schwarz, 1999). It took place during the sixth week of the course. At that stage, students had practiced the actions and passages between and within representations and representatives. They were aware of the fact that obtaining a graph representing a given phenomenon required an algebraic model.

The students in the Israeli class were first invited to suggest hypotheses without using the computerized tool, then to use it to check them. Students tried to figure out which family would eventually take the lead by using intuition and/or by computing "by hand" the areas of the three families of rectangles for a few values. Then students (in most of the groups) translated the situation into algebraic representations (with some difficulties in generating the function-area for Family C, $y=1/4*2^x$), and then obtained the graphs with their graphical calculators (see Figure 1 below for a stylized representation of the graphing window).

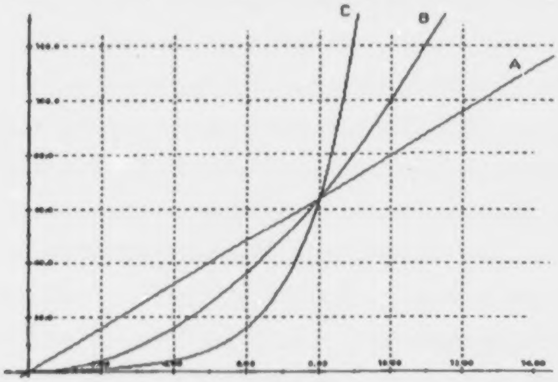


Figure 1

After the groups had finished, the teacher discussed with the class the mathematical findings and strategies. Students reported that in the eighth year (x=8), the three rectangles had the same area, and that from that year on, Family C took the lead from Family A. Family B remained in between. The evidence provided by the different representations was accepted even if, for some students, it was unexpected; no student declared the computer wrong. Nevertheless, they tried to reinterpret the situation, and even to overcome wrong intuitions, by matching together representatives from different representations; the algebraic, the numerical, the graphic, and the phenomenon itself. The sociomathematical norm of what constitutes evidence in problem situations was formed here as a consequence of students' interactions with the tool.

What happened in the case-study group

Before setting off to work in groups of threes on the problem situation, the research class from the Montreal school was asked also to read the problem and to vote on whether they thought that Family A, B, or C would have the largest area over the long term. A couple of students voted for Family A; most voted for Family B; and a few abstained. No one predicted that Family C would eventually have the largest area. Teams then started to work on the problem situation. They also had to produce a written report on the problem-solving process, as a common product of the team, and then to present it orally in front of the whole class during the concluding discussion. This activity was two periods long (an hour and a half in all).

The team to be featured will tell the story of the dialectical process between the mechanistic and the meaningful, with the search for meaning as the guiding thread of the dialectical process.

Round 1: Making the technological tool generate the algebraic expression for the situation.

After reading the problem questions, the group of Kay, Ema, and Sam (two girls and a boy) began immediately to create a table of values on paper. They entered the numbers 1 to 10 in the left-most column, headed "year." They labeled the next three columns "Family A," "Family B," and "Family C." To fill the Family A column, they calculated 8 x 1, 8 x 2, 8 x 3, and continued by increasing each entry by 8. Kay remarked: "*It is going by 8 each year.*" To fill the Family B column, Sam suggested "*the year where it is, just square that number and you will get the area.*" When the Family B column was completed, by computing the area of the growing rectangles in their heads, Sam began to compare the 1st and 10th entries for Families A and B. Kay remarked that "*so far, B has more over the long term.*" She then stated: "*We can do equations for each one and compare on the graph of the calculator to see where they all intersect.*" Sam nevertheless wanted them to first complete their table.

To fill the Family C column, they multiplied the given length of 2 for the first year by 1/4 (with the help of the calculator); they then took the length of the previous year, doubled it, and multiplied by 1/4. Thus, for the initial values of each of the columns, the procedure used for filling them reflected the operations suggested by the text of the problem situation (e.g., *"in C the length doubles each year and the width remains equal to 1/4"*). But the filling-in of the table was not done with the aim of detecting the relationship between x and y values so as to yield an expression for the function. It was done simply to have 10 values for each Family so as to be able to calculate the differences between the 1st and 10th values, and check their initial hypothesis as to which Family had the largest area over the long term. Their attention was on the global differences. It is worth noting that while the calculations for Families A and C were done in a recursive fashion, the calculations for family B were based on an explicit generalization for x (the year number) – Sam said: *"the year where it is, just square that number and you will get the area,"* and yet he did not reach or use a closed-form algebraic expression.

After completing column C, Sam announced that they had been wrong in their prediction that Family C would be ahead. He then said that *"by having all this, you can now just make up equations for each one."* It is hard to know what Sam meant by this; either he was ready now to generate from the situation and the tables the algebraic expressions for the growth of the three families, or something else. In any case, the objective of the three of them seemed to be to look for intersection points on a graph. Kay, at that point, seemed to have a very clear strategy in mind: *"Let's do a linear regression."* Sam added: *"So, if they want to know when they take over, you have to do an equation and find out where they meet."* Kay continued his sentence: *"And you have to find the point of intersection."*

The group had finished answering the front-end question that we had posed regarding their initial hypothesis and could now get on with the main task at hand. As in the Israeli class, the objective of all of them seemed to be to draw the graphs in order to be able to compare the three families by having intersection points. They were also aware that, in order to obtain a graph representing a given phenomenon, it is necessary to have an algebraic model. However, the way in which the group members went on to find the equation for each Family was quite unexpected. With the experience we had from the first class, we had thought that they would either analyze the given problem situation with its patterns of rectangles, or look at the numerical values they had generated for their tables in order to find an expression for the relation between the x values (the years) and the corresponding y values (the areas). The way in which they chose to join the paper-and-pencil and technology-based representatives in order to obtain the graphs and the intersection point/s is a crucial part of our story and is described in the following paragraphs.

After rapidly entering the first five values of each column from the paper table of values into STAT-Edit lists of her graphing calculator, and then selecting LinReg ($ax + b$) of the STAT-Calc menu, Kay obtained the following three LINEAR functional expressions for Families A, B, and C: $8x$; $6x - 7$; $1.8x - 2.3$. She suggested to her team-mates that they enter these "equations" into the *Y= editor* of the calculator and then ask for the graphs, which they did. She, who was the fastest at using the calculator, soon announced, *"OK, guys, these are the graphs,"* while turning her calculator toward them so that they could see her screen (see Figure 2).

It is noted that this team was not alone in its use of the calculator's linear regression tool as a means of generating the area expressions for the three families of rectangles. The teacher, when questioned, disclosed that the class had used regression for two or three weeks, about half a year earlier when they were dealing with problems involving real-world data. Students had learned to model (i.e., obtain equations for) these "real-world" situations with the help of the linear and quadratic regression tool of their calculator.

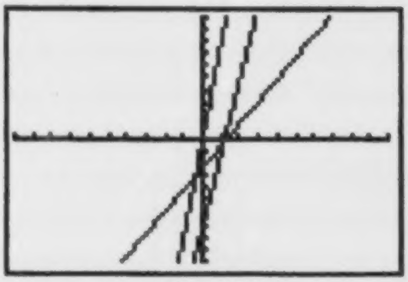


Figure 2

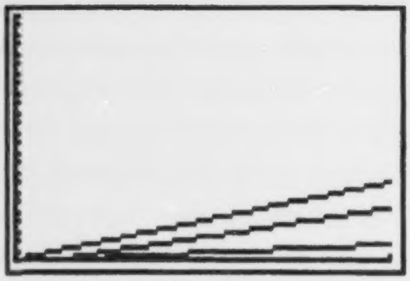


Figure 3

We can conclude that the above regression procedure, which is often the only way to get an algebraic rule for the so-called regularities of real-life phenomena, was adopted by these students as the most efficient way to get the algebraic model of a problem situation, even if this particular problem situation involved idealized mathematical data. The students appeared to trust the algorithmic routine involved in using the regression tool of the calculator and did not feel any immediate need to verify empirically the correctness of the expressions that had been generated by the calculator with the data that they already had in their paper-and-pencil table.

Nevertheless, one is led to ask why the group mechanistically used the calculator to generate and to join together the representatives (the numerical and algebraic representatives) instead of looking more carefully at their table where the answer to the question "WHEN?" was there all along (in Year 8, all the families had an area of 64). Did they consider the graphical representation more reliable? Or is a functional algebraic rule, and its joined-together graph, a stronger argument than mere numerical evidence? Or was it important for them that what they had been able to conclude by means of the numerical table (that Family C had the larger area over the long term) be supported by confirmatory and matching evidence from the graphical representation as well, that is, that they be able to join together the paper-based numerical representatives with the graphical tool-based representatives? Or were they just carried away by the above algorithm that begs to be executed once one has generated tables of data for a given phenomenon?

Round 2: Failure at getting the technological tool to work for them: a turning point.

From the beginning of applying the regression option of the calculator tool and up to this moment, the group had appeared to be marching along a mechanistic path in a kind of automatic fashion, without doing very much in the way of reflection. When the graphs they generated did not look as they had expected them to (Sam: "*They don't all intersect at the same point* "--as seen in Figure 2), Kay changed the scales immediately. But, that did not help her to obtain the meeting point she wanted to see (see Figure 3). Repeated scale changes (even up to 1500 years) did not produce the elusive point of intersection that they had intuitively been expecting. Kay then suggested they try the CALC-Intersect option to have the calculator provide some information with respect to the point of intersection. The error message of "no sign change" suggested that there was no point of intersection in the given window and confirmed for them what had been evident from their reading of the graphs, graphs that had been produced by the expressions of the linear regression option. Yet, there was a sense of unease. They had failed in their attempt to get the technology to work for them. They had expected a single point of intersection somewhere in the first quadrant.

Round 3: A shift of attention.

This round starts with Kay beginning to think more about the situation: *"We try to find how much it increases, which one grows the most."* They returned to the text of the problem situation and reread it: *"Which family (or families) takes over the other families (or family) and when?"* Kay emphasized the *"AND WHEN."* Sam began to look at the paper-and-pencil table of values and said: *"We already figured out who is going bigger, but we can't answer when, because ..."* Then Sam and Kay noticed in the table that, at 8 years, all three families had an area of 64, and so Sam asked three times: *"So why aren't they meeting?"* They now paid attention to the hard numerical evidence that there was a single point of intersection. Their inability to match this evidence and the graphical representatives they had obtained became very clear and waited to be resolved and explained.

Kay substituted 8 into their algebraic expressions for Families B and C and obtained 41 and 12.1, instead of the desired 64, and she said: *"We did something wrong."* In an attempt to make sense of what was going on, the students restarted the process of joining-together-representatives, again with the regression option of the calculator tool. But, this time they were equipped with the critical thinking they had developed from the above comparison of the numerical and graphical representatives. They recalculated the linear regression for Family B and this time noticed that the value of the correlation coefficient was not 1, but .98. Sam remarked that the resulting expression was therefore not 100% sure. Kay insisted: *"Even so, we were way off."*

Sam began to systematically substitute values into the expression for Family B:*"6(1) - 7 is -1: that's wrong; it's not having a negative area."* With his substitution of 2 into the same expression, he became even surer that they were wrong. Kay wondered aloud: *"So do we make up our own equation; maybe it's not a linear regression."* Sam, who was still substituting, said, *"We're getting further apart; even C goes off."*

In this round, their mechanistic approach to joining-together-representatives was being put into question by a more meaningful one. It is not clear if they started to suspect the mechanistic routine itself or the way they had performed it. However, as will be seen in the next round, rather than abandoning the process carried out with the technology, they insisted that it be made to work for them.

Round 4: Insisting that the technology be made to work for them.

Kay decided to check the way that they had used the regression algorithm by trying other forms of regression available on the calculator tool. She went to cubic regression and realized that the correct equation was $y = x^2$ -- because of the neat parameters and the correlation coefficient of 1. She added: *1 squared is 1, 2 squared is 4, 3 squared is 9.* Soon Sam reacted: *"I don't know why I didn't figure it out from the beginning. I said it, why didn't I see it?"* Even Ema expressed that they had been carried away by the regression routine: *"It's because we were so absorbed by our calculators."*

Meanwhile, Kay ran through other regression choices for Family C until she hit upon the exponential regression, which yielded $y = 0.25 \times 2^x$ with a correlation coefficient of 1. Just to be sure, she entered an x-value of 1 to see if the calculator would produce the same y value as she had in her paper-and-pencil table. One value seemed to satisfy her that the equation was correct, but she stated that she did not really understand the equation.

Sam too seemed a little uncomfortable with the expression for Family C. They had never before experienced exponential equations. Kay seemed prepared to go on to the next question. After all, they had earlier realized from their paper-and-pencil table of values that the point of intersection was (8, 64), and they now knew that they had equations that corresponded with the values of their paper table of values. However, Sam wanted more. He wanted to be sure that their newly-found equations did, in fact, yield graphs that intersected at the point (8, 64). He suggested that they enter the new equations in the *Y= editor* and graph them with a scale involving a y-maximum of 70: *"We should see that they are all meeting at 64."* He smiled visibly at the result. It is not clear, however, that Kay paid any attention to this last exercise in consistency.

The technology had now been made to work for them. The graphs, the equations, the paper-and-pencil table of values, and the situation all fit together. Doerr and Zangor (1999) have emphasized the importance of leading students to "develop a reasonable skepticism about calculator-generated results" and encouraging the establishment of classroom "norms that require results to be justified on mathematical grounds, not simply taken as calculator results" (pp. 271-272). It is noted that these students did not throw aside their calculator tool when the graphs it produced could not be justified on mathematical grounds; they continued with it until it could be made to deliver correct mathematical representations.

In short, we can conclude that the group completed its answer to Question 1 by means of actions involving "joining together representatives" which, at the starting point as well as at the end, were controlled by the need to have meaning. But the sequence as a whole was an intertwining one where both the mechanistic and the meaningful were dialectically connected. The regression routines were the mechanistic parts of the sequence. The to-ing and fro-ing between the mechanistic and meaningful joining-together-of-representatives was at times characterized by lengthy segments of mechanistic activity. Nevertheless, the search for meaning always prevailed.

Discussion

As mentioned before, when we speak of computerized tools in learning we usually speak of their «positive» potential in mediating learning. The above example showed that such mediation might raise dilemmas for learning. A crucial dilemma is how much and in what way we would like the tool «to do the work» for the students. And more specifically, do we value that students be able to express a problem situation with algebraic models, and that they produce themselves this algebraic model?

Both classes were driven by a search for meaning in comparing the growth of the three families and were looking for the three graphs and their intersection points, knowing that the algebraic models were the keys for obtaining the graphs. The first class, which had a more limited tool (TI-81), started immediately to construct the algebraic models of the three families from the problem situation itself. This was not so easy for Family C and a few groups in this class failed. Our group from the second class trusted from the beginning the mediation power of their more advanced tool (TI-83 Plus) to do the work of generating the algebraic models for them. But in the first round they failed to get the right algebraic models, even for Family B. During that stage, they had used the tool in a mechanistic way, without engaging critical thinking, and thus obtained «false representatives.»

What is the source of the differences between the two classes? It is obvious that the tool itself may explain part of the difference because the TI-81 has only a quite limited regression option. But, the dissimilar history of mathematics learning in the two classes is likely a major contributing factor. As we mentioned above, our second class had the experience of dealing with real world problems with non-idealized data, which usually do not fit perfectly an algebraic model. This encourages the use of regression techniques as a means of obtaining an algebraic model. So the students imposed the same kind of modeling technique on the «Growing Rectangles» problem situation. For students who have difficulty in modeling the growth of an exponential function from the situation itself, such as was experienced with the Family C rectangle, this technique may serve as a temporary scaffolding.

Students may rely on a similar kind of scaffolding when using spreadsheets for modeling. We have observed younger students investigating similar problems of exponential growth with EXCEL (Hershkowitz, 1999). Some of these 7th graders tended to generalize phenomena like the growth of Family C into a recursive model by carrying out actions such as the following:

Enter 2 into cell A1, and then in cell A2 punch in "=A1+A1" and drag down to produce the lengths of the Family C rectangles; then in column B enter into the B1 cell "=1/4*A1" and drag down to obtain the area of the Family C rectangles. In this way, the spreadsheet tool allowed students to combine the use of recursive formulae and dragging, and thus to overcome the local property of recursion.

In fact, if we look at the process which our group used to fill the table for Family C in Round 1 -- doubling the previous length and then multiplying it by 1/4 -- we may conclude that the recursive approach is much easier than trying to generate a closed-form rule where the independent variable is the number of the year. Recursive approaches might even be more natural, at least in the case of exponential growth. In this sense, the tool--either the TI-83 or EXCEL--with its scaffolding affordances enables students to act meaningfully on quite high-level objects, such as exponential functions, even before they have learned formally about them.

The other side of this coin is that this scaffolding may stay longer then we, as mathematics educators, would like in the process of learning algebra. We had examples of students working on a problem situation in which exponential growth was investigated, where students were quite close to generating a closed-form exponential formula. But, when they discovered the EXCEL option of generating a recursive expression + dragging, as a quite efficient alternative for obtaining a whole set of data for the phenomenon, they curtailed their efforts to mathematize the problem situation in a higher level manner. While it might be argued that these students were engaging in a kind of algebraic thinking (Kieran, 1996), might the use of computerized tools in learning algebra reduce students' needs for high level algebraic activity?

In addition, we face an even more crucial dilemma; the algebraic representation and its form seem minimized in importance for students. Our case-study group, like groups in the other class, knew that the algebraic formula was the key, but perhaps because of their more advanced tool, and because of their learning history, the shape of the algebraic model seemed unimportant. In fact, students can now go from entering lists to a graphical representation without ever seeing or having to examine the algebraic representation of the situation. Does the use of these tools in algebra signal the beginning of the loss of the algebraic representation from our mathematical classes at the secondary level?

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REACTION TO "ALGORITHMIC AND MEANINGFUL WAYS OF JOINING TOGETHER REPRESENTATIVES WITHIN THE SAME MATHEMATICAL ACTIVITY"

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Almost 20 years after I carried out my first mathematics education research on students' learning of algebraic ideas within a Logo environment I am again starting a new project on teaching and learning with new technologies. What have I learned in this intervening period? Firstly I know that although in the UK we are moving towards a ratio of 1 computer to 8 students in every school, mathematics teachers are amongst the most resistant of teachers to incorporate computer activities into their teaching. I know that although in our earlier research we theorised learning we did not adequately take into account the student's history of learning mathematics, the culture of the classroom, the culture of the school and the culture of the mathematics curriculum as influenced by both mathematics and national policy. I know that however beautiful the new technology there is nothing essential about this technology which implies it will be used in a particular way (although I do believe that it is possible to talk about the 'affordances' of a technology (Gibson, 1977)). Unfortunately in our consumer culture those who advocate new technologies tend to 'sell' them as if there is a cause and effect relationship between the use of the technology and student's learning. In particular I have sat through many presentations on the use of graphics calculators in which the presenter focuses on 'what the calculator can do' as opposed to 'what the student-plus-calculator might do when using a calculator'. Thus I welcomed an invitation to participate in the PME Research Forum on the potential and pitfalls of technological tools in learning mathematics.

In Hershkowitz and Kieran's paper they discuss the ways in which students in both Israel and Canada work with a graphics calculator on an investigative activity which involves predicting the growth patterns of the areas of three families of rectangles (which have been represented to them figuratively). The group of students in Israel chose from the outset to represent the areas of each family of rectangles as algebraic functions and then graphed these functions with their graphics calculators. This led to a graphical representation which they could interpret from the point of view of deciding on the relationship between the growth of each family of rectangles.

The group of students in Canada took a different approach. They used 2 x 10 tables to represent the growth for each family of rectangles. For each table they firstly entered a column 1,2,3…10. The task then became one of filling in the correct area for each family of rectangles for the first 10 years of growth. They managed to do this correctly. The rules to generate the 1st and 2nd family of rectangles were computed in a recursive manner 'in their heads' as opposed to using an algebraic rule in the calculator. However they did use a universal rule for the 2nd family of rectangles (which was increasing squares), although it is not clear in the paper whether they represented this rule in their calculators. By the end of this beginning activity they had produced a table of values for each family of squares. They knew that they needed to produce an algebraic model in order to plot graphs of growth and one student suggested "let's do a linear regression".

Hershkowitz and Kieran ask why the students so mechanistically used the calculator to produce an algebraic rule when the tables they had produced could be 'read' for the information which they had been asked to obtain. However the authors point out that these students had previously worked for several weeks on problems which involved finding models for real-world data and they had used linear regression in this respect. My interpretation of what is happening here is rather different. The students are firstly making sense of the problem presented to them by drawing on their previous experience of finding rules to represent data generated from real-world situations. In these previous situations fitting a straight line was a way of modelling a situation which may or may not have been underpinned by a linear relationship. Their approach to these messy real-world situations had been to start from data and then find a model which fitted the data. So for me it is not surprising that when presented with an investigative problem (which was underpinned by explicit algebraic models) they first decided to generate data from the situation and then tried to find rules to fit this data. They did not use the visual representations of the families of rectangles to find algebraic rules and they did not read the tables of data which they produced in order to find the relationship in the growth patterns. I would describe their behaviour as making-sense-for-them although it might not have made sense to an observer who had expected another approach. In this respect I disagree with Hershkowitz and Kieran's analysis that "from the beginning of applying the regression option of the calculator tool and up to this moment, the group had appeared to be marching along a mechanistic path in a kind of automatic fashion, without doing very much in the way of reflection".

When this group of students had produced graphs using linear regression they then began to question their approach, because the graphs produced did not tie up with how they were reading the table and the figurative representation of the problem. For me the story is one of students' actively making sense, working with the calculator, the problem and multiple representations generated until they can make sense of the whole. The first phase of the problem seems to have been a getting-started phase which drew on previous experience of working with real-world messy data, in which it is not usually possible to read patterns from the data. They needed to generate algebraic rules from their data in order to produce graphs with their graphics calculators and the calculator's regression function was how they had previously learned to do this. In the final section of the paper Hershkowitz and Kieran depart from their earlier analysis and start to discuss the students' approach in a similar way: "Our second class had the experience of dealing with real world problems with non-idealized data, which usually do not fit perfectly an algebraic model. This encourages the use of regression techniques as a means of obtaining an algebraic model".

The starting point of the paper centred around the idea of two kinds of representatives "representatives that do not represent the properties of the mathematical objects involved at all, and representatives that do". This perspective, I suggest invests too much power in the representatives and not enough in the person plus representative (Perkins, 1999). It also suggests that 'if only students produced the 'prototypical' representative then they would be able to 'see' what the teacher wants them to see". The case discussed throughout the paper has raised very important issues about learning mathematics with new technologies. The comparison between the students in Israel and the students in Canada tells us that students with different backgrounds can approach the same problem and the same technology in very different ways (similar issues have been discussed in Sutherland & Balacheff, 1999).

The question for me is how can we understand student's sense making in these situations? The paper has provoked me into thinking whether it would be possible to develop a theoretical model of teaching and learning mathematics which would enable us to predict that the Canadian and Israeli students might have been likely to approach the problem in the ways in which they did. For me, such a model would place the student as a central sense-making agent who works with whatever resources and theories are available (technologies, representations, language) and brings a history of working with resources and theories in order to solve the problem presented to them. In this model there would be no 'best' representation and no 'best' technology although each representation and each technology would 'afford' a range of different potentialities.

Working with both the graphics calculator and real-world data problems as a way into algebra is likely to lead to an under-emphasis of the power of algebraic formulae for representing patterns. It could also be argued that this type of work is more like science than mathematics. The paper does not ask why the students in Israel took such a different approach and there is no discussion of the previous experiences of these students. My final question to the authors is where was the teacher in all this work? A teacher on-hand could discuss with the Canadian students the difference between data which derives from real-world situations and data which has already been generated by an algebraic formula. There is much work to be done here and I anticipate a very fruitful discussion at the PME Research Forum.

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A META STUDY ON IC TECHNOLOGIES IN EDUCATION.
Towards a multidimensional framework to tackle their integration

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A survey of literature about educational uses of IC Technologies in mathematics education was done by a team issuing from five French laboratories working in various fields. A quantitative analysis of a corpus of 662 papers and two qualitative analyses of a sub-corpus helped to specify "dimensions" for the analysis of this very varied mass of innovation and research. Then, cluster analysis performed on each of the dimensions led to informative partitions. The method of analysis and the picture resulting from the partitions are offered as means to tackle the complex integration of IC Technologies into teaching and learning.

Introduction

This paper originates from a call by the French Ministry of Education to make use of existing published works to answer questions about educational use of Information and Communication (IC) Technologies. The opportunity of doing a meta-study was appealing to us because of the mass of published works existing in this field, contrasting with a general lack of precise understanding of what really happens when introducing IC Technologies. We thought that it was interesting not to do another research project, but to try to see what synthesis could be drawn and how it could be drawn[1].

The ministry's questions were: "How are IC Technologies used inside the educational system? Do they change the nature of learning? Do they modify the notions and the methods? What is their influence on students and teachers?" As for us, we were concerned by the discrepancy between promising IC Technologies and their difficult integration in schools. Our view was that teaching and learning involve complex processes, that bringing in elaborate technology adds even more complexity and that technologies proposed for educational use are quite varied. For these reasons we did not expect that a meta-study could bring the ministry's questions direct answers. Our aim was to build tools for the understanding of integration rather than to collect findings on objective changes brought by technologies.

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[1] A complete documentation on this research, including the material for the statistical analysis is available on the Internet: http://www.maths.univ-rennes1.fr/~lagrange/cncre/rapport.htm. A longer version of this contribution is also available for download. The longer version presents detailed numerical data summarised here.

Aims and approach

Building a methodology for analysing innovation or research, and using this methodology to identify trends in a corpus of works were two aims that we thought about together. A classical meta-study "translates results to a common metric and statistically explores relations between study characteristics and findings" (Bangert-Drowns & Rudner, 1991). Because of the above complexity and because of the diversity of the characteristics and findings of papers in the field of IC Technologies, we did not try to define a single metric. It appeared to us that looking at the integration of technology requires a plurality of perspectives or approaches and that a first stage of this "meta-work" would consist in a precise identification of these perspectives. Our methodology was to conceive of possible perspectives or factors (that we named "dimensions") using our knowledge of research work. Then we had to analyse writings on the introduction of IC Technologies. As a writing focuses on a necessarily limited number of perspectives, we wanted to be able to identify in which dimension(s) its contribution is and what specific approach and results it brings into this(these) dimension(s).

We had to make the dimensions operational by identifying each of these by a set of adequate questions. In our mind, this identification could not be done only from a theoretical reflection but had to be drawn from the existing literature. This is why we decided to carry out a qualitative analysis of a corpus as large as possible, covering the diversity of fields defined by a specific technology: (dynamic) geometry, computer algebra and other algebraic software, arithmetic and graphical calculators, computerised learning environments... We chose also to consider French publications in order to specify our own cultural and institutional context and also as many papers as possible from other countries to identify alternative approaches.

Methodology

The specification of the dimensions was the goal of a first stage analysis and a statistical analysis of a corpus in each of the dimensions was the goal of a second stage analysis. In the first stage analysis, we considered a whole range of publications regarding the nationality of the author(s), the type of work, the type of software and hardware used, the domain of knowledge... We decided to cover years 1994 to 1999, which appeared to include enough diversity. We used a variety of international sources as well as French works. This resulted in a corpus of 662 published works.
The exploitation of this "first stage" corpus was:

1. a quantitative identification of the repartition with respect to nationality, mathematical field, school levels, type of study,
2. a qualitative classification of questions (or "problématiques") that we did on a topical subset (we chose Computer Algebra), in order to establish the various dimensions in the approach of the use of technology,
3. the selection of a corpus for the "second stage" analysis.

The selection for the second stage analysis was necessary because not all papers in the first stage corpus had sufficient matter. The selection was to be large enough to respect the diversity of approaches and to avoid biases, but we had also to consider that the second stage analysis was expected to take several hours working on each paper. The principles of selection were:

- to have publications with sufficiently developed "problématiques", methodology and findings,
- to avoid papers too close in their analysis and findings,
- to have a distribution of nations, types of hardware and software, domains and levels similar to the initial corpus.

We selected 79 papers, judging this selection to be a good compromise between in depth analysis and respect of diversity. For each of these papers, one of the participants of the project established a detailed review. The exploitation of this "second stage" corpus was:

1. a qualitative analysis of a dimension (we choose the cognitive dimension) in order to get a first evaluation of the diversity of the approaches in this corpus
2. a statistical classification giving informative partitions of the corpus with regard to the various dimensions.

A quantitative identification of the corpus

We started this study with the hypothesis that the great variety of types of publications in the field of the use of IC Technologies gives evidence of a lively field but also makes difficult the search for convergent findings. Of course we had also an intuition of what this variety could be. Thus, the simple quantitative findings reported in this paragraph were intended to provide evidence of the diversity of the publications as well as provide a check of the conformity of the corpus as compared to what we thought of the general production.

Type of analysis and mathematical field. Research publications were not a majority (37%). This is relevant for our aim to analyse the use of IC Technologies as a whole, looking at classroom innovation, and speculation as well as at research studies. Geometry is well represented. Few papers are about arithmetic and algebra alone, probably reflecting the smaller amount of writings about the pre-college level, outside geometry, in the corpus. A number of papers do not specify a mathematical field, focusing on the support of technology in "general" mathematical learning.

Type of technology. Varied technologies were represented. Their repartition shows that, although simple and graphical calculators are everyday instruments for many students all over the world, papers tend to focus on "smarter" technologies, like computer software or symbolic calculators. Emergent technologies (Internet, etc…) although very present in discourses are rarely addressed by papers in our corpus. We thought this repartition was not a misleading picture of the varied writings about the introduction of IC Technologies in the period of time that we considered.

Countries. Our corpus represented also a great variety of countries. Of course there was a bias towards France because it was easier to get hold of publications in France like thesis, professional teachers' journals and congress and professional meetings reports that we wanted to include beside papers from international journals. The part of all the "Latin" and "Latino-American" world appeared to us reduced compared to what we had expected, a bias that we attributed to the sources we had in hand.

Two qualitative pictures

As indicated above we did two qualitative analyses, firstly to have a general picture of the publications in the initial corpus before selecting the second stage corpus, and secondly to get an evaluation of the diversity of the approaches in this latter corpus.

A qualitative classification of "problématiques" in the first stage corpus

To prepare for the second stage analyses, we considered the "problématiques". The "problématique" of a publication is the field of connected problems or questions that the publication addresses. This idea of "problématique" is important in our research because it aims at the study of the way questions are posed in the field of IC Technologies. A reason for the discrepancy between the favourable opinions on their use and their poor integration lies in the difficulty of addressing appropriately the problems really involved in a given experiment.

For each publication in the first stage corpus, we established a small text summarising its "problématique". An analysis of the "problématiques" of the whole corpus would have been too time-consuming for a qualitative picture. So we decided to restrict this analysis to a specific field: the educational use of Computer Algebra Systems (CAS). We chose this field because it appeared with a high percentage, reflecting the great number of papers written in this field. Another reason was that the corpus included all the papers issued from a journal dedicated to CAS educational use, the IJCAME. As this journal encourages research papers as well as articles about teaching issues, activities for class use and opinions, we were particularly sure of having a variety of papers in this field. The CAS sub-corpus also included thirty percent of publications on CAS use from a great variety of other journals or books.

We found that the publications of the CAS sub-corpus could be classified into five types of "problématique" that we summarise below.

Technical descriptions of possibilities of CAS (53 %). These papers stress the capabilities of CAS that they find relevant for educational use. Optimism is a very common feature of these papers and only a few articles stress the need for students' training to avoid pitfalls.

Descriptions of innovative classroom activities (13 %). These papers go farther, presenting actual CAS use in classrooms. Presentations of national projects for experimenting and integrating CAS (like French or Austrian projects) and of new curricular projects influenced by CAS capabilities are examples of these.

Papers starting from assumptions (18 %). These papers are more research-oriented studies, with hypotheses, experimentation and conclusions. Hypotheses come from general views on mathematics teaching and learning and on technology and are often backed by a theoretical cognitive approach.

Papers starting from questions about the use of CAS (31 %). In contrast with the above papers they do not presuppose advantages of this use. They present innovations, experimentation or examples of use not for themselves, but as a tool to address the questions.

Papers focusing on integration (7 %). These papers address the issue of the conditions for CAS to be used, such as paper and pencil in the everyday practice of teaching and learning in existing school institutions.

This classification reflects the diversity of reflection, research and experiment on the use of IC Technologies. The papers with a mere technical approach of possible use of IC Technologies (type 1) prevail. The mass of papers produced in this approach is representative of the interest raised by a technology like CAS among a part of the teachers. Papers arguing in favour of classroom innovations (type 2) are much less common, but seem useful especially when they report on long-time experiment of students' use of CAS. A weakness is that they generally do not discuss the teacher's options.

Among research papers, our classification distinguishes publications starting with assumptions on expected improvements resulting from students' use of CAS (type 3) from others starting with questions about this use (type 4). Ten years ago, in the first stream of research about CAS use, many papers were of type 3. In the period 1994-1999, the papers of type 4 are nearly twice as frequent as papers of type 3. We can take it as an indication that at this stage of development of research about IC Technologies, formulating and trying assumptions on improvements is no longer very productive. The assumptions and questions have in common their focus on the epistemological and semiotic dimension: they generally consider the mathematical knowledge at stake in technological settings and the possible effects of its computer implementation. Type 3 and 4 papers share also a common interest for a "cognitive" dimension: the assumptions of improvements and the question about the use of IC Technologies are generally based on a theoretical framework about the student's functioning and learning processes.

The last class (type 5) gathers a minority of papers explicitly addressing the issue of integration, which implies a study of questions like those in type 4 papers, but with a specific approach of an ecologically sustainable use. Questions on tasks, procedures and conceptualisation, as well as on CAS as an instrument exist in the type 4 papers. In the type 5 publications they appear as not to be missed dimensions that we briefly characterise below:

- The "instrumental" dimension of IC Technologies distinguishes a technological artefact and the instrument that a human being is able to build from this artefact.

While the artefact refers to the objective tool, the instrument refers to a mental construction of the tool by the user. The instrument is not given with the artefact, it is built in a complex instrumental genesis and it shapes the mathematical activity and thinking. (see Trouche, 2000).

- The "institutional" dimension investigates to what extent content to be taught as well as tasks and procedures (or "techniques") are affected by the institution in which they are taught. Institution has to be understood in a broad sense: a specific classroom with a specific teacher may be considered as an institution as well as a general school level for a country (see Lagrange, to appear).

So this classification helps to consider the varied types of "problématiques" and dimensions that can be used to analyse the use of IC Technologies. It helps also to look at the selection we did for the second stage corpus. Because they generally lack sufficient data and analysis, we could not integrate most of the papers of type 1 and 2 in the second stage corpus.

A qualitative analysis of the theoretical frames in the second stage corpus

As most publications in the second stage corpus are research papers, they generally offer a theoretical basis for their analysis. We chose to look qualitatively at this basis as means for preparing the quantitative analysis by showing convergence and diversity in the mass of papers that we selected.

As expected from the above study of "problématiques", we found a strong convergence of the theoretical frames in the predominance of cognitive approaches. Where constructivist views prevail, papers very generally base their analysis on theories of the student's functioning and learning processes. Within this general picture, diversity appears through more local theories, attached to specific research trends, such as for instance, the "process / object" approach, or the emphasis put on Piagetian processes of assimilation and equilibration. It appears also in specific theoretical constructs issued from the research on the use of IC Technologies like the notion of cognitive tool and conceptual reorganisations. Specific environments such as for instance dynamic geometry software, graphic calculators, CAS… tend to focus on specific approaches.

In spite of this diversity, we could find convergent elements of an evolution. This evolution reflects the progress of global educational research, especially towards socio-constructivist, socio-cultural and anthropological approaches, showing how the research on IC Technologies depends on the global context. Beyond this dependence, the evolution of the theoretical approaches of the use of IC Technologies marks also a sensitivity towards specific dimensions and provides for specific elaboration.

A specificity of research on the use of IC Technologies is a special sensitivity to the role played by perception in cognitive processes. It is interesting to notice that, even within the short period taken into account, some convergent evolution seems to occur in the way perception and visualisation are approached. Some texts remain still in what we could call the "naïve phase": the visual potential of technology is emphasised, illustrated by judicious examples, but is seen as a means for improving mathematics understanding and conceptualisation, per se. Most texts in the corpus are situated beyond this naïve attitude. The cognitive power of visualisation tools, the underlying cognitive processes, have become a matter of investigation. In some papers, this leads to the opposition of perceptive and conceptual approaches.

Nevertheless, most recent texts tend to present the relationships between perception and conceptualisation in more dialectic ways. For instance, accessing geometrical knowledge is no longer presented as resulting from the rejection of some perceptive apprehension of geometrical objects, but from the ability of relying efficiently both on spatial and geometrical competencies. More emphasis is put on the characteristics of problems and situations which can foster the dialectic interplay between these competencies of different nature and thus contribute to the development of geometrical expertise (Laborde 1998). The same sensitivity and evolution can be observed as regards the role played in conceptualisation processes by the interaction between the different semiotic registers of representation and as regards the "contextualisation of knowledge" (Noss and Hoyles, 1996).

From this qualitative analysis, we see that even in a specific dimension like the theoretical cognitive framework, diversity makes research advances not easy to synthesise and that relying on such synthesis for a development of the field will be difficult. A useful convergence can be found in the consistent evolutions noticed above.

The quantitative second stage analysis: data and method

Eight dimensions for the analysis of the integration of technology were derived from the above qualitative analysis. Five were briefly described above: the general approach of the integration ("problematiques"...), the epistemological and semiotic, the cognitive, the institutional and the instrumental dimensions. Three other dimensions were considered: the "situational" dimension which refers to the changes that the introduction of technology brings into the didactical situations, the dimension of "human-machine interaction" which analyses students' activity and interaction with the technological tools and the teacher dimension which looks at the teacher's beliefs and at the way (s)he organises the classroom activity.

For each of the dimensions a set of questions was designed resulting in a questionnaire of 96 questions. The answers to the questionnaire for each publication of the second stage corpus was assumed to give a picture of how this work takes each specific dimension into account. The identification of trends in writings about the introduction of technology could then be carried out through a statistical procedure. This procedure was designed on the basis of a cluster analysis. Applying this procedure to each dimension, we obtained specific partitions, data to explain clusters and one or two papers at the centre of each cluster.

The quantitative second stage analysis: results and interpretation

In this section we will detail and discuss the results of the "instrumental" partition as an example of the data that the procedure handles and of the type of insight that it provides for. Then we will draw more briefly on other partitions.

The "instrumental" partition

The instrumental approach was briefly introduced above. It is a rather new approach; older studies took into account neither the learning processes attached to the use of the tool, nor their evolution over time. This approach may be relevant for investigating how students make use of technologies, how the way they use it evolves over time under the influence of evolution of their knowledge in mathematics and about the technology. This approach may also be relevant for investigating how the teaching takes into account the construction of the instrument and its relationships with the learning of mathematics. Several points of our analysis are concerned with this approach.

The types of questions attached to this dimension in the questionnaire are:
- is time evoked as important (q1)?
- are features of the technology or constraints of the technology evoked (q2)?
- are possible organisations of students' work taken into account in the analysis (q3)?
- is the distinction social/individual part of the analysis (q4)?
- are the instrumentation processes part of the analysis (q5)?

This partition is made of six clusters but only three of them are informative.

Cluster 1 (14 papers from 8 countries, 6 Anglo-american and 5 French papers)

This cluster gathers papers that take into account
- the availability of technology, its features (type q2),
- the organisation of work (type q3),
- the social dimension (type q4),
- but not always the time (types q1 and q5).

Two papers, which are very different under [illegible]al aspects, are in the centre of this cluster and may thus represent it.

Chacon P.R. & Soto-Johnson H., 1998, *The Effect of CAI in College Algebra Incorporating both Drill and Exploration*
This paper dealing with the use of CAS in college algebra emphasises the role of permanent availability of technology for all students. Students having permanent access to technology acquire a better ability to analyse graphical representations and symbolic expressions. Just partial availability may imply frustration and antagonism toward technology. The paper calls for the integration of technology not only in activities but also in the course and for considering the interrelationship of the procedural knowledge developed about technology and mathematical knowledge.

Pratt D. & Ainley J., 1997, *The Construction of Meanings for Geometric Construction : Two Contrasting Cases.*

The paper describes a two-year experiment in which pupils from primary school had laptop computers. They used Logo and a dynamic geometry environment (Cabri) on their own, deciding themselves about their projects. The pupils used mainly Cabri as a drawing tool rather than as a construction tool and were not eager to produce drawings whose shape is preserved by the drag mode. Only explicit teacher interventions in specific tasks made the pupils move from a drawing activity to a construction activity.

Although these two papers deal with different technologies, school level, and mathematical topics, they share key features with other papers of the cluster, like empirical data in a long-term experiment of students' activities, fine-grained observations, accounts of the characteristics of technology, the fact that positive results of IC Technologies are not taken for granted but questioned through empirical data. Their balanced findings differ from more general papers which are imbued with optimistic appreciation.

Cluster 2 (6 papers, 4 American, 1Australian, 1 French)

Papers of this cluster all focus on the importance of time in the instrumental process (types q1 and q5) and do not consider the organisation of teaching (types q2, q3 and q4). Five of the six papers deal with geometry. The centre of this cluster is

Goldenberg P., 1995, *Ruminations about Dynamic Imagery (a strong plea for research)*
This paper analyses how the continuous transformation of diagrams in geometry may completely change geometrical properties (a geometrical theorem may become a property of a function) and conceptualising processes of students. This cluster gathers papers which do not consider the classroom but mainly analyse features of the technology and their implications on possibilities of action and on conceptualisation. From this analysis, they find that, even with technology, instant mathematical conceptualisation cannot happen and thus time is necessary for students to understand the mathematical implications of the use of the instrument.

Cluster 3 (8 papers, 4 French, 2 Austrian, 1 English, and 1 American).

The only explicit type in this cluster is time (q5) but it is considered in very different manners among the papers. Half of them consider that technology provides a wealth of opportunities for a "more conceptual" use of time while the other half is doubtful and claims that this issue of time deserves more reflection. The two central papers reflect this dichotomy of the cluster.

Kuntz G., 1998, *Une transformation oubliée qui sort de l'ordinaire : l'inversion.*
This paper reports on a teaching experiment at high school level on geometrical transformations using a graphic package. In the author's view, the use of technology makes classroom time more conceptually productive. Students' mathematical activity is reported as dense and involving deep mathematical ideas. Even the constraints of the computer representation are, in the author's view, pedagogically beneficial.

Mayes R., 1994, *Implications of Research on CAS in College Algebra.*
This paper describes a curriculum in algebra based on DERIVE, aiming at developing abilities in modelling, problem solving and conceptual understanding. In the author's hypotheses, DERIVE was crucial to allow "a reduction of the amount of time spent doing tedious manipulations..." An experimental group and a control group are contrasted. This comparison gives no evidence of significant difference among computation and manipulation skills and a marginal improvement in problem solving competence. The only difference is the critical point of view developed by the experimental group: "they felt the increased burden of problem solving and multiple approach".

In our interpretation of the instrumental dimension in this paper, the issue of time becomes central when the author's initial hypothesis is that technology is able to save time for conceptualisation and when the results are students feeling an increased burden and claiming for more "hours credit". We have to recognise that more complex problems generally associated with a "conceptual" introduction of IC Technologies bring students heavier cognitive load and technology does not, by itself, solve this difficulty. As the author concludes, time and explicit intervention of the teacher on strategies (or "techniques", as the institutional approach now calls them) would be essential for conceptual improvements. So Mayes is representative of a trend in this cluster: an evolution towards a more balanced view of the influence of technology upon conceptualisation and towards the necessity of a careful organisation of time in order that students benefit from technology.

Discussion

The above presentation of the "instrumental" partition shows that each cluster contains papers from varied countries and fields. We can see also that papers in a cluster share common concerns for key variables in the dimension. Their findings are varied but are clearly influenced by these concerns. Looking at the interpretation, we see that instrumentation is a big issue with at least three main entries. The time that technology could save or not is probably the more naive entry (cluster 3), but, seriously questioned, it provides interesting discussions. The time requested by a real mathematical instrumentation is another entry (cluster 2) raising deep interrogations on the nature of conceptualisation. It is otherwise interesting to see that papers focusing on the instrumental dimension (cluster 1) do not consider time as the main variable. Our interpretation is that in the experimental settings, the technology was always available; time constraints are not so important in such a case.

A synthesis of partitions

Looking at other partitions, we found that the largest clusters focus on the students, providing a mass of interesting data and results on the learner, certainly insufficient in themselves for a study of the integration but also a basis for this study. We found especially that the evolution of the cognitive theoretical framework generally used for the analysis of the introduction of technology is sustained by these data and results.

Large clusters are also centred on the epistemological dimension. This is evidence of the attention paid to the relationship between the content knowledge at stake and the new teaching means provided by IC Technologies. On the other hand evidence appears that epistemological relevance is not sufficient in itself when no attention is paid to instrumental constraints and ecological viability.

One of the clusters starts with varied postulates converging on the fact that technology enhances teaching or learning. A wider and more varied set of clusters questions this claim and investigates to what extent technology improves learning. With this posture, researchers consider the students' difficulties in the use of technologies as well as the demands that technology brings into the educational system. These clusters call for a wider approach which should include the instrumental, institutional and teacher dimension. References to these dimensions are sparse, although promising as we could notice in the detailed analysis of the "instrumental" partition. A similar analysis of the "institutional" partition should point out the questions of the tasks offered to the students (tasks especially designed to be solved with technology or "ordinary" tasks), of new instrumental techniques and their mathematical productivity, of the teacher's organisation of classroom activities, as important for the future.

Biases and limits

The most obvious sources for biases were ourselves, the researchers, because we came from five different research teams working in various fields of the use of IC Technologies. At the beginning, this brought many distortions in the coding, and we had to discuss every item of the questionnaire. From the observation of statistical indicators, we are now reasonably confident that the initial heterogeneity was overcome. A critical circumstance is that we were researchers from a single country and "cultural" biases may exist. Another limitation was that we could not analyse methods for learning mathematics on CD-ROM or on the Internet because we found only short presentations which were not sufficiently informative.

Conclusion

We saw the huge literature on the use of IC Technologies for teaching and learning mathematics as data representative of the efforts of a big and varied community of teachers, innovators and researchers. We had the intuition that knowledge about integration could be derived from an analysis of this data. Starting from a corpus as large as possible we got a first picture of what this literature is. Technical presentations and reports on innovative classroom use make the bigger part of this corpus. Because they look at the new applications appearing day after day, these papers are potentially interesting contributions on the use of up to date technology. On the other hand, the qualitative study of papers in the CAS sub-corpus shows that the ideas in these presentations and reports are generally weakly supported by reflection and experimentation and cannot address the complexity of the educational situations. Looking qualitatively at the more research-oriented second stage corpus, we saw a slow evolution towards attention to more varied aspects of educational use of IC Technologies and more dialectical cognitive approaches. So the difficult integration can be seen through this picture: innovations present a wealth of ideas and propositions whose diffusion is problematic; research struggles to tackle the complexity of the integration of evolving technologies.

Our aim was to build a framework as a method to look at research and experimentation. This framework was done using several dimensions which arose from a theoretical reflection, that the qualitative study of the global corpus helped to specify into a questionnaire. A statistical procedure was used to identify clusters of studies sharing a common concern in one of the dimensions. Then, looking at outcomes in each cluster, we could identify concerns and findings. In practice, this framework should help innovators or researchers to characterise a project or a field of research by means of the questionnaire and then to get insight on its specific contribution by looking at its position in the partitions. In our mind, this method is a tool to grasp the complexity of the introduction of technology or, more precisely, to tackle the integration of IC Technologies with a "multidimensional problématique".

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REACTION TO
"A META STUDY ON IC TECHNOLOGIES IN EDUCATION; TOWARDS A MULTIDIMENSIONAL FRAMEWORK TO TACKLE THEIR INTEGRATION"

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1. Global reaction

The work of Lagrange, Artigue, Laborde and Trouche is impressive. The French team surveyed an immense number (662!) of publications on technology use in mathematics education. For a researcher in this field, it is very fascinating to read this synthesis of such a massive corpus of literature. The virtue of the study is that it helps in identifying trends, developments and progress in research into technology use in mathematics teaching and learning. Looking back at the work of the last decade enables us to better see the headlines, the dead ends, the topics 'in fashion' and the issues to be elaborated. Also, as the authors indicate, the described framework can help innovators or researchers to characterise a project or a study and to position it in a broader perspective. To me, those are the most important merits of this study.

However, some questions can be raised. My first 'question mark' concerns the aims of the study. Related to this is the point of the dimensions that the study wants to identify. What exactly are these dimensions and what are they used for? Furthermore, depending on the aims of the study, the value of the conducted cluster analysis can be questioned. To summarize, I feel that the quantitative part of the study is too far away from the concrete integration of technology in the classroom to contribute to the 'understanding of integration' that the authors are searching for.

Let me address each of these issues briefly and then come back to the theory of instrumentation.

2. The aims of the study

The first issue I would like to raise is that of the aim of the study. What are the goals of the research presented here? In their introduction, the authors mention their concern about the discrepancy between promising Information and Communication Technologies (ICT) and low actual integration in the classroom. 'Our aim was to build tools for the understanding of integration ...'. Further on in the paper, two aims are distinguished: building a methodology for analyzing research, and identifying trends. I think these last two goals are achieved, although maybe it is better to speak about an investigation of diversity than about a synthesis. However, the study, as it is presented here, did not succeed in contributing to the understanding of the integration of technological tools. The subject of the study, as I see it, is not the integration of technology in the mathematics classroom, but *literature* on this issue. In this sense, it is really a meta-perspective, as is indicated in the title of the paper. The paper is far away from the mathematics classroom; it is a survey of literature on the teaching and learning of mathematics using ICT. Of course, it is quite legitimate and useful to serve a meta-goal. But as long as results and details of research projects are not considered, a cumulating understanding of what happens in the heads of the students cannot be expected.

3. What are dimensions?

The second point, the meaning of the word 'dimensions', is related to the question of the aim of the study. To me it remains unclear what exactly is meant by a dimension. Is it 'a perspective', is it 'an approach', is it set of questions in the questionnaire that was used in the analysis? If I look at the eight dimensions that are defined and were applied to a subset of 79 papers, it is not clear to me what they stand for and if there are any relations between them.

Furthermore, the way in which the dimensions are identified by the researchers is not explained. I regret that the researchers did not (explicitly) take their impressive expertise in this field as a point of departure for such an identification in order to compare this with the massive data they gathered. The only dimension that is discussed somewhat in detail, the dimension of the 'problematiques', seems to be related to the CAS research where it is derived from. The question arises whether dimensions inferred from studies concerning CAS are also valid for research into technology in education in general. Nevertheless, the identification of the most important aspects or themes that play a role in the integration of technology in the mathematics classroom is an interesting goal.

As far as the development of research is concerned, I recognize two of the mentioned trends from my personal perspective:

- The trend from a rather naïve and optimistic perspective ('technology improves teaching and learning') to a view with more nuances and with attention to the complicating aspects of technology use, the pitfalls as they are mentioned in the title of this research forum. The presentation of Hershkowitz and Kieran provides a nice example of this approach.
- The development of research into technology that stands more and more in the tradition of research on mathematics education in general. The presence of a constructivist theoretical framework in many studies, which is mentioned by the authors, illustrates this.

4. The cluster analysis

After the dimensions were identified, they were made operational in a questionnaire. The 'instrumental' dimension that is elaborated in the paper illustrates some of the difficulties with this. For example, the question q5 addresses the instrumentation process over time. My impression is that in the papers that are discussed, the time element concerns the question of efficiency and time savings, which is in my opinion not relevant in the instrumentation process; more important are the changes in student behavior over time. For example, it is not clear what the papers cited in cluster 1 have to do with instrumentation. Also, the role of time in the paper of Mayes in cluster 3 is not made explicit. I was also surprised to see that the three clusters that are described as informative together only contain 28 out of 79 papers.

The questionnaire is applied to a second stage corpus of 79 papers. It would have been interesting to know a bit more about the way the team selected the second corpus papers out of the first corpus. Of course one can't explain all the details in a paper like this, but I would appreciate having some more information on the criteria for this crucial selection.

5. The relevance of instrumentation

The exemplary partition of the instrumental dimension reveals the background of the authors: they have contributed much to the development of the concept of instrumentation of ICT tools. In my perception, the relevance of this theory is that it can help to interpret and to understand the behavior of students using technology. The first paper of this research forum, presented by Hershkowitz and Kieran, can serve to illustrate this.

In the contribution of Hershkowitz and Kieran, students use the linear regression procedure on a graphing calculator to fit a line through five data points that were calculated. The underlying algebraic relationship between the variables, however, was non-linear in some of the cases. Apparently, the students mastered the (non-trivial!) technical part of the instrumentation scheme. However, they are unaware of the mathematical meaning of this command. The accompanying mental scheme of what it means to apply a linear regression and when it is an appropriate technique seems to be lacking.

Thanks to the graphing calculator, teachers may be inclined to use regression with their students, maybe even for fitting a line through two given points, thus freeing the students from some algebra. However, if the teachers forget to pay attention to the limitations of the method and to the difficulties of curve fitting, the resulting instrumentation scheme will be incomplete. I suppose many teachers of a curve fitting course have experiences with students obtaining a perfect fit through n given data points using a polynomial of degree (n – 1) without wondering if the algebraic model is appropriate. In my opinion, the dialectic relationship between ICT technique and mathematical concept, the interplay between the two, is very important in the theory of instrumentation and in the understanding of student behavior while using ICT.

RESEARCH FORUM 2

Theme
Early algebra

Coordinator
Janet Ainley

Session 1

- *"Can young students operate on unknowns?"* David Carraher, Analúcia D. Schliemann, Bárbara M. Brizuela

Reactions

- *"Operating on the unknowns: what does it really mean?"* Liora Linchevski
- *"Of course they can!"* Luis Radford
- *"Reflections on early algebra"* David Tall
- *"Unknowns or place holders?"* Anne R. Teppo
- *"The unknown that does not have to be known"* Elizabeth Warren and Tom J. Cooper
- Audience discussion in small groups

Session 2

- Brief introduction and re-cap of questions
- Audience discussion in small groups continue
- Plenary panel

RESEARCH FORUM ON EARLY ALGEBRA

Co-ordinator: Janet Ainley
MERC, University of Warwick, UK

The purpose of this Research Forum is to debate, in the context of both well established and more recent research evidence, issues about the relationship between arithmetic and algebra, including their mathematical similarities and differences, their relative places in the school curriculum, the apparent difficulties of the transition from arithmetic to algebra and pedagogic approaches which address this transition.

To create a context for this debate, David Carraher, Analucia Schlieman and Barbara Brizuela were asked to write a stimulus paper based on their research into algebraic thinking with 8-9 year olds. Their paper, which follows, challenges some assumptions embodied in school curricula. It was made available on the PME25 website, and anyone planning to attend the conference was invited to contribute a short written reaction.

Because of limitations of space in these Proceedings, I had the difficult task of selecting from these reactions five which represented the range of issues raised. These reactions, from Liora Linchevski, Luis Radford, David Tall, Anne Teppo, and Eizabeth Warren and Tom Cooper, are printed here. I am very pleased that most of those reactors whose papers could not be included have accepted the invitation to participate in other ways in the Forum, and to make their papers available to other participants. Reactions were also received from, Laurinda Brown and Alf Coles, Stephen Campbell, Wim Van Dooren and Lieven Verschaffel, Nicolina Malara and Giancarlo Navarra, and Ana Pasztor.

The organisation of the Forum

In the first session, David Carraher and his colleagues will make a presentation based on their paper, including some video footage, to give a shared context for discussion. The authors of the five Reaction papers will then each make a short response, and raise some specific questions for discussion. There will then be a substantial time for group discussions, running over into the second session.

The second session will also contain an opportunity for groups to report on their discussions, and a 'plenary panel' discussion with the paper authors.

If you plan to participate in the Forum, please try to read the papers in this section before the first session.

CAN YOUNG STUDENTS OPERATE ON UNKNOWNS?[1]

David Carraher, TERC
Analúcia D. Schliemann, Tufts University
Bárbara M. Brizuela, TERC and Harvard University

Algebra instruction has traditionally been delayed until adolescence because of mistaken assumptions about the nature of arithmetic and about young students' capabilities. Arithmetic is algebraic to the extent that it provides opportunities for making and expressing generalizations. We provide examples of nine-year-old children using algebraic notation to represent a problem of additive relations. They not only operate on unknowns; they can understand the unknown to stand for all of the possible values that an entity can take on. When they do so, they are reasoning about variables.

Mathematics educators have long believed that arithmetic should precede algebra because it provides the foundations for algebra. Arithmetic presumably deals with operations involving particular numbers; algebra would deal with generalized numbers, variables and functions. Hence instructors of young learners focus upon number facts, number sense, and word problems involving particular values. Algebra teachers pick up at the point where letters are used to stand for unknowns and sets of numbers. Although there are good reasons for this natural order it lends itself to discontinuities and tensions between arithmetic and algebra.

The difficulties adolescents show in learning algebra (Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985, 1989; Sfard & Linchevsky, 1994; Steinberg, Sleeman & Ktorza, 1990; Vergnaud, 1985) has led to an even starker separation of arithmetic from algebra. Many have believed that algebraic reasoning is closely tied to and constrained by students' levels of cognitive development. For them, algebraic concepts and reasoning require a degree of abstraction and cognitive maturity that most primary school students, and even many adolescents, do not yet possess. Some have suggested that it would be developmentally inappropriate to expect algebraic reasoning of children who have not reached, for example, the period of formal operations (e.g. Collis, 1975). Others (Filloy & Rojano, 1989; Sfard, 1995; Sfard & Linchevsky, 1994) have drawn upon historical analyses such as Harper's (1987) to support the idea that algebraic thinking develops through ordered and qualitatively distinct stages. Filloy and Rojano (1989) note that western culture took many centuries to finally develop, around the time of Viète, a means for representing and operating on unknowns; they propose that something analogous occurs at the level of individual thought and that there is a "cut-point" separating one kind of thought from the other, "a break in the development of operations on the unknown (op. cit., p. 19)". Herscovics and Linchevski (1994) proposed that student's difficulties are associated with a *cognitive gap between arithmetic and algebra*, "the students' inability to operate spontaneously with or on the unknown" (p. 59). Function concepts and their associated algebraic notation are postponed until adolescence for similar reasons.

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We do not wish to deny that there are developmental prerequisites for learning algebra. And we agree that there is a large gap between arithmetic and algebra in mathematics education from Kindergarten to grade 12. The question is: does it have to be this way? Is the gap set by developmental levels largely out of the sphere of influence of educators? Or is it to a great extent a matter of learning? Or of teaching itself? Does the fact that there presently *is* a large gap signify that there m*ust always be* such a gap?

It is crucial for students to learn to represent and manipulate unknowns. However, we believe it is a mistake to attribute the late emergence of this ability to developmental constraints. We believe it emerges late because algebra enters the mathematics curriculum too late and at odds with students' knowledge and intuitions about arithmetic.

Arithmetic As Inherently Algebraic: Functions and Unknowns

Arithmetic derives much of its meaning from algebra. The expression, "+ 3", can represent both an operation for acting on a particular number and a relationship among a *set* of input values and a *set* of output values. This is borne out by the fact that we can use functional, mapping notation, "$n \rightarrow n + 3$", to capture the relationship between two interdependent variables, n and n plus three (Schliemann, Carraher, & Brizuela, 2000; Carraher, Schliemann, & Brizuela, 2000). So the objects of arithmetic can be thought of as both particular (if $n = 5$ then $n+3 = 5+3 = 8$) and general ($n \rightarrow n + 3$, for *all* values of n); arithmetic encompasses number facts but also the general patterns that underlie the facts. Word stories need not be merely about working with particular values but working with *sets of possible values* and hence about variables and their relations.

Arithmetic also involves representing and performing operations on unknowns. This is easy to forget since arithmetic problems are typically worded so that students need invest a minimum of effort to using written notation to describe known relations. The relations tend to be expressed by students in *final form*, where the unknown corresponds to empty space to the right of an equals sign. Were arithmetic problems sufficiently complex that students could not straightaway represent the relations in final form, it would become easier to appreciate how central algebraic notation is to solving arithmetic problems.

We are suggesting that arithmetic can and should be infused with algebraic meaning from the very beginning of mathematics education. The algebraic meaning of arithmetical operations is not an optional "icing on the cake" but rather an essential ingredient of the cake itself. In this sense, we believe that algebraic concepts and notation are part of arithmetic and should be part of arithmetic curricula for young learners.

During the last three years we have been working with children between 8 and 10 years of age to explore how to bring out the algebraic character of arithmetic (see Brizuela, Carraher, & Schliemann, 2000; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 1999). Our work focuses on how 8 to 10 year-old students think about and represent functions and unknowns, using both their own representations and those from conventional mathematics. This work is guided by the ideas that: (1) children's understanding of additive structures provides a fruitful point of departure for "algebraic arithmetic"; (2) additive structures require that children develop an early awareness of negative numbers and quantities and to their representation in number lines (3) multiple problems and representations for handling unknowns and variables—including algebraic notation—should become part of children's repertoires as early as possible; and (4) meaning and children's spontaneous notations should provide a footing for syntactical structures during initial learning even though syntactical reasoning based on the structure of mathematical expressions should become relatively autonomous over time.

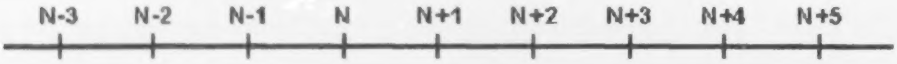
Here we will look at evidence that young children can represent and operate on unknowns. Our examples are taken from our longitudinal investigation with the students of three third-grade classrooms in a public elementary school from a multi-cultural working-class community in Greater Boston. When the children were 8 and 9 years of age, we held eight 90-minute weekly meetings in each of three classes, working with additive structures. Descriptions of our class materials are available at www.earlyalgebra.terc.edu. Our examples come from the seventh lesson we held in one of the three classrooms. There were 16 students in the class that day. Our team of researchers included a teacher, Bárbara, and two camerapersons who occasionally interviewed children as they worked through problems. The students' regular classroom teacher was also present. The following problem served as the basis for discussion and individual work:

> Mary and John each have a piggy bank.
> On *Sunday* they both had the same amount in their piggy banks.
> On *Monday*, their grandmother comes to visit them and gives 3 dollars to each of them.
> On *Tuesday*, they go together to the bookstore. Mary spends $3 on Harry Potter's new book. John spends $5 on a 2001 calendar with dog's pictures on it.
> On *Wednesday*, John washes his neighbor's car and makes $4. Mary also made $4 babysitting. They run to put their money in their piggy banks.
> On *Thursday* Mary opens her piggy bank and finds that she has $9.

We initially displayed the problem in its entirety, so that the students could understand that it consisted of a number of parts. But then we covered up all days excepting Sunday.

Representing An Unknown Amount

The student's first problem sheet contained information only about Sunday. It also contained the following variable number line (or *N*-number line):



After reading what happened each day, students worked alone or in pairs, trying to represent on paper what was described in the problem. During this time, members of the research team walked up to children, asking them to explain what they were doing and questioning them in ways that helped them to refine their representations.

Sunday: After Kimberley reads the Sunday part for the whole class, Bárbara asks whether they know how much money each of the characters in the story has. The children state a unison "No" and do not appear to be bothered by that. A few utter: "N" and Talik states: "N, it's for anything". Other children shout "any number" and "anything".

When Bárbara asks the children what they are going to show on their worksheets for this first step in the problem, Filipe says "You could make some money in them [the piggy banks], but it has to be the same amount". Bárbara reminds him that we don't know what the amount is. He suggests that he could write N and Jeffrey says that this is what he will do. The children start writing and Bárbara reminds them that they can use the *N*-number line on their worksheets if they so wished. She also draws a copy of the line on the board.

Jennifer used *N* to represent the initial amount in each bank. She draws two piggy banks, labeling one for Mary, the other for John, and writes next to them a large *N* along with the statement "Don't know?" David points to "*N*" on her handout and asks:

> David: Why did you write that down?
> Jennifer: Because you don't know. You don't know how much amount they have.
> David: [...] What does that mean to you?
> Jennifer: N means any number.
> David: Do they each have N, or do they have N together?
> Jennifer: (no response).
> David: How much does Mary have?
> Jennifer: N.
> David: And how about John?
> Jennifer: N.
> David: Is that the same N or do they have different N s?
> Jennifer: They're the same, because it said on Sunday that they had the same amount of money.
> David: And so, if we say that John has N, is it that they have, like, ten dollars each?
> Jennifer: No.
> David: Why not?
> Jennifer: Because we don't know how much they have.

The children themselves propose using *N* to represent an unknown quantity. We had introduced the convention before in other contexts but now it was making its way into their own repertoire of representational tools. Several children seem comfortable with the notation for an unknown and with the idea that they could work with quantities that might remain unknown. Some start by attributing a particular value to the unknown amounts in the piggy banks but, as they discuss what they are doing, most of them seem to accept that this was only a guess. Their written work shows that, by the end of the class, 13 of the 16 children adopt *N* to represent how much money Mary and John started out with. One of the children chose to represent the unknown quantities by question marks and only two children persist using an initial specific amount in their worksheets.

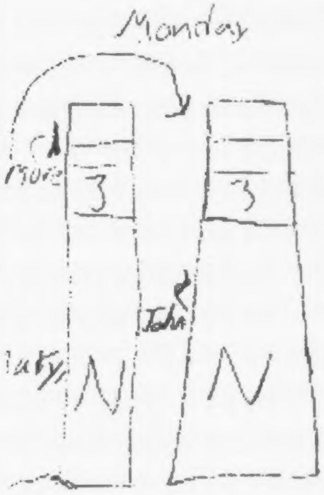
Talking About Changes in Unknown Amounts

Monday: When the children read that on Monday each child received $3, they inferred that Mary and John would continue having the same amount of money as each other, and that they both had $3 more than the day before. As Talik explains:

> Talik: ... before they had the same amount of money, plus three, [now] they both had three more, so it's the same amount.

Bárbara then asks the children to propose a way to show the amounts on Monday. Nathan proposes that on Monday they would each have *N* plus 3, explaining:

Nathan: …because we don't know how much money they had on Sunday, and they got plus, and they got three more dollars on Monday.



Talik proposes drawing a picture showing Grandma giving money to the children. Filipe represents the amounts on Monday as "? + 3". Jeffrey says that he wrote "three more" because their grandmother gave them three more dollars. But when David asks him how much they had on Sunday he incorrectly answers, "zero". Max, sitting next to him then says, "you don't know." The drawing in this page shows Jeffrey's spontaneous depiction of N +3. Note the 3 units drawn atop each quantity, N, of unspecified amount.

James proposes and writes on his paper that on Sunday each would have "N + 2" and therefore on Monday they would have $N + 5$. Carolina writes $N + 3$. Jennifer writes $N + 3$ in a vertical arrangement with an explanation underneath: "3 more for each". Talik writes $N + 3 = N + 3$. Carolina, Arianna, and Andy write $N + 3$ inside or next to each piggy bank under the heading Monday. Jimmy, who first represented the amounts on Sunday as question marks, now writes $N + 3$ with connections to Mary and John's schematic representation of piggy banks and explains:

Jimmy: Because when the Grandmother came to visit them they had like, N. And then she gave Mary and John three dollars. That's why I say [pointing to $N + 3$] N plus three.

Bárbara comments on Filipe's use of question marks. He and other children acknowledge that N is another way to show the question marks. She tells the class that some of the children proposed specific values for the amounts on Sunday. Filipe says nobody knows and James says that they're wrong. Jennifer says that it *could* be one of those numbers.

Only three children do not write $N + 3$ as a representation for the amounts on Monday.

Operating on Unknowns with Multiple Representations

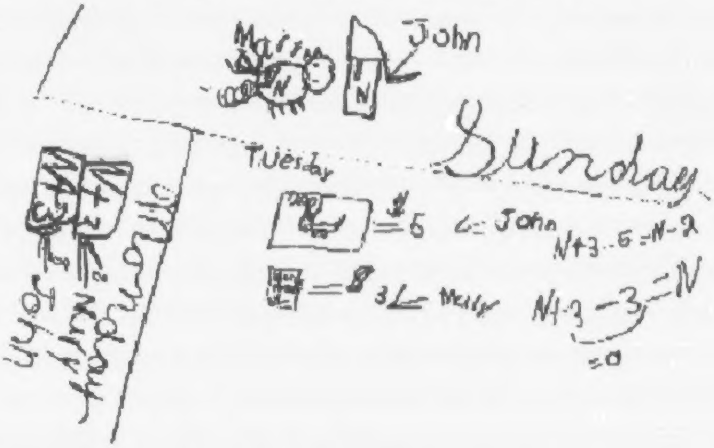
Tuesday: When they consider what happened on Tuesday, some of the students begin to feel uncomfortable because the characters have begun to spend money and the students feel the need to assure themselves that the characters have enough in their piggy banks. A child says that they probably have ten dollars. Most of the children assume that there must be at least \$5 in their piggy banks by the end of Monday; otherwise John could not have bought a \$5 calendar (they seemed uncomfortable with him spending money he didn't have).

Bárbara recalls for the class what happened on Sunday and Monday. The children agree that on Monday they had the same amounts. When she asks Arianna about their amounts on Tuesday, she and other children agree that they will have different amounts of money because John spent more money, leaving Mary with more money.

Jennifer then describes what happened from Sunday to Tuesday, concluding that on Tuesday Mary ends up with the same amount of money that she had on Sunday, "because she spends her three dollars." At this point Bárbara encourages the children to use the N-number line on the board. She draws green arrows going from N to $N + 3$ and then back to N

again to show the changes in Mary's amounts. She shows the same thing with the notation, narrating the changes from Sunday to Tuesday, step by step, and getting the children's input while she writes *N* +3-3. She then writes a bracket under +3-3 and a zero below it, comments that +3-3 is the same as zero, and extends the notation to *N* +3-3=N+0=N. Jennifer then explains how the 3 dollars spent negates the 3 dollars given by the grandmother: "Because you added three, right? And then she took, she spent those three and she has the number she started with."

Using the *N*-number line Bárbara then leads the students through John's transactions, drawing arrows from *N* to *N* +3, then *N* -2, for each step of her drawing. While sketching each arrow, she repeatedly draws upon the student's comments to arrive at the notation *N* +3-5. Some children suggest that this is equal to "N minus 2". Bárbara continues, writing *N* +3-5=N-2. She asks Jennifer to point to, on the number line on the board, the difference between John and Mary's amounts on Tuesday. Jennifer first points ambiguously to a position between *N* -2 and *N* -1. When Bárbara asks her to show exactly where the difference starts and ends, Jennifer correctly points to *N* -2 and to *N* as the endpoints. David asks Jennifer how much John would have to receive to have the amount he had on Sunday. She answers that we would have to give two dollars to John and explains, showing on the number line, that, if he is at *N* -2 and we add 2, we get back to *N*. Bárbara represents what Jennifer has said as: *N* -2+2=N. Jennifer grabs the marker from Bárbara's hand, brackets the sub-expression, "–2+2", and writes a zero under it. Bárbara asks why it equals zero and, together with Jennifer, goes through the steps corresponding to *N* -2+2 on the number line showing how *N* -2+2 ends up at *N*. Talik shows how this works if *N* were 150. Bárbara uses his example of *N* =150 and shows how one returns to the point of departure on the line.



Nathan's drawing (right) shows Sunday (top), Monday (bottom left), and Tuesday (bottom right). For Tuesday, he drew iconic representations of the calendar and the book next to the values $5 and $3, respectively, with the images and dollar values connected by an equals sign. During his discussion with Anne, a member of the research team, and using the number line as support for his decisions, he writes the two equations *N* +3-5=N-2 and *N* +3-3=N. Later, when he learned that *N* was equal to 5 (after looking at the information about Thursday) he wrote 8 next to *N* +3 on the Monday section of his worksheet.

Wednesday: Filipe reads the Wednesday step of the problem. Bárbara asks whether Mary and John will end up with the same amount as they had on Monday. James says "No." Arianna then explains that Mary will have *N* +4 and John will have *N* +2.

Bárbara draws an N -number line and asks Arianna to tell the story using the line. Arianna represents the changes for John and for Mary on the N -number line. Bárbara then writes out the notations, $N +4=N+4$, then $N -2+4 = N +2$. Talik volunteers to explain this. He says that if you take 2 from the 4, it will equal up to 2. To clarify where the 2 comes from, Bárbara represents the following operations on a regular number line: $-2+4 = 2$.

Bárbara asks if anyone can explain the equation referring to Mary's situation, namely, $N+3-3+4=N+4$. Talik volunteers to do so and crosses out the +3-3 saying that we don't need that anymore. This is a significant moment because no one has ever introduced the procedure of striking out the sum of a number and its additive inverse (although they had used brackets to simplify sums). It may well represent the meaningful emergence of a syntactical rule.

Bárbara brackets the numbers and shows that +3-3 yields zero. She proposes to write out the "long" equation for John, $N +3-5+4 = N +2$. The students help her to go through each step in the story and build the equation from scratch. But they do not get the result, $N +2$, immediately. When the variable number line comes into the picture they see that the result is $N +2$. Bárbara asks Jennifer to show how the equation can be simplified. Jennifer thinks for a while, Bárbara points out that this problem regarding John's amount is harder than the former regarding Mary. Bárbara asks her to start out with +3-5; Jennifer says –2. Then they bracket the second part at –2+4, and Jennifer, counting on her fingers, says it is +2.

Talik explains, "N is anything, plus 3, minus 5 is minus 2; N minus 2 plus 4, equals (counting on his fingers) N plus 2. He tries to group the numbers differently, adding 3 and 4 and then proposing to take away 5. Bárbara helps him and shows that +3+4 yields +7. When she subtracts 5, she ends up at +2, the same place suggested by Jennifer.

Thursday: Amir reads the Thursday part of the problem, stating that Mary ended with $9.00, to which several students respond that N has to be 5. Bárbara asks, "How much does John have in his piggy bank?" Some say (incorrectly) that he has two more; other children say that he has 7. Some of the students figure out from adding 5+2, others from the fact that John was known to have 2 less than Mary, since $N +2$ is two less than $N +4$.

Bárbara ends by filling out a data table that included the names of Mary and John and the different days of the week with the children's suggestions for how much money each one had on each of the different days. Some students suggest using expressions containing N and others suggest expressions containing the now known value, 5.

Some Reflections

Many students began by making iconic drawings and assigning particular values to unknowns. But over time, in this lesson, and in others like it, the students increasingly came to use algebraic and number line representations to describe the relations in stories.

We should be careful not to interpret their behavior as totally spontaneous; in fact, children's behavior, even when indicative of their own personal thinking, expresses itself through culturally grounded systems, including mathematical representations of the various sorts we introduced.

Number line representations are a case in point. By the time our students had reached the class we analyzed above, they had already spent several hours working with number lines. They also learned to express such short cuts or simplifications notationally: "+7 –10" could be represented as "-3" since each expression had the same effect. We introduced the variable number line (*N*-number line) as a means of talking about operations on unknowns. "Minus four" could be treated as a displacement of four spaces leftward from *N*, regardless of what number *N* stood for. It could equally well represent a displacement from, say, $N+3$ to $N-1$. At the projector students interpreted values of *N* when the *N* -number line was set just above and slid over the regular number line. They also gradually realized they could infer the values of, say, $N+43$ (even though it was not visible on the projection screen) from seeing that $N+7$ sat above 4 on the regular number line. The connections to solving algebraic equations should be obvious to the reader.

We have found that children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers without having to assign values. To conclude that the sequence of operations "$N+3-5+4$" is equal to $N+2$, and to be able to explain, as many children were able to, in lesson 7, that *N* plus 2 must equal two more than what John started out with, *whatever that value might be*, is a significant feat—one that many people would think young children incapable of understanding. Yet we found such cases to be frequent and not confined to any particular kind of problem context. It would be a mistake to dismiss such advances as mere concrete solutions, unworthy of the term "algebraic". Children were able to operate on unknown values and draw inferences about these operations while fully realizing that they did not know the values of the unknowns.

In addition, we have elsewhere (Schliemann, Carraher, & Brizuela, 2000; Carraher, Schliemann, & Brizuela, 2000) found that children can treat the unknowns in additive situations as having multiple possible solutions. For example, in a simple comparison problem where we described one child as having three more candies than another, our students from grade three were able to propose that one child would have *N* candies and the other would have $N+3$ candies. Furthermore, they found it perfectly reasonable to view a host of ordered pairs, (3,6), (9,12), (5,8) as *all* being valid solutions for the case at hand even though they knew that in any given situation, only one solution could be true. They even were able to express the pattern in a table of such pairs through statements such as, "the number that comes out is always three larger than the number you start with". When children make statements of such a general nature they are essentially talking about relations among variables and not simply unknowns restricted to single values.

By arguing that children can learn algebraic concepts early we are not denying their developmental nature, much less asserting that any mathematical concept can be learned at any time. Algebraic understanding will evolve slowly over the course of many years; but we need not await adolescence to help its evolution.

Final Remarks

Over the last several decades several mathematics educators have begun to suggest that algebra should enter the early mathematics curriculum (e. g., Davis, 1985, 1989; Davydov, 1991; Kaput, 1995; Lins & Gimenez, 1997; Vergnaud, 1988; NCTM, 2000). Some have initiated systematic studies in the area and begun to put into practice ideas akin to those expressed here (Ainley, 1999; Bellisio & Maher, 1999; Blanton & Kaput, 2000; Brito Lima & da Rocha Falcão, 1997; Carpenter & Levy, 2000; da Rocha Falcão & al., 2000; Davis, 1971-72; Kaput & Blanton, 1999; Schifter, 1998; Slavitt, 1999; Smith, 2000).

Still, much remains to be done. "Early algebra education" is not yet a well-established field. Surprisingly little is known about children's ability to make mathematical generalizations and to use algebraic notation. As far as we can tell, at the present moment, not a single major textbook in the English language offers a coherent vision of algebraic arithmetic. It will take many years for the mathematics education community to develop practices and learning structures consistent with this vision.

We view algebraic arithmetic as an exciting proposition, but one for which the ramifications can only be known if a significant number of people undertake systematic teaching experiments and research. The ramifications will extend into many topics of mathematical learning, teacher development, and mathematical content itself. It will take a long time for teacher education departments come to realize that the times have changed and to adjust their teacher preparation programs accordingly. We hope that the mathematics education community and its sources of funding recognize the importance of this venture.

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OPERATING ON THE UNKNOWNS: WHAT DOES IT REALLY MEAN?

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Carraher, Schliemann, and Brizuela raise in their paper a fundamental question: when is a decision to teach a subject at a certain age or in a certain sequence soundly based on some developmental prerequisites and when it is simply a result of a long unquestioned educational tradition? For them this question is crucial not because of time wasted or students' potential not fully developed but because a delayed introduction, they believe, in and of itself might be the very reason for future conceptual difficulties. They claim that sometimes we postpone the introduction of a mathematical topic until the emergence of some theoretically required ability while the late emergence of this ability is a direct result of the late introduction.

The mathematical topic Carraher *et al* deal with in this paper is the transition from arithmetic to algebra and more specifically the ability of beginning algebra students to operate on the unknowns. They believe that the research reported on "cut-points" separating arithmetical from algebraic thought (Filloy and Rojano, 1989), on "cognitive-gaps" between arithmetic and algebra (Herscovics and Linchevski, 1994), and other well-documented difficulties in early algebra (e.g. Collis, 1975; Kieran, 1985, 1989; Steinberg, Sleeman and Ktorza, 1990; Sfard and Linchevski, 1994; Sfard, 1995) has played a major role in educators' decision to introduce algebra only at higher grades. They question this universally accepted decision claiming that these well-documented cognitive-gaps are so widely detected among beginning algebra students *because* algebra enters the mathematics curriculum too late and with odds with students' knowledge and intuitions about arithmetic.

The lesson

The children's performance while negotiating the event of the story is interesting and it, indeed, challenges the idea of a "gap". The problem presented to the children consisted of several parts where the departure information is an unknown number. According to the paper some of the children treated this unknown initial value as a "known" and were even able to use the algebraic notation – the N – to represent it. Moreover, the later parts of the story – which contain only specific numbers – were translated into the mathematical language and added sequentially to the initial value (e.g. N + 3), the obtained expressions were treated as numbers albeit not yet known. Intermediate manipulations were carried out by the children (e.g. N + 3 –3 = N, or even the more impressive example N + 3 –5 = N - 2) leading to equivalent expressions.

The authors report that their teaching interventions included an explicit introduction of the letter as representing an unknown (or any) quantity and the "N-number line" as the model to be used for representing operations on letters, numbers, or combinations of the two. It can be concluded from the paper that the researchers, who served also as teachers/interviewers, also brought the mathematical voice into the class. For example, sentences like: "N, it's for any number" were initially introduced by them.

Some Reflections

It will be helpful to further elaborate on some of the theoretical ideas this paper challenges and to reexamine these ideas in the light of the reported teaching experiment.

Using letters to represent unknown numbers

It is widely accepted that letters can be used meaningfully within children's arithmetic experience. Carraher*et al* takes this idea far beyond what is considered the norm in many classes. The debate, however, is whether the presence of letters in and of itself guarantees that algebra has been introduced. Some people tend to see algebra everywhere and claim that whenever letters or missing addends are present the children are "doing algebra". Others claim that the presence or absence of letters cannot be considered as the indicator for algebraic thinking, claiming that the criterion is not to be found in the displayed task (whether it contains letters or not) but rather in the solver strategy. If letters are part of the expression at hand, one of the requirements the solver has to meet is the ability to perceive "letters" as numbers. This ability is a combination of several aspects.

I. The Lack of Closure

Collis (1974) observed that children at the age of 7 require that two elements connected by an operation be actually replaced by a third element. From the age of 10 onwards, they do not find it necessary. This observation lead to the conclusion that algebraic expressions cannot be introduced to children before the age of 10 (generally speaking) since the operations performed on letters cannot be closed as in arithmetic. The research conducted by Carraher *et al* seriously challenges these observations; their young pupils referred to N + 3 (for example) as a number and did not rejected it as uncompleted or unclosed process. From this perspective these children perceived the unknown as a number. However, Herscovics and Linchevski (1994) note that Collis' age levels have to be taken in some caution since the algebraic expressions used in his work were formal and detached from any context.

II. Operating on and with the unknown

In Herscovics and Linchevski (1994) we write: "...the idea (of Collis) of a pronumeral evolving into a generalized number is quite enlightening. However, it is not sufficient to endow it with "the same properties as any number", for this can be interpreted quite passively, as for example "let *n* be an even number". (*let N be the initial amount of money in the piggy bank.*) In fact, the pronumeral must also be endowed with the operational properties of number; the unknown must be perceived as a generalized number that can be subjected to all operations performed with or on the numbers. Perhaps the expression "operational generalized number" describes this necessary evaluation…".

What does it mean? Operating on and with the unknown implies understanding that the letter is a number. It does not only symbolize a number, stand for a number, and it does not only a tag/label/sign for an unknown number; it is a number. And from this understanding the ability to operate on and with the unknown is emerged. The ability to perform operations on the letters is derived from this perception. Thus, student that has constructed this concept has the ability to add, subtract, bracket…, unknown numbers exactly as he or she has in the context of numbers. To transform, for example, X + 3X, into 4X realizing that while doing it he or she were adding numbers and not just executing formal rules. Moreover, these pupils are expected to be able to use, for example, inverse operations on variables as naturally and as spontaneously as they do it on numbers. For example, the solution of an equation like $32 + X = 3X$ should trigger the use of inverse operation on the X on the left hand-side of the equation thus transforming this equation to $32 = 3X - X$, exactly as it occurs with an equation like $X + 15 = 31$ where they intuitively say that X equals to 31-15. (This last sentence puts Filloy and Rojano's notion of the didactic-cut in context). Thus, transforming an expression like $N + 3 - 3$ to N (as appears in the current paper), does not satisfies this criterion since these transformations do not involve the unknown. It is what we labeled as "static view of the literal symbol" (Linchevski and Herscovics, 1996) or "working around the literal symbol". The ability of the children in Carraher, Schliemann, and Brizuela's research to manipulate the numbers in an expression with one occurrence of the unknown and a string of numbers, is described in details in Herscovics and Linchevski (1994), and Linchevski and Herscovics (1994, 1996). Moreover, in these papers it was reported that sixth and seventh graders operated on the numbers in algebraic expressions of this sort spontaneously, without any prior instruction in algebra. The fact that the research population of these studies was sixth and seventh graders does not imply that younger students would not react in the same way. However, it certainly implies that the choice of the target population was heavily influenced by the current curriculum where algebra is usually introduced in the seventh grade. From this perspective Carraher*et al's* research is a major step forward. Nevertheless, we have to bear in mind that in their study an intensive and direct teaching interventions took place while the other papers reported on spontaneous development.

III. The cognitive gap

The existence of a cognitive gap (as defined in Herscovics and Linchevski, 1994) implies that the students were encouraged to proceed on their own as far as they wish and as long as their own procedures and approaches satisfied them. And a teaching intervention took place only at the point where their intuitive methods, drawn from their existence knowledge and their interaction with the new material, reached an upper limit and they became aware of the limits of their methods thus looking for new points of view. The notion cognitive gap is reserved to these steps in the pupil's learning experience where without a teaching intervention (to our best judgment and research methodology) he or she would not make a certain step. Some of these junctures might be different for different people and some are shared by many. Operating on and with the unknown as discussed in the previous paragraph is one of this junctures. Thus, after identifying a cognitive gap it is trivial to find it in (almost) every age if teaching has not taken place and it is less surprising not to find it after teaching took place.

However, this explanation still leaves the question with regard to the "optimal" age in which the teaching should take place, what are the desired interventions and what differences would be traced in the future is still unanswered. Carraher*et al's* research is a trial to start answering these questions. They do not reject the need for an explicit teaching intervention that provides the students with new tools and new mathematical language: thus they are actually accepting the existence of the cognitive gap.

In fact their study does not refute the existence of the cognitive gap. It explores the possibilities of crossing it earlier. It does not prove that indeed the gap has been crossed but it definitely shows that to a certain extent it might be crossed earlier and that young children can speak in the "algebraic voice".

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OF COURSE THEY CAN!

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In their enlightening paper, D. Carraher, A. Schielmann and B. Brizuela ask several questions and mention some problems that have been at the core of the research conducted in the learning and teaching of algebra for many years. The central question that they ask – namely, Can Young Students Operate on Unknowns?– is rooted in the idea that novice students find it difficult to operate on unknowns. This idea, however, has been refuted experimentally and historically many years ago. Hence a more appropriate question and more in line with their research intentions would be: When can young students start operating on unknowns? Carraher *et al.* suggest that arithmetical operations bear an algebraic meaning and that an early contact with algebra can help infuse this meaning into the children's arithmetic. I will discuss in Section 2 of this reaction the possibilities of such an enterprise, when I will comment, from a semiotic-cultural perspective, on some salient aspects of the classroom episode. In Section 1, before mentioning some of the historical and contemporary experimental data that show that the operation on unknowns is not an intrinsic problem arising in the transition from arithmetic to algebra, I will argue that, in adopting a traditional view in which algebra relates to arithmetic only, Carraher *et al.* restrict the scope of their endeavor and miss important chances to infuse arithmetic concepts from other fields and to talk about e.g. geometrified arithmetic.

1. Operating on the unknown

Without denying developmental prerequisites for the learning of algebra, Carraher *et al.* seem reluctant to accept that a gap set by developmental levels could be the reason for which children cannot operate on the unknown in algebra and that such a gap would be out of the sphere of influence of educators. They suggest that it is probably a question of teaching and that it is wrong to attribute the gap to developmental constraints.

Carraher *et al.*'s refusal of a teleological idea of development and its entailed kind of determinism ("that there must *always be* such a gap") is well tuned with current anthropological views on cognition and conceptual development, where closer attention has been paid to the role of the context and of the others in the conceptual growth of the child, leading to the elaboration of new approaches in which biological lines of development appear dialectically interwoven with cultural ones, so that development becomes inseparable from context (Radford 2000). In their reference to the history of mathematics, they leave without questioning, however, the fact that to operate on the unknown in a certain historical period is not necessarily equivalent to operate on the unknown on a different period. They failed to notice that mathematical concepts are framed by cultural modes of knowing and that several cultures have conceptualized numbers and unknowns in different ways –sometimes more arithmetically (as in Diophantus' work), sometimes more geometrically (as in Babylonian mathematics). In adopting the traditional view that algebra relates to arithmetic only, they reduced the scope of their endeavour and narrowed the possibilities to deal with what is one of their more important submitted problems, that is, the infusion of new meanings into arithmetic. Many years ago, I presented a communication in a meeting held at the CIRADE, in Montreal. I titled my presentation: "Why does algebra not come from arithmetic?" (It later appeared with a slightly different title in a volume edited by Bednardz, Kieran & Lee: see Radford 1996). In that paper I wanted to show that algebra is much more than a generalized arithmetic and that the algebra that we know owes a lot to geometry too (to witness the term *square* root) and argued for a broader view in considering the students' introduction to algebra.

However, the main point that I want to discuss now is the operation on the unknown. Historico-epistemological research has evidenced that the operation on the unknown did not seem to have presented particular difficulties to past mathematicians. This is attested to in the Old Babylonian period, the Antiquity, the Middle Ages and the Renaissance. As concerning the rhetoric pre-Vietan period of the Renaissance, this can found in many abacus treatises. For example, in Raffaello Canacci's *Ragionamenti d'algebra,* we find different ways to operate on and with the unknown. In solving a problem with the means of rhetoric algebra, Canacci (a Florentine algebraist of the second half of the 15th Century) was led to an equation that, for brevity, we can put into modern notations as follows: $t+12=35t-60$. Canacci operated the unknown and easily solved the problem. In an earlier book, Fibonacci's *Liber Abaci* (1202), we find Fibonacci (also working within the representational possibilities of rhetoric algebra) solving the equation $12t-t^2$=54-9t. He transformed it into $t^2+54=21t$ and then solved it by canonical procedures (the problems are discussed at length in Radford 1995. Concerning the operation on the unknown in the Antiquity, see Radford 1991/92; and for examples in Babylonian mathematics see Radford 2001, p.35 footnote 36). In each case, the way the unknown was handled was different: it depended, in particular, on the concept of number.

The successful operation on the unknown has also been reported in contemporary students with no prior knowledge of algebra. This is what Pirie & Martin did in 1997. In light of their experimental research they suggested, referring to the operation with the unknown, that "Rather than an inherent difficulty in the solution of linear equations, the cognitive obstacle is created by the very method which purports to provide a logical introduction to equation solution." (Pirie & Martin 1997, p. 161). A similar conclusion was reached in a previous teaching setting inspired by the history of mathematics. The engineering of the lessons was based on a use of manipulatives that allowed the students to act on concrete objects and then undergo a progressive semiotic process affording the production of meaning and the elaboration of more and more complex representations of the unknown and its operation (Radford & Grenier 1996a, 1996b).

To sum up, the question of the operation of the unknown in algebra has received a positive answer for many years, from a historical and from an educational point of view. When can young students start operating the unknown is, in contrast, a new question. Carraher *et al.* relate this question to their idea that arithmetic can be infused with algebraic meaning in early mathematics education. I would like to comment on this idea in terms of the kind of algebraic meanings that novice students may attain and how it relates to their symbols and repertoire of representational tools. To do so, I will refer to the students' mathematical activity as provided in the paper.

2. Some remarks on the mathematical activity

The paper describes the teacher's attempt to bring the students into contact with some elements of algebra and the way the students gained insights and underwent a process of progressive understanding of key concepts of algebra. I will limit my discussion to some aspects of the children's conceptualization, representation and operation of the unknown.

In general terms, the students seem to have reached a certain level of algebraic understanding. Time and movement were two vital ingredients in the activity. The problem itself was set in terms of steps, where amounts of money were changing. However, time and movement were intermingled with speech, gestures, written symbols, arrows and cultural artefacts –such as *geometrical* N-number line. These elements constituted the arena where the activity and the production of meaning unfolded.

2. 1 The concept of unknown, its representation and operation

The activity provided the students to conceptualize the unknown in a meaningful way. Indeed, the idea of using a piggy bank permitted the students to think of the unknown as a hidden amount of money. Yet this was not enough. A semiotic act still had to be accomplished: the unknown had to be named or represented. The representation of the unknown is a very important step because, through this representation, the students objectify a new mathematical entity that can be applied not only to the piggy bank context but to other completely different contexts as well. An 'all-purpose-or-so' name/sign was hence needed. In the teaching episode, we saw that many students suggested the letter N. Of course, it was not through an individual well-inspired creative act of thought that the students suggested N. The activity was preceded by other activities where the idea of using a letter to represent an unknown number was introduced. The choices, of course are many. Diophantus used the term *arithmos* (number), Al-Khwarizmi used *root* and the Italian algebraists used *res*, and later *cosa* (the thing). But what did the letter N represent? The dialogue suggests that for the students the difference between *any number* and *a-not-yet-known number* was not completely clear. Furthermore, even if the students realized that N is *a-not-yet-known number*, some of them showed a strong tendency in adjudicating to N one of the possible numbers in the range of possible values, as in the Monday episode. As to the children's operation on the unknown, strictly speaking, in the classroom episode there was no operation on the unknown. For instance, unknown terms were not added or subtracted. The operations were performed on numbers (3, 5 etc.).

2.2 The geometrico-algebrafied arithmetic and the rise of meaning

Carraher et al.'s idea of starting algebra earlier than usual is related to the infusion of algebraic meaning in arithmetic. If we were to answer the question: Where does arithmetic really become geometrico-algebrafied? I would suggest that it is in reaching the expression N+3. The *movement* along the *N*-number *line*, is crucial for the construction of meaning. And it acquires a more dramatic tone when the students arrived at N+3-5 and identified as N-2 (on the Tuesday episode) and arrived at N+3-3+4 that they identified as N+4 (Wednesday episode). These experiences are impossible to reach within the confines of arithmetic. Let us analyze the meaning arising from these experiences. Concerning the first one, Arabian mathematicians would make sense of N-2 in thinking of N as being deprived of 2 units (and then, in the process that we now call 'the isolation of the unknown', they would have hurried up to 'repair' N that they would have imagined as a 'broken' segment. To 'repair' it, they would have then applied the rule of *al-gabr*, from where the name algebra derives). Concerning the second one (referring to N+3-5), a Babylonian scribe would have said that 5 is 'detached' from N+3, and would have associated the latter to the width of a field that he would have imagined in the mind or would have drawn on a clay tablet. There is a marvellous Mesopotamian problem in which the scribe arrives at a subtraction of equal terms, and to remove them he says: "not worth speaking about" (transcription and analysis in: Høyrup, 1994, p. 9). Talik, in the video-taped episode, thinking of 'n', '3' and '-3' in terms of money, and coordinating the symbolic expression with *movements* along the *N*-number line, says that +3-3 is not needed anymore. We see how cultural conceptualizations and their meanings, rooted in different semiotic systems, enacted in mathematical activities and objectified in speech, may be different. Still the point is that as different as they may be, the richness of the conceptualizations results from the variety of contexts and the management of varied semiotic resources (speech, gestures, drawings, etc.) to produce meaning. The students' grasping of certain algebraic ideas in Carraher *et al.*'s lesson is related precisely to the richness of the cultural representational repertoire with which the students were provided and to the students' and teacher's progressive integration of such a repertoire in the mediated space of interactions.

Conclusion

We saw that Carraher *et al.* started asking an incorrectly founded question. Their research, however, opens up new avenues. The idea of infusing algebraic meaning into arithmetic appears appealing. Yet it still has to be demonstrated how the learning of arithmetic is really enhanced or if it is merely a question of starting algebra earlier. Their work suggests that 8 and 9 year-old students can attain a certain understanding of the algebraic unknown. The scope of this understanding requires further research. Another point that deserves more reflection is the status of negative numbers. I don't want to venture saying that the students secured a strong concept of negative numbers. Nevertheless, it is clear that the beginning of a conceptualization was started in the course of the lessons.

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REFLECTIONS ON EARLY ALGEBRA

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Introduction

At this conference we celebrate 25 years of research meetings organised by the International Group for the Psychology of Mathematics Education. It is therefore fitting to place this response within this context. Carraher, Schliemann and Brizuela (2001) refer broadly to a range of previous research, partly to report observed difficulties and partly to respond to suggestions to 'bring out the algebraic character of arithmetic'. This means that, apart from using the advice to 'algebrafy' arithmetic, the 25 years of previous research is not used in any foundational way. My analysis of their paper therefore uses a global theory of developing symbolism to place the research in context.

Analysis

Carraher et al (2001) base their research in a 'typical' class of 9 year-olds and is implicitly an approach to teach 'algebra for all'. I begin, therefore, by looking at their data to see if they are actually reaching every child in the class and also to analyse precisely what kind of algebra the children appear to be learning.

The class is presented with a story in which two children start with the same unspecified amount of money on Sunday and spend and receive specific amounts on successive days. When the researcher Bárbara asks the class if they know how much money they have, 'the children state a unison "no"', but 'a few utter "*N*" and Talik states "*N*, it's for anything."' Thus we have some children who have already met the idea of using a letter to stand for a number and some who presumably have not. Our first piece of evidence is that, faced with adding 3 dollars to the initial unspecified amount, we are told that 'only three children do not write $N+3$ as a representation for the amounts on Monday.' What is missing is an analysis of what the children individually bring to the class from their previous development and why some children are more adept at algebraic thinking than others. This in turn requires a theory of longer-term development that is consonant with empirical evidence.

Tall et al (2000) present a theory of symbolic development arising from earlier work of Gray & Tall (1994) and many others. This reveals a *bifurcation* in performance in arithmetic between those who become entrenched in a procedural mode of counting and those who develop *proceptual* thinking involving the flexible use of symbols as both process and concept. This is not to be interpreted as a naïve prescription that the successful always get better and the less successful get worse.

The case of Emily (Gray and Pitta, 1997) reveals a child growing from counting procedures to flexible number concepts by being given support using a calculator that carries out the procedures for her so that she can concentrate on the conceptual relationships. However, the theory does intimate that what children bring to a given situation—depending on their preceding development—radically affects how and what they learn. It can have a profound effect on early algebra.

For instance, the National Curriculum for England and Wales intended to use arithmetic problems such as the following as a precursor of algebra:

(1): $3+4=\square$, (2): $3+\square=7$, (3): $\square+3=7$.

Although these *look* like algebra, they are certainly not. Children perform them using their repertoire of methods of counting and deriving or knowing facts. Question (1) can be done by any counting method, (2) can be done by 'count-on' from 3 to find how many are counted to get to 7. Equation (3) is more subtle. If the child senses that the order of addition does not matter, the problem is essentially the same as (2); and can be solved by count-on from 3. If not, the child who counts has a far more difficult task to find out 'at what number do I start to count-on 3 to get 7?' This involves trying various starting points to count-up using a 'guess-and-test' strategy.

Foster (1994) used these three types of question in a study of 'typical' children in the first three years of an English Primary School. He found a significant spectrum of performance in the first year where the lower third were almost totally unable to respond to questions of types (2) and (3). By the third year the top two-thirds of the class obtained almost 100% correct responses but the lower third obtained 93% correct on type (1), 73% correct on type (2) and 53% on type (3). Seen in the light of procept theory, this suggests that the lower third operate more in a procedural than a flexible proceptual level. This would be consistent with the lower third of a class in Grade 3 in the USA including children who are more procedural than proceptual, which, in turn is consistent with difficulties with algebraic qualities of arithmetic exhibited by some children in this study. I would counsel, therefore, that in carrying this activity out in a classroom context, some children are already struggling and need special individual care. Even those who succeed in writing down the symbolism '$N+3$' are likely to be using it in a manner different from that observed by an expert.

When the symbols introduced into the work of Carraher *et al* are analysed, they are all of the form of an unknown quantity followed by successive numerical additions and subtractions, such as $N+3-5$. (On the web-site related to the paper, there are also considerations of equivalence of expressions such as $N+3-5$ and $3+N-5$.) The children can, and will, make their own interpretations of the meaning of the symbolism.

The researcher Bárbara leads a discussion using 'the number line centred on N' to visualise the symbols $N+3-3$ in terms of shifts along the line starting from N. The paper describes how she 'writes a bracket under 3–3 and a zero below it […] and extends the notation to $N+3-3 = N+0 = N$.' This is the description of what she—the expert in this case—sees. *But does each individual children see and think in this way?*

A wide array of literature reports children conceptualize the 'equals' sign as an *operation*, not as an equality between two expressions. It is here that procept theory helps. Bárbara and her co-researchers have the ability to switch between seeing the symbol $N+3-3$ as an expression for a single mental concept on the one hand and as a process of successive steps on the other. She can see the 'equality' of the two concepts. As the discussion unfolds, it is *Bárbara* who writes $N+3-5 = N-2$ and the *authors* of the paper who call 3–5 a 'sub-expression', claiming that Jenny 'writes a zero under it'. However, we have a reproduction of the work of Nathan, who writes not zero, but '=0'. An alternative, and more likely, explanation is that some (perhaps most) of the children are interpreting the symbols as *processes* to be performed rather than as *expressions*.

As *all* the formulae in the paper consist of an unknown N followed by number operations, the context allows the children to operate essentially in an *arithmetic* mode. They are not asked to *operate* directly on the unknown, rather this is the starting point from which arithmetic operations occur and can be the main focus of attention.

There is evidence that some children work with N as an unknown. For instance, 'Talik shows how this works if $N=150$.' This inhabits an intermediate stage that Thomas & Tall (2001) call *evaluation algebra* in which expressions are used to represent a general arithmetic operation (as, for instance, they do in a spreadsheet). This is an earlier stage than full-blown *manipulation algebra* where the symbols are freely manipulable entities as expressions and sub-expressions. In evaluation algebra, symbolic expressions are seen as *processes* of evaluation. Manipulation algebra sees them as *procepts* representing either process or manipulable concept.

Carraher *et al* (2001) ask in their title: 'can young children operate on unknowns?' The evidence they provide reveals that their approach has absolutely *no* operation *on* unknowns in the sense of symbol manipulation. There *is* evidence of *evaluation* by substitution (as a by-product rather than a direct focus of the activity). In general, the children's activity involves arithmetic operations on arithmetic symbols.

Is this a problem? Absolutely not. Some children are evidently becoming familiar with the use of a letter to represent a specific but (to them) unknown number. Thus at least one aspect of the development of algebra is beginning to take root. However, the journey through evaluation algebra and on to manipulation algebra is a long one and for many *but not all* children it will involve difficult cognitive reconstructions.

All the symbols used in the activities are read in the usual left-to-right direction in Western languages. There is still a long way to go for children to cope with expressions such as $2+3x$ where the product of 3 and x must be performed *before* 2 is added to it.

The work of Carraher *et al* is certainly a first step, but it needs to be explicitly aware of what individual children might bring to the task and where they might go later.

Concluding Remarks

I have suggested that the study of 'early algebra' needs to be seen not only as an activity in itself but also as part of a longer-term development. The activities need to be carefully analysed to see as clearly as possible what it is that the children have to build on and what it is that they are likely to be thinking. In the analysis presented here I have indicated conceptions that children may bring to the enterprise that may hinder or help them (for example, arithmetic as procedures or as flexible process-and-concept). I have analysed what some children might be doing with the symbols (operating with them as processes, rather than seeing them as expressions). I have emphasized the chosen limitations (symbolism read from left to right, starting with an unknown that may be left on its own, allowing a focus on the arithmetic operations that follow). There is the evidence that some children (eg Talik) have taken the first step into evaluation algebra by substituting a number for the unknown. However, it is less clear as to who sees the symbolism $N+3-5 = N-2$ as an arithmetic *process* and who see it as an *equality* of mental expressions (concepts).

A step has been taken by some (many?) of the children. It is a significant step. But it has implicit properties (reading an expression left to right, perhaps seeing the expression as 'a process to do' rather than 'a concept to manipulate', perhaps coping by working only at an arithmetic level). Such properties may become part of the child's mental structure that needs reconstructing at a later stage. The major cognitive obstacles of manipulation algebra that afflict many, but not all, children in the bifurcating spectrum of performance still remain to be addressed in the future. If the bifurcation we have observed continues to occur (and it seems to be very persistent), it may be that some may be profitably focused on evaluation algebra that has powerful uses in computer contexts whilst others develop proceptual flexibility required for manipulation algebra.

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UNKNOWNS OR PLACE HOLDERS?

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This reaction paper examines the cluster of algebraic concepts associated with variables used either as unknowns or as place holders. *Unknowns* are variables that represent particular numbers. As such, these variables appear in equations, whose solutions are a particular number, and the equations, themselves are about that number. In contrast, *place holders* are variables that represent any number. Such variables appear in identities and functions. These algebraic sentences are generalizations about particular operations and the order in which these are applied.

In the paper by Carraher, Schliemann, and Brizuela, a distinction is not clearly made between the particular algebraic meanings associated with each of these two variables. The term "unknown" is sometimes loosely used by the authors, and the children in the study make little distinction between "N" as a representative of an unknown quantity or as a generalization for "any number."

Unknowns and Equations

The solution of the piggy bank problem utilizes unknowns. Even though the initial amount of money in each child's bank is not given, the context of the problem makes it clear that this amount, while not specified, can only be a certain value. Thus, the reasoning that takes place as the children work through the activity is that of determining the exact value of this amount. The words that several of the children use as they talk about the problem do not align with this notion of variable.

For example, as the children discuss the quantity of money in the banks on Sunday, several students state that "N" could stand for "anything" or "any number." Conceptually, this is not correct. N stands for a particular amount of money that is actually in the bank, an amount that the children can find if they reason through the given situation.

The classroom emphasis on N as a number that is not known is important. It is clear that the children are able to perform operations on such an entity and can comfortably explain the quantitative relationships that exits on the different days described in the piggy bank problem. However, it is also important to help the children understand that, while unknown, the number in the given context cannot be "anything."

The solution to the piggy bank problem embodies the algebraic concept of *equation*. The implicit understanding of the activity is that it is possible to perform operations on an unknown number, within the constraints of a given situation, in order to determine the value of this number. In the classroom, the use of expressions involving N and the N-number line provide an alternative representation for the conventional symbol string that comprises an algebraic equation in one unknown.

It would be useful to have held a discussion, before the children began their number line representations, about whether it could be possible to find the exact amount of money contained in each bank on Sunday. Children should have been able, given the result that Mary had $9 in her bank on Thursday, to have decided that the original amount would have had certain limits placed on its possible value. For instance, Mary would have needed to have spent a great deal more money if she had started with $100 in her bank on Sunday and ended up, on Thursday, with only $9 left.

Then, the activity could proceed as it did, with comments on the fact that the amount of money is not known at that time. After the specific value had been found at the end of the activity, it would also be appropriate for the teacher to remind the children that, by working through all the changes to the unknown that were given in the problem, they were able to find the exact amount originally in each piggy bank. It would have been a valuable extension to the activity if some reflective discussion could have been included to lay the foundation for the notion of an equation as a useful algebraic tool for finding the value of an unknown.

Dummy Variables and Functions

A second example of children dealing with an algebraic situation is briefly mentioned at the end of the paper. Here, children are discussing a comparison situation in which one child has three more candies than another. This example utilizes *place holders* rather than unknowns. However, the authors' use of the term "unknown" in the comment "that children can treat the unknowns … as having multiple solutions," has the potential for mixing conceptual entities. The authors do continue their discussion by commenting that "when children make statements of such a general nature they are essentially talking about relations among variables and not simply unknowns restricted to single variables."

In the comparison example, the children, as the authors correctly point out, are dealing with different conceptual entities than those used in the piggy bank problem. Here, the undetermined amount can take on any value, and the expression "N+3" represents a function. This expression provides information about a particular order of operations ("add three"), and the focus of the activity is not on finding a number, but on stating a relationship.

Discussion

It is important to use separate names to distinguish the two different uses for algebraic variables, since each use represents a different type of algebraic entity. Unknowns occur in equations that are essentially numeric in focus. The goal of using such symbols is to model a quantitative situation in which certain constraints make it possible to reach a numerical result. On the other hand, dummy variables are used as place holders to make general statements about mathematical relations, expressed as a particular sequence of operations.

The authors' descriptions (in the section "Some Reflections") of the ways in which the students used the number line illustrate how easy it is to blur the distinctions between the two clusters of algebraic entities. In a single paragraph, examples are given of representations for both functions and equations. In one case, students treat "minus four" as a displacement of four spaces, and recognize that the result of this operation can be represented as N-4 and as the movement from N+3 to N-1. In conventional algebraic symbolization, this is equivalent to, "if $f(x) = x - 4$, then $f(x+3) = x - 1$." In a second instance, students "infer" the value of N+43 from observing that N+7 is above the number 4. This is equivalent to solving the two-variable system of equations "$x + 7 = 4$ and $y = x + 43$."

In their concluding remarks, the authors state that "little is known about children's ability to make mathematical generalizations and to use algebraic notation." This statement is ambiguous in light of the previous remarks. It is not clear how the authors link "making generalizations" to particular uses of algebraic notation. I would prefer to make a clear distinction between using notation to make generalizations (place holders and functions) and using notation to find numerical results (unknowns and equations).

The authors' examples of young children using algebraic notation and number-line representations raise some interesting questions.

- Should the mathematical distinction between unknowns and place holders be a matter of concern in introductory algebraic experiences? (This distinction is one that many college-level students do not necessarily understand.)
- At what point in their educational experience should students first implicitly, and then explicitly deal with these conceptual issues?
- How deeply should teachers understand the algebraic structure behind particular introductory activities?

I agree with the authors that research is needed to investigate the extent to which difficulties in the development of algebraic understanding are either developmental or the result of educational practice. I would like to extend the focus of such investigations further, however, to include a careful understanding of algebraic entities themselves. Unless we as mathematics educators can fully unpack, for ourselves, this complex field of reasoning, we can only partially address the concerns of educating others.

THE UNKNOWN THAT DOES NOT HAVE TO BE KNOWN

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In response to the misconceptions students are experiencing in the algebraic domain, there has been a call to begin algebraic thinking early. Kaput (1999) believed that algebraic understanding evolves from viewing algebra as the study of structures abstracted from computation and relations, and as a study of functions (a static and dynamic dimension). The arithmetic knowledge base that is needed for algebra comprises an understanding of (i) arithmetic *operations,* (ii) the *equal* sign as equivalence, (iii) the operational *laws*, and (iv) the concept of *variable* (Ohlsson, 1993). Usiskin (1988) argued that the notion of variable could be introduced through three approaches: solving equations with *unknowns*; generalisations of *patterns*; and *relationships* between quantities. He contended that, in the long run, these notions had to be combined and abstracted to develop the concept that variable was a *member of an abstract system.* While some researchers (e.g., Chalouh & Herscovics, 1988) argued that unknown was not an appropriate algebraic conception for variable as it does not represent multiple meanings, Graham and Thomas (1997) contended that an appreciation of unknown could allow students to better assimilate later concepts. For this to happen, they argued that activities with unknown should cover a wide variety of situations including recognising unknown situations, substituting for unknowns, considering solutions as values that make the equation true, and finding solutions through arithmetic and algebraic methods.

Carraher, Schliemann, and Brizuela's paper explores young students' ability to operate on and represent unknowns in a relational situation as a precursor to developing understanding of the variable. They describe an example of classroom activity (two children starting with the same unknown amount of money) to provide evidence that "children as young as eight and nine years can learn to comfortably use letters to represent unknown values, and can operate on representations involving letters and numbers without having to instantiate them" (p. 7).

Power, formal letters and the limits of number lines

The power of the activity. We applaud the power of Carraher, Schliemann, and Brizuela's classroom activity. First, it presents arithmetic as *change* rather than solely as relationship; that is, it is a *dynamic* form of arithmetic in that it represents +3 as a movement on the number line from 2 to 5. Second, because of this focus on change, it allows the notion of *backtracking* (undoing the changes) to be introduced. Although limited (Stacey & McGregor, 1999), backtracking is a useful procedure for solving equations with one instance of unknown. Third, it involves sequences of operations which, initially, are not capable of *closure*. Fourth, it encourages children to interrelate a rich array of representations when articulating their understanding (e.g., verbal and written language, diagrams, number lines, and symbols), thus developing rich representational scheme. All these signify a significant move away from the traditional approach most children of this age experience in their everyday classroom.

The use of N. However, the use of the symbol N in the activity is a concern and raises questions. Even though the activity reflects Herscovics and Linchevski (1994) suggestion that the transition to formal algebra involves considering numerical relations of a situation, discussing them explicitly in simple everyday language, and eventually learning to represent them with letters, we wonder whether the use of N in the activity reflects a limited view of algebra (as "arithmetic with letters" rather than as the "mathematics of generalization"). Do you need letters to do algebra? To us, the answer is no; we see algebraic thinking as predominantly the ability to operate in generalized abstraction and prefer that the students use normal language (e.g., "the beginning money") or their own invented "symbolisation/notation" (e.g., "apb" - *amount in piggy bank*). We believe this use of language or invented terms are as algebraic as "N", and maybe less dangerous. We are not convinced that the children in the activity are not seeing the "N" as "the piggy bank" or as a specific number, understandings of N that are inappropriate for algebra (Küchemann, 1978).

The number line representation. The use of the number line in the activity also raises questions and concerns. While the number line contextualisation of the activity is very powerful, the change represented is linear movement (back and forward on the number line for addition and subtraction). How do we deal with situations where the change involves multiplication and division (e.g., he doubles the amount of money in the piggy bank or shares it among three friends)? How do we prevent prototypic thinking (Schwarz and Hershowitz, 1999). The number line also seems to restrict the type of problem that children can explore. It is difficult to see how the number line will model a problem where the unknown is not the starting point or it appears more than once. For example, Mary had $15 in her piggy bank on *Sunday*, was given some money on *Monday*, spent $6 on *Tuesday*, and opened her piggy bank to find $30 on *Thursday*? And with the unknown as the starting point, closure is available for all other computations. In fact, it is possible to simply ignore the unknown.

Unknowns, young students' understanding and equals

The approach to variable used in the Carraher, Schliemann, and Brizuela's classroom activity falls into the category of *unknown* (Usiskin, 1988). While the activity uses a dynamic broader understanding of arithmetic that offers opportunities for developing algebraic ideas not available in traditional classrooms, we have concerns with young students' capacity to understand unknown. We have recently investigated this with a sample of 87 children of average age 8 years and 6 months attending four schools across metropolitan Brisbane. In an interview, the children were asked to explain how they could find the unknown in the following two situations:

16 +☐ = 49

54 =☐ - 12

The script was as follows: What is the card asking you to do? How can you find the missing number? What is the missing number?

Initial analysis of the scripts indicated that all the children understood that the task was to *find the missing number*, and most could do this for the first example. For this example, the common strategy was counting on and the common response was: *You go 16 and then you count from 16 to 49. Can I do it in my head - can I just count in my head 17, 18, 19, 20.* Most difficulties involved keeping track of how many had been counted on, *I just keep losing track of it.* Some simply counted from 6 to 9 and from 1 to 4 giving the solution of 33, *put a 3 on the 6 equals 9, put a 3 on the 1 equals 4.* Only four students found the unknown by using subtraction, *you can find the something by taking 16 from 49.* Very few children could find the unknown in the second task, for a variety of reasons that all seemed to relate to everyday classroom experiences. A common obstacle was the *non-standard* formatting of the question: *12 minus can't equal this. This is a wrong one because 12 minus can't equal 54. It's backwards.* Many children could not go beyond this point. This seemed to occur for two reasons. First the position of the = sign caused difficulties. When directed to explain what the problem was asking them to do, many said: *Fifty four take something equals 12 - you have to find the something.* Second, the position of the unknown also seemed to cause problems: *It is all mixed around. You can't have 12 minus something.* Most believed that the unknown should occur on its own after the equal sign: *12-54 = something. It would be a little number – it would be 0 because if you get 12 and take away a big number you would only get 1 and then an extra number would be 0 it would be 1 or 0.* Some simply dealt with this problem by: *flipping it over, 54-12=42. Or you could just do 54-42=12.* Only two students found the missing number by converting the problem into the correct addition situation: *You find the missing number by adding 12 onto 54.* Understandings abstracted from classroom experiences seemed to be acting as cognitive obstacles to solving equations with *unknowns.*

Conclusions

Commonly, classroom activities present arithmetic equations in the form 3+4=7, computation on the left and solution on the right. Unknowns presented in the same format (e.g., ☐+6=11) are simpler for young children. Different formats, sequences of operations that cannot be closed (e.g. 4+☐+9) or two or more instances of unknowns (e.g., ☐+7+☐=15) are much more difficult. "I am a number" activities can work at quite young ages because the "unknown" is first and can be ignored while the numbers are computed (e.g., *I am a number. I have been multiplied by 3 and 5 has been added. I am now 23, what was I?).* Does replacing ☐ by "N" make the activity more algebraic? Does drawing a number line with N in the middle mean that the students are handling unknowns and understand the meaning of N+3?

Carraher, Schliemann, and Brizuela's classroom activity prepares students for algebra in its dynamic presentation of sequences operations, its potential to prevent closure (at least in the first operation), and its integration of problems, language and activity. However, the activity's use of N is not compelling and the number line places limitations on the position of the unknown that means it does not have to be known during the remainder of the operations. The activity should be extended to include all the components suggested by Graham and Thomas (1997) and combined with activity on operations that prepares students for non-standard formats and a variety of positions of unknowns. The challenge is to develop a number and operations sense that leads to algebra (and algebraic sense).

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RESEARCH FORUM 3

Theme
Comparative views of mathematics goals and achievements

Coordinator
Thomas Romberg

Session 1

- Introduction to the Research Forum by Thomas Romberg
- Brief information about the four papers
  - *"A description from Germany"* Gabriele Kaiser
  - *"Even college students cannot calculate fractions: Mathematics goals and students' achievement in Japan"* Yoshinori Shimizu
  - *"Mathematics goals and achievements: The case of Lebanon"* Murad Jurdak
  - *"Mathematics goals and achievement in the United States"* Thomas Romberg
- Open discussion with the audience

Session 2

- Reflection on the information presented and raising questions
  by Gabriele Kaiser, Yoshinori Shimizu, Murad Jurdak, and Thomas Romberg
- Open discussion of the questions with the audience

COMPARATIVE VIEWS OF MATHEMATICS GOALS AND ACHIEVEMENTS

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Murad Jurdak, American University of Beirut, Lebanon

This Research Forum in educational policy research focuses on the differences in perspectives across countries with respect to mathematics goals and achievement. Each of the four participants will present a brief paper on the mathematical goals for three groups of students in their country (non-college-bound students, college-bound liberal arts students, and college-bound mathematics and science students) and on the evidence on achievement of mathematics goals with respect to three levels (knowledge of concepts and procedures, understanding of the relationship of mathematical ideas in specific domains, and the use of mathematics to mathematize unfamiliar problem situations).

These papers provide background information related to the policy implications of the recent international comparative studies (e.g., TIMSS) and of OECD's forthcoming PISA project. Too often politicians and other policymakers fail to understand that differences in student achievement in mathematics need to be seen in light of each country's goals for its students and the quality of evidence related to achieving those goals.

A DESCRIPTION FROM GERMANY

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Abstract: This paper concentrates on the goals of mathematics education in Germany after compulsory education. The school system is first explained briefly. In what then follows, it is demonstrated that, based on an analysis of the curricula and results from the TIMSS study, German general education and the various kinds of vocational education differ very much from each other concerning goals of mathematics teaching as well as in the students' achievements. Finally, it is stated that mathematics teaching only barely attains the goals formulated in the curricula and that in international comparisons the weaknesses of German mathematical teaching are becoming obvious.

The Structure of the German School System

This paper concentrates on mathematics education after compulsory education has been completed. In order to better understand what follows, the German school system is first briefly described.

In Germany, compulsory schooling commences at the age of 6 and finishes at 18. Nine or 10 of these years, depending on the school system of each federal state, must be spent in full-time schooling, and the remaining 2 or 3 years spent either in full-time schooling or in part-time vocational schools in conjunction with a trade or apprenticeship programme.

Lower secondary level for students aged 10 to 16 offers differentiated teaching in accordance with student ability, talent, and inclination. Students are placed into one column of a three-column-system, within which there is no streaming.

Upper secondary level, for students aged 16 to 19, takes place, as already mentioned, in two systems: a general education system and a vocational system.

Overview of the General Education System

In the general education system, a three-year course qualifying students to enter university is offered. Courses are offered at two different levels in terms of academic standards and teaching time: basic or advanced courses (*Grundkurse* and *Leistungskurse*, respectively). The basic courses are intended to ensure that all students acquire a broad general education of the subject in question whereas the advanced courses are meant to provide further specialist knowledge and serve as an in-depth introduction to academic study.

Many limitations are placed on students' choices to ensure that all students achieve a broad range of knowledge. Students must select at least two advanced courses, one of which must be either German, a foreign language, mathematics, or a science subject.

Mathematics as advanced course is chosen as a second course most often by 35% of the students, but with a great difference between boys (47%) and girls (26%); as well as between the western (32%) and the eastern (41%) federal states. There is a significant difference in teaching time (3 vs. 5 hours weekly) between basic and advanced courses. Currently, mathematics is a compulsory subject and must be taken at least at a basic level until Grade 13. Until recently, students in some states were able to drop mathematics in Grade 13, which on average 10% of the students did.

Those students whose aim after school is to study mathematics, science, or technical sciences choose mathematics as an advanced course; those oriented more toward linguistics and social sciences tend to take basic courses in mathematics. Those attending basic courses of mathematics can be described as college-bound liberal arts students; the students participating in advanced courses are generally college-bound mathematics and science students.

The upper secondary level also encompasses full-time and part-time vocational education. The Western German dual system of vocational education involves cooperative apprenticeship at two learning sites: the school and the workplace. Enterprise-based vocational training has two sponsors: the governments of the federal states, which establish and finance vocational schools; and the enterprises themselves, which finance and provide apprenticeships. Full-time vocational education comprises many mixed forms of schooling, which shall not be differentiated here. These students may be considered non-college-bound.

Of students 17–19 years old in the upper secondary level, 31% are in the gymnasium and comprehensive schools, Grades 11 to 13; nearly 16% are in full-time vocational education; and about 53% are in part-time vocational education.

Goals of Mathematics Education for General Education

The curriculum for mathematics in Germany is laid down in curricula for each state and for each of the different types of schools.

In general, the curricula state that the general aims of mathematics education are to—

- Provide fundamental knowledge and skills in important areas of mathematics.
- Provide mastery of the techniques, algorithms, and concepts necessary for everyday life in society.
- Develop the ability to describe facts mathematically, to interpret the contents of mathematical formulae, and to enhance the solving and understanding of nonmathematical or environmental phenomenon through mathematics.
- Teach pupils to think critically and to question.
- Give examples of mathematics as a cultural creation in the historical development of civilisation.
- Provide terms, methods, and ways of thinking useful in other subjects.

During the last few years, the curricula have undergone changes, but the trends of changes are more apparent in approach than in content. Thus, the new orientation for mathematics teaching is to—

- Present mathematics both as a theoretical study and as a tool for solving problems in the natural and social sciences.
- Provide experience with the fundamental mathematical idea of generalisation, the need for proofs, structural aspects, algorithms, the idea of infinity, and deterministic versus stochastic thinking.
- Use inductive and deductive reasoning, methods for providing proof, axiomatics, normalisation, generalisation and specification, and heuristic work.
- Provide variation in argumentation and representation levels in all fields and aspects of mathematics teaching.
- Teach historical aspects of mathematics.

These goals, indicated as common learning objectives of mathematics teaching, can be regarded as consensus among mathematics educators and, among others, are therefore explicated in the expertise on mathematics teaching in the upper secondary level (Borneleit et al., 2000). The main difference between basic and advanced courses is that objectives formulated for advanced courses are oriented more toward mathematics as science, whereas those for basic courses stress algorithms and mathematics formulae. Both courses are based on the same three pillars of mathematics education: calculus, linear algebra/analytical geometry, and probability and statistics. Among these, calculus is the most important and gets the largest portion of teaching time. The first two strands were found in curricula already at the beginning of the 20th century, but probability and statistics have been added only in the last 10–15 years. These unshakable pillars of mathematics teaching are firmly laid down in the German baccalaureate (*Abitur*) standards for all states of the German Republic. Further specifications and differentiation of goals vary very much from state to state.

Goals of Mathematics Education for Vocational Education

In vocational education, the importance given to mathematics varies very much, and there is a wide spectrum of goals in mathematics teaching. The TIMSS study on the upper secondary level divides the fields of vocational education into three groups: (a) professions closely related to mathematics, including professions in the trades; (b) professions closely related to technology, including professions in the fields of metal, electrical/electronic, and construction industries; and (c) professions not closely related to mathematics or technology professions, including those found in the fields of agriculture, housekeeping, social services, and nursing (see Baumert, Bos, & Lehmann, 2000, Vol. 1). In each of these three categories, mathematics is of different importance. Often, in vocational education not closely related to mathematics, mathematics does not exist as a separate subject, but is integrated into content-related subjects. In nearly all fields, in fact, the learning objectives depend on content-related contexts. Therefore, mathematical literacy should be taught, as it has been tested by the TIMSS study, focusing on the following competencies:

- Everyday real-world-related conclusions.
- Application of basic routines.
- Elementary modeling and linking of mathematical operations.
- Mathematical reasoning, especially through graphs.

Interviews with experts show that, in technical education and education for manual work, mathematical modeling is given special emphasis and that, in the commercial fields beyond those dealing with graphs, that emphasis is also needed. Mathematics in social services and nursing mostly focuses on everyday real-world reasoning and conclusions as well as application of routines.

Achievement of German Students

The above mentioned TIMSS study provides an overview of the real achievements of German students. In the following, I refer to these results.

Achievement of Students in General Education Schools

In basic course lessons, only a small portion of the students (almost one fifth) reached a level that enabled them to apply safely and independently what they had learned to solve standard problems. If the context of items they were familiar with was changed, nearly all students had great problems in solving the problems. The results of more than four fifths of the students in the basic course did not exceed the level of application of simple mathematical concepts and rules. Students who dropped mathematics almost never stepped beyond this threshold. As one would assume, in advanced courses, a significantly higher level of student achievement was found. Nevertheless, fewer than one in eight students showed an ability to deal successfully with mathematical problems when the solutions were not directly evident (see Table 1). With the tested topic fields divided as for TIMSS, in geometry German students showed above-average results, whereas in algebra (number and equations) and calculus, they perform more or less below average. In international comparisons, women obtained clearly worse results (Mullis et al., 1998, pp. 145ff).

> Altogether, from interviews of the students it became clear that German mathematics lessons are structured strongly receptive and focusing on practising skills. Unlike the learning objectives formulated in the curricula, the support of understanding and applying of mathematics to everyday problems plays only a minor role. However, obviously the basic courses did not develop a didactic form of its own and give the impression as being advanced course on a reduced level of achievement demand. Furthermore, teaching in these courses is less application and understanding oriented, which leads students to take a more strongly receptive attitude (Baumert, Bos, Lehmann, 2000, Vol. 2, pp 275ff).

Table 1

Student Results, General Education Schools

Ability Components	Dropped Basic Course	Continuously Attended Basic Course	Attended Advanced Course	Total
Elementary reasoning and conclusions	57%	29%	8%	24%
Application of simple concepts and rules	37%	53%	38%	46%
Application of upper secondary level mathematics for standard problems	6%	17%	45%	25%
Individual problem solving on upper secondary level	9%	1%	12%	5%

Note. From Baumert, Bos, & Lehmann, 2000, Vol. 2, pp. 193ff)

Achievement of Students in Vocational Education

The TIMSS results make clear that in the broad tendency, German students face greater difficulties with items that demand more complex operations, the application of mathematical models, and individual mathematical argumentation. Generally, the strongest capabilities of German students were found in solving routine mathematical items and items close to their mathematical experiences. Specifically, 22% of the students achieved only a competency level of everyday reasoning/conclusions, 43% remained at the level of applying simple routines, 32% reached the level of simple modeling, and only 3% achieved the highest standard of mathematical argumentation. There was a great gap between classroom reality and the curricula claiming conceptual understanding, the ability to combine elementary operations, and the transfer of trained skills to new contexts (Baumert, Bos, & Lehmann, 2000, Vol. 1, pp. 199ff).

An analysis of achievement in the three groups of vocational education show substantially great leaps between the levels, and four achievement groups can be recognised: Bankers achieved the best test results, followed by garage mechanics and industrial mechanics. The third group consisted of retail merchants and trained office clerks; hairdresser trainees obtained the worst results. The differences between achievement levels strongly depended on the educational level in the lower secondary level. Students achieving better results generally had a higher education qualification than those who obtained worse results. Furthermore, great gender-dependent differences, to the disadvantage of women, can be recognised. Foreign students whose families were still strongly rooted in a non-German culture also produced clearly worse results (Baumert, Bos, & Lehmann, 2000, Vol. 1, pp. 227ff, 295ff).

Final Remarks

Summarising, it can be stated that German mathematics teaching in reality only barely attains the goals formulated in the curricula. This applies to students of all three education groups: students in vocational education/non-college-bound students and students in general education with its two levels of differentiation, college-bound liberal arts students and college-bound mathematics and science students. In international comparisons, in which German students ranked in the lower or average achievement level, the weaknesses of the German mathematical teaching are also becoming extremely clear.

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EVEN COLLEGE STUDENTS CANNOT CALCULATE FRACTIONS: MATHEMATICS GOALS AND STUDENTS' ACHIEVEMENT IN JAPAN

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Abstract: The goals and content of the current "Core and Option Module Curriculum" at the upper secondary schools in Japan are briefly reviewed to discuss mathematics goals for the different groups of students. Students' achievement in mathematics is discussed, referring to the results from large-scaled international comparisons of mathematics and smaller-scaled national achievement tests. It appears that the academic achievement of Japanese students is satisfactory with respect to knowledge of concepts and procedures at the lower secondary school level. At the upper secondary and tertiary level, however, there are signs that students' achievement has declined with the growth of enrollment ratio of students to those levels.

Japanese mathematics educators, like their counterparts in other countries, recognize mathematics as the basic discipline for science, technology, and other areas of human activities. At the time of transition to the new curricula based on the revised curriculum guidelines just released, however, it is not so clear that school mathematics meets the expectations of our society.

In this paper, the author discusses goals of mathematics education in Japan for various groups of students and reports on the evidence that students are achieving those goals with respect to three levels of achievement.

The goals and content of the current "Core and Option Module Curriculum" at the upper secondary schools in Japan are briefly reviewed to discuss mathematics goals for the different groups of students at this age. Then, students' achievement in mathematics is discussed at three levels, referring to the results from both large-scale international comparisons of mathematics and smaller-scale national achievement tests. Finally, some issues of mathematics education remaining to be explored will be discussed as a basis for setting research questions in this area.

Goals of Mathematics Education in Japanese High Schools

In the Japanese educational system, a national curriculum standard, the Course of Study, describes the goals of school education and basic guidelines for school curricula (Ministry of Education, Science, Sports, and Culture, 1989). The educational expectations are communicated through the Course of Study, as well as through other additional materials issued by the Japanese Ministry of Education.

A Core and Option Module Curriculum has been implemented in mathematics at the upper secondary school level in Japan since 1989. There are three major factors for using curricula of this type rather than the former uniform curricula.

First, about 96.8% of graduates of lower secondary students entered upper secondary school in 1998 (94.7% in 1989). This means that there were great differences among students in their aptitudes, interests, career plans, and so on. It is very difficult to cope with such diverse differences in the uniform curriculum.

Second, the growth of enrollment of students in upper secondary schools has led to polarization among students. Namely, we can typically find two major streams of students in the upper secondary schools. The proportion of the age group going on to universities and junior colleges was 42.5% in 1998. About 22.7 % (male 25.0 / female 20.5) of upper secondary school graduates went to the work force. About one fourth of the group were college-bound mathematics and science students; the remaining were college-bound liberal arts students.

Third, there are the requirements of the information age. The role of technology in mathematics education should be considered for setting mathematics goals with respect to the students of different groups.

The "philosophy" about the goals of mathematics education in Japanese high school underlying the current Course of Study is as follows. Cultivation of "mathematical intelligence" should be emphasized as the goal of mathematics education. "Cultivation of mathematical intelligence" includes fostering sound “mathematical literacy” for the majority of students, as well as the development of deep mathematical potential among the brighter students (Fujita, Miwa, & Becker, 1990).

In the revised Course of Study that will be implemented beginning in 2003, a new subject, “Basic Mathematics,” which incorporates mathematical history and statistical processing of daily events, will be introduced (Ministry of Education, Culture, Sports, Science, and Technology, 1998)). It will be an elective required subject; that is, students will be required to take one of two courses: “Mathematics I” or “Basic Mathematics.”

Students' Achievement in Mathematics

In this section, students' achievement in mathematics is discussed with respect to three levels of achievement: knowledge of concepts and procedures, understanding of the relationship of mathematical ideas in specific domains, and use of mathematics to mathematize unfamiliar problem situations.

Relatively little survey research has been done to find the levels of Japanese students' achievement in mathematics at the upper secondary school level. A few large-scale international comparisons of mathematics achievement like SIMS included Japanese students in the final year of upper secondary school. An overview of the results from both large-scaled international comparisons of mathematics and smaller-scale national achievement tests is given here with respect to the following three groups of students: (a) non-college-bound students, (b) college-bound liberal arts students, and (c) college-bound mathematics and sciences students.

A Quick Review of the SIMS, TIMSS, and TIMSS-R

One of the findings of the SIMS was the superior mathematical achievement of students from eastern countries like Japan compared with countries like the United States, Canada, and New Zealand.

Under the present curriculum, the academic achievement of Japanese students seems to be satisfactory with respect to knowledge of concepts and procedures at the lower secondary school level. On the other hand, Japanese students' negative attitude toward mathematics has been found through the SIMS and the TIMSS. Many students want to do well in mathematics and feel that their parents also hope they will, but in reality many of them are not very diligent about learning mathematics and do so passively. Further, not a few students do not realize the power of mathematics in applied work and see mathematics merely as exercise for solving given problems.

At the upper secondary and tertiary level, there are signs that students' achievement has declined in recent years, possibly with the growth of the enrollment ratio of students in higher education.

An eight-year longitudinal study on science and mathematics education was conducted by the National Institute for Educational Research (NIER, 1998). Using the data from three groups of students ($n > 2000$ for each group) in 11th grade, in 1989, 1992, and 1995, we can compare student achievement levels. Achievement has slightly declined from 1989 to 1995 on some items:

- *Item 55, 20% of 125, "Knowledge"*

Correct answer: 85.9% in 1989, 84.0% in 1992, 83.2% in 1995.

- *Item 17, Range of the square root of 75, "Understanding"*

Correct answer: 62.0% in 1989, 57.8% in 1992, 55.4% in 1995.

- *Item 66, linear function, "Use"*

Correct answer: 86.4% in 1989, 80.2% in 1992, 79.6% in 1995.

Another study done by a group of mathematicians shows that achievement of liberal arts students even in prestigious universities is relatively low (Tose & Nishimura, 1999). In particular, the achievement of college students who did not take a mathematics test as part of the entrance examination was inferior to those who did. It should be noted that students simply do not take further mathematics than required, if the entrance examination to the university to which they wish to go does not include mathematics.

Final Remarks

The current goals of mathematics education in Japan for various groups of students are described in the national curriculum guidelines. However, the evidence that students are achieving those goals with respect to three levels of achievement has not been explored systematically. We need to assess students' understanding of the relationship of mathematical ideas in specific domains and their use of mathematics to mathematize unfamiliar problem situations.

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Mathematics Goals and Achievements: The Case of Lebanon

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Abstract: Though the goals of the new mathematics curriculum in Lebanon include the three levels of mathematical abilities (procedural knowledge, conceptual understanding, and problem solving), they are content based and decontextualized. In basic education (ages 6–15), the focus is on mathematical literacy whereas, for college-bound secondary school (ages 16–18), the goals are more specialized and differentiated by scope and level of treatment. Evidence from a national study indicates that the overall achievement of goals is partial and low (mastery level is less than 40%). The achievement index for individual abilities, in descending order, was procedural knowledge, conceptual understanding, and problem solving. Private schools (autonomous, with middle and higher socioeconomic students) had higher achievement indices in each of the three abilities.

One striking peculiarity of education in Lebanon is the predominance, both in size and quality, of private schooling compared to public (or state) education (65% of students are in private schools compared to 35% in public schools). Since the early 19th century when Christian missionaries started to establish schools, private schools have multiplied and developed to include schools belonging to other religious groups or to secular groups or individuals. The private schools, being tuition based, normally attract students from the middle and high socioeconomic classes. On the other hand, public schools, which started much later, have grown at a much slower pace. The public schools are under the direct control of the central Ministry of Education (MOE) whereas the private schools are nominally supervised by the MOE. The three primary tools of government control over private schools are the licensing of the schools, the general structure and content of the curriculum, and the public examination and certification. English or French was the language of instruction of mathematics and sciences in the early missionary schools, and that tradition became a general practice protected by state policies. The issue of the language of instruction is heavily entangled with cultural and political controversies (Jurdak, 1988). The independence of private schools is so valued by the different groups in Lebanon that it was incorporated in the most recent constitutional amendment in 1990.

The year 1993 marked the beginning of a planning process for the rehabilitation of post-war Lebanon. The focus of this process was the rehabilitation of the country's infrastructure, including that of the education sector. These efforts materialized in producing a national curriculum document (CERD, 1997), which was implemented during 1998–2001. The new curriculum introduced structural as well as content changes. The educational ladder was changed from 5 elementary, 4 intermediate, and 3 secondary, to 9 basic and 3 secondary. The basic stage is further divided into three substages: first cycle (Grades 1–3), second cycle (Grades 4–6), and intermediate (Grades 7–9). By the end of basic education, students can go to technical secondary education or to general secondary education (normally college-bound). The tracks in the general secondary education were changed from three (math, science, philosophy) to four (literature and humanities, sociology and economics, general sciences, life sciences). New subjects were introduced for the first time (informatics, technology, cultural studies, sociology, economics). The contents of existing subjects were updated and detailed in terms of general, special, and instructional objectives. Mathematics was maintained as a common core subject in basic education (Grades 1–9), as well as a required subject, though differentiated in scope and level according to the track, in all three classes of the four tracks in secondary school.

The new curricula were implemented over a period of three years: the new curricula for Grades 1, 4, 7, and 10 in 1998–1999; those for Grades 2, 5, 8, and 11 in 1999–2000; and those for Grades 3, 6, 9, and 12 in 2000–2001.

Mathematics Goals Before 1997

Prior to the last attempt at educational reform in the country in 1997, mathematics goals were not made explicit except in terms of mathematical content. The only change occurred in 1969–1970, and its purpose was to align the mathematical content in the curricula with the then popular "new mathematics" movement. However, it is not difficult to infer the goals of mathematics at that time from the structure and goals of the general curriculum plan then in effect. The predominance of content reflected the value attached to mathematics as a critical subject for academic purposes embodied in successful promotion of students through the school system. The purpose of mathematics in the elementary stage (Grades 6–9) was basically arithmetical literacy, injected after 1970 with a low dose of mathematical literacy. The middle stage (Grades 6–9), including mathematics, was to prepare student for secondary school, which catered primarily to college-bound students.

Mathematics Goals in the New Curricula

The goals of the new mathematics curriculum are different from those of the old ones in many ways (CERD, 1997). First, the mathematics goals are made explicit and include mathematical reasoning, problem solving, connections and applications, mathematical communication, and the valuing of mathematics. Second, specific instructional objectives were formulated and made part of the curriculum documents. Third, standards were defined in the form of competencies. Fourth, mathematical literacy was widened in scope as part of basic education and in depth in terms of preaching the constructing of meaningful learning.

There were many changes in the content of the curriculum. Table 1 gives the distribution of content strands over the cycles. New strands such as statistics and solid geometry were introduced starting from the second cycle. Measurement concepts were given more attention in the first and second cycles. Probability was treated more systematically in the secondary cycle. However, the new math curricula remain within the confines of the traditional paradigm in being content based within a closed system of the concepts and skills of mathematics, decontextualized, and not responsive to the demands of mathematical literacy in the information age.

Table 1

Distribution of Content Strands Across Cycles

Content Strands	First Cycle (Grade 1–3)	Second Cycle (Grade 4–6)	Intermediate (Grade 7–9)	Secondary (Grade 10–12) (all tracks)
Spatial	X	X	X	X
Numerical	X	X	X	X
Measurement	X	X	X	X
Statistics		X	X	X
Algebraic		X	X	X
Probability				X
Calculus				X

Competencies

In 1999, two years into the implementation of the curriculum, a new competency-based student assessment system was introduced. The competency used in the system is a performance-based objective encompassing a number of related instructional objectives that cut across lessons, units, and even subjects. As an example, the competency "perform operations on algebraic expressions" (Grade 8) cuts across different units in Grades 7, 8, and 9. However, the textbooks, teacher guides, and assessment had been already implemented on the basis of the objective-based curriculum.

Achievements of Goals

Achievement data are available from a national study on student achievement (Jurdak, 2001), which was one in a series of studies to assess the new curricula. This study was conducted in spring 2000 as the first phase of a two-phase project. The purpose of the first phase was to provide quantitative data on the achievement of the competencies in seven basic subjects (including mathematics) of the new curricula in Lebanon. Specifically, this study aimed at—

1. Assessing and comparing achievement profiles of each of the seven subjects in four grades: Grades 5 and 8 (studying the new curricula) and Grades 6 and 9 (still studying the old curricula).

2. Identifying the strengths and weaknesses in the achievement of the competencies in the seven subjects.
3. Identifying the differences in the achievement of competencies associated with educational–organizational variables, teacher variables, and school variables.
4. Establishing baseline data for phase two of the study, to be conducted in May 2001, at which time the curriculum would be fully implemented.

Criterion-referenced tests based on the competencies as defined and published by the Center for Educational Research and Development (CERD, 1999) were constructed for each of the four grades. The tests were content-validated by experts and piloted before being used in the target sample. A national sample of 5350 students in Grades 5, 6, 8, and 9 was selected by stratified cluster sampling techniques; of these, 950 students in the four grades took the mathematics tests.

Figure 1 presents the achievement profiles by mathematical ability (procedural knowledge, conceptual understanding, problem solving) and by achievement mean index (defined as the ratio of the mean score of a set of items to the maximum score of that set). The results are as follows:

1. The overall achievement index for all grades combined and for each of the four grades is less than 0.
2. The achievement index for procedural knowledge was the highest followed by conceptual understanding and problem solving (which was the lowest).
3. In the great majority of cases, grades that have studied the new curricula (Grades 5 and 8) had a higher mean achievement index than grades that had studied the old curricula (Grades 6 and 9).

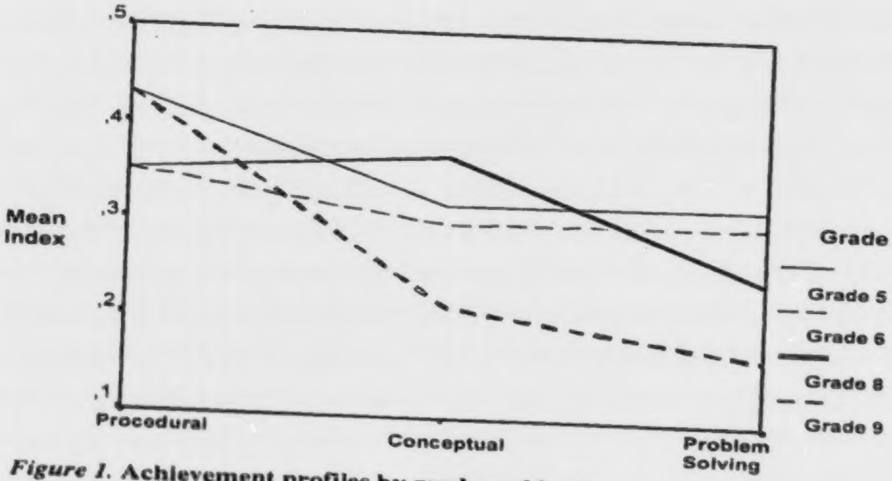

Figure 1. **Achievement profiles by grade and level.**

Of the many independent variables studied, differences associated with "sector" were largest in favor of the private sector (see Figure 2). Needless to say, this bundle of variables cannot be disentangled easily from each other and include factors related to students, teachers, facilities, school culture, degree of centralization, and others.

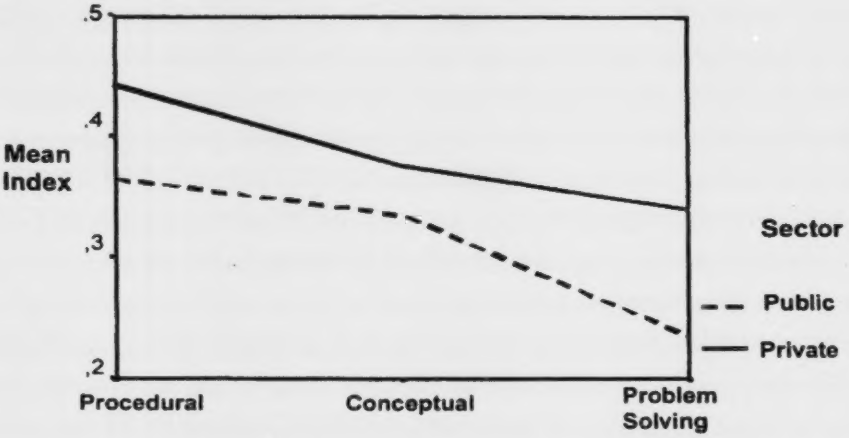

Figure 2. **Achievement profiles by sector and level (Grades 5, 8).**

One variable that showed an impact on achievement in Grade 5 was the language of instruction in mathematics. The results support the inference that instruction in Arabic was associated with higher achievement index than instruction in a foreign language (English or French; see Figure 3). It was not possible to make a similar comparison for other grades because in these the language of instruction in mathematics was either English or French.

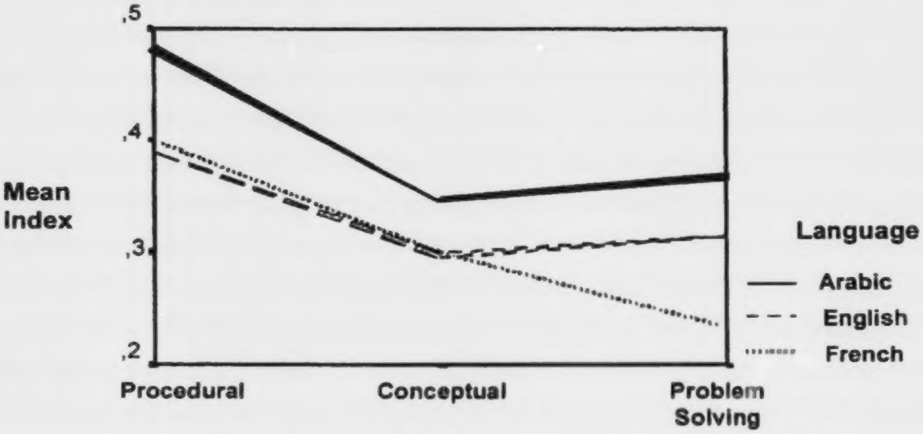

Figure 3. **Achievement profiles by level and language of instruction.**

Conclusion

The declared goals of the new mathematics curriculum in Lebanon include the three levels of mathematical abilities: procedural knowledge, conceptual understanding, and problem solving at all levels of education. In basic education (ages 6–15), the emphasis is on mathematical literacy whereas in secondary school (ages 16–18), the goals for the four tracks are more specialized and are differentiated by scope and level of treatment. The goals of mathematics, whether for mathematical literacy or for specialization, remain within the confines of the traditional paradigm in being content based within a closed system of the concepts and skills of mathematics, decontextualized, and not responsive to the demands of mathematical literacy in the information age.

Evidence from a national study for the basic education stage indicates that the achievement of mathematical goals is partial and low (mastery level is less than 40%). Improper alignment of instruction with the assessment competencies and unfavorable school teaching/learning conditions may have significantly contributed to the low level of achievement.

The level of mastery was higher in procedural knowledge than in conceptual understanding, with both higher than the level of mastery in problem solving. Private schools, which normally attract students from middle and higher socioeconomic levels and which enjoy autonomy, had higher achievement indices in the three levels (procedural, conceptual, and problem solving). Anecdotal and impressionistic reports confirm the same pattern in secondary school and also confirm that mathematization is a goal that is neither targeted nor achieved.

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MATHEMATICS GOALS AND ACHIEVEMENT IN THE UNITED STATES

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Abstract: In this paper, the goals of both traditional and contemporary reform school mathematics are first situated in the complex public school system of the United States. This is followed by a discussion of the methods of gathering achievement in schools and states. The conclusion is that there is no way one can aggregate data across districts or states to summarize achievement for the three different categories of students at the three levels of achievement for the nation.

Describing the mathematics goals and achievement in U.S. schools is not easy for two reasons: (a) the complexity of the public school systems and (b) the current efforts to reform school mathematics. One of the most striking features of U.S. schools to foreign visitors is the diversity of schooling practices, particularly with respect to governance and to the ways policy decisions are made, a result of the fact that educational policy is not national. The writers of the Constitution of the United States, in omitting any reference to education, left decisions about education to the states. With the exception of Hawaii, the states have, in varying degrees, turned over the control of schools to local communities with locally elected school boards. Today there are over 15,000 school districts that hire administrators and teachers, approve programs, select texts, and so on. As a consequence of shared state and local control, and shared state and local taxes to support schools, there are vast differences in the quality of programs, facilities, staff, and teachers both across and within states. There is no national curriculum, no national set of standards for the licensing or retention of teachers, no common policies for student assessment of progress or admission to higher education, and so forth.

Traditional Mathematics Goals

Until the past decade, in spite of the diversity in school governance, there was considerable similarity in practice and expectations for school mathematics. The curriculum reflected the 19th century compromise, described by Jahnke (1986), to formalize school mathematics as a closed unified system rather than as a sequence of methods for analyzing and understanding our world. Scientific management of this system resulted in a fragmented, hierarchical classification of mathematical concepts and skills. Scope-and-sequence charts, which specify behavioral objectives to be mastered by students at each grade level, were commonly produced. The goal for all students was that they sequentially master one concept or skill after another, and their primary task was to get correct answers to well-defined problems or exercises. This method of segmenting and sequencing school mathematics led to the assumption that there was a strict partial ordering to the discipline.

Another way of describing what mathematics has been taught to students in schools is to describe the topics covered in various grades. The emphasis in the elementary and middle schools was on computational proficiency in arithmetic. The standard topics included addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals; some experience-based geometry; and a few word problems. At the secondary school level, there was a 4-year "layer-cake" sequence with a year of algebra at Grade 9, followed by a year of Euclidean geometry at Grade 10, another year of algebra at Grade 11, and a year of pre-calculus mathematics at Grade 12. The goals for different groups of students varied only in how far along the sequence of topics students were expected to study. The college-bound mathematics and science students were to complete the entire 12-year sequence. Students planning to go to college but not planning to study mathematics or the sciences were expected to complete 10 years of the same sequence of courses. For students not college bound, eight or nine years were usually required. Finally, in some schools a few "accelerated" students started algebra in Grade 8 and took a calculus course in Grade 12. This portrayal of school mathematics was universal in the United States until the past decade and is still the dominant picture in the majority of schools today.

Reform Mathematics Goals

With the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983) and *Educating Americans for the 21st Century* (National Science Board Commission, 1983), it was apparent that our schools were not adequately preparing most of our students to participate meaningfully in the real world of work, personal life, and higher education; or in the country's social and political institutions. The initial response to these concerns in most states and school districts was to continue the same sequence of mathematics courses but to shift the expectations for the college-bound mathematics and science students to algebra in Grade 8 and a year of calculus in Grade 12 and to add an additional year of mathematics to the education of both college-bound liberal arts students and non-college-bound students.

However, on reflection it was clear that the traditional course sequence in our schools was designed to meet the demands of an earlier industrial age. The transition to an information society has created new demands on U.S. citizens. The mathematical sciences education community argued that learning must be generative: that students learn mathematics in ways that provide a basis for lifelong learning and for solving problems that cannot be anticipated today. The "standards-based reform movement" now under way in many states and schools is based on NCTM's three standards documents (NCTM, 1989, 1991, 1995) and its recently published *Principles and Standards for School Mathematics* (2000). The vision espoused in these documents involves a shift in the epistemology of learning mathematics, systemic notions about schooling that follow from that shift, the need for appropriate evidence related to the notions of schooling practices, and new assessments. The central tenet underlying the shift in epistemology is that students should become mathematically "literate." For a person to be literate in a language implies that the person knows many of the design resources of the language and is able to use those resources for several different social functions (Gee, 1998). Analogously becoming mathematically literate implies that students must not only learn the concepts and procedures of mathematics (its design features), but must also learn to use such ideas to solve nonroutine problems and to be able to mathematize a variety of situations (its social functions).

A set of assumptions about instruction and schooling practices has been associated with this vision of mathematical literacy. First, all students can and must learn more and somewhat different mathematics than has been expected in the past in order to be productive citizens in tomorrow's world. In particular, all students need to have the opportunity to learn important mathematics regardless of socioeconomic class, gender, and ethnicity. Second, some of the important notions we expect students to learn have changed due to changes in technology and new applications. Thus, at every stage in the design of instructional settings one must continually ask, Are these ideas in mathematics important for students to understand? Third, technological tools increasingly make it possible to create new, different, and engaging instructional environments. Finally, the critical learning of mathematics by students occurs as a consequence of building on prior knowledge via purposeful engagement in activities and by discourse with other students and teachers in classrooms. The point is that, with appropriate guidance from teachers, a student's informal notions can evolve into models for increasingly abstract mathematical and scientific reasoning. The development of ways of symbolizing problem situations and the transition from informal to formal semiotics are important aspects of these instructional assumptions.

What is envisioned in the reform documents is that all students will study an integrated mathematics program for at least 11 years. Integration includes activities from such strands as number, algebra, geometry, and statistics, and so forth from the early grades through Grade 11. This provides students with an opportunity to explore, in an informal manner, topics traditionally taught in high school and to proceed from such informal notions to more formal mathematics in the later grades. It also provides teachers with considerable flexibility to organize instruction to meet the specific needs of their students. In this vision, there is no distinction in goals between the non-college-bound and the college-bound liberal arts students. For the college-bound mathematics and science students, an additional year of more formal mathematics is required.

The problem with the reform vision of school mathematics is that it is based on ideas put forward by educational leaders, policymakers, and professors about what mathematical content and pedagogy *should* be. Implementation of such ideals can be undermined by a number of factors. For example, not everyone agrees with the goal of mathematical literacy for all; some influential people believe that the current course of study works reasonably well (particularly for their children), and so forth. In fact, as Labaree pointed out, during the past century, calls for reform have had "remarkably little effect on the character of teaching and learning in American classrooms" (1999, p. 42). Instead of changing conventional practices, the common response to calls for reform has been "nominal" adoption of the reform ideas. Schools adopted the reform *labels* but not most of the *practices* advocated, and it is often a political necessity for schools and teachers to claim they are using a standards-based, reform program even if classroom practices have not changed. Thus, to document the impact of any reform efforts in U.S. classrooms, one needs to examine the degree to which the reform vision has actually been implemented.

In summary, there are no consensual mathematics goals in the United States for the three groups of students However, what traditionally was and is proposed for the college-bound mathematics and science students seems reasonable. Traditionally, no special provisions were considered for college-bound liberal arts students or non-college-bound students. The reform recommendations, on the other hand, are tailored to meet the assumed needs of these groups.

Student Achievement

Traditionally, U.S. schools judge students' knowledge of mathematics either from quizzes and tests made and administered by teachers in order to prepare a formal report (usually to give a grade) or from externally developed (and often mandated) tests. Although grades are commonly used for a variety of purposes, including admission to higher education, there are no common criteria for assigning grades and no way to summarize achievement for the three categories of students based on the grades.

Most school districts periodically administer an external norm-referenced standardized test, and all but one state administers some form of state test at one or more grades. The typical test used by school districts in the United States measures the number of correct answers to questions about knowledge of facts, representing, recognizing equivalents, recalling mathematical objects and properties, performing routine procedures, applying standard algorithms, manipulating expressions containing symbols and formulae in standard form, and doing calculations. Such tests reflect the fragmentation of content and the corresponding emphasis on low-level objectives of the curriculum. Multiple-choice questions on concepts and skills emphasize the independence rather than the interdependence of ideas and reward right answers rather than the use of reasonable procedures. Unfortunately, none of the existing instruments commonly used to judge student performances in mathematics were designed to assess mathematical literacy. As such, at best they measure a student's knowledge of some of the "design features" associated with mathematical literacy. Some items on these tests may measure understanding of such features, but none make any serious attempt to assess student ability to mathematize. Thus, because of these characteristics and the variety of different tests used, there is no way to aggregate data across districts or states to summarize achievement for the three different categories of students at the three levels of achievement for the nation.

To be consistent with the standards-based vision, the quality of student performance should be judged in terms of whether students are mathematically literate. Information needs to be gathered about what concepts and procedures students know with understanding and the ways students use such knowledge to mathematize a variety of nonroutine problem situations. Only then can one judge whether student performance meets the reform vision and, in turn, whether the curriculum and teaching changes meet society's needs. To assess the intended impact of standards-based reforms in mathematics education, new assessment systems are now being developed. For example, the new international assessment framework emphasizing literacy (reading, mathematical, and scientific) prepared for the Programme for International Student Assessment (PISA) by the Organisation for Economic Cooperation and Development (OECD, 1999) was designed to monitor on a regular basis the mathematical literacy of students as they approach the end of secondary school.

The only summary data for the nation comes from the periodically administered National Assessment of Educational Progress (NAEP; OERI, 1997) and from international studies such as the Third International Mathematics and Science Study (TIMSS; OERI, 1996). These external tests are administered to a national sample of students via matrix sampling. They provide a general profile of achievement and can be summarized for different groups of students (e.g., by gender, ethnicity), but not for districts, schools, or classrooms. In 1995, on the TIMSS tests, U.S. students tested slightly above the international average in mathematics at Grade 4 and below at Grades 8 and 12. On the NAEP tests in 1996, students in 44 (of 50) states were tested and showed improvement in scores at Grades 4, 8, and 12 when compared to scores in 1990 and 1992. It should also be noted that although these assessments included a few open-response items designed to assess understanding, they do not test mathematizing and, thus, do not provide information about the three levels of achievement. Neither do they provide information about the three groups of students. However, in TIMSS, the results of a sample of the top 10–20% of students who had taken or were taking precalculus or calculus (generally college-bound mathematics and science students) were compared to those of advanced mathematics students in other countries. The U.S. students scored considerably lower than the international average.

Overall, no summary data is available to judge how well the three groups of students were achieving the three levels of performance for either the traditional or reform goals for school mathematics.

Conclusion

For the United States, characterizing the goals (both traditional and reform) for three groups of mathematics students and their achievements at three performance levels is not possible, at least with any confidence. This fact, however, does not stop policymakers, administrators, and politicians from making inferences about U.S. students.

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RESEARCH FORUM 4

Theme

Designing, Researching and Implementing Mathematical Learning Environments

—

The Research Group "Mathe 2000"

Coordinator

Erich Wittmann

Session 1

- *"Drawing on the richness of elementary mathematics in designing substantial learning environments"* Erich Wittmann
- *"Understanding—The underlying goal of teacher education"* Christoph Selter
- Reaction by Kenneth Ruthven *"Between psychologising and mathematising"*
- Reaction by Lieven Verschaffel *"Design and use of substantial learning environments in Mathe 2000"*
- Audience discussion

Session 2

- *"Children's understanding of number patterns"* Anna S. Steinweg
- *"Mathematical knowledge and progress in the mathematical learning of children with special needs in their first year of school"* Elisabeth Moser Opitz
- *"Analyses of mathematical interaction in teaching processes"* Heinz Steinbring
- Reaction by Kenneth Ruthven and Lieven Verschaffel
- Audience discussion

Designing, Researching and Implementing Mathematical Learning Environments
—
The Research Group "Mathe 2000"

Co-ordinator: Erich Ch. Wittmann (Dortmund, Germany)

Presenters: Elisabeth Moser Opitz (Freiburg, Switzerland), Christoph Selter (Heidelberg, Germany), Heinz Steinbring, Anna S. Steinweg and Erich Ch. Wittmann (Dortmund, Germany)

Reactors: Ken Ruthven (Cambridge, UK), Lieven Verschaffel (Leuven, Belgium)

Origin

In 1985 the State of Nordrhein-Westfalen adopted a new syllabus for mathematics at the primary level (grades 1 to 4). This syllabus is essentially due to Heinrich Winter, one of the leading German mathematics educators, who chaired the commission preparing the document. For three reasons this syllabus marked an important turning point in the history of mathematical education in Germany:

- The list of objectives also contains the so-called general objectives "mathematizing", "exploring", "reasoning" and "communicating" which reflect basic components of doing mathematics *at all levels.*
- The complementarity of the structural and the applied aspect of mathematics is stated explicitly and its consequences for teaching are described in some detail.
- The principle of learning by discovery is explicitly prescribed as the basic principle of teaching and learning.

In order to support teachers in putting this syllabus into practice the project "mathe 2000" was founded at the University of Dortmund in 1987 as a joint venture of the chairs "Didactics of Mathematics at the Primary Level" (Gerhard N. Müller) and "Foundations of Didactics of Mathematics" (Erich Ch. Wittmann). In 1993 Heinz Steinbring joined the project and brought in the missing empirical component. Since its inception "mathe 2000" has been based mainly on the brains of its members and so has been independent of funds although some of its research was funded.

Basic philosophy

According to a conception of mathematics education as a "design science" three areas of research and development are closely linked and pursued simultaneously:

- the design of substantial learning environments and curricula,
- practical work (pre-service and in-service teacher education in both mathematics and didactics, school development, counselling),
- empirical studies into children's thinking and into communication in the classroom.

The five papers presented at the research forum try in an exemplary way to cover the whole range of the project work and to display a special feature of the project, namely the systematic reference to learning environments: Erich Ch. Wittmann gives an example of design ("Arithmogons") and points to its implications for teacher education. By referring to "Number chains" Christoph Selter explains how teacher education can be systematically related to developmental research. The other three papers are concerned with empirical research: Anna S. Steinweg's paper indicates how number patterns from the "mathe 2000" textbook can be used for qualitative research into children's thinking. Elisabeth Moser Opitz reports on a quantitative study which revealed that the "mathe 2000" approach is feasible also for children with special needs. Heinz Steinbring gives an example of his qualitative studies into classroom interaction in which he used "number pyramids" as research instruments. A special mark of his studies is that they relate psychological processes and communication in the classroom to the epistemological structure of subject matter.

The five papers also shed light on four principles which are at the heart of the project:

- Fundamental ideas of mathematics as guidelines (epistemological orientation)
- Learning as a constructive and social process (socio-psychological orientation)
- Teaching as organising learning processes (practical orientation)
- Co-operation with teachers (systemic orientation).

The arch fathers

The roots of the present reform of mathematical education date back to at least the late 19th century. "mathe 2000" is understood as part of this process and draws heavily on ideas developed in the past, in particular on four "arch fathers":

- John Dewey (1859-1953), for his clear decision for education in a democratic society and for conceiving of theory as a guide to an enlightened societal practice at all levels (as developed, for example, in "Democracy and Education", "The Child and the Curriculum", "The Relation of Theory and Practice in Education", "The Sources of a Science of Education")
- Johannes Kühnel (1869-1928), for his precise description of the shift from "guidance and receptivity" towards "organisation and activity" (cf., his classical books "Neubau des Rechenunterrichts" [Reconstructing the Teaching of Arithmetic] and "Die alte Schule" [The Old School])
- Jean Piaget (1896-1980), for displaying the overwhelming importance of the learner's "constructive actitvity" in his pioneering work in genetic epistemology and psychology ("Psychology of Intelligence", "Mathematical Epistemology and Psychology", "Biology and Knowledge", "Theories and Models of Modern Education")
- Hans Freudenthal (1905-1990), for his fundamental contributions to understanding mathematics as a human activity and as an educational task as well as for his insistence on developmental research as the core of mathematics education ("Mathematics as an Educational Task", "Weeding and Sowing", "Didactical Phenomenology of Mathematical Structures", "Revisiting Mathematics Education").

Publications

The research conducted by project members is documented in numerous articles and some books. As mentioned before a special mark of almost all papers is that they are systematically related to learning environments. A booklet with selected papers in English will be available at the conference.

Materials

It was a strategic decision at the very beginning to present the basic message of "mathe 2000" in a "Handbuch produktiver Rechenübungen" [Handbook of practicing skills in a productive way] (2 vols. published in 1990 and 1992). The "Handbuch" contains a systematic epistemological analysis of arithmetic from grades 1 to 4 in the form of substantial learning environments which combine the practice of skills with higher mathematical activities.

The "Handbuch" inspired many teachers to conduct teaching experiments. These experiments were very successful. So it was teachers who demanded a new textbook consistent with the "Handbuch". In collaboration with a group of teachers the four volumes of "Das Zahlenbuch" were developed and tested from 1993 to 1997.

Since the middle of the nineties the book has spread over Germany and crossed the borders to some neighbouring countries (Switzerland, Belgium, and the Netherlands).

In addition to the "Handbuch" and the "Zahlenbuch" the "Programm mathe 2000" published by Ernst Klett Grundschulverlag contains other materials for teaching primary mathematics (including the CD ROM "Blitzrechnen"/ "Calculightning", with an option in English, and the booklet "Double Mirror Magic" in English). The complete "Programm mathe 2000" will be exhibited at the conference.

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CHAPTER 1: DRAWING ON THE RICHNESS OF ELEMENTARY MATHEMATICS IN DESIGNING SUBSTANTIAL LEARNING ENVIRONMENTS

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By means of an example (arithmogons) it is shown how the design of substantial learning environments for both students and student teachers is based upon the progressive elaboration of the epistemological structure of elementary mathematics.

> Progress in group theory depends primarily on an intimate knowledge of a large number of special groups.
> Graham Higman (1957)

In the first volume of the "mathe 2000" textbook the learning environment „Rechendreiecke" is introduced in the following way:

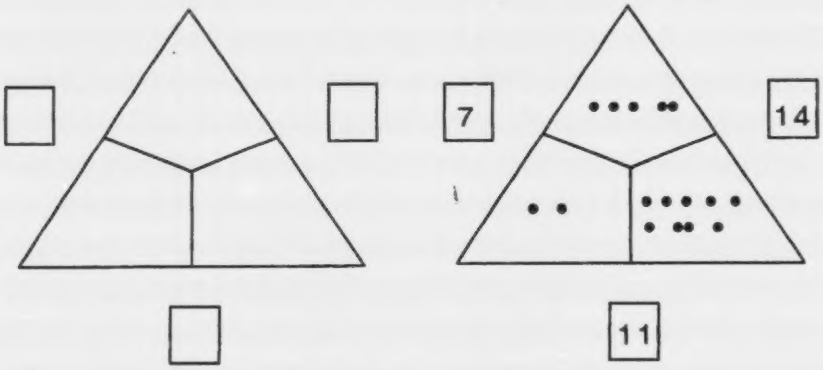

Fig. 1

A triangle is divided in three fields. We put counters or write numbers in these fields. The simple rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side (see Fig. 1).

Various problems arise from this context: When starting from the numbers inside, the numbers outside can be obtained by addition. When one or two numbers inside and respectively two or one number outside are given, the missing numbers can be calculated by addition or subtraction. When the three numbers outside are given, we have a problem that does not allow for direct calculation but requires some thinking. Firstgraders can find the solution by (more or less) systematically varying the number of counters in the inner fields. There are, however; also systematic solutions. By of counters in the inner fields. There are, however; also systematic solutions. By studying the behaviour of a "Rechendreieck" one understands why there is exactly one solution. Interestingly, the solutions are not always natural numbers. If the sum of any two of the outer numbers is smaller than the third number at least one of the inner numbers is negative. If one outer number or all three are odd the inner numbers are fractions (denominator 2). So "Rechendreiecke" is a nice context to cross borders even in grade 1.

It is obvious that "Rechendreiecke" forms a substantial learning environment in the following sense (Wittmann 2001):

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.

(2) It is related to significant mathematical contents, processes and procedures ***beyond*** *this level, and is a rich source of mathematical activities.*

(3) It is flexible and can be adapted to the special conditions of a classroom.

(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

In order to illustrate the driving forces behind the design process it will be explained in the following how in "mathe 2000" learning environments come to life and find their way into the curriculum: elementary mathematics provides the raw material and the above properties of a substantial learning environment serve as a checklist.

The following problem of elementary geometry is well-known (Fig. 2): For a given triangle with sides a, b, c construct three tangent circles around the vertices.

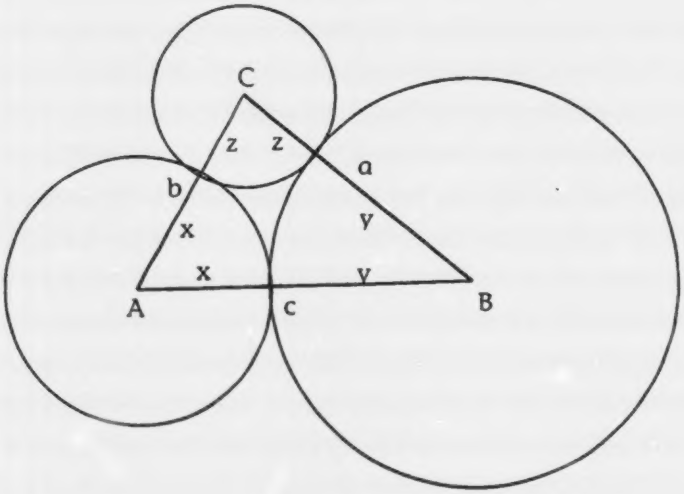

Fig. 2

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Solution: The unknown radii x, y, z fulfil the relations $x + y = c$, $y + z = a$ and $x + z = b$. Therefore $c + a = x + z + 2y = b + 2y$ and $y = \frac{1}{2}(c + a - b)$. Correspondingly $x = \frac{1}{2}(b + c - a)$ and $z = \frac{1}{2}(a + b - c)$ are derived.

The analogous problem for quadrilaterals has a somewhat different structure: It is solvable if and only if the quadrilateral has a circumcircle. In this case the set of solutions is infinite. A closer investigation shows that the case of triangles is typical for polygons with an odd number of vertices and the case of quadrilaterals typical for polygons with even vertices.

This geometric problem appeared in Sawyer 1963 (p. 150 ff.) in the disguise of the "bears problem": Someone wants to mount gates on each of three posts A, B, C such that gates x and y close side c, gates x and z close side b, and gates y and z close side a (in order to keep bears coming from various directions away).

The problem is also generalized by Sawyer to more than three posts.

McIntosh&Quadling (1975) chose another algebraic setting for this problem: An equal number of circles and squares are arranged like the vertices and sides of a polygon. Numbers have to be filled in such that each square carries the sum of the numbers in he adjacent circles (Fig. 3).

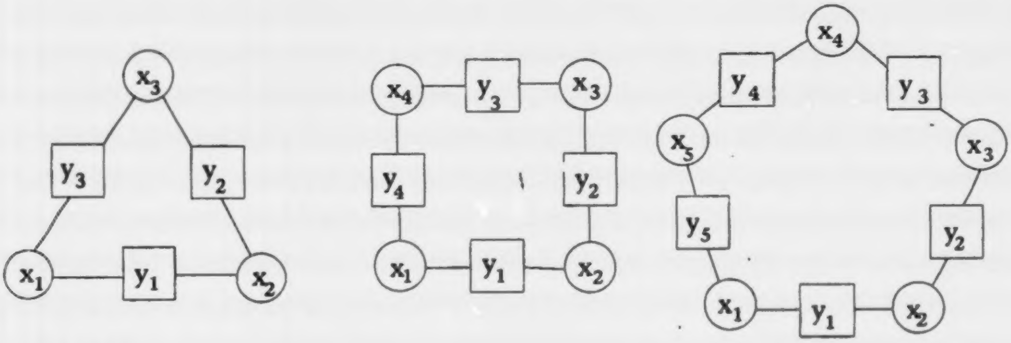

Fig. 3

The authors elaborated the mathematics underlying this structure: The mapping $(x_1,...x_n) \rightarrow (y_1,...y_n)$ is a linear mapping from R^n into R^n. The corresponding matrix is non-singular for odd n and singular for even n (rank n−1).

Another important step forward was made by Walther 1985 who transformed arithmogons into "arithmetic chains" in order to get a richer variety of activities: When the sums (lower row) are given children can investigate how the target number (and the other numbers) depend on the start number (Fig. 4). In particular children can find out if it is possible to have the same number as start and target.

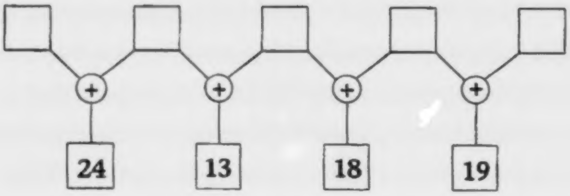

Fig. 4

Like McIntosh&Quadling Walther traced arithmetical chains through the grades up to the upper secondary level by unfolding richer and richer algebraic structures.

In subsequent papers the structure of "arithmetical chains" was generalized to other chains and the analytical background (fixpoints of functions) was elaborated further.

There was no question that arithmogons because of their mathematical substance should become part of the mathe 2000 curriculum. However, the question was at which place and in which form they should be introduced. As in the project it is a general principle to introduce fundamental ideas as early as possible the above version of a "Rechendreieck" was chosen as an appropriate format for grade 1: counters can easily be moved around and thus favour children's exploratory activities and early reasoning.

The mathematical substance behind arithmogons (property 2) allows not only for revisiting this context over the grades but also for using them in teacher education. In a course on "Elementary Algebra" for primary student teachers with mathematics as a major subject the structure of arithmogons has been generalized to polyhedra in an obvious way: Numbers are assigned to vertices and faces such that the numbers assigned to the faces are the sum of the numbers assigned to the vertices of this face.

In this way a rich universe of examples arises which can be explored by student teachers. All phenomena and all concepts relevant for the theory of systems of linear equations occur and can be studied and explained in this context: linear mappings, independence of a system of vectors, kernel of a linear mapping, image space, rank of a matrix, rank of the augmented matrix, basis and dimension of a subspace, Steinitz' theorem, dimension theorem. The theory, however, does not hover in the air. It is developed "just as far as is necessary to frame a certain class of problems" (Giovanni Prodi).

In this context even systems of linear equations over a finite field make sense: As shown above a "Rechendreieck" does not allow for an integer solution if one outside number or all three are odd. As the properties "even" and "odd" can be represented by the numbers 0 and 1 of the field Z_2 we get a system of three linear equations over Z_2 (Fig. 5). The rank of the corresponding matrix (Fig. 6) is only 2, not 3 as before.

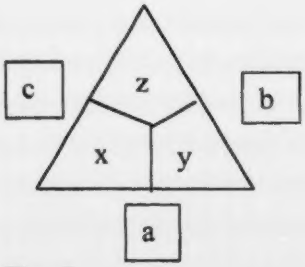

Fig. 5

$$x + y = a$$
$$y + z = b$$
$$x + z = c$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Fig. 6

$$\begin{pmatrix} 1 & 1 & 0 & c \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{pmatrix}$$

Fig. 7

When Gaussian elimination is applied to the augmented matrix (Fig. 7) we get $a + b + c = 0$ as a necessary and sufficient condition for the solvability of the system. In ordinary language this means that either none of the numbers or two of them are 1 (which means "odd"). For student teachers it is intriguing to see how "abstract" algebra applies to "concrete" numbers.

Concluding remark

"Arithmogons" is an example of local design. However, the "mathe 2000" research is not restricted to designing isolated learning environments. In fact the project's central publication is the "Handbuch" (Wittmann&Müller 1990/92) in which a comprehensive concept of teaching arithmetic in the primary grades is given in the form of coherent systems of substantial learning environments. The main emphasis in the "Handbuch" is on the epistemological structure of the subject matter. This does not mean that we consider psychological and social studies into learning processes as less important, but that we consider a profound knowledge of the epistemological structure of the subject matter as a pre-requisite for teaching as well as for these studies. In addition we strongly believe that providing teachers with substantial learning environments and the necessary epistemological background strengthens their role as reflective practitioners.

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CHAPTER 2: UNDERSTANDING –THE UNDERLYING GOAL OF TEACHER EDUCATION

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Teachers' background knowledge is surely a precondition for their professionalism. However, they actually become professionals *while* they are teaching (Bromme 1994; Thompson 1992; Cooney & Krainer 1996). Consequently, the present paper sketches how (prospective) teachers should be encouraged to build up *background knowledge* as well as to develop *awareness* regarding (1) their own mathematical activity (*mathematical* component), (2) children's thinking and learning (*psychological* component), (3) subject-matter-specific didactics (*didactical* component) and (4) teaching practice (*practical* component). Activities from the context of one substantial learning environment are used as representative examples to illustrate this conception.

2.1 Understanding mathematics

An important goal of mathematical components in elementary teacher education is to contribute to breaking a vicious circle. Many (prospective) teachers do not feel confident with mathematics due to their own prior negative learning experiences. Thus, they are likely to perpetuate their limited understanding to their own students. In this context, (prospective) teachers' encounters with mathematics courses play a crucial role, as they offer opportunities to encourage them to develop a lively relation to the activity of *doing mathematics.*

Number chains - challenging mathematical acitivities

The context 'number chains' shall serve as a representative example (Price, Hoskin & May 1991, 12; Selter & Scherer 1996): Choose two start numbers and write them down beside each other; then note their sum as the third number. Now add the second and third number and write down this sum. Finally, put down the sum of the third and fourth numbers (the so-called target number), e.g., 1, 4, 5, 9, 14 or 66, 23, 89, 112, 201. One activity for third graders consists of finding two (non-negative integer) start numbers that lead to the target number 50. Besides this problem, there are many different ones up to university level, such as: (1) What happens, if you take a different target number? (2) If you work on number chains of a different length (4, 6, 7, ..., n numbers)? Use examples first! (3) Is there a relationship between different target numbers and the number of solution pairs? (4) Is there a solution pair for each target number? (5) If not, is there a largest one that cannot be reached? (6) Is there a relation between the length of the chain, the target number and the number of solutions?

Doing mathematics (sensu Freudenthal)

Number chains are a typical example within the conception developed in the 'mathe 2000' project. In this context, one of its arch fathers, Hans Freudenthal, should be quoted who criticized what he called anti-didactic inversion: taking the ready-made system as the starting point of the teaching-learning process. In his terms, learning is not to be understood as duplicating, but as guided reinvention (Freudenthal 1991, 48). Children should learn *mathematizing* instead of consuming the finished product 'mathematics'. Thus, he developed his didactical phenomenology (*Freudenthal* 1983):

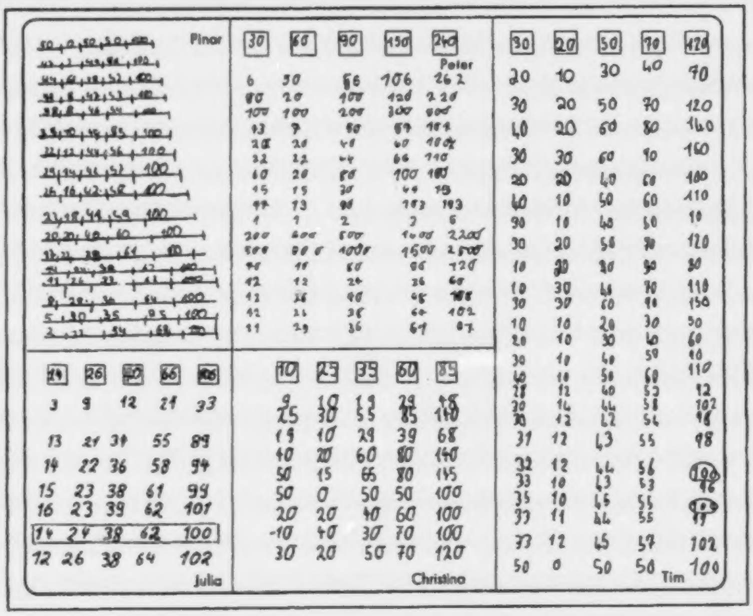
starting with phenomena that are meaningful for learners and thus provoke learning processes. Consequently, the mathematical components of teacher education courses should deal with an informal, process- and problem-oriented mathematics (Müller 1997; Wittmann, in press; Müller et al. 2001). In addition, the courses need to be organized in a way that provides (prospective) teachers with learning experiences that they will want *their students* to experience. The culture of teacher education should be similar to the favorable culture of teaching practice (Wittmann 1989).

2.2 Understanding children

With respect to the *psychological* component one is confronted with another vicious circle. Quite a couple of (prospective) teachers have learned as students, that it was of crucial importance to understand the ways their own teacher thought, whereas their own thinking often was regarded as not really important. They are likely to adopt this point of view for their own teaching as well. Thus, (prospective) teachers should learn about children's ways of thinking and learning.

Solution strategies for number chains

To illustrate this, the 'number chains' are revisited (Selter & Spiegel 1997, 68). Third graders were asked to find all number chains with target 100 (see besides). (1) How do/did you solve the problem yourself? (2) In how far did they work systematically? (3) Which similarities and differences do you notice between their ways of working and your own approach?


Pinar
Peter
Julia
Christina
Tim

Looking through children's eyes (sensu Piaget)

Activities of this kind were designed for (prospective) teachers in order to learn to look through children's eyes, a crucial idea inseparably connected with Jean Piaget's work (1972). An extensive number of his writings has convincingly shown that even very small children interact with their environment and actively construct their own knowledge by modifying already existing schemes. Piaget was among the first who scientifically proved that children have their own ways of thinking, that make sense from their perspective, although it does not seem to be so at first sight.

Reflecting on children's documents by analyzing documents like videos, transcripts or own productions can sensitize to their perspectives. (Prospective) Teachers can learn to understand that children often think differently from (1) how we think, (2) what we assume how they think, (3) how other children think and (4) how they think in different situations dealing with basically the same (Selter & Spiegel 1997).

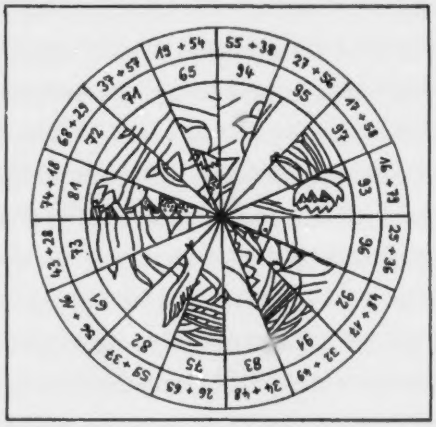
2.3 Understanding didactics

A third vicious circle comes to the fore with respect to the didactical component of teacher education. The (prospective) teachers' own learning experiences shape their beliefs regarding what it means to teach and to learn. Formerly the teacher gave them the relevant information that had to be learned in order to pass the next test. Consequently, nowadays the mathematics educator is expected to tell them the subject matter needed to get good marks in the exams. Although (prospective) teachers often have a critical scepticism with respect to didactical ideas, they often do not make their doubts public. As they understand the mathematics educators' remarks as academic truth, they often accept it as part of their teacher education knowledge to be learned, without sufficiently making it part of their own thinking. Thus, a fruitful discussion should be encouraged instead in which arguments are exchanged and teachers as well as teacher educators can learn.

Number chains and arithmetic domino – two types of practicing

An example is given in the following. Work on the two tasks and analyze them according to following questions: (1) What are the goals of each of them? (2) Which are the advantages of each of both? (3) Which difficulties do you expect in teaching practice? (4) Which variations are possible in order to cope with different abilities of students? (5) Which task would you prefer, if addition in the domain up to 100 has to be practiced in your class? Give arguments for your decision!

Arithmetic domino: Cut out the domino cards. Take any card and work out the respective problem. Lay down the card containing the correct result next to it, work out its problem, and so forth. If you do not find a card that fits, you have to work out the problem again.



Number chains: Write down two start numbers one beside the other, add them and put down their sum as third number. Finally, write down the sum of the second and the third number, e.g., 36, 23, 59, 82. Add 1, 2, 3, … to the *first* (the *second, both)* start number(s). What happens to the other numbers? Explain!

Taking didactical theories as a basis for own decisions (sensu Kühnel)

Activities of this kind were designed to stimulate (prospective) teachers to critically relate knowledge input from different sources to their own theories. In this context, it is worthwhile to quote the German pedagogue and mathematics educator Johannes Kühnel (1925, 137) who postulated "Not *guidance* and *receptivity*, but *organization* and *activity!"* as central principles for teaching as well as for teacher education. His goal was to educate active and reflective personalities who do not just blindly copy didactical decisions others have made, but who are able to develop their own or modify existing conceptions based on different didactical theories (Kühnel 1923, 88).

2.4 Understanding teaching

The fourth and last vicious circle with respect to teacher education to be dealt with in this section is the following one: As German (prospective) teachers normally do not have any opportunities to experience something different, they continue to more or less teach as they were taught. This often happened according to a traditional approach counter-productive to modern conceptions of teaching and learning. One example shall be given to illustrate, how a higher degree of awareness can be developed with respect to a so-called progressive conception of teaching.

Number chains - analysis of a teaching episode

A group of (prospective) teachers commonly observes a lesson on number chains that one of them is giving. In order to focus their perceptions as well as their interpretations they have developed several criteria in collaboration with the teacher educator, such as (1) Did the teacher lucidly explain the problem? (2) Are the children aware of what (s)he expects? (3) Were the numbers chosen in the explanation appropriate (or too small, too big, too similar)? (4) Did (s)he give enough time for the children's own work? (5) Did (s)he maintain an atmosphere that, in principle, enabled all students to think for themselves and to contribute to whole-class or small-group discussions? (6) Did the teacher have advice available for children who experienced difficulties? (7) Did (s)he provide challenging activities for children who solved the problem faster than others?

Learning to teach according to the laboratory conception (sensu Dewey)

The goal of activities like these ones is to focus the (prospective) teachers' observations. This conception relates to a paper by John Dewey (1904), in which he distinguishes two different approaches: On the one hand, practical components can provide teachers with necessary tools of their profession, like the technique of whole-class instruction. With this aim in view, practical work is of the nature of an *apprenticeship training*, the aim is to form the actual teacher. On the other hand, practical work can make (prospective) teachers reflective and attentive, by relating theoretical aspects to what they can observe within the classroom – the *laboratory conception*. Here the goal is to equip the teacher with the intellectual methods and with materials of good workmanship instead of creating the good workman on the spot. Dewey does not reduce these two points of view to an 'either-or'. According to him, the apprenticeship and the laboratory conception give the limiting terms within all practical work falls.

2.5 Coherence of components

The present paper describes important principles of teacher education developed within the project 'mathe 2000' (see Selter 1995; Wittmann, in press). As teaching and learning are complex phenomena, it is obvious that a coherent view integrating all four components is needed. In this context, Wittmann (1984) has elaborated how substantial learning environments – like number chains – can fulfill this multiple function and form the integrating core of teacher education.

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CHAPTER 3: CHILDREN'S UNDERSTANDING OF NUMBER PATTERNS

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One of the main aims of the maths 2000 project is to take both the child and the subject seriously (cf. Dewey, 1974). According to Devlin (cf. 1997), mathematics is seen as the science of patterns. Patterns can arise in geometrical themes as well as in arithmetical ones. The latter are called number patterns.

Looking through the maths 2000 textbooks (cf. Wittmann et al., 2000/01) one can find several examples of number pattern tasks offered to the students. The adaptation of mathematics as the science of patterns into textbooks is innovative for German schools and is not found in daily schoolwork. Reservations against patterns formed by 'pure' numbers are common. Mathematics is regarded as a boring subject and tasks with supposed connections to the child's daily life and its daily needs are believed as the appropriate approach. My dissertation research project (cf. Steinweg, 2001), which was inspired by the pattern in the maths 2000 textbooks and the special situation in German math lessons, focused on two major questions:

I How do children across ages react to the given number pattern, which are unfamiliar to them?

II To what extent can a kind of natural or genetic development of understanding of number patterns with advancing age be found?

Design and Theoretical Framework of the Research

The different tasks should offer children of all skill levels the opportunity to show their competence in dealing with number patterns. The themes polygonal numbers, sequences, number pairs with functional relation, number pyramids, and so called beautiful 'packages' were selected as tasks for the research because they show typical characteristics of mathematical patterns, and different appropriate approaches are possible. The various tasks can be divided into three types. Type A describes patterns formed by numbers, type B patterns formed by shape, which have to be recognised and continued. Type C are a series of calculation tasks (beautiful 'packages'), which are connected by patterns. One of the tasks in the 'package' does not fit into in the pattern and has to be replaced by a fitting one.

The understanding of number patterns is theoretically structured in three levels:

I Recognising and (intuitive) Continuing of the pattern
II Describing the pattern
III Explaining the pattern

These levels do not mirror a development schema but are understood as interdependent. They serve to both analyse the result and design the research. On the one hand the construct of levels allows different reactions of the students categorised qualitatively. On the other hand the research project has to be designed in such a way that the participating students have the opportunity to show their competencies of all levels. In consequence two different designs were created. The first one to determine the level I – competencies of the children in a paper-and-pencil-test and the second one, an interview study, to offer the students the opportunity to show their competencies in all three levels.

A sample of 257 students (111 female / 146 male) across ages (6/7 to 9/10 year–olds) of the German primary school was tested in the **paper-and-pencil-tests**. They came from three different schools with very different social backgrounds. Number patterns were not taught in any of the participating schools prior to the test. The test took place during the daily maths lessons. There was a limitation of time to one school lesson (45 min.). The students were free to do the tasks in their individual order.

In the **interview study** 60 children (34 female / 26 male) took part in the interview study, 15 of each age group (6/7, 7/8, 8/9, 9/10 year–olds). The interviews were held during school in a separate room. Each interview was videotaped. The interviews were designed for a 20 min. time period for the youngest ones (6 to 7 year–olds) and up to 45 min. for the oldest students (9 to 10 year–olds).

Analysis of the Children's Solutions

Every number pattern task could be solved in different mathematically correct ways. Nevertheless the solutions had to and could be categorised in an epistemologically evident way. Four categories were fixed, which will be described by examples of children's solutions to a 'beautiful package' (a type C – task for 2nd graders, 7 to 8 year–olds).

(1) *solved with the solution which suggests itself*

51 - 6 = 45
62 - 16 = 46
73 - 26 = 47
84 - 37 = 47, 84-36=48
95 - 46 = 49

Kim replaces the fourth task by 84 – 36 = 48. She comments: "everywhere here is 6, 6, 6 ... and there is a 7."

(2) *solved with aspects of the pattern having been considered*

51 - 6 = 45
62 - 16 = 46
73 - 26 = 47
~~84 - 37 = 47~~
95 - 46 = 49
50-2=48

René tries to find the non-fitting task by searching the results. He recognises the double 47 and replaces the fourth task by 50 – 2 = 48.

(3) *solved without an apparent connection with the pattern*

51 - 6 = 45
62 - 16 = ~~46~~
73 - 26 = ~~47~~
84 - 37 = 47
95 - 46 = 49

Jana suspects the calculations to be wrong. She comments: "2 – 6 aren't 6 that are 0". First she crosses out the 46 than the 47 of the third task.

(4) *not worked on*

First the results of the paper-and-pencil-test and the interviews were quantitatively analysed whether the task was worked on or not. In the second step the solutions offered were categorised as shown above. The video documents were also qualitatively analysed with special emphasis on level II and III.

Results of the Research Studies

The main results of the **paper-and-pencil-tests** are in summary:

(1) The percentage of children whose answers belonged to category 1 or 2 averaged 60 % (i.e. the results of the type C – tasks see fig. below).

(2) Category 4 (not worked on) was more often found in solutions of older children.

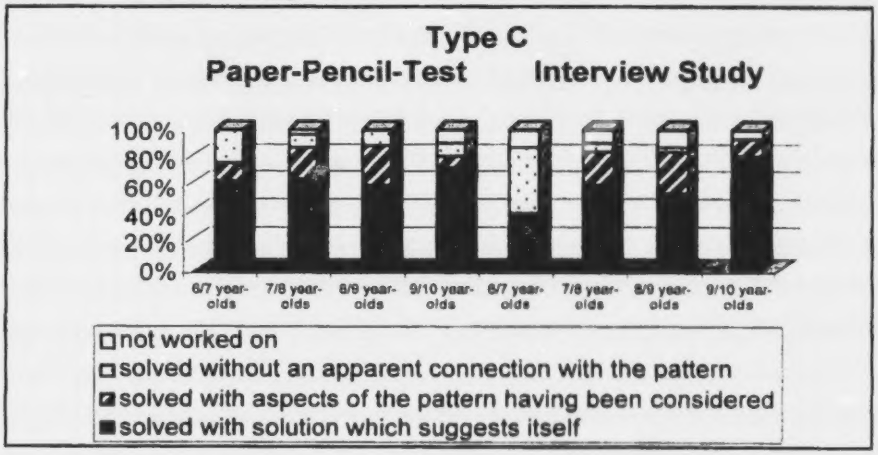
(3) A genetic or continuous increase of category 1 and 2 solutions across ages could not be found.

The paper-and-pencil-test showed a great acceptance of number pattern tasks. The design of the following interview study allowed the children to show all three levels of understanding (recognising & continuing, describing, and explaining).

The main *quantitative results* of the **interview study** can be summarised as follows:

(1) The percentage of category 1 and 2 solutions varied from type to type. A continuous increase with advancing age could not be found (type C – results).

(2) The percentage of category 4 (not worked on) solutions showed no uniform tendency across ages and types.



The *qualitative results* have shown i.a. that

(1) the older children were more able to describe the pattern more generally. The younger ones often referred to examples in their descriptions.

(2) with advancing age the students were more able to recognise different patterns that simultaneously occurred in one complex structure, e.g. patterns of the sequence of addends and the one of the sums.

(3) calculation skills were useful in solving number pattern tasks.

(4) the children showed no spontaneous need to explain the structures and occurring patterns (level III).

Conclusions

The research project gained insight into the children's understanding of number patterns. The results should not be overestimated. The shown and assessed solutions only mirror the present strategy the individual student regarded suitable, and do not give any information about the ability to use alternatives (cf. Litherland, 1997, p. 13). As a tendency the results of the studies are evident, because "... every particular is also a sample of a larger class. In this sense, what has been learned about a particular can have relevance for the class to which it belongs" (Eisner, 1998, p. 103). The tendency "provides us with a guide, not a guarantee" (Eisner, 1998, p. 105).

The research gave answers to the two questions posed in the beginning.

I *How do children across ages react to the given number patterns, which are unfamiliar to them?*

The children do not have any reservations about the unfamiliar tasks. 60 to 88% of the solutions offered in the paper-and-pencil-test and 40 to 94% of the reactions in the interviews could be categorised as 1 or 2. The number patterns were fully accepted by the students and did not need any extrinsic motivation.

II *To what extent can a kind of natural or genetic development of understanding of number patterns with advancing age be found?*

A genetic development of understanding with advancing age could not be found on principle, even though some of the older children have shown a qualitatively better insight into complex structures and were more able to describe patterns generally. Other researches, i.e. the pattern researches in Leeds, came to similar results: "It could not be expected to occur 'naturally' in all classrooms, and would have to be taught more actively" (Threlfall 1999, p. 27). The understanding of number patterns has to be motivated by the teacher in school lessons. The active thinking approach, the SLEs (cf. Wittmann, Chapter 1 and Steinbring, Chapter 5), and the patterns offered in the textbooks of the maths 2000 project provide the settings in which the teacher and the children can unaffectedly access mathematics as the science of patterns.

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CHAPTER 4: MATHEMATICAL KNOWLEDGE AND PROGRESS IN THE MATHEMATICAL LEARNING OF CHILDREN WITH SPECIAL NEEDS IN THEIR FIRST YEAR OF SCHOOL

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The paper is a summary of an empirical study which showed that the "mathe 2000" approach is feasible also for children with special needs.

A research project conducted by the Institute for Special Education at the University of Freiburg, Switzerland, has evaluated the mathematical knowledge of children with special needs at the start of their school education. One definition of "special needs" is children with learning disabilities. In Switzerland, these children are assigned to special classes where, as a criterion, an intelligence quotient between 75 and 90 is employed. Alternatively, so-called "introductory classes" are common in Switzerland. This kind of class is meant for children with partial delay in their development and, in some cases, when the maturity level required for entering the first class is questionable. Here, the subject matter of the first class is extended over two years. The children then enter either the second year of the mainstream class, or a special class for children with learning disabilities.

Current approaches to initial mathematical teaching are strongly marked by an understanding of the concept of numbers developed by Piaget. It recommends introducing numbers step-by-step after a lengthy pre-numerical practice (several months, up to one year). Classification, seriation and number-conservation are considered as prerequisites for understanding numbers. However, the "mathe 2000" approach (Wittmann; Müller, 2000) offers the sphere of numbers between one and twenty simultaneously. Exercises, which precede concrete calculation, are completely lacking. In addition, the emphasis lies on working with images of quantities with a given structure. The reasons for this are as follows:

There are many studies showing that the numerical knowledge of first-grade children in mainstream classes is much higher than traditional schoolbooks assume. Lengthy pre-numerical practice, based on the concept of number by Piaget, is ignored because such an approach to mathematical learning is under question. For example, many studies show that number-conservation is not a prerequisite for the development of mathematical abilities (Wember, 1989; 1998; Moser Opitz, 2001, p. 48f). Nowadays, it is known that children acquire mathematical knowledge by solving meaningful mathematical problems and not by solving tasks like conservation and class-inclusion. Furthermore, it is important to present the sphere of numbers from 1-10 or 1-20 immediately. A separate number can only be understood as a part of a whole, in the context of a larger area of numbers. A step-by-step introduction of numbers hinders such an overview and, consequently, understanding. To help children to represent numerical quantities, it is important to use sets with a given structure of five or ten. This structure should help them to internalise the concept of numerical quantities.

In the practice of special education, some critical questions about this new approach have arisen. It is doubtful if children in special classes have the prerequisites necessary to work in the sphere of numbers from one to ten or one to twenty immediately. In addition, it is difficult for teachers to accept that pre-numerical practice, which used to and still does characterise the teaching of mathematics in special education, is questionable. Furthermore, there are doubts over whether children in special classes are actually able to acquire the concept of numerical quantities from one to twenty. This situation has led to the questions the intended research project seeks to address:

- What kind of numerical knowledge do children in special classes bring into school?
- Do children in the first year of a special class, who are taught according to the "mathe 2000" approach, make more, less or similar progress in mathematical knowledge than children taught according to current (special education) methods?
- What conclusions can be drawn from this information for initial mathematics teaching?

Method

The subjects of the study were 162 children (59 female, 103 male), schooled in special classes at the start of their school education. The average age was 6 years 9 months. The test comprised two parts, "prerequisites" (pre-numerical practice; comprehension of quantities; grasp of numerical quantities; counting; number words and writing of numbers) and "calculation" (addition and subtraction from 1-20 with and without counting aid). The test was given in the form of a gold coin game and the children were tested individually (cf. Moser Opitz, 2001, 126f).

Results

The results are given as percentages. Where a percentage range is given, different scores were given for different tasks.

The results show that the children's numerical knowledge at the start of their school education is higher than current approaches to special education presuppose (table 1). Most of children managed the pre-numerical tasks and had a comprehension of quantities from 1 to 6. More than half of them knew the number words from 1 to 10 and a similar number were able to write numbers from 1 to 5 (cf. Moser Opitz, 2001; 1999a; 1999b). The addition tasks, with the possibility of counting within the first ten, were completed by 43-66% of the children, the subtraction tasks by 32-40%. It shows that the ability of children with learning disabilities is lower than in mainstream classes, where the score for the addition is 80% and for the subtraction 40%. Only a few children were able to solve addition and subtraction without counting aids.

Task	% (N = 162)
Prerequisites	
Pre-numerical Practice	
Classification (multiple)	66.7
Seriation	72.0
One-to-one-correspondence	89.5
Comprehension of quantities	
Take n objects	88.3-98.8
Number words from 1-10	69.8-92.6
Writing of numbers	
Numbers from 1-5	46.3
Numbers from 1-10	29.6
Counting	
Counting forward to 20 (and further)	55.6
Counting backwards from 6	49.4
Calculation	
Addition with counting aid from 1-10	43.2-66
Subtraction with counting aid from 1-10	32.1-40.8
Addition without counting aid from 1-10	20.3-27.1
Subtraction without counting aid from 1-10	5.5-6.2

Table 1: Numerical knowledge of children with special needs at the start of their school education

The second part of the research project examined the improvement in mathematical knowledge in the first year of school. Based on written reports by the teachers, three different groups were matched. One group was taught according to the current approaches to mathematical teaching with lengthy pre-numerical practice, the second group by the "mathe 2000" approach (including working in the sphere of numbers from 1-20 and laying emphasis on the recognition of sets with a given structure) from the beginning of their school education and the third group started working with the "mathe 2000" approach during the first year of school. Eight months after the start of their school education, the children were tested again.

The scores of the second and third group were, in some areas (number words, writing of numbers, recognising sets with a given structure) but not overall, significantly better than those of the first group. Interestingly, the group which has worked with images of quantities with a given structure from the beginning of their school education used (finger) counting strategies significantly less in addition and subtraction than the other groups. This is an important finding because it is known that one symptom of children with learning disabilities in mathematics is the frequent use of finger counting strategies (Geary; Brown; Samaramayake, 1991).

The results of the study presented here show that the numerical knowledge of children with special needs at the beginning of their school education is much higher than current approaches in special education presuppose. These common teaching materials, which prescribe pre-numerical practice during several months or a whole year without using numbers, can be considered questionable. Moreover there are good reasons to conclude that working with the "mathe 2000" approach helps children to develop their concept of number. Particularly, working with sets with a given structure seems to supply number representation and the practice of addition and subtraction. Early mathematical teaching for children with special needs should take these results into account and adapt its methods.

In addition, interviews carried out with special education teachers (Moser Opitz, 1999c, 172) show that they lack knowledge as to how to adapt the "mathe 2000" approach to children with special needs (e.g. with regard to perception, spatial organisation, memory problems). This should be taken into consideration. More guidance to special education teachers for teaching the "mathe 2000" approach is necessary.

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CHAPTER 5: ANALYSES OF MATHEMATICAL INTERACTION IN TEACHING PROCESSES

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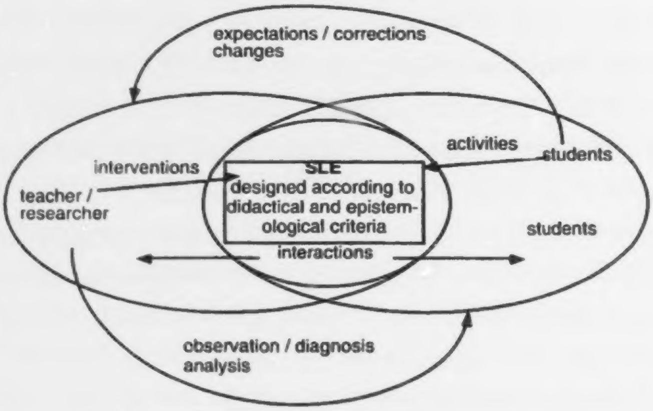
About the Epistemological Nature of Substantial Learning Environments

Mathematical concepts are no empirical things, but represent relations. "... there is an important gap between mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary ... The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations." (Duval, 2000, p.61). With regard to this epistemological position, mathematical knowledge is not simply a finished product. The (open) concept-relations make up mathematical knowledge, and these relations are constructed actively by the student in social processes of teaching and learning.

Mathematical learning environments concern the particular character of mathematical knowledge in the following way: On the one hand, it is a matter of concrete mathematical problems which are given in a situative - not formalized - context as immediate learning offers (cf. Wittmann, 2000 & chap. 1). On the other hand, it also becomes obvious that it is not only a matter of concrete mathematical activity, but with the exemplary treatment, something should be learned at the same time, something which is not directly visible, e.g. a mathematical relation or a generalized structure. Essentially, mathematical learning environments develop only in the active construction by the learner. Also, one is not dealing with concrete objects which one could touch, but with *embodiments of mathematical structures*.

The Role of Learning Environments in the Analysis of Classroom Interaction

Mathematical learning environments essentially represent an open, structural system which was constructed according to didactic and epistemological design criteria, and which can then become an environment filled with life only in classroom interaction through the students' activities and the teacher's interventions. The wired structure of learning environments is illustrated in the following diagram:



When interactively treating a learning environment, a similar problem arises in communication: The fact that the invisible mathematical relations in learning environments cannot be directly communicated by naming visible attributes means that the students have to construct relevant mathematical relations in the present exemplary learning environment with their own mathematical conceptions. This construction by the students is not directly readable in their descriptive statements; one has to analyse the intentions meant in the children's descriptions, and to discover which general mathematical relations the children aim at with their exemplary words.

In the interaction, the children have to deal with the not directly palpable mathematical knowledge and with the hidden relations by means of exemplary, partly direct interpretations - and not by means of abstract descriptions, notations, and definitions. By means of epistemological analysis (cf. Steinbring, 2000a; 2000b) it is to be found out whether the exemplary description used in the documented statement aims at a generalizing knowledge construction or whether it is a statement in the frame of the old, familiar knowledge facts.

Exemplary Analysis of a Classroom Interaction in a Learning Environment

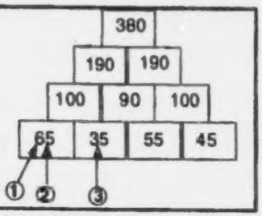
On the basis of the following short interaction scene, the joint and situative construction of an "invisible" mathematical relation in a learning environment about number walls (in a 4th grade class) shall be illustrated exemplarily. The children are working on four-stage number walls; the four base stones 35, 45, 55, and 65 have been interchanged several times, which led to number walls - written on the board - with different top stones. The teacher asks about particularities, whereupon the children give descriptions and first reasons for big or small top stones. The student Timo explains his reasons first:

102 Timo	The reason is also that if the sixty-five is in the middle, it is counted twice. Once with the outmost and once with the one next to, that is with both next to it. If it is at the edge, it is only counted once.

For the time being, Timo stays at his seat; he talks about the base number 65 "in the middle". This is not a mere description, but Timo develops linguistic-conceptual designations: "in the middle", "the outmost", "both next to it", "at the edge". These are not names for the objects ("stones" or "numbers"), but relations between these number-places which are important for the application of the rules for number walls.

103 T	Timo, I would like you to come up to the board and to show that to the other children

Timo goes to the board, talks, and shows:

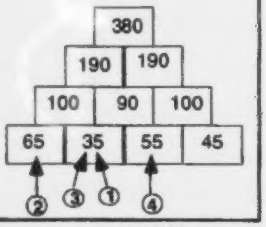
104 Timo	Here, (*points at the first stone "65" of the lower number wall with the top stone "380"*) when the sixty-five is here, it is counted only once. (*points at the first and second stone, "65" and "35"*) Here. One moment, plus thirty-five. #  380 190 190 100 90 100 65 35 55 45 ① ② ③

Now, Timo does not continue using his "concepts" constructed before; he points at the stones and talks about the numbers. Thereby, his description becomes more concrete and tied to the example. The importance of the attribute "count a base number once" can only become clear to other students through additional, active allusions by Timo, who points at the stone "65" and designates an edge stone at the same time.

Timo constructs new "mathematical signs" and symbolic relations with the sequence of designating numbers and showing stones in a wall.

105 T	# Please show the children where you are counting to!#

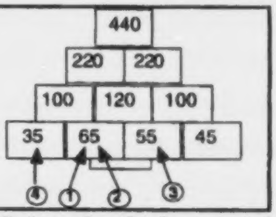
After the teacher's request, Timo points at the calculated "result", the number "100" in the 1st stone of the second stage of the wall.

106 Timo	# Yes, there (*points at the first stone "65"*) is the sixty-five, there (*points at the second stone "35"*) plus that are a hundred. (*points at the stone "100"*) But if the sixty-five stands here, (*points at the second stone "35"*) that one there and there the thirty-five, (*points at the first and second stone, "65" and "35"*) this one here plus thirty-five and there plus fifty-five (*points at the third stone "55"*) is counted once more.  380 190 190 100 90 100 65 35 55 45 ② ③ ① ④

In the second step of his argumentation, Timo refers to the 2nd stone - a middle stone - of the same number wall. He uses the same number wall, but varies the position of the "65": he moves it mentally from the left edge stone to the middle stone next to it. This is what he intends by pointing at the corresponding stones. By means of the possibility that "65" and "35" could switch positions, Timo stresses that it is mainly a matter of the different positions (edge or middle stone); in this way the example contains aspects of a general interpretation. Furthermore, Timo emphasizes that this number now has to be added twice: "... counted once more." The teacher asks Timo to show this possibility for "65" as a number in the middle in an appropriate example.

107 T	Great. But now, please, choose an example where the sixty-five really is in the middle, then we can imagine even more easily how you mean that. (*Timo points at the second stone "65" of the lower number wall with thetop stone "440"*) yes. Mhm.

Timo points at the lower wall with top number "440":

108 Timo	Plus fifty-five (*points at the second and third stone "65" and "55"*) and plus thirty-five. (*points at the first and second stone "35" and "65"*)  440 220 220 100 120 100 35 65 55 45 ④ ① ② ③

Here, the rather "general" interpretation "if the 65 stands here" (on the 2nd stone) is concretized. Timo now only names the readable addition tasks. The possibility that, not only the concrete calculations and results are meant hereby, but that e.g. the middle number "65" is counted *twice*, cannot be inferred from the statement, but has to be constructed actively by the other participants.

Timo begins a partially general interpretation of the *relations* between the positions of the base stones, which affect the number of calculations and therefore the size of the top stones, and which could lead to a mathematically full reasoning in the frame of the exemplary environment. The teacher requests Timo to concretize his considerations several times in the progress of the interaction. Timo follows this request, also where he in thoughts flexibly places the "65" instead of the "35", by now taking a number wall with the appropriate numbers. The span between the original exemplary conceptual generality in Timo's description and the teacher's request for concretization is narrowed down reciprocally until the general description has been substituted by concrete numbers. Timo‘s declaration seemed too abstract and therefore incomprehensible - although the teacher aimed at the just not directly visible structural relation in the environment "number walls", which cannot be described directly. In contrast to this, the designation of calculations with the given, concrete numbers and the written results, as they follow according to the construction rules of number walls, could, in tendency, only have led to a simply verifying confirmation of mathematical facts, without offering insights to the invisible relations. The "complementarity" of intended generality - in Timo‘s first description - and reducing concretization to the given numbers and examples - as requested by the teacher - seem to lead to an optimum of constructive interpretation and communicative understanding in this classroom interaction.

The Specific Social Epistemology of Interactively Constituted Mathematical Learning Environments

The particular epistemological character of mathematical knowledge consists in the concentration on *relations* which are neither openly visible nor directly palpable. (Duval, 2000). In order to develop these relations and to be able to operate with them, they have to be represented by signs, symbols, words, diagrams, and references to reference contexts (Steinbring, 2000c), learning environments, or experiment fields. Thereby, the scientific status of the mathematical knowledge does not depend on the choice or the abstractness of the means of representation; neither are there any universal means of illustration distinguished a priori which would automatically guarantee the epistemological quality of the mathematical knowledge (cf. Ruthven, 2000). The development of mathematical knowledge always occurs - be it in the academic discipline or in classroom learning processes - in social contexts which can, however, differ concerning their objectives and particular constraints (Steinbring, 1998).

Mathematical, substantial learning environments - the core element of the research project "mathe 2000" - represent such experiment fields which are suitable for interactive, social learning and developing processes in different situations of learning and acquiring mathematical and didactic knowledge in the classroom or in the training and in-service training of teachers. Which epistemological conception of mathematical knowledge becomes relevant is not simply determined beforehand and objectively, but this is subject to the active construction in the communication and in the proceeding interactive processes. Exemplary, epistemological analyses show that conceptions of the knowledge of the following kind can be constituted at this point: Mathematical knowledge as a collection of single facts, or mathematical knowledge as isolated, formal structures (cf. Steinbring, 2000b).

In the exemplary episode, a third and essential epistemological attribute of mathematical knowledge appears: A situatively tied form of describing and constituting the relations of mathematical knowledge in the frame of the exemplary learning environment; using exemplary, independent descriptions and words, but with the intentions - identifiable in the analysis - of generalizing exemplary attributes of the situation to the invisible general mathematical relations. In this regard, substantial learning environments represent a productive base for the interactive acquisition of knowledge, on which knowledge about mathematical knowledge can be acquired through the interaction at the same time, i.e., in the interaction, a specific, partly situation-bound, social epistemology of mathematical knowledge constitutes itself - which is not given by an independent authority from the outside.

This particular social epistemology constitutes itself in the proceeding of the according situation, for example during the treatment of a learning environment, and for this purpose, it needs situative, exemplary context conditions as well as words and relations already known and familiar for communication. In order to understand how relations in mathematical knowledge - which are not directly, empirically palpable - can actually be expressed and communicated in this way, a thorough epistemological analysis is required (Steinbring, 2000b). Such qualitative analyses of different situative epistemological interpretations of mathematical knowledge in interactive treatments of learning environments have different objectives and react upon the perspective of mathematical knowledge taken in the different chapters. So, feedback to the design and construction of learning environments, especially such modifications which make these environments become living systems, occur; furthermore, testing analyses of environments by teachers or students can increase the awareness (Selter, 1995) concerning the complex (professional) application conditions as well as the classroom interaction with mathematical learning environments.

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REACTION 1: BETWEEN PSYCHOLOGISING AND MATHEMATISING

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This necessarily brief response will focus on the central idea informing the rich body of work developed within the Mathe 2000 project: that of 'substantial learning environments' as organising systems for mathematical and didactical transactions between students and teachers, and between teachers and teacher educators.

As Wittmann illustrates, while common epistemological, psychological and pedagogical principles underpin these environments, what characterises each of them is a distinctive flexible mathematical artefact around which classroom activity can be organised. Steinbring and Selter variously show how such environments are brought to life through classroom activity in which mathematical structure latent in an artefact is actively (re)constructed by learners with assistance from their tutor. Underpinning successful learning through this co-operative and communicative activity are the complementary processes by which codified mathematical knowledge and reasoning are -in Dewey's term- 'psychologised' to recast them in forms more accessible to, and usable by, students; and through which student experience is -in Freudenthal's term- 'mathematised' to recast it in disciplinary terms. Evident too, in the design principles and exemplary environments, is an orientation towards -in Bartlett's terms- 'simplification by integration' over 'simplification by isolation'. Indeed, Moser Opitz conceives her study in this way, and provides some empirical support for the integrative position through evidence of successful pre-school learning as well as of distinctive trends in the development of student competence and strategy over the first year of schooling.

Some of the complexities of these processes emerge from Steinweg's study which explores -and largely rejects- the conjecture of 'natural or genetic' development in understanding of number patterns 'with advancing age'. Intriguingly, an important socio-cultural dimension is suggested by the illustrative student responses to the unfamiliar type of 'beautiful package' task. When Jana 'suspects the calculations to be wrong', she seems to be interpreting the task simply as one of checking the solutions to a conventional school exercise. From the way in which Réné 'repairs' the exercise, he appears to be interpreting it as a puzzle devised so that the answers form some pattern (somewhat analogous to the picture pattern formed by the arithmetic domino illustrated by Selter). Finally, Kim forms the 'epistemologically evident' interpretation that not only the answers but the problems themselves are intended to conform to an overarching mathematical pattern. Arguably, then, Jana, Réné and Kim were not tackling the 'same' task, and only Kim was tackling the task as envisaged by the designer. 'Success' in the task seems to depend on students' anticipation of new didactical norms as much as their grasp of mathematical structures. While over half the spontaneous responses fitted the maximally mathematically formatted interpretation of the task, one suspects that further analysis would reveal some troubling relationships between social position and task interpretation. Considerations of equity in making these (new) rules of the classroom mathematics game explicit lend further support to Steinweg's suggestion that such ideas may need to be taught more actively.

Nevertheless, the idea that teacher and students are working on 'the same task' and seeing 'the same structure' is a useful strategic fiction where classroom participants are seeking to co-ordinate and articulate their interpretations of a task and their resulting constructions. Steinbring's microgenetic analysis of the production of a public explanation shows such processes in operation. Through interaction between teacher and student, the focus of this episode of classroom communication shifts from the articulation of a generic mathematical structure which the student perceives as fundamental to an artefact, towards the concretisation of this idea in specific examples inscribed on the board. Far from the front of the classroom, Timo uses positional language to evoke the common structure he perceives in number walls. But as he moves closer to the board and starts physically pointing, 'the outmost' becomes 'here', and his account becomes more tied to a designated example. In response to teacher solicitations to concretize his ideas, his accounts become progressively more focused on the manifest content of specific number walls. As Steinbring notes, this layered explanation casts latent structure into a form of relief, making it more readily accessible to other participants.

These accounts lead one to appreciate the considerable demands of managing these forms of classroom activity so as to promote effective learning, and both Moser Opitz and Selter emphasise the change and challenge this presents for many teachers. In this respect, one admires the sense and subtlety with which substantial learning environments seem to extend and reshape the familiar didactical form of the exercise rather than wholly rejecting it. An important aspect of this is the way in which tasks seek to 'combine the practice of skills with higher mathematical activities'. Correspondingly, Selter emphasises the potential of substantial learning environments as organising structures within teacher education around which concerns to develop the mathematical, psychological, didactical and practical expertise of primary teachers can be co-ordinated.

One crucial question is how to conceive mathematical expertise for primary teaching. For the honourable tradition of 'elementary mathematics from an advanced standpoint' may lead to an over-extension of a single-minded idea of progression. Take the treatment of a generalised form of arithmogon in terms of the theory of systems of linear equations. Whilst appreciating that the theory is developed 'just as far as is necessary to frame a certain class of problems', and noting student teachers interest 'to see how abstract algebra applies to concrete numbers', it is not clear how such ideas - estimable in their own right- might readily be brought to bear in shaping children's mathematical enquiry and argumentation. There, the teacher is required to operate with great ingenuity, flexibility and fluency within the construction zone of the students. So, while a matrix representation neatly captures the relationship between 'inner' and 'outer' values of the Rechendreiecke, a more didactically pertinent representation might highlight the complementary arithmetic relationship between each 'inner' value and its spatially opposite 'outer' counterpart, and between 'inner' and 'outer' sums. In short, the question is one of finding a sound balance between 'psychologising' and 'mathematising' in professional education.

REACTION 2: DESIGN AND USE OF SUBSTANTIAL LEARNING ENVIRONMENTS IN MATHE 2000

Lieven Verschaffel, University of Leuven, Belgium

This short reaction is organized around the four principles at the heart of the MATHE 2000 project that are listed in Wittmann's introduction to the Research Forum. However, I want to begin by expressing my admiration for the project, which entails a unique, provocative and successful concept of how to organize and do research in the field of mathematics education. The decision to place substantial learning environments (SLE) at the core of the research and the developmental work and to use these SLE as the common starting point for the design of curricula and textbooks, empirical studies into children's thinking and communication, and teacher education, has proven to be a very valuable and effective strategy both from a theoretical and a practical point of view. Although the project has many similarities with other projects, such as the Realistic Mathematics Education (RME) project in The Netherlands, there seem to be some differences with these other approaches in terms of aims, content and research strategy, -- some of which will be pointed out below.

Fundamental ideas of mathematics as guidelines. As stressed and illustrated in all papers, SLE are at the very heart of the MATHE 2000 project. Throughout the papers it is shown how the design of SLE -- both for pupils and for student-teachers -- is based upon the progressive elaboration of the epistemological structure of elementary mathematics. Major characteristics of SLE are that they represent central objectives and contents at a certain level of instruction, and that they are related to significant mathematical objectives and contents beyond that level. Quite strikingly, almost all examples of SLE relate to pattern finding (see, e.g., Wittmann's "rechendreiecke", Selter's "number chains", Steinweg's "series of beautiful calculation tasks", and Steinbring's "number walls"). I consider this as one of the clearest manifestations of taking the (quintessens of the) subject of mathematics as a "science of patterns" -- already from the very first years of elementary mathematics education on -- very seriously. Equally strikingly, these pattern-finding tasks are given in a purely symbolic format and not embedded in a real-life context. Although it should be acknowledged that some of these pattern-finding tasks are briefly introduced through a realistic or fantasy context and that these symbolically presented pattern-findings activities are complemented with more "applied" kinds of problem-solving activities in the textbook series "Das Zahlenbuch", the early and frequent recurrence of these context-lean pattern-finding activities -- is remarkable. Interestingly, Steinweg's study of 6-10-year old children's understanding of patterns reveals that these children do not have any reservations against such unfamiliar, context-lean tasks, that they fully accept them, and do not need any extrinsic motivation.

Learning as a constructive and a social process. The view of learning that lies at the basis of the MATHE 2000 project is in line with current conceptions in instructional psychology in general, and in the psychology of mathematics education in particular, wherein learning is considered an active, constructive, cumulative, self-regulated, goal-oriented, situated and social process of knowledge building and meaning construction. Throughout the papers, it becomes clear that in the MATHE 2000 project not only the subject (i.e., mathematics) but also the learner -- with his or her own constructions, interpretations, beliefs and goals -- is taken very seriously. The documentation that this process of knowledge building and meaning construction is fundamentally social in nature and that it is shaped by the norms, rules and agreements that constitute the culture of the mathematics classroom (especially in Steinbring's paper) fits with recent analyses by others (e.g., Brousseau, Cobb).

Teaching as organizing learning processes. Complementary to the above view of learning, teaching is conceived in the MATHE 2000 project as inducing and supporting children's active process of knowledge building and meaning construction, rather than providing them with the necessary knowledge and skills to pass the next test. As in most other countries where similar new conceptions of teaching and learning mathematics are being developed and implemented, MATHE 2000 risks of being criticized for giving children too much freedom in developing their own strategies, for paying too less attention at the basic skills, for not progressing quickly enough towards abstraction and formalisation... For many reasons, these criticisms do not hold to the MATHE 2000 project. The carefully designed SLE, grounded in a thorough epistemological and historical analysis of the domain and supported by empirical research on pupils' reactions to the materials, provide ample opportunities for the well-balanced combination of the practice of skills with higher mathematical activities. Moser-Opitz's evaluation study of children with special needs provides some empirical support for this claim, -- even among those children for whom this new approach is deemed to be most detrimental.

Cooperation with teachers. Finally, in the MATHE 2000 project, not only the subject and the child, but also the teacher is taken very seriously. This is evidenced, among other things, by the great attention given to teacher pre-service and in-service education and to cooperation with teachers. Selter's approach to pre-service teacher education, with its multi-dimensional view of the components of teacher professionality, its emphasis on "de-constructing" student-teachers' erroneous beliefs about and negative attitudes towards mathematics (education), and its focus on the development of teachers' background knowledge and awareness rather than of their executive skills, convincingly demonstrates the value of SLE as the keystone, not only for elementary mathematics education, but also for teacher education.

RESEARCH FORUM 5

Theme
Realistic Mathematics Education Research:
Leen Streefland's work continues

Coordinator
Norma Presmeg

Session 1

- Introduction by Norma Presmeg
- Presentation 1
  "A focus on experiences—my experiences with Leen" Willibald Dörfler
- Presentation 2
  "Social interaction as reflection: Leen Streefland as a teacher of primary school children" Ed Elbers
- Presentation 3
  "Learning from history to solve equations" Barbara van Amerom
- Synthesis by Norma Presmeg
- Audience discussion

Session 2

- Introduction by Norma Presmeg
- Three simultaneous Round Tables based on
  "Didactising: Continuing the work of Leen Streefland"
  Erna Yackel, Diana Underwoood, Michelle Stephan, Chris Rasmussen
  - Round Table 1: Diana Underwoood
    Theme: Roles of symbolizing (Activity: Stacked cubes task)
  - Round Table 2: Chris Rasmussen
    Theme: Models of students' thinking (Activity: Tangents task)
  - Round Table 3: Michelle Stephan
    Theme: Arguments and discourse (Activity: Analysis of video clip)
- Short presention by the round table leaders
- Reaction by Koeno Gravemeijer and Marja van den Heuvel-Panhuizen
- Audience discussion
- Synthesis by Norma Presmeg

REALISTIC MATHEMATICS EDUCATION RESEARCH: LEEN STREEFLAND'S WORK CONTINUES[1]

Leen Streefland

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Willibald Dörfler, University of Klagenfurt
Ed Elbers, Utrecht University
Barbara van Amerom, Freudenthal Institute
Erna Yackel, Purdue University Calumet
Diana Underwood, Purdue University Calumet
Chris Rasmussen, Purdue University Calumet
Michelle Stephan, Purdue University Calumet
Koeno Gravemeijer, Freudenthal Institute (Reactor)
Marja van den Heuvel-Panhuizen, Freudenthal Institute (Reactor)

On this appropriate occasion of the 25th annual meeting of PME, at the Freudenthal Institute where Leen Streefland carried out his work, this Research Forum describes and commemorates some of the aspects of the Realistic Mathematics Education research in which he was engaged, and illustrates projects in which that work is being continued in The Netherlands and in the USA. Core constructs that Streefland's legacy addresses include the centrality of the learner, rooting of learning in prior or current experiences of the learner, mathematics education research as a developmental process, learning communities and interactive learning environments, learning from history, didactising and guided reinvention in developmental mathematics courses, argumentation and discourse in classroom mathematical practices, social learning trajectories and the role of teachers in guided reinvention.

Introduction: A focus on experiences – my experience with Leen. Willi Dörfler

It has been very good luck for me to have the opportunity for enjoying a close cooperation with Leen Streefland. We got to know and appreciate each other in the context of PME. When I was entrusted with editing Educational Studies of Mathematics (ESM) I decided to share this demanding task. It was absolutely clear to me that Leen had to be my first choice and I was very happy that he agreed. Thus we had six years from 1990 until 1995 jointly dedicated to the journal and to the enhancement of maths education research. Over this time, of course partly together with Gila Hanna as the third editor and with the members of the editorial board, Leen and I had a fruitful and productive exchange of ideas and opinions about questions like: main issues of mathematics education research, basic orientations, criteria for quality of research, role of theories in maths education, policy of the journal. Mostly these questions and issues were discussed with respect to submitted papers and thus they were not just esoteric deliberations but had a strong relevance for concrete action.

It is against this background of shared experiences that I will now turn to some traits of the personality of Leen which could be found in his personal life, in his academic life and in his so highly valuable work in mathematics education. Leen`s interest in and concern for people had a strong influence on what he did, what he said, what he wrote, in short on how he lived his life. Foremost there was the feeling that you really matter as a person and a human being which you had when sharing your time with Leen. This was never an abstract or detached interest, say, just in your academic work but it took into account the complex living conditions, the emotions, fears and wishes of the respective other. One can read this general attitude from many letters by Leen to authors of papers submitted to ESM. In those he tried to establish a communicative basis on which to negotiate about the content, style or quality of the paper. Leen never forgot that it always is a concrete person who writes and that a judgement on a paper implies a judgement on the author. This should not be misinterpreted as a tendency to lowering standards. Quite to the contrary, it was the struggle to guide authors in matching high standards.

The concern for people be they pupils, authors, colleagues or friends in my view most prominently resides in Leen's own work in mathematics education. This research in the context of Realistic Mathematics Education (RME) genuinely and seriously puts the learner into the center. And again it is not an abstract or epistemological subject from which maths education is stipulated to start. The subjective, individual and personal experience of the learner is on the one hand the background and the basis for all (mathematical) learning. On the other hand learning mathematics according to RME is to be organized by making new experiences possible, by engaging students in reflections on their personal experiences in mathematical ways of thinking. This reflects a deep respect for the individual student and his or her faculty for building up mathematical meaning provided adequate experiential situations are offered. It also expresses a view on mathematics as originating out of human experiences and their description, organization, structuring and planning. That this nowadays is a broadly accepted stance can be attributed to the insistence with which Leen and other representatives of this school of thought have expounded their position: mathematics is relevant for the future lives of our students and can be experienced as being meaningful by them. But for this to occur the widespread separation and isolation of school mathematics have to be overcome. Leen's work impressively shows us a way to attain that goal and convinces us that it really is attainable. The related demands Leen did not only impose on his own work, but he always tried to urge the authors of papers submitted to ESM to think about consequences of their work for the students in the classroom and how it relates to their struggles with making their mathematics meaningful to themselves. When admitting that there is a multitude of possibilities for approaching that goal this could and should mean a basic guiding framework and orientation for future research in mathematics education: to make the mathematics experientially real to the learner.

A related feature of Leen's thinking about research and scientific theories in general and specfically in maths education is the following one: As the learning of mathematics itself, also the development of theories about it has to be experientially grounded. This attitude of Leen's showed itself in a kind of doubtfulness and suspicion of what he sometimes called a jargon. By this he labeled texts that used vague or opaque concepts too much detached from the concrete realities of the mathematical classrooms to have any sensible implications for the organization of the latter. In other words the basic tenets of RME in Leen's view have also to be applied to maths education research. As the teaching of mathematics in school has to be rooted in prior or current experiences of the learner, a valuable piece of research and its presentation in a paper has to be related to the experiences of the readers. It must make sense by making clear the meaning of used notions and terms and should have the potential to change the experience of the reader. As the relevance of most mathematical concepts and methods resides in their potential to structure and organize the experience and activity of the learner, research and theories in mathematics education should have analogous implications for the practice of school mathematics. I remember various vivid discussions on this issue that showed us both that such general tenets have to be substantiated in each single case, such as a specific contribution to ESM. It also became clear that experience is not a given which passively is imposed on the individual but that it is something that is actively constructed and developed by the latter in his/her social context. This inherent indeterminacy of individual experience, be it by the pupil in the classroom or by the reader of a journal article, brings in notions such as affordances and constraints. Whatever a teacher does in the class or an author writes in a paper establishes affordances and constraints for the thinking and understanding on the part of the students or of the readers. This might make teaching and writing a daunting endeavour; but only if one believes in fixed and absolute meanings (in mathematics and mathematics education), which have to be acquired and transmittered adequately. Contrary to that, Leen‘s conception of RME and of mathematics education research is that of a developmental process that leaves room for interpretations, negotiations, inventions, deviations and the like, which yet constantly and consciously is devoted to sense-making. And in this framework a jargon is a way of teaching, speaking or writing which inhibits or even prevents the above cognitive and communicative processes. And I take it as a kind of legacy from Leen to avoid jargon in this sense because it acts against the interest in people as learners, readers and researchers.

Social interaction as reflection: Leen Streefland as a teacher of primary school children. Ed Elbers

Leen Streefland did pioneer work in creating a community of inquiry in the mathematics classroom. He encouraged students to "do research" and to adopt the attitude of researchers. The task of the teacher was to guide and assist students who had been given considerable responsibility for their own learning. Streefland was convinced that creating a learning community in which students had ample opportunity to produce and discuss ideas would allow their mathematical creativity to blossom. Interaction and collaborative learning would stimulate children to make their own mathematical constructions and to discuss them in what amounted to a social process of reflection. Streefland was involved not only as a researcher but also as a teacher. He worked with the primary school teacher Rob Gertsen for about 15 years.

I shall present a case study of a lesson co-taught by Leen Streefland and Rob Gertsen. In this lesson Streefland alternated whole class discussions with individual or group work. I want to use this particular case as an illustration of Streefland's ideas about mathematics education. Moreover, I shall analyse the relationship between whole class discussions and learning processes of individual students and the tension between teacher's guidance and students' invention.

Three principles of realistic mathematics education form the basis of this lesson. Firstly, the starting point is the problem instead of the mathematical strategy or solution. The teacher introduces a meaningful problem, which the students use as a source for constructing mathematical understanding. Secondly, a basic element in realistic mathematics education is to motivate students to 'mathematize', to turn everyday issues into mathematical problems and use the mathematics resulting from these activities to solve other problems. Thirdly, the students do not depend on the teacher to find out whether their ideas are correct. Part of their task is to develop good arguments to support their approach and solutions.

Streefland's lessons had a basic format (Elbers & Streefland, 2000a, b). They started with a statement of the principle of a community of inquiry: "We are researchers, let us do research." The students were given a topic or problem as the subject of their research (often with reference to some example from everyday life, a newspaper clipping, a photograph, etc.). After the introduction of the topic, the students were invited to formulate research questions and develop answers to these questions. Work in small groups of 4 or 5 students, and sometimes individual work, alternated with class discussions in which the results were made available for discussion in the whole class. Leen Streefland and Rob Gertsen worked as a team. They introduced themselves not as teachers, but as senior researchers. This role allowed them to participate in the discussion themselves. The students knew, of course, that they could expect assistance and guidance by their teachers. However, by acting as they did the teachers gave the students responsibility for their work and made it clear that the validation of their solutions comes from mathematical argument and not from the teacher's authority.

The case study

The lesson was taught in a combined seventh and eighth grade class at a primary school in the Netherlands (children between 11 and 13 years of age). It was part of a short series of lessons in June, at the end of the school term, when the students had already completed their regular mathematics curriculum for that year. An activity sheet was used with the problem and its variations printed on it, leaving space for the students to write down their own solutions. My presentation is based on an analysis of the video recording of this lesson, a transcript and a short Dutch article Leen Streefland wrote about this lesson in 1997. The case is divided into a number of Episodes. The problem introduced in the classroom was set in a pharmacy and involved calculating the number of tablets prescribed by a physician.

"Elisa works at a pharmacist's. She is preparing medicines prescribed by Doctor Sterk for Mrs. Jansen.

> For Mrs. Jansen: 40 falderal tablets.
>
> 6 tablets a day for 2 days;
>
> then 5 tablets a day for 2 days;
>
> then 4 tablets a day for 2 days;
>
> then 3 tablets a day for 2 days;
>
> then 2 tablets a day for 2 days;
>
> then 1 tablet a day for 2 days.
>
> Elisa thinks: 'This isn't right! The doctor has made a mistake.'
>
> What do you think? Is Elisa right?"

During the lesson variations on the original problem were given which amounted to changes in the number of tablets. The students, first, had to solve the original version of the problem (starting with 6 tablets), next they had to calculate the number of tablets in a prescription starting with 8 tablets (8 tablets for the first two days, 7 for the next two days, etc.) and then the number of tablets in a prescription starting with 10 tablets. For reasons of space, I shall restrict my presentation here to Episodes 5, 6 and 7. In the preceding Episodes 1 to 4, the students, in order to solve the problem in its original form, had developed two approaches: (1) multiplying the various numbers of tablets (2x6; 2x5; etc.) and adding them up (12 + 10 + etc.), and (2) adding up the numbers (6 + 5 + etc) before multiplying them (2 x 21). In Episode 4, which consisted of individual and group work, students worked on the version of the problem starting with 8 tablets. Episode 5 is a class discussion immediately following Episode 4.

Episode 5. One of the children showed his solution to the whole class. He used the solution to the original version (starting with 6 tablets) as a starting point for solving the new problem (starting with 8 tablets): 42 + 2x7 + 2x8. After this, the teachers asked who had applied a different approach. One of the children showed that she had made combinations of tens (8+2=10, 7+3=10 etc.) in order to add up quickly: "I took out the tens". In the ensuing discussion about this approach, Streefland asked: "What shall we call this approach? Can we invent a name for it?" The children proposed expressions such as "making tens", "jumping to tens" and "bridges of ten". Leen Streefland concluded this discussion by suggesting they call this method: "making combinations" and "making clever combinations".

The third variant of the problem was then introduced, starting with 10 tablets. But before the students started their work on the activity sheets, the teachers presented a challenge:

Sequence 1.

Gertsen: You can solve the next problem. Just try to solve it: if I start with 10 tablets, how many do I need? The children who have found the answer quickly, then think about this extra question: can I find the answer without making the calculation? I'll give you a hint: compare the three numbers. When I start with 6 tablets..., when I start with 8..., when I start with 10... When you compare, you may come to a conclusion, a discovery. After all, you are researchers, aren't you?

Streefland: I would like to add that it may be fun to try out other combinations. Try to experiment with combinations. Maybe you'll discover something surprising. If you discover that, the problem is a piece of cake.

Episode 6. In this Episode, some students invented a new solution to the problem. Their invention was the result of a discussion in a group of four boys.

Episode 7. In the subsequent class discussion, these boys' solution was presented by one of them in the following way:

Sequence 2.

Gertsen: Researcher Tom, show your solution on the blackboard.

Tom: If you start with 42, it is 6x7. If you look at 72, that is 9x8. Beginning with 10, that is 10x11. (He writes these numbers on the blackboard).(...)

Streefland: Be consistent: 6x7, 8x9, 10x11.

Gertsen: Put the smaller number first.

Streefland: It is very nice to do it this way.

Gertsen: I can know the next one, too, because, look, (pointing at the numbers on the blackboard) here is 6, 8, 10, and the next one should be: 12x13. (...)

Streefland: That is very good, but I think that he should show it by writing it out in full. Because it does not appear out of the blue, of course!

Gertsen (addressing Tom): Show it, prove it.

With some help Tom succeeded in showing on the blackboard that 6+5+4+3+2+1 = 3x7 (6+1, 5+2, etc.), and because the numbers should be doubled, the result is 6x7. Next, Gertsen demonstrated this way of calculating for other variants of the problem. The prescription starting with 8 tablets can be solved by making combinations of 9. With many students participating, the teachers demonstrated the outcome of the problem starting with 12 and with 14 tablets.

Analysis of the activity sheets demonstrates a clear and direct influence of the solutions discussed in the whole class on the individual work. The majority of students adjusted their solutions on their activity sheets and adopted the strategies developed just before in the whole class. The sheets show that, in three episodes of individual work, 26, 19 and 16 (of 28) students appropriated the strategies discussed in the whole class. The activity sheets also show that students did not stick to one strategy once they had invented it and found it to be correct. They acted in line with the teachers' encouragement to continue finding other, more efficient and clever, solutions.

Discussion.

At first sight the results would seem to fit into a two level approach (suggested by Inagaki et al., 1998): understandings are first constructed collectively, and then appropriated by individual students. This theory would seem to apply here, since the case study showed that the majority of students accommodated their solutions on the worksheets to the previous collective argumentation in the whole classroom. The difficulty with an account in terms of a two level approach is that it is not so easy to tell where the collective work ends and individual learning begins. Both collective argumentation and individual work took place within a discursive structure with rules such as: find out for yourself, choose a practical solution, present it understandably. These rules structured the students' work, both in the whole class discussions and during their work on the activity sheets. In their individual work, students applied the same kind of arguments which they had to use in the class discussions. Students' achievements are best understood by referring to this discursive structure. Individual work is to be considered as an anticipation of a class discussion or a reconstruction of it. Given this discursive structure, there is no priority for individual or collective work; they are two sides of the same coin.

Students did not just internalize or incorporate the outcomes of the class discussion, they had to reconstruct them (cf. Elbers, Maier, Hoekstra & Hoogsteder, 1992). Even the use of the outcomes of class discussions for writing an answer on their sheets was not a reproduction, but demanded creativity. Students' creation of novel solutions, as in Episodes 6 and 7, can illustrate this. The discursive structure of the interaction was the outcome of the teachers' transformation of the mathematics class into a community of inquiry. At the beginning of the lesson Streefland told the students: "I am convinced that, if you have the courage to figure out something, you can do much better than you thought you could!" Addressing the class as a learning and researching community created roles and responsibilities for the teachers and the students which differed from a conventional classroom (Ben-Chaim et al, 1998; Elbers and Streefland, 2000b).

Because of the students' contribution to the class, the teachers faced a problem originating from their double role. On the one hand, they were in charge and responsible for the students' activities. They decided what topics would be worked on and they had their ideas of what knowledge students should acquire during the lessons. On the other hand, they wanted the students to find out for themselves: to invent solutions to problems and to prove their validity. They did not want to frustrate children's creativity by using their authority for supporting certain answers instead of others. For solving the problem originating from this double role, the teachers used three strategies to channell the discussion. Firstly, the teachers selected students to give a presentation in front of the whole class. During the parts of the lesson in which students worked on their worksheets the teachers walked around and sometimes asked individual students to show and explain their work. During these episodes, the teachers observed what solutions the students were trying to work out and they singled out students with novel solutions rather than with familiar ones to present their work to the whole class. Secondly, the teachers stimulated variation in solutions. Students trying to discover a different solution from one already found were rewarded with compliments and enthusiasm by the teachers. Thirdly, the teachers made general suggestions to help students to view the problem from a different perspective. An example of this can be seen in Episode 5 above. The children, who at this stage had only made combinations of ten, proposed calling this procedure: taking out tens, etc. Streefland taught them to use the term: making combinations. In this way he paved the way for students to find out that they could make combinations which add up to numbers other than ten. Using these strategies, the teachers could direct the discussion and at the same time leave the students ample freedom to find out and make inventions. After having worked out a correct solution, there was no reason for students to stop, since there was always an even more efficient solution to be found.

The case demonstrates how students in an atmosphere of collaboration and interaction contributed to their learning and how the teachers exploited the various productions and constructions made by the students to structure the learning process.

Learning from history to solve equations. Barbara van Amerom

Several research projects of recent years report on learning difficulties related to algebraic equation solving (Kieran 1989, 1992; Filloy & Rojano 1989; Sfard 1991, 1996; Herscovics and Linchevski,1994, 1996; Bednarz et al. 1996). These difficulties include constructing equations from arithmetical word problems, as well as interpreting, rewriting and simplifying algebraic expressions. According to some researchers part of the problem is caused by fundamental differences between arithmetic and algebra. A good starting point for an investigation into this issue could be a return to the roots. By looking into the past we shall try to gain insight into the differences and similarities between arithmetic and algebra and learn from the experiences of others. Streefland emphasized the value of 'reciprocal shifting': changing one's point of view, looking back at the origins in order to anticipate (Streefland 1996). Such a change of perspective can propel the learning process of the researcher, the teacher and the student alike.

Algebra and arithmetic

A closer look at the similarities and differences between algebra and arithmetic can help us understand some of the problems that students have with the early learning of algebra. Arithmetic deals with straightforward calculations with known numbers, while algebra requires reasoning about unknown or variable quantities and recognizing the difference between specific and general situations. There are differences regarding the interpretation of letters, symbols, expressions and the concept of equality. For instance, in arithmetic letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. According to Freudenthal (1962), the difficulty of algebraic language is often underestimated and certainly not self-explanatory: 'Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory' (p. 35). He continues:

> The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic 3 + 4 means a problem. It has to be interpreted as a command: add 4 to 3. In algebra 3 + 4 means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. a + b cannot easily be interpreted as a problem" (Freudenthal 1962, p. 35).

Several researchers (Kieran 1989; Sfard 1991) have studied problems related to the recognition of mathematical structures in algebraic expressions. Kieran speaks of two conceptions of mathematical expressions: procedural (concerned with operations on numbers, working towards an outcome) and structural (concerned with operations on mathematical objects). But despite the contrasting natures of algebra and arithmetic, they also have definite interfaces. For example, algebra relies heavily on arithmetical operations and arithmetical expressions are sometimes treated algebraically. Arithmetical activities like solving open sentences and inverting chains of operations prepare the studying of linear relations. Furthermore, the historical development of algebra shows that word problems have always been a part of mathematics that brings together algebraic and arithmetical reasoning.

Cognitive obstacles of learning algebra

An enormous increase in research during the last decade has produced an abundance of new conjectures on the difficult transition from arithmetic to algebra. For instance, with regard to equation solving there is claimed to be a discrepancy known as cognitive gap (Herscovics & Linchevski 1994) or didactic cut (Filloy & Rojano 1989). They point out a break in the development of operating on the unknown in an equation. Operating on an unknown requires a new notion of equality. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal sign changes from announcing a result to stating equivalence. And when the unknown appears on both sides of the equality sign instead of one side, the equation can no longer be solved arithmetically (by inverting the operations one by one). Sfard (1996) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She writes that rhetoric (in words) and syncopated algebra (involving abbreviated notations) is linked to an operational (or procedural) conception of algebra, whereas symbolic algebra corresponds with a structural conception of algebra. Da Rocha Falcaõ (1995) suggests that the disruption between arithmetic and algebra is contained in the approach to problem-solving. Arithmetical problems can be solved directly, possibly with intermediate answers if necessary. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved. Mason (1996, p.23) formulates the problem as follows: 'Arithmetic proceeds directly from the known to the unknown using known computations; algebra proceeds indirectly from the unknown, via the known, to equations and inequalities which can then be solved using established techniques.'

'Reinvention of algebra'

Recent research on the advantages and possibilities of using and implementing history of mathematics in the classroom has led to a growing interest in the role of history of mathematics in the learning and teaching of mathematics. Inspired by Streefland's work as well as the HIMED (History in Mathematics Education) movement, a developmental research project called 'Reinvention of Algebra' was started at the Freudenthal Institute in 1995 to investigate which didactical means enable students to make a smooth transition from arithmetic to early algebra. Specifically, the 'invention' of algebra from a historical perspective will be compared with possibilities of 're-invention' by the students. The historical development of algebra indicates certain courses of evolution that the individual learner can reinvent. Word or story-problems offer ample opportunity for mathematizing activities. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with the solving problems situated in every day life, although they also showed interest in mathematical riddles and recreational problems. Fair exchange, money, mathematical riddles and recreational puzzles have shown to be rich contexts for developing handy solution methods and notation systems and are also appealing and meaningful for students. Another possible access is based on notation use, for instance comparing the historical progress in symbolization and schematization with the contemporary one.

The learning strand: pre-algebra as a link between arithmetic and equation solving

The barter context in particular appears to be a natural, suitable setting to develop (pre-)algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. The following Chinese barter problem from the 'Nine Chapters on the Mathematical Art' has inspired Streefland (1995a, 1996a) and the author to use the context of barter as a natural and historically-founded starting point for the teaching of linear equations:

> *By selling 2 buffaloes and 5 wethers and buying 13 pigs, 1000 qian remains. One can buy 9 wethers by selling 3 buffaloes and 3 pigs. By selling 6 wethers and 8 pigs one can buy 5 buffaloes and is short of 600 qian. How much do a buffalo, a wether and a pig cost?*

In modern notation we can write the following system:

$$2b + 5w = 13p + 1000 \qquad (1)$$

$$3b + 3p = 9w \qquad (2)$$

$$6w + 8p + 600 = 5\mathrm{b} \qquad (3)$$

where the unknowns b, w and p stand for the price of a buffalo, a whether and a pig respectively. The example is interesting especially when looking at the second equation, where no number of 'qian' is present. In this 'barter' equation the unknowns b, w and p can also represent the animals themselves, instead of their money value. The introduction of an isolated number in the equations (1) and (3) therefore changes not only the medium of the equation (from number of animals to money) but also the meaning of the unknowns (from object-related to quality-of-object-related). Streefland (1995) has found in his teaching experiment on candy that the meaning of literal symbols is an important constituent of the vertical mathematizing process (progressive formalization) of the pupils. "The changes of meaning that letters undergo, need to be observed and made aware very carefully during the learning process. In this way the children's level of mathematical thinking evolves." (Streefland 1995, p 36, transl.).

We also intend to investigate how notation and mathematical abstraction are related. The categorization rhetoric – syncopated – symbolic is the result of our modern conception of how algebra developed, and for this reason it is often mistaken for a gradation of mathematical abstraction (Radford 1997). When the development of algebra is seen from a socio-cultural perspective, instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. Perhaps our students will reveal similar needs for efficiency when they develop their own notations (from context-bound notation to an independent, general mathematical language), but this may or may not coincide with the conceptual development of letter use.

Classroom examples

The first version of the experimental learning strand for primary level was tried out in 1997-1998 in two primary school classes grade 5-6 pupils (combined). The general topic of the primary school lesson series is recognizing and describing relations between quantities using different representations: tables, sums, rhetoric descriptions and (word) equations. No prior knowledge was required apart from the basic operations and ratio tables. The study is based on data collected through the observation and analysis of classroom work and the evaluation of a written assessment test taken by the students after the last lesson.

Shortened notations form one of the spear points in the learning strand. One of the units for grade 5-6 is centered on the context of a game of cards. In one of the lessons children suggested what could be the meaning of the expression 'pA = 3 × pJ'. Our decision to use this kind of symbolism is based on other pupils' free productions in a preliminary try-out. The letter combination maintains the link with the context: the letter p stands for 'number of points belonging to' and the capital letter stands for the person in question, in this case Ann and Jerry. In the expression, such a letter combination behaves like a variable for which numbers can be substituted. When the score of one of the players is given, the expression becomes an equation which can be solved. The teacher asked the children for an example that will illustrate that the relation between the variables pA and pJ is '3 times as much':

Teacher: 'If we think of points, what would be possible? You have to write it down in a handy way, just like in *Pocket Money*, which numbers are possible.'
Yvette: '3 and 9'.
Teacher: 'Who has 3 and who has 9?'
Yvette: 'Annelies has 9'.
Teacher: 'How would you write it down? Why don't you show us on the blackboard.'
Yvette writes: A – 9 p j – 3 p
Sanne: 'I would write an equal sign, not a line.'

Figure 1: inconsistent symbolizing

Figure 1 illustrates three samples of inconvenient symbolizing: Yvette's choice to write a capital letter A and then a small letter j, her use of the letter p as a unit even though it is already part of the variable, and Sanne's suggestion at the end. Apparently it was not a problem to the children that letters mean different things at the same time. As long as the pupils and the teacher are all conscious of this fact, the development and refinement of notations is a natural process. On the other hand, it is not our intention to cause unnecessary confusion regarding the meaning of letters. It was decided that if children have a natural tendency to use the letter p as a unit, p should not be included in the expressions and formulas.

The lesson materials were adjusted and tested again in 1999. The dual character of the learning strand – to develop reasoning and symbolizing skills in the study of relations – was maintained but placed in a more problem-oriented setting and with a more explicit historical component. We have selected two examples of student work from the final classroom experiment to demonstrate that (pre-)algebraic symbolizing tends to be more difficult for students than reasoning.

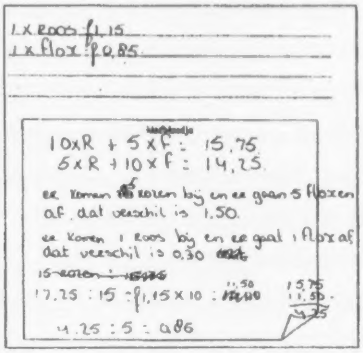
Symbolizing versus reasoning

The project's final experiment was conducted in three primary schools (grade 6) and two secondary schools (grade 7). Encouraged by the ideas and results of the classroom experiment on candy by Streefland (1995), a grade 7 unit on equation solving was designed based on the mathematization of fancy fair attractions into equations. One of the tasks in the written test was:

> *Sacha wants to make two bouquets using roses and phloxes. The florist replies: 'Uhm ... 10 rozes and 5 phloxes for f15,75, and 5 roses and 10 phloxes for 14,25; that will be 30 guilders altogether please.'*
>
> *What is the price of one rose? And one phlox? Show your calculations.*

One of the outcomes of the experiment is that algebraic equation solving need not necessarily develop synchronously with algebraic symbolization. For instance, we have observed student work where a correct symbolic system of equations was followed by an incorrect or lower order strategy, or where the student proceeded with the solution process rhetorically. The student in figure 2 mathematizes the problem by constructing a system of equations, and then applies an informal, pre-algebraic exchange strategy which is developed in the unit. Below the equations she writes: 'We get 5 roses more and 5 phloxes less, the difference is 1.50. We get 1 rose more and 1 phlox less, the difference is 0.30.' The calculations show that she continues the pattern to get 15 roses for the price of 17.25 guilders, and then she determines the price of 1 rose and 1 phlox. The level of symbolizing may appear to be high at first due to the presence of symbolic equations, but the student does not operate on the equations. The equations may have helped her structurize the problem but they are not a part of the solution process. And even though the unknown numbers of flowers are an integral part of her reasoning, the letters representing them are not needed in the calculations. There is a parallel here with the historical development of symbolizing the solution. In the rhetoric and syncopated stages of algebra the unknown was mentioned only at the start and at the end of the problem; the calculations were done using only the coefficients.

Figure 2: symbolic notations and informal reasoning



1 x roos f1,15

1 x flox f0,85

kladblaadje

10xR + 5xf = 15,75

5xR + 10xf = 14,25

er komen 5 rozen bij en er gaan 5 floxen af. dat verschil is 1.50.

er komen 1 roos bij en er gaat 1 flox af dat verschil is 0,30

~~15 rozen = 15,75~~

17,25 : 15 = f1,15 x 10 = 11,50

15,75

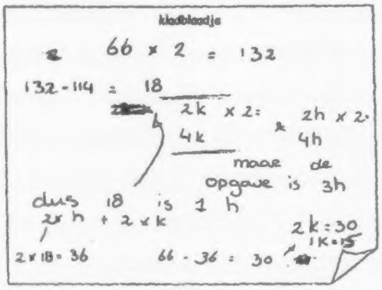
11,50 -

4,25

4,25 : 5 = 0,86

reasoning

Figure 3: informal notations and algebraic



kladblaadje

66 x 2 = 132

132 - 114 = 18

2k x 2 = 4k

2h x 2 = 4h

maar de opgave is 3h

dus 18 is 1 h

2x h + 2 x k

2 x 18 = 36

66 - 36 = 30

2k = 30

1k = 15

Alternatively the solution in figure 3 illustrates how the level of reasoning can be higher than the level of symbolizing. This student solves the system of equations

$$2 \times h + 2 \times k = 66$$

$$3 \times h + 4 \times k = 114$$

by doubling the first equation and then subtracting the second from it. First he deals with the right hand sides of the equations (66 × 2 and 132 – 114). In between the two horizontal lines we observe how he multiplies the terms with the unknowns. Then he writes ‘but the task says 3h so 18 is 1 h’. Finally he substitutes the value 18 to solve for k. A remarkable contrast presents itself. This student successfully applies a formal algebraic strategy of eliminating one unknown by operating on the equations, while his symbolizing is still at a very informal level. Again the unknown is only partially included in the solution process; it appears only where necessary. In other words, both examples of equation solving illustrate that competence of reasoning and symbolizing are separate issues.

Conclusion

Difficulties in the learning of algebra can be partially ascribed to ontological differences between arithmetic and algebra. The project 'Reinvention of algebra' uses informal, pre-algebraic methods of reasoning and symbolizing as a way to facilitate the transition from an arithmetical to an algebraic mode of problem solving. We have shown some examples where informal notations deviate from conventional algebra syntax, such as inconsequent symbolizing and the pseudo-absence of the unknown in solving systems of equations. These side effects bring new considerations for teaching: how can we bridge the gap between students' intuitive, meaningful notations and the more formal level of conventional symbolism. The observation that symbolizing and reasoning competencies are not necessarily developed at the same pace – neither in ancient nor in modern times – also has pedagogical implications. It appears that equation solving does not depend on a structural perception of equations, nor does it rely on correct manipulations of the equation. In retrospect we can say that knowledge of the historical development of algebra has led to a sharper analysis of student work and the discovery of certain parallels between contemporary and historical methods of symbolizing. Streefland's notice to look back at the origins in order to anticipate has turned out to be a valuable piece of his legacy.

Didactising: Continuing the work of Leen Streefland

Erna Yackel, Diana Underwood, Michelle Stephan, & Chris Rasmussen

When we think of the work of Leen Streefland we think of his seminal work in developing prototypical courses and instructional sequences (fractions, negative numbers and algebraic expressions and equations). In developing these courses and sequences Streefland was not only putting into practice the general principles of Realistic Mathematics Education (RME) that had been set forth by Freudenthal and Treffers but he was demonstrating how these principles might be realized in practice over an extended period of instructional time. In doing so, he went beyond earlier work that demonstrated one or more of the principles for individual problems, such as the van Gogh sunflower problem (Treffers, 1993). However, Streefland viewed his work as much more than the development of prototypical courses and sequences. In the abstract of his paper, *The Design of a Mathematics Course, A Theoretical Reflection,* Streefland (1993) pointed to what he saw as the major contribution of this work, namely operationalizing RME instructional design theory and thereby raising it to a higher level. As the title of the paper indicates, Streefland's purpose was to reflect on the development process and identify strategies used in the design of the exemplary materials. To this end, he analyzed his fractions course and other examples of prototypical instructional sequences that were developed following the principles of RME. As Streefland noted,

> In consequence an important theoretical change of perspective looms ahead. Where the theory was first an *after*-image, it can now act as a *pre*-image, i.e., as a *model for* realistic mathematics education in advance (p. 109).

The activity of developing such after-images that then can be used as pre-images in future work is what Streefland called *didactising*.

In one sense, the work of our research group might be thought of as applying the model that Streefland has articulated since we are developing prototypical courses in mathematics for various audiences, including university students. However, the intention of our work extends beyond applying Streefland's model. We prefer to view our work as having the same character as Streefland's in that as we engage in the process of the developmental research that is required to develop prototypical courses, we are continually analyzing aspects of our own activity for potential after-images that might be useful as pre-images in other situations. In this sense, we, too, are engaged in didactising.

Each of the researchers in this session will describe their didactising activities within the context of their respective research. First, Underwood will discuss designing instructional sequences so that students' mathematical understanding grows out of their development of symbolic representations while at the same time contributes to the development of those representations. Next, Stephan argues that argumentation analyses are useful not only for analyzing students' learning as they engage in prototypical courses, but also as a tool for the designer in her attempts to anticipate the quality of the social interaction and discourse associated with the instructional sequences under development. Stephan's work is a form of didactising in the sense that she is using argumenation theory as a tool for describing how the conditions for learning the desired mathematics can be created and sustained in social interaction (Streefland, 1993). Rasmussen uses different modes of "listening" as a conceptual tool for describing aspects of the activity of analysing the vast amounts of data collected from developmental research for the purposes of informing and revising the development of instructional sequences. As an after-image, these different modes of listening have the potential to be useful pre-images for others engaged in RME-based instructional design.

Thus, each of the three researchers demonstrates a form of didactising. In doing so, each goes beyond treating the development of prototypical courses for various mathematical content areas and various audiences as a simple matter of applying Streefland's model. In each case, the researcher gives explicit attention to reflecting on critical aspects and strategies of the design process which includes: developing means of recording and notating that can describe informal activity and that have the potential to lead to formal and/or conventional mathematical notation, anticipating the classroom discourse that can emerge as students solve problems, and selecting or designing "realistic" contexts that have the potential to lead to formal mathematizing.

Emergent Models in a Context of Linear Equations

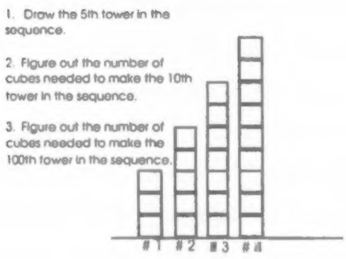
The purpose of this section is to illustrate how mathematics instruction might be designed to facilitate the emergence of conventional symbolism for linear equations from students' ways of representing and notating their reasoning in situations of linear change. This approach is in contrast to much of the recent reform curricula concerning linear functions that focus on facilitating students' ability to move flexibly within and across tabular, symbolic, and graphical representations. While it is important that students are able to interpret linear functions in a variety of ways, a problem with this approach is that the student still needs to integrate them. For example, even when students are able to describe slope graphically as "rise over run" and are provided opportunities to "discover" that the number representing the slope is the coefficient of x in the equation for a line, they usually cannot explain a basis for this relationship.

One explanation for this difficulty is that students are asked to create and use the graphs of functions on a Cartesian plane as a *model for* reasoning about quantities without first facilitating development of the plane as a *model of* anything. The Cartesian plane is a symbol system used in creating a visual (dynamic) representation of the relationship among quantities. According to the principles of RME, this symbolism should emerge from students' mathematical activity (Gravemeijer, 1994) rather than be given to them prepackaged.

The Stacking Cubes instructional sequence attempts to promote students' understanding of a coordinate system while simultaneously facilitating their understanding of linear relationships. Our inspiration for creating this sequence grew out of noting students' solutions to a data recording and graphing activity. In this activity from the Connected Mathematics series (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), students were asked to flip a coin for 90 seconds, record the cumulative number of flips at 10-second intervals, and make a graph of the data. Many students generated bar graphs to represent their data. In an earlier instructional sequence adapted from the Mathematics in Context series (Wijers, Roodhardt, van Reeuwijk, Burrill, Cole, & Pligge, 1997), students had successfully developed formulas for growing spatial patterns. By combining the idea of writing formulas for growing patterns with the idea of bar graphs, our goal was to design an instructional sequence that would help students develop a conceptually-based understanding of linear relationships, thereby enabling the symbolic representation of a linear function to grow naturally out of graphical representations.

The instructional design was based on the tenets of RME emphasizing that through engaging in realistic tasks, students create models of/for their mathematical activity (Gravemeijer, 1994). In the initial activity students were presented with a sequence of "towers," asked to draw the next few towers in the sequence, and figure out the number of cubes that would be in the 100th tower in the sequence (see Figure A). The spatial arrangement of the towers resembled the bar graphs that students had drawn in their prior data recording and graphing activity. As the instructional sequence progressed, students were asked similar questions, but were given less information to answer them. At the same time, the notation used to represent the stacking cubes evolved to look more like ordered pairs. For example, towers were first replaced by "sticks" and the sticks were ultimately replaced by points on the Cartesian grid:

Figure A



1. Draw the 5th tower in the sequence.

2. Figure out the number of cubes needed to make the 10th tower in the sequence.

3. Figure out the number of cubes needed to make the 100th tower in the sequence.

Figure B

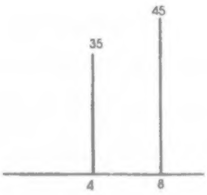
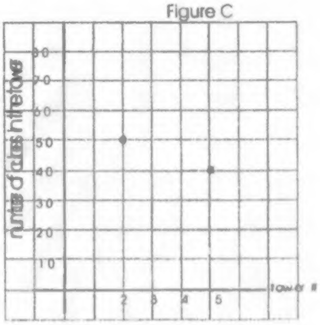


Figure C



The students' descriptions of the spatial patterns also evolved. Initially they gave detailed verbal descriptions such as, *I figured out that the change between the towers was 2 and then I counted back to the zero building, which is one. So I know that the* 100^{th} *building is 2 times 100 plus 1, or 201 cubes high.* Later, this type of description was generalized to *height of the* n^{th} *building = zero building + pattern number * change.* Eventually students developed the symbolic notation $H = a + bP$ to represent the relationship that was illustrated in the graph.

While the bar graphs/towers initially served as models of students' thinking in the graphing activity, the sticks/points became models for their reasoning about the relationship between height and pattern number. This symbolic progression can be thought of as *a chain of signification* (Cobb et. al, 1997; Gravemeijer, 1999):

$$\{\text{pictures of cubes}_{\text{signified 1}}\text{-pictures of sticks}_{\text{signifier 1}}\}_{\text{signified 2}} \;-\; \text{graphs of points}_{\text{signifier 2}}$$

In a similar way, the verbal/symbolic descriptions of the patterns that the students developed were models of their reasoning about the graphical representations of the patterns. Here again, the progression from an extended verbal description to a verbal formula to a symbolic formula can also be viewed as a second chain of signification.

The two chains of signification are intertwined in the sense that the symbolic representation of the relationship grew out of the students' thinking about the different graphical representations (towers of cubes, sticks, points) while at the same time the graphical representations enabled students to develop meaning for the different components of the equation $H = a + bP$. In contrast to the multi-representational approach, the two chains of signification are not two separate ways to represent a linear relationship. Instead, the instructional goal is that students' concept of a linear relationship will grow out of their development of symbolic representations while at the same time contributing to the development of those representations. That is, they evolved together as a dynamic, interactive system.

Under the guidance of the RME emergent models heuristic, we designed the Stacking Cubes sequence to support students' moving beyond using symbolic descriptions as models of patterns in towers to using them as models for reasoning about linear relationships. We believe that the approach that we take can be viewed more broadly as a *pre*-image for designing instruction that provides students with opportunities to create and reason with conventional symbols.

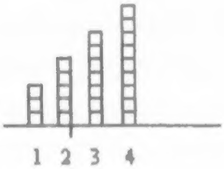
Argumentation as a Tool for Didactising

The purpose of this section is to show how analyzing students' argumentations impacts the design of instructional activities within a broader instructional sequence. Generally, argumentation has been used to analyze students' learning (e.g., Krummheuer, 1995; Yackel, 1997). In addition to this function, we will argue that argumentation analyses can serve to provide feedback to the RME designer by informing her of the nature of the justifications that students provide as they engage in the designed activities. The justifications may not be those that are anticipated by the designer and thus, she can revise the sequence by constructing tasks that better provide students the opportunity to construct mathematical justifications that are more in keeping with the overall mathematical goals for the instructional sequence. To begin this conversation, we first describe Toulmin's (1969) scheme for analyzing argumentation.

For Toulmin, an argument consists of at least four parts: the data, claim, warrant and backing. In any argument, the speaker makes a claim and, usually presents evidence or data to support that claim. Even so, a listener may not understand what the data presented has to do with the conclusion that was drawn and, therefore, challenges the presenter to clarify the role of the data in making a claim, a *warrant*. Perhaps the listener understands why the data supports the conclusion but does not agree with the mathematical content of the warrant used. The authority of the warrant can be challenged and the presenter must provide a *backing* to justify why the warrant, and therefore the entire argument, is valid mathematically.

In general we have found that students' warrants consist of further elaboration of their methods for solving a problem and that backings involve justifying why their method or interpretation should be mathematically acceptable in the classroom. In this section we would like to explore the usefulness of Toulmin's model of argumentation from a design perspective. In other words, what kinds of reasoning might the designer/teacher find useful to capitalize on in whole class discussions and what warrants and backings for a particular type of task are productive for learning? Do the instructional tasks she has designed provide the opportunity for such justifications to arise? Anticipating the nature of the warrants and backings that we think would be useful for supporting the classroom argumentation can aid in the development of mathematically productive instructional tasks. We will illustrate this with an example from the Stacking Cubes sequence. On the first day that students engaged in the Stacking Cubes sequence, the diagram shown in Figure D was drawn.

Figure D



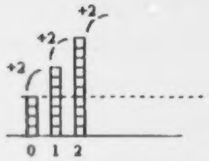
The teacher explained that the picture showed a series of buildings constructed by a company and asked, "How many more little blocks do I need to make the 13th building?" While some students attempted to count how many increases of two there would be to get to the 13th building, one student explained that she found a formula for finding the number of floors in any building.

> *Abby: I tried to figure out how many blocks would zero have. And it would be 1. Each building as you go down is decreasing by 2, so I just subtract the 2 [to get the 1]...2P + 1.*
>
> *Teacher: How did you get the 2P + 1 part of it though? That's the part I don't understand.*
>
> *Abby: They are each increasing by 2. So I just figured we're going to multiply by 2. And I know there is just 1, because the zero. That one has 1. So, 1 is already there. So you would be adding 1.*
>
> *Teacher: She's going to be adding 1. Do you know what she means when she says that she is going to be adding 1? Do you know? Do you want to say something about that, Terry?*
>
> *Terry: She is adding the 1 because it is...[inaudible]...you still have to add the 1 that was like the odd man out...1 times 2 is 2, but you have to add that 1 man in building 2 because of the one block in the zero building. 2 times 2 is four, plus 1 which is 5 in building 2.*

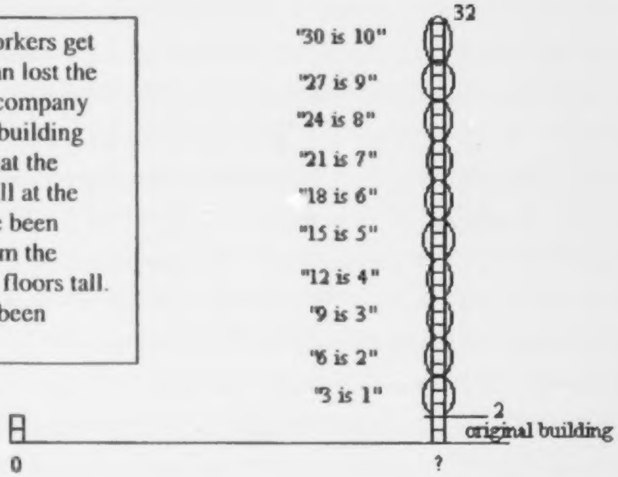
Analyzing the structure of the argumentation above, we see that Abby provided a claim that consisted of her formula, 2P + 1. She provided data for her claim when she explained, "I tried to figure out how many blocks would zero have." Once Abby explained her claim, the teacher challenged Abby to explain how finding the plus one led her to the formula 2P + 1. In other words, she was asking for Abby to make the warrant, how the data "+1 leads to the formula 2P + 1," more explicit. Abby responded by explaining the origin and necessity of each term in the formula. In Toulmin's terms, Terry provided a backing regarding why multiplying by 2 and adding the "odd man out" each time led to the desired results in each pattern ("2 times 2 is four, plus 1 which is 5 in building 2").

This type of argumentation for generating a formula was typical early in the instructional activity. The backing provided justification for the origin of the formula, but it was grounded in the specific example of the first and second buildings. As students' understanding becomes more general, we would expect that the backings to become more general in nature, i.e., the buildings are increasing by 2 floors per pattern number (*a rate*) from the original pattern (the "zero building"). However, this type of backing never arose in the course of instruction. As a consequence, we created two new tasks designed to provide students the opportunity to construct justifications for the formula that were based more explicitly on rates of change.

Task 1: Show students a picture of a building pattern that increases by 2 floors per day and starts at 4 floors. Only show the first three days. On which day will you have 14 more floors than the original? The teacher can draw a horizontal line on top of the 0 building and also notate each jump of two as pictured:



Task 2: Some construction workers get paid by the day but the foreman lost the record of how many days the company has worked since they started building onto the original. He knows that the current building is 32 floors tall at the end of the day and that they've been putting 3 floors on per day from the original building that was two floors tall. How many days has the crew been working?



In new Task 1, students may either draw subsequent buildings, counting the extra floors each time until they have 14 more floors than the original, or they might double count, keeping track of the number of twos and the number of days increased until they have counted 14. Double counting in this manner might allow students to draw attention to the 2 floors per day justification. The dashed line might support students constructing the backing described above: a rate of change of 2 floors per day added onto the original 4 floors. The dashed line might also support students seeing the original building as nested in each subsequent building.

In Task 2, some students might actually draw every building between the original and the last day while others may simply double count again to find how many counts of size three can be found between the last day and the original. The teacher might even symbolize those double counts by circling 10 groups of three blocks on the last day only (see figure accompanying Task 1). She might also ask students to explain what each number means as they double count. In this case, the intent again is to bring attention to the constant rate of change as it goes on from the original number of floors.

The point of this short excerpt is to bring attention to the role of argumentation in the designer's activities. By examining students' argumentations as they engage in tasks, we can discover the nature of the students' current justifications *and the range of potentially productive justifications*. This type of examination can lead to revisions in the instructional activities within a sequence.

Listening as a Conceptual Tool

The purpose of this section is to describe how the construct of "listening" to data can be used as a conceptual tool for the purposes of instructional design. As Streefland's work demonstrated, paying close attention to students' reasoning has always been a fundamental characteristic of RME based instructional design work. Using our developmental research efforts in differential equations as an example, we use different modes of "listening" as a lens to reflect on our retrospective analysis of the data collected during a semester-long classroom teaching experiment. In particular, a mode of listening that we call *generative* listening helps shape and clarify our thinking about realistic starting points for instruction that are mathematical in nature.

Within the theory of RME, realistic starting points for instruction refer to situational contexts that can serve as a building block for students' mathematical development. For example, Streefland (1990) discussed how distribution situations, like sharing 3 candy bars among 4 friends, can serve as a realistic starting point for students' learning the concept of fraction. Although this example of a realistic starting point is characterized by a real-world situation, the term "realistic" is intended to be broad enough to include mathematical situations that are experientially real for students. Relatively few examples, however, of realistic starting points that are themselves mathematical currently exist.

As researchers begin to explore ways in which the theory of RME can inform instructional design at the university level, it will be useful to have images of strategies that others find useful in locating realistic starting points that are mathematical. To begin to address this need, we use the notion of generative listening as a means to bring to light aspects of our developmental research activity that has yielded mathematical starting points that are experientially real for students.

Generative listening is intended to reflect the negotiated and participatory nature of interacting with data. This type of listening, which Davis (1997) called hermeneutic, is "an imaginative participation in the formation and transformation of experience" (p. 369). The notion of generative listening can be clarified by comparing it with what Davis calls interpretive listening and evaluative listening. In comparison with generative listening, where the purpose is to learn something new about one's own thinking, interpretive listening is to decipher the sense that students appear to be making of the mathematics under discussion. Davis posits that within interpretive listening, mathematics is still about constructing conventional associations between signifiers. Finally, evaluative listening is characterized by the fact that the listener is listening for something in particular. The motivation for evaluative listening is to evaluate the correctness of the contribution by judging it against a preconceived standard (Davis, 1997).

During the spring of 2000, we conducted a 15-week classroom teaching experiment in differential equations. At the commencement of the teaching experiment, we conjectured that population situations would serve as an experientially-real starting point for the development of students' concept of the solution space for differential equations, where solutions to these differential equations are functions of time. Note that the nature of this starting point has the same real-world character as Streefland's distribution situations for fractions. After engaging in extensive retrospective listening to the data collected during this teaching experiment, what students had to say transformed our thinking about what we could take as an experientially-real starting point.

To illustrate the notion of listening generatively to data, we use an excerpt from an end-of-the-semester interview with Marta, one of the students in the class. In the excerpt that follows, we asked Marta if she now thinks about the concept of function differently than she did before taking the differential equations course. We asked her this question because we take the viewpoint that solutions to differential equations are functions and thus the study of differential equations may provide opportunities for students' to deepen their notions of function. At the time of the interview, we were curious about students' evolving notions of function through their study of differential equations. That is, our listening was more interpretive and evaluative. Only later, upon retrospective analysis, did we listen generatively to this piece of data.

> *Marta: I can think of it more as, when you say this function, I can think of it more as instead of three x squared, I can think of it more as a motion, more as some kind of change. More as something that's actually going on opposed to, yeah, these are some numbers and this is what it looks like on a piece of paper...when I say 3 x squared, what I'm really talking about is, I'm talking about this marble moving from here to here and how it got there, you know?*

Although we think it is useful for developmental researchers to listen evaluatively and interpretively to data, we restrict the discussion to generative listening because when we listened generatively to this piece of data, we began to think differently about the possibilities for starting points in differential equations. In particular, our own thinking was transformed by engaging imaginatively in Marta's description of motion and of a "marble moving from here to here." We began to think about how the movement from "here to here" stems from conceptualizing rate of change and how solution functions can, for students, grow out of their mental and bodily experiences with rate of change. That is, the mathematical construct of rate of change, when coupled with population situations, can serve as an experientially-real starting point.

To take rate of change as an integral component of an experientially-real starting point is not to say that all students have a full conception of rate of change that is in line with expert notions. It is to say, however, that students' at this level have some way to conceptualize rate of change as a mathematical construct so that they can proceed with a problem situation. For example, students might conceptualize rate of change as an intensive quantity by_which a different quantity changes over time or they might view rate of change as a ratio of two co-varying quantities that gives rise to motion or movement and involves directionality. Although beyond the scope of this paper, we should note that the transformation in our thinking about taking rate of change as an experientially-real starting point has also led to revisions in the sequence of instructional activities.

The intention of this short example was to describe how we can use listening as a conceptual tool for reflecting on our developmental research efforts at the university level. Although we used the construct of listening to crystallize our efforts at locating experientially-real starting points that are mathematical in nature, the different modes of listening, generative, interpretive, and evaluative, may serve as a broader image for others engaged in developmental research.

Conclusion

As these three examples show, didactising can take a variety of forms depending on the mathematical content, on the student audience for the prototypical courses or instructional sequences, and on the interests of the researcher. At the same time, the researchers' interests evolve as the developmental research progresses. In this way there is an evolution of the nature of the after-images that researchers develop that then become pre-images for future work. Thus, in a sense Leen Streefland's work has set in motion a cyclic process that has the potential to move the field of mathematics education forward in substantive ways.

Reaction Koeno Gravemeijer

The work on RME at the Freudenthal Institute constitutes the heart of what people at the institute call, "educational development". The term educational development has been introduced to indicate an all-embracing process of educational innovation, encompassing both the actual enactment of the innovation in the classroom and an open dialogue between researchers and practitioners. Developmental research is conceived of as a catalyst of innovation. The results of developmental research are meant to function as a source of inspiration for practitioners. What we aim for is re-enactment of instructional sequences in various situations, adapted to those situations, and shaped according to the insights and preferences of the practitioners involved. This then is seen as an extension of the original developmental research, which produces feedback that will contribute to an enrichment of the original findings. Leen Streefland would be pleased to see how efforts to explicate the RME design theory with help of exemplary materials have led to a similar process in the mathematics education research community. To see that other researchers are inspired, experiment with, and expand RME theory. in this respect, the Purdue-Calumet research group has much to offer. Moreover, they address issues that were near to Leen Streefland's heart: - the perspective of a reflexive relation between the development of symbols/models and meaning, which fits so well with the quotation of Leen Streefland on page 1; - the focus on the role of argumentation within whole-class discussion, which is in line with Leen Streefland's research activities on the basis of the idea of a community of researchers; - the proposal to integrate generative listening as a conceptual tool in developmental research, which he would have welcomed as a valuable contribution to his effort to bring the RME design theory to a higher level. In conclusion, we may truly say: Leen Streefland's work continues.

Epilogue Marja van den Heuvel-Panhuizen

As Freudenthal stressed once, it was Leen Streefland who opened our eyes to the anticipatory learning of concepts that will develop in full at a later time. From the very beginning of his work in the field of mathematics education Leen was focused on where and how education can anticipate the learning process that is coming into view in the distance. This anticipatory perspective was not only true for Leen's ideas about how to teach mathematics to students, but is as true for Leen's role within our research group at Utrecht University and the international community of researchers of mathematics education. The concepts, the language, the way of thinking with which Leen provided us, turned out to be strong and continuing guides for deepening our understanding of the learning and teaching of mathematics. The contributions to this PME Research Forum prove this.

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[1] This Research Forum was initiated by Marja van den Heuvel-Panhuizen, who worked together with Leen Streefland on many projects.

DISCUSSION GROUPS

Imagery and Affect in Mathematical Learning

Coordinators:
Lyn English, Queensland University of Technology (Australia)
Gerald A. Goldin, Rutgers University (USA).

Representation in learning mathematics includes not only external structured physical configurations, but also internal systems that encode, interpret, and operate on mathematical image and symbol configurations (Goldin & Janvier, 1998). Continuing the discussion begun at PME-24 in Hiroshima, we focus on imagery, affect, and their interplay with natural language, formal notations, heuristics, beliefs, and especially with each other. Traditional views of mathematics as an abstract, formal discipline have tended to relegate visualization, metaphor and metonymy, emotions, and the relation between feeling and mathematical imagination to incidental status. Yet we have a case for the centrality of imagistic reasoning, analogies, metaphors, and images in mathematical learning (English, 1997; Presmeg, 1998). Lakoff and Nunez (2000) even aim to recast the foundations of mathematics in terms of metaphorical image schemas. The essential role of affect, encoding information and influencing learning and performance, has also been stressed (McLeod, 1992, Goldin, 2000). It may even be the most fundamental system in powerful mathematical learning and problem solving (DeBellis & Goldin, 1997; Gomez-Chacon, 2000).

Our general purpose is to explore the nature and role of affective and imagistic representational systems in mathematical learning and problem solving. The first session will begin with brief presentations by the coordinators, focusing on: analogies, metaphors, images, affect, meta-affect, and beliefs in mathematical reasoning and problem solving, including highlights of last year's discussion. Participants are encouraged to cite examples of imagery, affect, and their interplay in children and adults doing mathematics, for group discussion and interpretation. We shall discuss some of the difficult issues in the empirical investigation of these topics through classroom observation and structured clinical interviews; identify key research issues, and shape plans for future development of these ideas.

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DISCUSSION GROUP:
STOCHASTICAL THINKING, LEARNING AND TEACHING

Coordinators: Kath Truran, University of South Australia
Brian Greer, University of San Diego, USA
John Truran, Private Practice, Adelaide, Australia

The group's discussions will be on *the relationship between stochastical and mathematical thinking, learning, and teaching*. We consider that there is more on this theme which can usefully be discussed. It is our intention to approach this theme from multiple perspectives, including:

- Philosophical, in terms of the perceived boundaries of the disciplines.
- Historical, in terms of the developments of the disciplines.
- Educational, in terms of the positioning and implementation of the teaching and learning of stochastics within school and tertiary curricula, including such fundamental issues as teacher development, assessment, and technology.
- Psychological, in terms of the specific cognitive and sociocultural processes involved in the teaching and learning of stochastics.
- Research, in terms of cross-fertilisation of theoretical frameworks and methodologies.

The following members will present a short contribution which will be followed by discussion, questions and feedback from the other participants.

- Angustias Vallecillos Jiménez (Spain) who will speak about her work in statistical inference learning.
- Rene Ritson (Northern Ireland) who will speak about her current research study into possible relationships between young children's understanding of probability and their understanding of fractions.
- John Truran (Australia) whose subject is "The Place of Probability in Stochastical Thinking Learning and Teaching"

DISCUSSION GROUP
SEMIOTICS IN MATHEMATICS EDUCATION

Coordinators: Adalira Sáenz-Ludlow, University of North Carolina, USA.
sac@email.uncc.edu
Norma C. Presmeg, Illinois State University, USA,
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Mathematics Education is a unique field in which philosophy, psychology, anthropology, sociology, and numerous other fields find a place alongside education and mathematics. Each of these fields provides a different perspective on teaching and learning of the subject matter. A recent addition to this interdisciplinary list is semiotics, which is the science of signs, signification, and sense-making. Mathematics and the teaching and learning of mathematics make use of different sign systems to encode and decode concepts. It is through a constant interpretation of conventional and non-conventional signs that one comes to construct personal mathematical meanings. Researchers in mathematics education, all over the world, are trying to incorporate semiotic perspectives to shed light on classroom teaching-learning processes and learners' interpretations of mathematical concepts.

The purpose of the group is threefold: (a) to foster interaction among researchers using or wanting to use semiotics in the analysis of classroom data; (b) to share and discuss different semiotic perspectives; and (c) to share papers and analysis of data. The following contributions will be presented in the meetings to stimulate discussion and interaction.

First session

I) *Mathematical epistemology from a semiotic point of view*, Michael Otte, Germany. (20 minutes)

II) *Euclid's signs: A cultural-semiotic analysis of the theory of even and odd numbers*, Luis Radford, Canada. (20 minutes)

III) *Cognitive analysis of comprehension problems in the learning of mathematics*, Raymond Duval, France. (20 minutes)

IV) *General audience participation and discussion.* (30 minutes)

Second session

I) Summary of the first session. (20 minutes)

II) *Progressive mathematizing using semiotic chaining*, Norma Presmeg, USA. (20 minutes)

III) *Classroom discourse as an evolving interpreting game*, Adalira Sáenz-Ludlow, USA. (20 minutes)

IV) *General audience participation and discussion.* (30 minutes)

Presentations will be both theoretical and empirical in nature, and will include examples to elucidate semiotic perspectives. Participants are also invited to bring classroom episodes and share their work with the group. Each session will end with a discussion about the advantages of semiotics as a theoretical framework, its potential for analysis of data, outlets for dissemination, and partnerships for research.

Culture and Mathematical Cognition

Coordinators: Yeping Li, University of New Hampshire, USA
Tad Watanabe, Towson University, USA

Although there is no universal agreement as to whether mathematics is a culturally bound subject, few would disagree that the teaching and learning of mathematics is a cultural activity. The view of mathematics teaching and learning as a socio-cultural activity is reflected in and supported by various studies (e.g., Boaler, 2000; Stigler & Hiebert, 1999). Although researchers with different research perspectives tend to agree that both socio-cultural factors and individual cognition are two inseparable parts in the development of students' mathematical cognition, the nature of culture and mathematics cognition and the relationships between them are far more complicated than we may think of. As one further step towards a better understanding of culture and mathematical cognition, this discussion group is proposed to use cross-system studies as examples to examine relevant issues.

The discussion group will be organized as a two-section activity. During the first section, two researchers will present brief (10 minutes each) overviews and/or examples of relevant research on two issues: (1) the ways of assessing students' mathematics thinking, and (2) the nature and quality of mathematics curriculum and teaching in different settings. After the presentations, the participants will be organized to join small group discussions that will constitute the second section. Based on cross-system studies that have been discussed in the first section, the discussion in small groups will center on the following four questions:

1. Can mathematical cognition be examined cross-culturally?
2. Are cultural variations in context, tools, and practices (e.g., curriculum, teaching practices) related to the development of students' mathematical cognition?
3. Can a cultural practice be adapted in a different setting?
4. What similarities and differences may exist between cross-system studies and multi-cultural studies within an education system in understanding the issues relevant to culture and mathematical cognition?

After small group discussions, all participants will come together to generate a collective summary and synthesis of the small group discussions. A list of potential research questions will be generated/selected and interested participants will be organized to develop further research activities on this topic after the meeting.

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RESEARCHING THE SOCIAL IN MATHEMATICS EDUCATION: ISSUES OF INCLUSION

Co-ordinators: Hilary Povey, Tansy Hardy & Mark Boylan, Sheffield Hallam University, England

Mathematics education which seeks to promote inclusion has a number of different but interrelated facets. Firstly, there is the development of a critical content in the curriculum: this seeks to use mathematics as a means to help learners understand the world they live in. Secondly, drawing on some aspects of ethno-mathematics and situated cognition, the informal mathematics practices of various groups in society can be examined. This reminds us that the reality that the accepted curriculum content reflects is that of the dominant cultures and groups in society and is not the only model of mathematics available. More importantly, the gap between these informal mathematics practices and school mathematics helps to reveal the mechanisms by which the mathematics curriculum acts to disadvantage certain groups. Thirdly, building on concepts of communities of practice, we see the development of critical pedagogies that seek to democratise the mathematics classroom.

Against this background, we wish question what it means to conduct research into inclusion in mathematics education. Our inquiry will include the following.

The *choice* of what to research. Choices about what to research are inevitably ethical and political.

> Our decisions about what to research … are, at root, value-based decisions which we expect to have to defend. (Wiliam 2000, p124f)

The *purpose* of research. Whether or not research is critical or promotes inclusion is bound up with its potential to be used in transformative ways. This might be the purpose for which it is intended: could it also be the purpose to which it is put?

The *methodology* of the research. Are there essentially critical research methodologies or can research based on traditional methodologies promote inclusion? Are there types of research questions which can only be addressed through critical research methodologies and that cannot be answered by traditional methodologies?

The *practice* of the research. What can the relationship between/amongst research participants be like? How do different research relationships allow us to uncover different truths?

The discussion group will take as an initial focus a paper dealing with some of these issues.

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SEARCHING LINKS BETWEEN CONCEPTUAL AND PROCEDURAL MATHEMATICAL KNOWLEDGE

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The aim
is to discuss and share perspectives on the relation between conceptual knowledge and procedural knowledge. Research in this important domain seems to be neglected or at least eclectic, e.g. restricting mainly on terminological level. To what extent it can be revitalised and enhanced to develop a relevant overarching research paradigm - or even help to produce theories for concrete teaching?

Sessions
The first of the two 90-minute sessions is aimed to give an overview of researcher's views. The distinction between procedural knowledge and conceptual knowledge seems to be possible at a theoretical, epistemological and terminological level. However, many kinds of problems begin when this distinction is to be operationalized by acceptable tasks, and the relation between the two knowledge types is to be clarified.

The second session tries to share perspectives for resolving some of these problems by use of different mathematical content, age cohort to be taught, from historical perspective or other concrete contributions. The focus of the discussions is to search for links between the two knowledge types, and to check, whether research in this domain might have some relevant implications for teaching.

Preparation
for the discussion will be made via Internet. Potential participants are invited to send possible contribution for discussions via e-mail to the co-ordinators. This discussion within the next 2-3 months should offer a thorough warm-up for the group activities during the conference, as collecting theoretical studies on the topic. An example is the recent Haapasalo/Kadijevich paper in JMD 21 (2), 139-157.

Outcomes
Evaluation of the relevance of the topic will be made for the PME activities to come. Furthermore, a summary of the discussions will be made.

MODELING AND TECHNOLOGY

Helen M. Doerr Syracuse University
Dave Pratt University of Warwick
Michal Yerushalmy University of Haifa

The words "mathematical modeling" cover a broad range of theoretical and practical orientations to the teaching and learning of mathematics. Equally broad are the approaches researchers have taken to understand the role of technology in the teaching and learning of mathematics. In this session, we propose to bring these two major strands of work together by examining some of the issues and some of the approaches that have emerged over recent years. The main purpose of this group will be to discuss the nature of the learning activity when students engage in modeling with various technological tools. The discussion group will begin by discussing some epistemological and psychological perspectives on modeling. Then we will shift the discussion to examine the issues that are raised by some examples of research on modeling and technology.

Epistemological and Psychological Perspectives: There are two epistemological underpinnings to mathematical modeling: first, the model is separate from the world to be modeled and, second, modeling is a cyclic, iterative process. The essence of this epistemological stance is that the world of phenomena and the model world co-construct each other. A psychological perspective on modeling and technology more directly addresses issues related to the role of the learner and learning. A distinction we have found useful is that between the learner's activity in using existing model created by an expert and in building a model that reflects the learner's own emerging understandings of the phenomena. In the first case, the learner's task is to explore the consequences of actions taken within the boundaries of a content domain model. In the second case, the learner's task is to express their own concepts by making explicit the relationships among objects and variables and then to examine, interpret and validate the consequences of their idea.

Exploratory modeling through microworlds: Environments such as Cabri Geometry, probability microworlds and spreadsheets have all been used to provide environments for students to explore models that have been designed by experts. After presenting a brief example of some work by students in a probabilistic microworld, we will engage in discussion on questions such as: How do such tools mediate between reality and mathematics? How is it that learners come to understand the explored world?

Expressive modeling through function toolkits: Tools such as graphing calculators, spreadsheets, and qualitative function graphers have all been used to provide students with the opportunity to express their understandings of some experienced phenomena. Again, we will present a brief example from the Visual Mathematics project showing some work done by students as they strive to make sense of experience. We will discuss such questions as: How do the available tools support or constrain the expression of students' ideas? What is the nature of effective modeling tasks with particular tools? What is the role of the teacher in such expressive environments?

FACILITATING THE CONCEPTUAL CHANGE IN MATHEMATICS

Erkki Pehkonen, University of Turku, Finland
Kaarina Merenluoto, University of Turku, Finland

The crucial idea in the conceptual change is the radical reconstruction of prior knowledge which is not very well observed in traditional teaching. The theories of conceptual change (see Vosniadou 1999; Duit 1999; Reiner, Slotta, Chi & Resnick, 2000) define two levels of difficulty in the learning process. The easier level of conceptual change means *enrichment* of one's prior knowledge structure. In this case the prior knowledge is sufficient for accepting the new specific information. The student needs only to add the new information to the existing knowledge. The more difficult conceptual change is needed when the prior knowledge is incompatible with the new information but needs a reconstruction. There seems to be at least two basic kinds of directions for problems in conceptual change. The one is the knowledge and operations which are relevant on a certain domain, but need to be revised on the other. This is the case with the mistaken transfer from natural numbers to the rational numbers. For example the students seem to face considerable difficulties in trying or struggling to sort fractions (Hartnett & Gelman 1998) because of their spontaneous use of the logic of natural numbers. The other kind of problems are the beliefs and conceptions caused by experiences with mathematics (see Verschaffel, DeCorte & Lasure 1999). The need to make drastic changes to the prior thinking may not even occur to the students unless the needed change is made very explicit in the teaching.

The purpose of the discussion group is to explore the nature of conceptual change in mathematics concept formation, to discuss about the role of prior thinking and to find the principles that govern the facilitation of the conceptual change.

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Discussion Group on

Work-related Mathematics Education

(Coordinators: R. Straesser, Univ. of Bielefeld & J. Williams, Univ. of Manchester)

The Discussion Group will continue the discussion begun at PME 23 in 1999 (see the PME News of November 1999, pp.5-7[1]) by briefly recalling this activity and then looking into aspects of "Work-related Mathematics Education" not treated at PME 23. An additional reference will be the papers of a one day conference in Manchester in February 2001 on "Maths into Work" (for information on this conference's papers see http://www.man.ac.uk/CME/conferences/index.htm).

At PME 25, we start from a dictionary definition of work as "exertion directed to produce or accomplish something, productive or operative activity, employment, a job, esp. that by which one earns a living", and look into the role of mathematics in this type of human activity and the teaching/learning of work related Mathematics - not forgetting activities like "community services". As a salient feature of maths related to work we take the fact that the goal of the activity is ***not*** to produce or learn mathematics, but to use it for different purposes outside maths.

Keeping in mind that we discussed "current use of maths at the workplace" and "current ways to teach work-related maths" at PME 23, work at PME 25 will focus on three additional issues:

- the role of old and new technologies ("artefacts")
- theoretical perspectives on discourse and activity in work, college and research
- research methodologies.

The importance of artefacts is based on the use of technology (old as in Geometry, charts as in administration, technical languages ... and/or new as computers, software and programming languages) in workplace activities and teaching/learning for it. Discussions in the group should aim at better understanding the contradictory trends of hiding or revealing workplace maths by means of artefacts and discourses, and identify problems and potentials for teaching and learning work related mathematics.

Current research into the use of mathematics at work (with the spectrum from traditional statistics to ethnomethodology) seems to favour case studies in a participatory style, while different ways of "stimulated recall" are also in use. After discussing research perspectives and methodologies, the Discussion Group should decide on the feasibility and necessity of continuing the Discussion Group in future PME conferences and between conferences.

PME-delegates who want to actively participate in the Discussion Group are asked to contact the coordinators via email as soon as possible at
rudolf.straesser@uni-bielefeld.de or julian.williams@man.ac.uk

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[1] Reference: Hoyles, C., Mathematics in the Workplace Practice Discussion Group. PME News November 1999, pp. 5-7.

Discussion Group: Psychology of Computer Science Education

Coordinators: Dan Aharoni, Technion, Israel Institute of Technology, Haifa, Israel
Chronis Kynigos, Univ. of Athens and Computer Technology Institute, Patra, Greece
Tami Lapidot, Technion, Israel Institute of Technology, Haifa, Israel

Compared to Mathematics Education research, Computer Science Education (CSE) research is still in its infancy. Especially, the domain of Psychology of Computer Science Education (PCSE) lacks enough research; hence no specific community has been formed yet. But many domains of CS are mathematical in nature (e.g. the computational ones), and thinking in these domains is mathematically oriented. In fact, these domains of CS originated in mathematics. Thus it has been a few years now that PME is also the community of PCSE researchers. It has already been shown that PME theoretical and research background is adequate for PCSE (e.g. Aharoni & Leron, 1997). PCSE researches use PME theories like the Actions-Process-Object model (Sfard, 1991), theories on the role of affect (McLeod, 1989) etc., to name just a few. Papers on PCSE were presented in several conferences including PME (e.g. Aharoni, in press; Kynigos, 1995; Levy & Lapidot, 2000).

From past PME conferences we know that there were other PME members interested in the mathematical aspects of the PCSE domain. We would like this discussion group to be an opportunity for these people to share their experience in PCSE research, and discuss issues like:

- What is mathematical about learning computer science?
- Which PME theories were found adequate for PCSE?
- Which theories or parts of theories are specific to PCSE, beyond the scope of PME?
- What are the questions and issues PCSE should address?
- What are the CS domains or sub-domains that are adequate for PCSE research?
- What PCSE studies were done up to now?
- What PCSE studies should be done in the future?

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The Importance of Matching Research Questions and Methodology to the Reality of Researchers' Lives

Coordinators: K. Hart and P. Sullivan

One focus of the Discussion Group is the training of students studying for a doctorate in Mathematics Education, their expectations and their needs. The doctorate is the highest academic qualification we offer. But the nature of the preparation doctoral students receive raises several questions: Is it an adequate preparation for a lifetime of research, supervision of students and leading a University department? Should we consider what use the student is likely to make of the research degree? What courses or readings do we think essential for any person studying for this higher degree? What level of mathematics should we require before a student is accepted, or should we provide tuition in mathematics as part of the degree course?

Another issue has to do with the nature of the thesis. Typically, the student submits a thesis, which is examined by external moderators. Does the supervisor have a responsibility to advise his or her students on a choice of subject that reflects the work they will probably do when they have graduated? If the future occupation is likely to be dealing with teachers should the student work on topics that are likely to appeal to teachers? On what criteria should the submitted thesis be judged?

These questions form the basis of our debate.

Symbolic Cognition in Advanced Mathematics

Stephen Hegedus, University of Massachusetts, USA (Convenor)
David Tall, University of Warwick, UK.
Ted Eisenberg, Ben-Gurion University, Israel

The main aim in establishing a group for the discussion of *Symbolic Cognition in Advanced Mathematics* would be to enable a forum for international debate and exchange in a field which has received much attention recently at PME in a variety of formats (e.g. Tall, 1999)

We would aim to facilitate discussion of new ideas emerging in the field and to distinguish between research in undergraduate mathematics education and inquiry into more general psychological aspects of mathematical thinking.

The role of symbol and cognitive processes relating to the representation and manipulation of mathematical signs and symbols is a popular topic (e.g. Dehaene, 1997; Deacon, 1997). The discussion group would aim to discuss such works and related aspects of pedagogy as well as present technological appreciation of symbolic cognition. In discussing the role of symbol in advanced mathematical work we would aim to discuss the process of symbolic manipulation from a psychological perspective where one would investigate in more depth the transitional processes from elementary mathematical thinking to advanced mathematical thinking.

Agenda: Research in following areas for possible discussion

1. The role of symbol in mathematical thought and meaning making;
2. Syntactic progression from the evolution of signs into symbols;
3. The historic consequence of sociological devices which enable constructive meaning for symbolic development;
4. Emerging theories of symbolic cognition in the fields of mathematical psychology and neuroscience;
5. Discussion of symbolic processing with reference to specific mathematical topics, i.e. in addition to limits, functions, calculus, analysis, linear algebra develop topics in abstract algebra, topology, probability/statistics, etc.

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WORKING SESSIONS

EXPANDING RESEARCHERS' ABILITY TO STUDY STUDENT EXPERIENCES

Markku S. Hannula, University of Helsinki, Finland
Chris Breen, University of Cape Town, South Africa

ACTIVITIES

In this working session we will be involved in role play where participants will be asked to engage in learning situations where emotions are engaged. From these activities, implications for mathematics education researchers wanting to explore the emotions of learners will be discussed.

AIMS

Increasing interest has been devoted to students' experiences in the mathematics classroom, including the emotions they have during mathematics-related activities. In qualitative research, the researcher is the main instrument of data analysis. While trying to understand the experiences of students, the researcher is always reflecting upon their own experiences to interpret and understand the student's behaviour. While the researcher's own experiences provide the basis for interpreting any data, they also are a source of bias. The researcher can best interpret experiences similar to his/her own and easily ignores or misinterprets other kinds of experiences. The aim of this working session is to expand the participants' ability to identify and interpret different experiences that student's might have.

FOCUSSING ON MENTAL ARITHMETIC

Coordinators: Kees Buys & Dagmar Neuman

Description of the topic

Today, there are some interesting tendencies to be observed in primary school mathematics education in many countries. First of all there is a growing tendency to postpone or even abolish the teaching of standard written algorithms. Instead, there is a growing focus on the establishment of early number sense and on the development of mental strategies (mental arithmetic) as central topics for the mathematics curriculum. This is not only taking place within the field of numbers up to 100, but also within the field of numbers up to 1000 and, with 'nice numbers', above 1000. Secondly, there is growing attention for estimation and the development of estimating strategies as a vital part of the curriculum. And thirdly, the use of pocket calculators, especially with larger numbers and decimal numbers, is becoming more and more common in the higher grades of primary school. Altogether this means that developing number sense and mental strategies are becoming more and more the heart of the curriculum.

Activities

In this working session we focus on mental arithmetic and work on important characteristics of the learning and teaching of mental arithmetic as an essential part of the new mathematics curriculum in primary school. During the two 90 minutes slots of the session the participants will do several activities, including:

-- Analyzing and discussing typical mental arithmetic problems as they occur in recent textbook series from various countries

-- Analyzing and comparing students' work and assessment results in the field of mental arithmetic in various countries

-- Analyzing and discussing videotapes and interview protocols of the learning and teaching of mental arithmetic

-- Mapping the changing place of mental arithmetic in the primary school mathematics curriculum in different countries

-- Analyzing and constructing games that can stimulate the development and flexible use of mental strategies

People from several countries will inform the participants of the working session about the 'state of the art' in their country and will provide the participants with recent materials from their country. The aim is to analyse and compare developments in different countries, to reflect upon these developments and to envisage possibilities to stimulate and coordinate these developments.

Embodiment, Gesture, and Mathematics Education

Coordinators:

Janete Bolite Frant, CEDERJ, Rio de Janeiro
Jan Draisma, Catholic University of Mozambique, Nampula
Laurie Edwards, St. Mary's College of California

The study of mathematical thinking and learning has, for the most part, focused on how learners and teachers reason and communicate about mathematics using language and symbols. Data may include written inscriptions, diagrams, and/or verbalizations between teacher and learner, or among learners. However, the utilization of gesture for both the construction and communication of mathematical understanding has received much less attention in the research community. The purpose of this working group is to investigate the nature of the use of gesture in mathematical learning and knowledge. This focus will be situated within the context of recent work in cognitive science, including the theory of embodied mathematics (Lakoff & Nunez, 2000), research on gesture and thought (McNeill, 1992) and perspectives on number from cognitive neuroscience (Dehaene, 1997). In addition, specific research on gesture and number (Draisma, 2000; Gerdes & Cherinda, 1993; Fuson & Kwon, 1992) will be summarized to provide a background for the work that will take place within the two sessions.

In accordance with the purposes of a Working Session, the sessions will provide opportunities to participate in developing a research perspective and engaging in analysis of data within this perspective. The first portion of the initial session will be devoted to preparing a theoretical and practical grounding for the work. We will then present two video case studies, one focused on young children and gesture arithmetic (Draisma), and the other on motion graphics (Frant). A preliminary analysis of each case will be presented by the researchers, followed by an opportunity for participants to further analyze each case in small groups, as well as to engage, for example, in learning gesture arithmetic themselves. The groups will then present their analyses within the whole forum for further discussion. The session will conclude by considering possible ways to move forward in this area of investigation.

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OBSERVING SYSTEMS

David A Reid, Acadia University, Canada

Laurinda Brown, Alf Coles, University of Bristol, Graduate School of Education, UK

AIMS

As researchers we observe the dynamics of learners and groups of learners in communication with one another. Learners in communication can be said to form a social system. We will explore the differences and similarities in the three perspectives on the observation of social systems of Bateson, Maturana and Varela and Luhmann. In a reading which participants will get before meeting the author states:

> Gregory Bateson defines information, or the basic unit of information in communicational and mental processes, as "a difference which makes a difference" (e.g. Bateson, 1972:453). And in Niklas Luhmann's terminology, observation means nothing more than handling distinctions - making a difference in this context involves an observing system (Alrøe, 2000).

Maturana and Varela deal with the issue of observers making distinctions within systems of which they are a part. All three perspectives bring a systems view to the making of distinctions and cognition. The aim of this working session is for us to understand more fully the differences in these approaches, and their use in observing cognition in individuals and in social systems.

ACTIVITIES

The work of the session will revolve around analysis of a video-tape segment and associated transcript. Relevant selections from the literature will be distributed ahead of time to prospective participants, to establish a basis for our work. Multiple viewings of the video, small and whole group discussion of it from perspectives guided by the three approaches, and sharing interpretations of the data seen from these perspectives will be the main activities during the two 90 minute programme slots. Participants will be invited to continue our collaboration via email after PME25.

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WORKING WITH TRANSCRIPTS

Co-ordinator: Janet Ainley

As researchers in mathematics education, many of us make extensive use of transcripts, made from audio or video recordings of interviews, discussions and classroom interactions. We analyse these in many ways, to look at both the content and the social organisation of dialogue. We use them for many purposes, and yet we generally use a very small range of ways of presenting transcripts to serve all of these needs.

This Working session will be based around the exploration of some different styles of presenting transcripts, and how these might support different kinds of analysis and interpretation. I hope that we shall get into discussion of some important research issues, such as the notion of the 'truth' of a transcript, or whether more (detail) is necessarily better. However, the primary purpose of the session is to do some work together, and this will be organised as follows.

In the first session, I shall introduce some models for presenting transcripts, and invite any participants who are familiar with other models to add to these. Then we shall work in small groups, each taking a section of dialogue from an existing transcript, or better still from an audio tape, and transcribing this in different ways.

At some point in the second session, we shall come back together and invite groups to present what they have done, and their views about what may gained or lost in each of the styles. I am confident that this will lead us into valuable discussion of the sorts of general issues indicated earlier.

If you wish to participate in this working session you are encouraged to bring along your own data to work on. There will also be some examples of different kinds of dialogue provided. If you have a small audio recorder which could you used for the working session, please bring this as well.

HOW ICT AFFORDS THE REINVENTION OF MATHEMATICS: TWO EXAMPLES

Pijls, M. H. J.

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In reform mathematics education in general, and in the case of realistic mathematics education (RME) especially, the constructing activity of students plays an important role. In contrast with the traditional view in which teaching mathematics was seen as a process of teaching partial skills according to a pre-designed hierarchy, mathematics teaching in the reform view is seen as guiding students to let them reinvent mathematical knowledge.

The Dutch research program 'Mathematics and ICT' is concerned with the question how ICT can be utilized to facilitate the learning of mathematics according to the principles of 'guided reinvention'. This is elaborated in five projects with an emphasis on developmental research, a cyclical process of thought experiments, developing educational materials and classroom experiments.

In this working session we want to elucidate how ICT is used in experimental educational materials to both support a reinvention process by the students, and to ensure that formal mathematics is firmly rooted in the students' understanding of common sense knowledge. We like to discuss this usage of ICT and we want to discuss how we try to understand how ICT can be utilized for mathematics education.

The materials of two projects concerning upper secondary education, 16-year-old students will be presented. In one project a computer program is used as a tool in the process of modeling movement for the learning of calculus. It is expected that the affordances of the software elicit students to invent the concept of derivative. The other project concerns the learning of probability theory. Here the role of the software is to create an environment for explorations on the visual level that links up with students' experiences.

Participants of the working session get the opportunity to work with the learning materials of both research projects. Subsequently, student's answers will be analyzed on indications for reinvention. Finally, we will debate on the aims of the software, the experiments, the experiences and what we learnt from it.

Working Session

What Counts as Multiplicative Thinking in Younger Students?

Coordinators: Anne Reynolds, University of Oklahoma, USA
Tad Watanabe, Towson University, USA
Gard Brekke, Telemarksforsking-Notodden, Norway

When the Project Group, Understanding of Multiplicative Concepts met in Hiroshima, Japan last year the following questions emerged as in need of further study:

- If repeated addition is not multiplicative thinking, when does a child's understanding of multiplication operation become multiplicative?
- Where is the line between additive and multiplicative reasoning?

This working session will provide an opportunity for mathematics education researchers who are interested in these questions to develop criteria for identifying a child's actions as multiplicative rather than additive.

The group will examine data from student interviews focused on multiplicative reasoning and discuss how the data does or does not support an interpretation of multiplicative thinking. The coordinators will provide transcripts, interview protocols, and samples of written responses from their research with younger students for the group to consider. Participants are also encouraged to share episodes from their research into multiplicative thinking with the group to further extend our thinking on the topic.

Based on our analysis it is proposed to redesign the tasks to better probe young students multiplicative thinking. Participants who are interested will be invited to use these tasks with students in their various locations over the coming year and collaborate on further analysis of young students' multiplicative thinking from different countries and cultures to highlight similarities and differences.

The Ferris Wheel problem

Ricardo Nemirovsky and Jesse Solomon

TERC and City on a Hill

This working session will focus on a problem that is part of the Interactive Mathematics Program for the senior year in high school. Imagine a Ferris Wheel and a train travelling underneath. A diver is in one of the ferris wheel's cart waiting for the right time to jump in order to land safely on a cart full of water that is part of the train. The problem is when he should jump. The unit was conducted by Jesse Solomon in his class at City on a Hill public charter school in Boston, Massachusetts. In order to investigate new ways to work with the Ferris Wheel problem, we constructed a physical device that includes all the components (a ferris wheel, an electric train, a diver jumping at a pre-set time, etc.) although in a reduced scale to allow for its classroom use. The speed of the ferris wheel and of the train can be changed by turning two knobs on the controllers. The time at which the diver is going to jump is set by positioning a magnet on the train track, so that the jump will take place whenever the train goes over the magnet. Students worked on this unit for six weeks.

The working session will be divided in two parts. During the first half the participants will work on different mathematical aspects of the Ferris Wheel problem and on ways to connect it to the behavior of the Ferris Wheel device. During the second half the participants will analyze selected episodes filmed in Mr. Solomon's class. The coordinators will set up the ferris wheel device and bring videotapes and transcriptions for the selected episodes.

The research question for the session will be about the roles of physical experiences in mathematics learning. This question will be explored through the participants' engagement with the ferris wheel problem as well as through the analysis of classroom interactions.

SHORT ORAL COMMUNICATIONS

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A TEACHER'S TESTS CAN CHANGE

Jeanne Albert
The Weizmann Institute for Science, Rehovot
The David Yellin College of Education, Jerusalem

In this paper I want show how one teacher's tests changed over a period of four years. This case study is part of my doctorate, in which I wanted to show that if elementary school mathematics teachers were provided with good assessment tasks and were taught how to analyze their students' work on such tasks, then their mathematics teaching would improve. (Albert, 1999) I collected data on their teaching by questionnaires, interviews, and samples of assignments and tests.

This teacher was involved in an in-service project which put emphasis on changing the teachers' vision of mathematics. Mathematics assessment was not a part of this project. I had decided that School-based Assessment would be introduced first to the principals, and then to all mathematics teachers - not just the leading ones participating in the weekly workshops. In December, 1996 a workshop was held for the elementary school principals where they solved a mathematics investigation task, discussed its characteristics, and were introduced to other tasks and samples of student work. They were shown how these tasks enable teachers to assess students' mathematical achievements in a broader way than is possible with single-answer tests. During the spring term, I ran three workshops for **all** fourth through sixth grade mathematics teachers.

On the first questionnaire of this workshop, this teacher who I will call Sarah gave an example of an assignment which required students to explore fractions and use manipulatives to do this. Compared to other teachers not participating in the project, this task showed Sarah was teaching more in the direction of reform mathematics. Yet the test I collected from her that spring was extremely conventional, with twenty-six short exercises and two standard word problems. Thus there was no correlation between the way she was teaching and the way she was testing.

Four years later, the test Sarah brought me was radically different. It included different types of mathematical processes, not just calculations. It not just permitted the students to use manipulatives, but encouraged their use, and required generalizations based on this use. There were also questions requiring verbalizations and justifications. The way Sarah talked was different too. Her explanations of the test questions showed understandings she had not previously applied in her assessment tasks.

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In O. Zaslavsky (Ed.). *Proceedings of the 23rd PME Conference*, Vol. 1, p. 337, Haifa, Israel.

TYPES OF REPRESENTATION USED IN THE PRESENTATION OF TRIGONOMETRY IN A TEXTBOOK

Kristin Amundsen, Telemarksforskning- Notodden, Norway

This study reports on how representations are used in the presentation of trigonometry in a textbook written for upper secondary school. It is an analysis of how a tripartite use of representation is used in the presentation of the topic.

Representation is a complex concept used in different contexts. In proportion to learning and teaching there is a main partition between external and internal representations (Goldin & Kaput, 1996). In my work the focus was on external representations. One motivation for applying a tripartite view on mathematical topics came from the introduction of "The Rule of Three" in an American calculus reform in the 1990's. The main idea here was to emphasise a balance between numeric, algebraic and graphic representations in the teaching and learning of calculus (Hughes Hallet, 1991). Lesh (1999) uses a model for transfer between five types of representation in the learning situation, which is an extension of Bruner's enactive, iconic and symbolic representation forms.

The main aim of my study was to explore how concrete, visual and symbolic representations were used in the presentation of trigonometry. Part of the work was to adjust and define the difference between concrete, visual and symbolic representation, and especially what concrete representation is in the context of a textbook. My research questions were: What kinds of representations are used; in which order do they occur; and what connections are found between and within the different representations.

Visual and symbolic representation dominated. When concrete representation was used, it was so strongly connected to visual representation that it was difficult to separate them. Even if the three representations were used, pairs of representationforms appeared simultaneously. Generally there was a bipartite approach and not a tripartite approach to the topic that dominated. The order of the representation was mainly from visual to symbolic except from the graphic representation of the functions.

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THE COMPUTER AS MEDIATOR IN THE DEVELOPMENT OF MATHEMATICAL CONCEPTS

Charlotte Krog Andersen
IMFUFA, Roskilde University, Denmark

To motivate my ideas, I present an episode from observations of mathematics university students using a computer software package consisting of pre-programmed Maple procedures. These procedures may be used to create dynamical and interactive illustrations of central spatial concepts of differential geometry. In the episode, the students explore geodesics on an ellipsoid. They manipulate the parameters and create an ellipsoid with a curve running from the north "pole" to the south crossing itself on the equator and making a small circle at each pole. Subsequently, the students come up with a preliminary description of the curve as *Ribbon on an Easter egg* in order to formulate and anchor their experiences with this ellipsoid.

With the aim of investigating the role of such preliminary descriptions for mathematical concepts formation theoretically, I draw on Vygotsky's work on the development of scientific and spontaneous (everyday) concepts. A similar distinction proves valuable also in relation to university mathematics. Scientific concepts (here mathematical) are according to Vygotsky initially introduced through their verbal definitions but based on a relatively mature development of spontaneous concepts formed in an out-of-school context. This emphasises the importance of everyday concepts as they mediate meaning to the mathematical ones. However, in an educational context students form semi-spontaneous concepts originating in intuitive reactions and empirical experiences as evidenced by the episode. These are valued by their illustrative and sense making potential in relation to the mathematical concepts, and I shall call them preliminary intuitive mathematical concepts.

In conclusion, I suggest that the use of computer software may play a mediating role in the acquisition of mathematical concepts. This is so at least if the software is rich in interactive, dynamic visualisations that enable the students to form preliminary intuitive mathematical concepts.

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STUDENTS' UNDERSTANDING OF GRAPHICAL ASPECTS OF DERIVATIVE AND OF SOME OF ITS UNDERLYING CONCEPTS

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The purpose of the study, on which this paper is based, was to examine university students' understanding of fundamental calculus concepts, after the concepts concerned had been dealt with in first-year calculus courses. 630 students were subjected to the diagnostic tests (a pre-test and a post-test) pertaining to this study. Fifteen of these students also participated in individual interviews. The analysis of students' written and verbal responses to test items revealed a great deal of detail about the nature and characteristics of students' understanding of key calculus concepts and their symbolic representations (Bezuidenhout, 1998; Bezuidenhout and Olivier, 2000; Bezuidenhout, in press). This paper focuses on students' understanding of graphical aspects of the derivative and of some of its underlying concepts, including the derivative as a rate of change.

The following three kinds of function representations were most commonly used in the diagnostic tests: tabular and graphical representations and the function defined in algebraic symbols. The main reason for the utilization of different representations of functions, was to examine students' understanding of concepts within different modes of representation. Concerning this paper the graph of a non-linear function provides the graphical context for examining students' "graphical" understanding of aspects relating to derivative or rate of change. These aspects include the following: increase/decrease of a function at a point; rate of increase/decrease; increase/decrease at the greatest rate; the average rate of change over a small interval of the independent variable as an approximate value of the derivative at a point; slope of a secant line; slope of a tangent line; increase/decrease of the rate of change.

The interview format served to reveal the nature of some misconceptions of students. One such misconception suggests that "the average rate of change over an interval corresponds to the average of the rates of change at the two endpoints of the interval". Knowledge associated with the arithmetic mean was identified as a key source of this misconception.

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A CONSTRUCTIVIST APPROACH OF ERROR IN ADDITIVE ARITHMETIC PROBLEM SOLVING

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Theoretical Framework

The status of error in the teaching and learning process has been analysed from several points of view depending on the epistemological bent from which different authors stand. Within a constructivist perspective, the error produced by a subject when performing a task is not an ill to eliminate and forget, but a behaviour worthy of analysis and interpretation. (Inhelder *et al.*, 1974, Vinh-Bang, 1990, Astolfi, 1997).

This research has the goal of analysing errors commited by pupils, from kindergarden, first and second grades, on solving different kinds of addition arithmetic problems (e. g. composition and transformation) (see G. Vergnaud, 1981).

We intend to answer to three issues: firstly to determine if the error produced by the child results from a lack of structural mechanisms to solve the question or if it is only a functional difficulty. The second issue is to determine if the pattern of errors is connected with the child's psychogenetic development. The third issue is to determine the relationship between the additive problem proposed and the type of errors commited.

Methodology

In this investigation 30 pupils where questioned in four sessions and qualitative methodology formed the basis of data collection. After a operatory diagnosis being done with three Piaget's tasks (number's conservation, seriation of lenght and class inclusion), nine additive verbal problems were be presented at which the child shall answer using manipulative materials, followed by iconographic representation and finally symbolic representation (calculus) to determine the kind of errors produced.

Results and Conclusion

The first results analysed until now shows:

1. In general, pupils commited more errors in additive problems involving transformation.
2. These errors seem to be at the root of a lack of structural mechanisms of knowledge; in fact pupils with low results in Piagetian tasks, independently of their educational level, are those who commit more errors.
3. Errors comming out from functional difficulties, like computational errors, are not connected with psychogenetic development, but with the educational level of the pupil.

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AFFECTS AND BELIEFS IN SCHOOL MATHEMATICS: GENDER DIFFERENCES

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The TIMSS study found significant gender differences in achievement only in a few of the participating countries (grades 7 and 8). In general attitudes towards mathematics were positive for most countries, however gender differences in favour of boys were found in 31 out of the 40 participating countries (Beaton et al 1996). Norway was one of countries with the largest gender difference (Lie et al 1997). The largest gender difference in achievement in the mathematics test in TIMSS' population 3 (grade 13) was found in the Norwegian population.

Leder and Forgasz (2000) have presented a rational and methods which they used to develop a scale for investigating to what extent mathematics continues to be considered as a gender domain. The study presented here concerns issues from a large-scale project could offer another to this discussion.

Since 1995 the project has collected national data on students' understanding of key concepts in the national mathematics curriculum. A questionnaire that contained a wide range of issues related to the teaching and learning of mathematics (125 items) was administered to 1482 students in grad 6 (11.5 y) and 1183 students in grade 9 (14.5). Forty-two of the items were related to students' beliefs about mathematics, mathematics teaching and self, as well as attitudes towards the subject. A factor analysis of the 42 items formed five groups, *Interest*, *Usefulness*, *Self-confidence*, *Diligence* and *Security*.

In another eight items the students were asked to consider assertions such as: "*To enjoy asking questions in the mathematics lessons is most typical for: boys, girls, equal.* The responses to these items gave important information on how such gender stereotypes develop in different age groups.

A subgroup of the students, respectively 273 and 243, responded to a test which investigated students' conceptions of measurements and units. This made it possible to study relationships between scales above and gender, and in addition to investigate relations between students' performances to these scales.

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THE SOCIAL CONSTITUTION OF SCHOOL MATHEMATICS CURRICULUMS

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In whose interest are we working when we construct mathematical curriculums for schools? How does mathematics as a discipline lend itself to being categorised within curriculum documentation? Our attempts to describe the world always result in formulations that at best approximate the world we seek to describe. But we are always governed by specific purposes in trying to create such descriptions. Governments, mathematicians, teachers, economists would all have different motivations as to how mathematics should be described. The attempt to describe mathematics in a curriculum inevitably results in a caricature of traditional understandings of mathematics as a discipline. Should this sort of formulation be seen as a simplification, a misunderstanding or as an ideological distortion? The formulation can be viewed variously, for example, as a serious but imperfect attempt to describe mathematics to guide school instruction or as a cynical ploy to make teachers and children more accountable according to a particular institutionalised account of mathematics or a reconfiguration of the discipline itself to meet contemporary needs. Ricoeur (1981) would downplay intent in the construction of the formulation and see it more as a matter of subsequent interpretation, and action on the basis of this, as to how the formulation is viewed.

This presentation will review new work (Brown, 2001) discussing the social construction of mathematics for school instruction and how progress in learning mathematics might be viewed as a function of particular learning theories or evaluation strategies, and the particular characteristics they value. It will also consider strategies for enabling teachers to address such concerns within professional development programmes (Brown and Jones, 2001).

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HOW TO APPROACH NEGATIVE NUMBERS IN SCHOOL

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At the Pedagogical University in Beira, teachers of mathematics for secondary schools are being trained since 1996 under BLEM Programme (*Bacharelato e Licenciatura em Ensino de Matemática*). In the programme, I am lecturer of Didactics of Mathematics and my main aims are: a) to discuss secondary school mathematics topics with prospective teachers and find best ways of teaching them; b) to provide prospective teachers with basic knowledge for scientific research in mathematics education.

The first topic that we studied in 1998 was *Negative Numbers* because it is the first topic of the syllabus for secondary schools (grades 8 – 10), and because "Negative numbers have intrigued and confused some of the greatest mathematicians who have ever lived" (Davis and Maher, 1993).

Firstly, the prospective teachers answered a questionnaire with simple addition and subtraction tasks in which they were asked to explain how to calculate and to explain how they would explain these tasks to pupils. They were asked to indicate, which everyday life situations they could use to explain the tasks. After that, the prospective teachers conducted interviews in secondary schools to verify how pupils work with negative numbers.

In the study we wanted to find out which specific problems the pupils in school have with negative numbers and identify their origins, in order to find an approach which may help pupils and teachers.

With this topic we found that prospective teachers had many difficulties in formulating questions which could help the pupils to reflect on their own words, ideas and solutions. There were some very positive examples, where the students succeeded in conducting an interesting dialogue with a pupil and making a good report of the discussions. During the activities we required that the students learned to analyze the ways of thinking of pupils. They became conscious that in teaching mathematics, teachers should teach in the direction of what pupils think instead of what the teachers think.

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THE ROLE OF FIGURATIVE CONTEXT IN REALISTIC TASKS

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The real context a realistic task is embedded in is called *figurative* context (Clarke & Helme 1996).

Previous studies covered the role of figurative context in certain aspects: familiarity (Papastravidis et al. 1999), gender (Kaiser-Messmer 1993), involvement (Stillman 1998), individuality (Clarke & Helme 1996).

There is no comprehensive theory covering the role of figurative context. Especially questions concerning the subjective reception of figurative contexts have not been answered and not been investigated empirically yet.

Quantitative approaches to theses questions would not lead to a sufficient understanding of the students' individual processes. On the contrary, these questions require a qualitative approach.

The aspect of the subjective reception of figurative context was, among others, investigated in a qualitative-orientated case-study. Students were asked to solve different realistic tasks. Their solution-process was recorded on video. Afterwards the students watched the recordings and expressed their thoughts concerning the figurative context they had had during the solution-process (method of stimulated recall). In following interviews the students were asked additional questions.

First analyses allow, among others, the following conjectures:

- Figurative contexts given in the tasks can internally be represented in very different ways.
- Depending on the individual and the situation the role of figurative context can vary considerably
- In different phases of the solution-process the figurative context can have various functions

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RESEARCH IN MATHEMATICS EDUCATION AT THE NEXUS OF MATH, MIND, BRAIN, AND WORLD

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All mathematicians concern themselves with mathematics, and most scientists and engineers concern themselves with mathematical applications. Some psychologists concern themselves with mathematical cognition, and a few neuroscientists have recently begun to concern themselves with the physiology supporting mathematical thinking. This broad spectrum of study comes to a focus in mathematics education. It would seem appropriate therefore, that mathematics educators and researchers in mathematics education concern themselves with how these studies interrelate.

Understanding the relation between mathematics and the material world has long been a classical problem in the history of the philosophy of mathematics. More recently, the potential roles of psychology and physiology have been added into this equation. Can recent psychological developments in cognitive science, or recent physiological findings in the neurosciences shed light on this problem, or do they complicate the matter further? This talk is concerned with identifying and explicating some central issues that are emerging in understanding research in mathematics education at the nexus of math, mind, brain, and world.

What, for instance, are the implications for teaching and learning mathematics of limiting or defining mathematics and/or mathematical cognition in neural terms? Can, or should we try to extend our understanding of mathematical cognition downwards into the very fabric of the world out of which the human organism has emerged? In terms of mind-brain identity and interaction theories, how and at what level(s) do categorical structures of cognition relate to physiological structures?

At a more functional level, what mathematical operations are primitive and which are derived or adapted? What operations are innate, if any, and which result from the conscription of preexisting neural processes that may have evolved for different purposes? In terms of mathematical applications, what aspects of mathematical understanding, if any, are unconsciously embodied in the evolutionary past of human development (e.g., numerosity)? What aspects are normative? That is to say, what aspects may have been consciously developed solely on the basis of more recent human motivations and purposes (e.g., cryptology)?

The emerging need to address such issues requires critical evaluation the ability of our existing conceptual frameworks to do so. Some basic philosophical assumptions may need to be reevaluated, and some consideration of new ones may be warranted. Attempting to develop new roadmaps for navigating through these kinds of issues poses many new interesting questions and challenges for teaching and learning mathematics, and promising new directions for research in mathematics education.

FROM GEOMETRICAL FIGURES TO LINGUISTIC RIGOR: VAN HIELE'S MODEL AND THE GROWING OF TEACHERS AWARENESS.

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The work takes off from a curriculum design elaborated within a school-university working group for students aged 14-16. The observational study focused on how teachers manage the project in order to lead students to the full understanding of the significance of the *definition* meant as the expression of the minimal of a figure, i.e. as the nucleus of the idea of *necessary and sufficient condition*. We used van Hiele's Theory in order to encourage expert teachers to reflect on a curriculum previously implemented and on their own practise. We registered with evidence that teachers' awareness of the van Hiele's level has a positive influence on students' learning.

The itinerary has been developed through individual worksheets alternated with slots of collective discussion and of reorganization guided by the teachers. We considered that in a teaching prospective the scope for a correct definition is to develop awareness of *what a definition is* better than to explain what *an object is*; i.e. we undertook the difference between descriptive definition and prescriptive definition.

Exemples from the project materials, teachers' observations on students's written work, collective discussion and behaviour will be illustrated in the oral communication.

Observations. 1. a good homogeneity was reached in all the eight classes; 2. teachers recognized in the learning process many actions, i.e., in van Hiele' conception, the *essence of geometry*; 3. teachers declared to feel they have learned to adapt themselves better to the students; 4. we observed the two different registers into which action research methods leads to be active: the register of practitioners' reflections on their own action and the register into which researchers in Mathematics Education are active.

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IN-SERVICE CALCULUS COURSE: WHAT DO STUDENTS AND TEACHERS WISH?[1]

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Mathematics has penetrated into several areas of knowledge, seeking to comply the demands of professional formation and technical assistance. However, students and teachers of these technical areas complain that mathematics courses do not generally offer them what they need. According to Howson et al (1988) the problems arising from the so-called "courses in service" have been poorly understood and insufficiently analyzed by mathematics educators and researchers.

During 1999, we used a questionnaire and semi-structured interviews in order to determine the profile of the students' expectations in a freshmen calculus course for a Geology Program. Our leading question was: *what do they wish?* It turned out that their basic complain, as well as of other teachers in the Program, was the lack of applied problems whose importance for the subsequent professional courses could be recognized as relevant. We designed and carried out a freshmen calculus course for geologists attempting to fulfil this demand. Here we shall report on the outcomes of such attempt.

As soon as the applied exercises were introduced into the classroom, the students started complaining about their "complexity", about their unfamiliarity with the concepts involved, about the length of the problem statements. Some asked us to go back to the routine exercises in the textbook. A polemical situation was installed in the classroom. From this point on, our research focused on what was happening. We expected to appease the students' complaints, but our action only made them worse. Based on the psychoanalytical theoretical framework of Slavoj Zizek [3] and Tânia Cabral [1] we identified the students' contradictory behavior as apparently hysterical: *this is what we ask you but it is not that we want you to give us.* The theoretical framework led us to consider two concepts, the symbolic and the imaginary identifications that frame one's desire. We inquired the social models that the institution (university) offers to teachers and students of in-service calculus courses: to which gaze do the actors play their roles? From what point of view do they look at themselves?

In trying to answer these questions we finally discovered that our own speech had also been contradictory. With the applied exercises we expected that the students would recognize the importance of the operations of differentiation and integration that we were trying to teach them, but we failed to stimulate the discussion about the meaning of these problems for their future courses and profession.

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[1] This study is part of M.Sc. dissertation, initially supervised by Altair Polettini and, after her tragic death, by Miriam Penteado, Roberto Baldino and Tânia Cabral.

INCLUSIVE SCHOOLING: FROM POSSIBLE TO UNDELAYABLE

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In the past decade, the broadening of compulsory schooling in Portugal to 9 years and the inherent principles of inclusive schooling, defending a school for all – therefore, Mathematics for All – forced the educational community to look at a set of problems for which it (still) had no answers but which at the same time demanded the urgent discovery of forms of action. But the change processes are slow and not without contradictions and they only work if the different actors of the educational scenario are involved in this change process, feeling they can actively contribute towards what is happening.

Defending the principles of inclusive schooling (Ainscow, 1994, 1997; Porter, 1997; Ware, 1997) means believing in the educationability of all and in the educators' capacity to promote the full development of their pupils. It necessarily means believing that every child learns – albeit with different rhythms – that each one can develop his/her socialising skills, be able to build a life project and that the teacher must discover the best way to get him/her to achieve all this.

The case study we undertook refers to a class (8^{th} grade) that integrates several pupils either with special educational needs or from socio-cultural minorities and whose teacher is taking part in an action-research project that has existed for seven years and studies and promotes peer work as a way of enhancing pupils' self-esteem, promoting their socio-cognitive development and their mathematical performances. We shall highlight the cases of two pupils since we feel them to be paradigmatic and clearly show how the classroom practices, working instructions and the didactic contract that is established may contribute to go from the ideals of inclusive schooling to classroom practices that really take them into account, thus contributing towards Mathematics for All and preparation for full citizenship as a reality in our schools.

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Students' Concept Images for Period Doublings as Embodied Objects in Chaos Theory

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The aim of this research is to study how visualisation using dynamic computer software helps students construct conceptual knowledge in a context where symbolic proofs are difficult or unknown and numerical computation offers insightful data. We focus on the study of period doubling in chaos theory, situating our analysis in process-concept theory (Tall et al, 2000), relating this to the embodied theory of Lakoff and Nunez (2000).

The results presented here will reveal in what ways able mathematics undergraduates interpret visual pictures of the eventual behaviour of the iteration of $x=\lambda x(1-x)$ for various values of λ starting at $x \in (0, 1)$. For $1<\lambda\leq 3$ the iterations home in on a limit which bifurcates to an iteration of period 2 after $\lambda_0=3$, then to periods 4, 8, … at a sequence of increasing values λ_1, λ_2, … . which seem to approach a limit λ_∞ (the Feigenbaum limit). After this point the behaviour is chaotic and graphing the set of limit points against the corresponding value of λ gives a picture termed the Feigenbaum tree.

The investigation is performed with a second experiment that involves period doubling of a closed loops on an oscilloscope. (See Chae & Tall, 2001 for details.)

In this paper we focus on students' responses to the following post-test questions:

> What first comes to your mind when you think about 'period doubling'?
> Please draw/make an example of a period doubling in your mind's eye.

The pictures were diverse, including sketches of both of $x=f(x)$ iteration and loop doubling, which we analysed in terms of a new perspective of 'base object'–'process'–'concept'. We found most of the 20 students focused on the pictures as *base objects*; 7 drew a single picture representing the base object *after* bifurcation, 12 drew 'before and after' representing the *process* of bifurcation and only one drew a Feigenbaum diagram which we interpret as moving towards an overall view of the *concept* of bifurcation. Further analysis reveals that of the 8 students drawing loops, 7 concentrate on the move from a single loop to a double loop, but of the 10 students using $x=f(x)$ iteration only one draws the shift from a point to a period 2-orbit and only one draws a period 2-orbit. We will discuss how this intimates that students focus more on *visual* concepts as embodied objects than on symbolic/numeric representations at this stage of their investigation.
For the full paper, click the world wide web at http://warwick.ac.uk/staff/David.Tall.

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THE TAIWANESE TEACHERS' BELIEFS AND VALUES IN MATHEMATICS EDUCATION

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This article reports the pedagogical values of two Taiwanese teachers, Ms. White and Ms. Wang. They have been teaching mathematics in a junior high school for 16 years. They went back to the university and learned constructivism and constructivist teaching six years ago. However, as time went on, their teaching began to return to traditional teaching. Why did this happen? We suppose that they value something in traditional teaching rather than in constructivist teaching. This study was designed to reveal the teachers' beliefs and values in their teaching. We took the teachers interviews, observed their teaching in class, made discussions with them after class, and encouraged them to change their teaching. The findings show that White and Wang value differently above the surface, but they tend to have a few core or deep beliefs and values in common such as: score-ism in education, specialism in mathematics education, and absolutism in mathematics. Taiwanese teachers' score-ism was reported (Chang, 2000). This article focuses on the latter, Taiwanese teachers' specialism and absolutism.

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School Based Inservice Improvement as an Effective Instrument to Change Mode of Mathematics Teaching

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The national mathematical curriculum of Taiwan stressed that the learner should actively construct and understand mathematical concepts from his own experience; that the learner should understand, evaluate and respect his classmates' ways of problem solving and his viewpoints and the viewpoints of others. In addition, the teaching should inspire each student to communicate, discuss and debate. Since the curriculum changed so greatly, the change of the teachers needs support. At present, the inservice improvement programs are mainly lectures and rely on experts. This is not effective. The school is responsible for the realization of the curriculum, so the school has to push and carry out the reform.

I raises my six years action research in National Taipei Teachers College (NTTC) laboratory school as a focal case of 〝school-based〞 inservice improvement program (Chung, 2000). What was done in NTTC lab school in the reform of mathematics teaching, though it is a unique case (Hsiau, 1998), it realize the resocialization preached. From the beginning, I made the pedagogic study meeting on mathematics into pedagogic discourse session (PDS) which was the nucleus of the model. A stable model appeared with four aspects: workshops, newly tranferred teachers, inter-grade growth group, daily interactions. This model carried several characteristics. The key word are culture 、education 、humanization 、duality 、momentum 、spotlight.

Yet, from the success of the NTTC laboratory school with the new mathematical curriculum, I believe that setting up many school-based inservice improvement programs could be an effective measure to implement the new curriculum, but the following key points should be observed: 1.The functioning of the old pedagogic study meeting should contain reflective professional discourse to induce the teachers to participate actively. 2.The principal should be the action leader of the curriculum reform and teaching mode change. 3.The frequency and quality of the professional discourse should improve considerably as days going by. 4.The improvement activities should gain the consent of the teachers and be effective. 5.The administration staff should make wise planning so that the environment is supportive and safe. Competent expert or subject leader should be present when needed.

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CHILDREN'S THINKING ABOUT AREA AND PERIMETER IN THE CONTEXT OF A GARDEN THEME[1]

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As part of a research project aimed at establishing and analyzing links between "home mathematics" and "school mathematics," we developed a teaching innovation centered on a garden theme in a fourth-fifth grade classroom (9-10 year-olds). This theme was chosen based on the students' and their families' knowledge and experiences with gardening. Our goal is to engage children in sociocultural activities that are personally meaningful to them and "recognized as 'real' by the mathematical community" (van Oers, 1996, p. 106). The gardening context allowed us to explore students' informal understandings of area and perimeter as they faced authentic problems. For example, the need to cover their enclosed gardens with plastic led to finding the areas of these irregularly shaped gardens. The need to make their enclosed gardens bigger, yet using the same amount of chicken wire, led to an optimization problem. The discussion of these real life situations was followed by in-class tasks to further probe the children's understanding of area and perimeter. As Hoyles (1991) writes, "it is pedagogic intervention which imposes the mathematical structuring and provokes the pupils' awareness of the underlying mathematical ideas" (p. 149).

This presentation will focus on four children's thinking about area and perimeter as they worked on a series of tasks which included finding the area of a miniature garden (tools available were transparent grid paper, rulers, tiles and cubes) and exploring what shape would give the maximum area (for a fixed perimeter). Each child was individually interviewed and each interview was videotaped and audiotaped. Analyses of the interviews shed light on at least four areas: 1) the interplay between everyday knowledge and school mathematics; 2) the influence of tools (e.g., rulers vs. tiles) on the children's approaches (Nunes, 1996); 3) the effect of prior kinesthetic experiences on shaping a child's thinking about perimeter (the children had explored perimeter by making shapes with their bodies as units of length); 4) the use of "academic" approaches to the tasks (in terms of methods and language).

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PREFERENCE OF DIRECTIONS IN 3-D SPACE

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In the course of research dealing with misconceptions in 3-D geometry basic concepts, I could clearly see that students tend to prefer particular and typical directions, and disregard other directions. These preferred directions are not necessarily the obvious ones – the horizontal and the vertical. They may be other directions, which are chosen relatively to the directions given in the problem.

For example:

- Given a line ***a*** situated in a plane ***P***, the students are asked to place another line ***b***, perpendicular to ***a***. Most of them choose ***b*** to be perpendicular both to ***a*** and to the plane ***P***.

For some of the students the preferred directions are their "first choice", but they have no difficulties to see other directions as well. However, there are students who focus only on those preferred directions and are not able to see other options. For those students, awareness of their choices and of the possible reasons for these choices can improve considerably their visual ability and flexibility in 3-D space.

This study is an attempt to locate and to analyze those preferred directions among prospective teachers in a college of education. It is a part of broader research, which I presented last year in Japan: "Misconceptions in 3-D Geometry Basic Concepts".

Over a period of 8 years, I watched most of the 272 students who participated in the research. During their discussions (some of which were video or audio taped), I noticed very clearly that they almost always choose typical and predictable directions when they illustrate interrelations in 3-D space. Then I interviewed some of them in order to try to analyze more carefully my assumptions about the preferred directions.

In general, we can see three types of preferences of directions:

1. Preference of gravitational directions: horizontal or vertical.
2. Preference of "convenient" directions, in which there are no conflicts between different concepts related to the same terms: "perpendicular", "parallel" or "angle". (For instance: in the example above, we can notice a choice of direction that avoids the conflict of being perpendicular to the line but not to the plane.)
3. Preference of directions which are perpendicular to one another (like the axes x, y, z), or directions which have a "balance" (internal or gravitational).

In the oral presentation some of the findings will be presented and analyzed, with examples of episodes that took place during the interviews or the discussions.

Three Ways of Using the Arithmetic Mean

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A problematic issue in research on understanding the arithmetic mean has been to develop a satisfactory conceptual and instructional definition. My research in the area of statistics education has brought me to distinguish between three different uses of the arithmetic mean in statistics, each of which requires different sophisticated understandings of statistical analysis. Hence, I believe each of these uses should be introduced at a different time in the curriculum.

1) Measure of the aggregate. The mean is used to measure the aggregate of a group when the total becomes inadequate because of differences in group size. The salient property of the mean in this usage is that of addressing the multiplicative relation between total accumulation in a group, and the number of units that generated that accumulation. This way of using the mean is common in science and in social sciences to construct normalized units of measure (e.g. kilometers per liter). My research shows that 12 and 13-year-old students with limited exposure to statistics, can make sense of this way of using the mean (cf. Cortina et al., 2001).

2) Measure of the center. The mean is used to characterize a distribution. Its salient property becomes the indication of the point of symmetry (or balance) of the deviations of values in a data set. The mean in this case is used in combination with other measures to characterize a distribution. To use the mean in this way, students need to at least understand "center" as a stable characteristic of a data set. Recent research suggests that even after 6 months of instruction, many13 and 14-year-old students do not easily develop this understanding of center (cf. Cobb et al, in press). Hence, relatively long exposure to statistical instruction might be necessary in order for students to develop an understanding of the mean as a center.

3) Inferential tool. In this case the mean is used to infer, from a sample, characteristics of the aggregate and/or distribution of a population (i.e. central tendency). This use is based on the Law of Large Numbers, and requires relatively sophisticated understandings of sampling and probability. Hence, this use should be addressed in a rather advanced stage of statistical instruction.

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Attainment and Potential: Procedures, Cognitive Kit-Bags and Cognitive Units

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Why do two students, both of whom did well in one course, have vastly different experiences in the subsequent course? Does the successful student merely have more available procedures, or is there a fundamental difference in his or her cognitive activities?

Nancy and Kathy both earned a "B" in college algebra, and both enrolled in pre-calculus the following semester. Nancy had little difficulty in pre-calculus, but Kathy had a great deal. The course is a degree requirement, so she needed to pass it; she dropped once, re-enrolled and eventually passed, but with much work and anguish. Why do "so many of the population fail to understand what a small minority regard as being almost trivially simple?" (Gray & Tall, 1994). Why do two students with apparently similar attainment go on to perform so differently?

To seek insight into these questions, we explored the cognitive structure demonstrated by the two students working problems involving graphs of lines from the first algebra course, and compared their problem solving approaches. We have previously studied the diffuse cognitive structure of a less successful algebra student (Crowley & Tall, 1999). This study compares and contrasts the work of a similar student—who struggled with very straightforward algebra concepts—with that of a student who proved to be more successful in the succeeding course.

Barnard and Tall (1997) introduced the idea of "cognitive unit" as "a piece of cognitive structure that can be held in the focus of attention all at one time." We see cognitive units as forming the nodes of a cognitive structure linked to other units using the web metaphor of Hiebert and Carpenter, incorporating the varifocal element of Skemp. If various elements are not connected securely, the individual may not be able to consider the totality as a cognitive unit. Links are not made to a flexible conceptual entity, but to one procedure from a collection, the student's "cognitive kit bag". Interviews revealed quite different cognitive structures. The successful student had a variety of approaches to problems, checking mechanisms, and an overall grasp of linear equations as if it were a **cognitive unit**. The other student had a **cognitive kit-bag of procedural techniques** with no flexibility or checking mechanisms. She had the same attainment but very different potential to cope with the ensuing course.

PLEA FOR A SIMPLE CALCULATOR IN PRIMARY SCHOOLS

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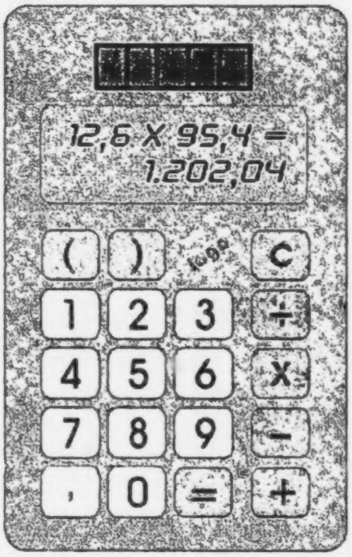
One of the goals in Dutch primary education (age 12) is about the electronic calculator: "Students are able to use a calculator with understanding". The additional description "with understanding" means that the teaching should be focussed on developing such an attitude that the students can decide theirselves about the help of using the calculator in a specific arithmetical problem. The calculator is a very helpful tool in applications with difficult numberdata (decimals, percentages), but it is not useful if the student doesn't know how to organise a method of calculation. There are also lots of problems in which the use of a calculator could work in a counterproductive way (The train leaves A at 10.37. Arrival in B at 12.11. How long does this trip last?). At last, the students should not use the calculator for simple problems like 12,3 x 100 and 1002 - 999. Mental computation and estimation activities are highly recommended in the Dutch curriculum (age 6 -12). These abilities are important conditions for a sensible use of the calculator.

At the moment calculators with very specific functions have come into the market. Texas Instruments now recommends for primary education a calculator with more than 50 functions. We think that this is a too sophisticated and too complicated apparatus with respect to the goal, mentioned before. What we need for the Dutch situation is a simple four-function calculator, but with a display that represents the formula that has been typed in.

That is why Van den Brink, Moerlands, De Moor and Vermeulen have written a report, in which they state their arguments and make a proposal for a very simple model, as represented in the picture below (Vermeulen 2001). During this oral communication I will explain this report in some detail.



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LEARNING TRAJECTORIES BASED ON PROTOCOLS

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A learning trajectory according to Simon (1995) is a theoretical design for a possible way of concept development of an intended learner. Using the notion of protocol as espoused in Dörfler (2000) such trajectories are designed for the concepts of permutation and discrete function or mapping. This is done by telling a narrative about a hypothetical student in an appropriate learning environment. A protocol of an action (actual or imagined) carried out by the learner is conceived of as a cognitive process which takes note of relevant steps, stages, conditions or outcomes of that very action. This is achieved by the learner through developing or applying graphic, diagrammatic or symbolic inscriptions which denote the aspects of interest of the action. Depending on the experience and available knowledge of the learner many different ways of producing a protocol and its inscriptions are possible. The guidance of a teacher will be helpful for arising interest in and focusing attention on the intended aspects and relationships created by the actions. In the current case as actions are chosen the placing of books on different places and the rearranging of books on numbered places. This context can be enriched by asking how many ways there are in both cases for carrying out those actions. The learning trajectory now proceeds along the development of protocols and their inscriptions in a process of stepwise refinement and change of the means of the inscriptions. They are conceived to develop from verbal reports to lists of places and books, to tables and to arrow diagrams in both action contexts. The arrow diagrams can be seen by the learner as expressing an association of elements of one sort with elements of another or in the second case as describing a kind of movement on a set of elements. The focus of attention gradually shifts to the inscriptions themselves and their properties. Thereby the diagrams and symbolic expressions and the operations on them become the objects of investigation instead of the original objects and the actions upon them. The operations by the learner on these inscriptions constitute the core of the notion of a mapping between two sets or within one set. By this process the notion of a mapping gets related to the learner's own activity thus avoiding the more common alienation of mathematics.

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The usefulness of mathematics studies through the eyes of electronics college teachers.

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There is a wide consensus about the necessity of studying mathematics in favor of engineering and hi-tech. It is clear that understanding and implementing electronics, for example, are based upon conceptual understanding of mathematics, and the ability to contextualize it. This explains why electronics practical-engineers take some calculus topics as an essential component of their training. It is interesting to examine to what extent the mathematics course supports the electronics studies in a practical manner.

Learning mathematics without a cognitive commitment results in pseudo-conceptual or pseudo-analytical modes of behavior rather than in true conceptual understanding (Vinner, 1997). These modes of behavior resemble the behavior that results from true understanding so accurately that most people do not distinguish between them. They are characterized by two elements. In pseudo-conceptual behavior, actions are mainly based on association of words with other words, rather than on association of words with ideas. In pseudo-conceptual behavior words are used and actions are taken without going through any reflective procedure, or self-critique.

The mathematics is contextualized if data is processed with an orientation to an external phenomenon (Janvier, 1996). This is expressed when a decision made how to solve a problem can be based both on the algebraic and numeral content of an expression, as well as on the external-concrete content.

Five Israeli college electronics teachers were interviewed, and opened a window to the ways they see mathematics education as part of electronics practical engineering training. They compared the students' knowledge with the knowledge needed, and pointed at the gap between them. They also pointed at the degree of instrumental behavior when solving problems. They argued against the lack of contextalization to their subject matter, and described the students' anxiety when calling on an advanced mathematical topic. The poverty of mathematical knowledge was described as a consequence of pseudo-conceptual and pseudo-analytical modes of behavior. These modes of behavior led to anxiety related to advanced mathematical topics, to non-conceptual understanding, and above all, to uselessness of these studies regarding a large portion of the students.

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A TEACHING MODEL FOR RATIO COMPARISON, PROPORTION AND DENSITY

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The work reported in this communication is part of a research project whose main objective is the building up of a local theoretical model of ratio, proportion and proportionality which enables the organization of the teaching of those mathematics contents from the first grades of primary school up. Such construction requires the characterization and integration of four main components: the formal competence component, the cognitive component, the communication component and the teaching component (Filloy, 1999). Freudenthal's (1983) phenomenological analyses of the mathematical ratio concept has been taken as a starting point for structuring a formal competence model.

A didactical phenomenology has shown that an important role is played in the track towards the constitution of the ratio and proportion mental object by mental objects precursors of this mental object. A number of those mental object precursors have a qualitative character and involve ratio comparison as the context in which the equality of ratios, that is, the proportion can be given sense. A particularly important one is the mental object called "relatively" by Freudenthal which has been used by Fernández (2001) to design parts of a teaching model. The didactical sequences structured about the density concept emphasizing ratio unities, as well as its experimentation by means of two teaching interviews that constitute case studies is the theme of this communication.

The performances in a paper and pencil test of both primary students, a forth and a sixth graders, were characterized by a classification scheme (see for example Fernández and Figueras, 1999). That characterization enabled us to identify a performance tendency: the use of qualitative compensations (according to Lammon, 1993). The teaching model improved children's knowledge in both cases. However, the fraction knowledge better mastery of the elder pupil provided a further understanding of density tasks as well as a more appropriate use of ratio unities.

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STUDENT TEACHERS' REFLECTIONS ON QUESTIONING, LISTENING, AND RESPONDING

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How to teach (secondary) mathematics is a question of increasingly problematic nature. Appropriate methods courses and the student teaching phase of teacher education programs can provide prospective teachers with opportunities to face numerous challenges, such as becoming reflective practitioners and skillful questioners, listeners, and respondents to their students as well as diminishing their anxiety about mathematics teaching and improving their sense of self-efficacy in teaching mathematics. This study intends to analyze the extent of the impact of the methods courses at a large Midwestern university on student teachers': propensity for reflective thinking; questioning, listening, and responding skills; and perceived anxiety and self-efficacy towards teaching mathematics. Initial data suggested that, in general, the student teachers, whose conceptions about mathematics and its teaching and learning conform to the recommendations of the NCTM, hold a high sense of self-efficacy as well as a high anxiety level towards teaching mathematics. However, they have disparate opinions regarding the usefulness of their methods courses for their student teaching and their future life as teachers. Based on the analysis of the initial data, three student teachers will be selected to be observed in their classrooms and interviewed afterwards.

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WHERE TO WITH HIGH SCHOOL GEOMETRY?

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In South-African high schools where the emphasis is on formal Euclidean geometry, one often hears complaints from teachers and learners that geometry is difficult, boring and irrelevant, i.e. has no real-life application (Glencross, 1998). This is not just the trend in South Africa, but also part of a much-talked about topic worldwide, and researchers are constantly searching for solutions to the problems surrounding the teaching and learning of geometry. It seems that in most countries geometry education has entered a period of low tide. Even in those countries where geometry still maintains its former central position in school curricula, this seems to be owing more to the persistence of tradition, than to a careful analysis of the impact of formalist Euclidean style geometry teaching on the "mathematical culture" of contemporary learners (Mammana, 1998). Hansen (1998:20) believes that when selecting geometry content in the secondary school mathematics curriculum, it will become increasingly important to choose units of geometry that foster the "right" skills, abilities and attitudes for meaningful and useful (further) education. The question that needs to be addressed then is: What should be the aims and outcomes of geometry education in primary and secondary grades?

In an attempt to address this question, an investigation in the form of an email enquiry among identified international experts in the field (n=30) was done to determine what kind of geometry, and geometry teaching and learning should be done in the high school mathematics curriculum? From this, key perspectives emerged, which will be discussed in the presentation. For instance, transformation geometry must definitely be included in the curriculum (e.g. Glencross, 1998; Bartels, 1998; Mariotti, 1998); formal proof must have a place in the curriculum (e.g. Carroll; 1998); in countries where Euclidean geometry had to make place for transformation geometry, it was not always exactly the right thing to do (e.g. Pressmeg, 1998).

In view of the outcomes of the email survey, a field survey followed among selected teachers (n=147) in the Northwest Province of South-Africa to determine the extent to which current classroom practices were in compliance with the identified perspectives. Results, which will be discussed too, suggest that, due to inappropriate teacher training, little else than traditional practices were prevailing. As a change of a curriculum has to start at the teacher-level (Mammana, 1998), the question about appropriate school geometry-related training for teachers needs to be, and will be accounted for, again, in view of the gained perspectives.

A COMPARATIVE STUDY ON THE INFLUENCE OF DYNAMIC GEOMETRY SOFTWARE ON THE ACQUISITION OF BASIC GEOMETRIC NOTIONS

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Background Dynamic Geometry (DGS) increasingly makes its way to the classroom. It is widely believed that its capabilities help learners to gain insights and develop heuristics. But one must also account for a *computational transposition*[1] – for instance, Hölzl[2] reports epistemological shiftings and increased cognitive demands as side effects of DGS usage. But his case studies have been criticized for being too demanding and not representative. Therefore it seems appropriate to investigate on a broader basis the impact of a more retentive DGS application on students' achievements in regular classroom situations.

Method At 3 senior high schools (one being a private school for girls), 12 lessons were taught in grade 7, developing basic notions like perpendicular or angular bisector from "real life" problem contexts. This was done either based on paper (P classes) or, by the same teacher, exploring "electronic worksheets" with DGS (C classes). In the control classes (V), ordinary lessons were given by the regular teacher.

Results Mean posttest achievements (adjusted by pretest) were about equal in P and C (and considerably higher than in V)[3], but we found some distinctive features:

- in C, *higher achievers* profited more than lower achievers – vice versa in P,
- P was (significantly) superior to C *for girls at the private school*,
- C was somewhat more effective than P *for girls at public schools*, but considerably less effective for lower achieving boys, so C lay slightly behind in totals.

Also, differences in achievements and strategies between P and C occurred with "dynamic" problems, but surprisingly also with some "static" ones.

Discussion It seems that when dealing with standard examples, the benefits of dynamic exploration can easily be outweighed by the extra costs of DGS, so we confirm that *dynamics is not a didactical advantage per se* (Hölzl). The use of DGS should therefore be preceded by thorough consideration – it will be most favorable when an objective requirement for the tool meets an advanced mathematical experience.
The interplay of environment and gender with the effect of DGS seems new and deserves further attention. In coeducative classes, girls can profit from DGS treatment, but special care should be taken of lower achieving boys, especially to prevent them from using DGS just as a plaything.

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[1] in the sense of Balacheff, N. (1993): Artificial Intelligence and Real Teaching. In: Keitel, C.; Ruthven, K. (eds.): Learning from Computers: Mathematics Education and Technology. Springer.

[2] See e.g. Hölzl, R. (1996): How does 'dragging' affect the learning of geometry? Int. J. Computers for Math. Learning, vol. 1(2), p. 169-187.

[3] This could have been expected, if one considers DGS just as media, as treatment is commonly believed to matter more than media, see e.g. Clark, R. (1983): Reconsidering research on learning from media. Rev. Educ. Res., vol. 53, p. 445–459.

MATHEMATICS, LANGUAGE AND POETRY

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Mathematics is a kind of language with its symbols, relations, and grammar. This language is perfect, abstract, universal and is used not only in the exact and the natur sciences. Mathematics is a very powerful tool in social sciences, liberal arts, arts and many activities of every day life. The aim of this paper is to show an humanistic approach to mathematics education as to increase motivation, satisfaction, interest and to reduce anxiety from learning this subject. Poetry uses also symbol systems, relations and grammar to express feelings, ideas and sometimes... axioms, theorems. A poem, like mathematics theorem, condense in few symbols - words, a whole idea. Omar Khayyam , the Persian mathematician, astronomer, philosopher and poet, wro his "Rubyat" to express in four lines his opinion about meaning of life. In parallel he confront with the parallel postulate as with the third order equations, the binomial coefficients and with a new sun calendar (Franceschetti, 1999). Omar Khyam is one some mathematicians that were involved in creative writing rather than in mathematics language. Hardy declared (1940) that mathematician's product is like t product of poet or painter. Russel (1903) wrote about the feeling of being more than human being, that can be found in mathematics as in poetry.

Ebn Ezra, Bachet, Dodgson ("Lewis Carroll"),Hamilton, Weirstrass, Leibnitz, Kowalevsky and the P.M.E. member Shlomo Vinner are examples to the duality rath than dichotomy, between mathematics and poetry. Joining poems and historical background with features of the mathematicians-poets may introduce some interest, color, esthetic and also fun to the almost, but not always, monotonous, dull lessons. Demonstrations of poems will be presented during the short oral communication.

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IMPROVING TEACHERS'CRITICAL THINKING IN A "ON LINE" GEOMETRY COURSE

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Introduction

The ability of teachers to develop better professional understanding is an evolving process presenting multiple faces and it's usually an "incomplete activity" (Ponte 1994). Professional development in a teacher practice is necessary for increasing metacognitive understanding about mathematical processes but other components will appear. Thus, in a research process, we want to analyze critical aspects (Smith 1991) of the situated (distributed) professional knowledge (Llinares 1998), and evolution of a critical thinking process (Kuhn 1999).

Developmental design.

Our developmental "on line" research framework and activity (Bairral, Giménez e Togashi 2000) is a part of a greater project for improving teacher training in geometry (for 11-14 years old students) developped since 2000 as a cooperation project by the Federal Rural Universty of Rio de Janeiro (Brasil) and Barcelona University (Spain). A group of teachers followed a a distance-telematic learning course of about 50 hours in sixth months by using a diversity of teleinteractive tools : email, discussion list and collaborative chats.

A case study is conducted with two of the teachers in order to (1) characterize some specific issues in this "on line" activity and (2) proposing a model for the critical component of a professional knowledge by means of an internet training experience.

Rationale and results

The initial, intermediate and final slots of the course were considered as data. We observed all registrated teachers' comunications in the course tasks, collaborative lists, narratives and self-regulation enquiries at the end of each lesson. Two semi-structured interviews and text writings were also used and videotaped experiences of their classroom activity has been anallyzed in order to recognize their changes-in-action.

The study reveals the importance of meta-strategic, metacognitive components (Kuhn 1999). In fact, semantic text analysis shows (i) affective influences as an important issue for development of professional critical thinking and (ii) specificity on the epistemological geometrical perspective. Eventhough just descriptive level was a starting point for many teachers, some of them could exhibit through the course, pseudoepistemological view when confronting geometrical situations in the on line environment.

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[1] Granted by IN3. Grup Edus. UOC Barcelona.

[2] Granted by CAPES BEX-1855-9 Brasilia. Brasil

COMPUTABLE REPRESENTATION OF CONTINUOUS MOTION IN DYNAMIC GEOMETRY ENVIRONMENTS

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In compulsory education, geometry is presented as a model to organise the physical environment, including considerations on the motion of objects in space (through the geometric transformations). Besides, we use drawings to represent, at the same time, geometrical concepts and physical objects, and the dynamic geometry environments give us the possibility of moving the drawings on the screen. This feature is used to promote in students the construction of geometrical knowledge from exploration and conjecture activities (Laborde, 1998) and has consequences on the behaviour of the users and their reasoning style (Hölzl, 1996).

In this work we analyse to what extent the computational model of the geometry implemented in a dynamic geometry environment provides models for continuous physical motion. In particular, we go over the utility of dynamic geometry environments to simulate the motion of mechanical linkages, as this activity allows us to compare, by means of dynamic drawings, the computable representation of geometric properties with the real motion of a mechanism.

Analysing a simple example, we provide foundations for the particular behaviours observed in the motion of a picture on the screen, which require a subtle interpretation to be understood in a purely physical context. This approach allows us to go deeply into the particular relations existing between the different contexts that come into play: physical, geometrical and computational. In this way, we reflect on the new epistemological and didactic questions derived from the computable representation of the knowledge (Balacheff, 1994).

We consider this work to be a previous step to deduce didactic consequences on the students' perceptions of the moving drawings; in particular those concerning the uses of the dragging mode as a tool not only for automatic drawing of many instances of the same construction, but also to produce continuous motion.

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REFLECTIONS FROM TWO-DAYS SEMINAR/WORKSHOP ON USING CALCULATORS IN TEACHING OF ELEMENTARY SCHOOL MATHEMATICS

Hulya Gur, Gozde Comlekoglu, Yasar Ersoy

The result of many studies point out that the hand-held personal technology such as calculators are the cognitive tools that must be used in mathematics classes. To have the widespread use of technology and effective mathematics learning, the teachers should be aware of the recent development, e.g. I^4 (introduction, integration, implementation and issues on using information and communication technology) in mathematics education (Ersoy, 1997). Therefore, a two-day teacher training seminar and workshop for pre-service and in-service teachers were organised with the participation of various groups of people at Balikesir University in Turkey. This activity is, in fact, a part of an ongoing project, which was conducted and coordinated by the director of the project, and namely Technology Supported Mathematics Teaching, and by local Directory of Ministry of Education.

In the present study, a group of teachers' and teacher trainees' perceptions of using calculators in mathematics instruction is searched, and their views were reflected to some extend. The planned activities were programmed and scheduled as seminar and workshop for the participation to teachers of maths. In the seminar part, information on the integration of calculators in mathematics instruction and the recent developments in the implementation were summarised and discussed. After the seminar, workshops on elementary school level mathematics were organised, and various topics in four sessions studied and discussed. At the end of the workshop a questionnaire about the use of calculators and teaching/learning mathematics was conducted by the sample of 90 teachers and 120 teacher trainees. Then a panel was organised in which the teacher and teacher trainees reflected their ideas, beliefs about calculators, mathematics education and the main issues.

The statistical analysis of the designed questionnaire is in the process, and some results have been obtained. The first result shows that although the teachers and teacher trainees had negative opinions about the use of calculators in mathematics instruction and student-centred activities at the beginning of the seminar and workshop, their attitudes and beliefs were changed towards positive at the end. The detail of two-days seminar and workshop will be reported, and teachers' and teacher trainees' perception and views will be reflected in presentation of the paper.

THE MULTI-LINE-SCREEN CALCULATOR AND THE EMERGENCE OF NUMERICAL STRATEGIES IN SECONDARY 1, 2, AND 3 STUDENTS

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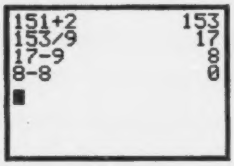
The graphing calculator with its multi-line screen served as a tool of exploration in a week-long sequence of activities developed around the "Five steps to Zero" problem.

"Take any whole number from 1 to 900 and try to get it down to zero in five steps or less, using only the numbers 1 to 9 and the four basic operations +, -, x, ÷. You may use the same number more than once."



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151+2          153
153/9           17
17-9             8
8-8              0
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Three classes of Secondary 1, 2, and 3 students, aged 13-16 years, participated. The analysis focused on the interaction between the epistemological power of the technology and the emergence of mathematical strategies. In all three classes, the main initial strategies observed were S1: Decrease or increase the number so that it ends in 0 or 5, and then divide by 5, and S2: Decrease or increase the number to bring it to the form *abc* such that one can identify a divisor of *abc*, either from *ab* or from *bc*. For Secondary 1, S2 gained in importance over the study. For Secondary 2, S2 became more complex; furthermore, many students moved toward S3: Decrease or increase the number to make it divisible by 9. For Secondary 3, S3 became the dominant strategy.

The multi-line-screen calculator, with its suppression of calculation details, thus keeping a proximity--physical as well as temporal--between the numbers and results of the operations, served to heighten students' number sense along several important dimensions. But it played a different role at each grade level. For Secondary 1 and 2, the calculator was used as a tool for calculating and as a medium favoring the formulation of conjectures. Reasoning in a local manner, these pupils based their judgments on an analysis of the form of the numbers. In contrast, for Secondary 3 pupils, who reasoned in a global manner, it also served as a search tool. This allowed them to develop more powerful algorithms than were seen in Secondary 1 and 2.

Acknowledgments

Our appreciation to the Social Sciences and Humanities Research Council of Canada (grant # 410-99-1515) and CONACYT of Mexico (grant # I32810-S) for their funding of the research described herein.

A longer version of this paper is available from the first author at jguzman@mail.cinvestav.mx

LEARNING MATHETMICS IN SMALL GROUPS: CASE STUDIES FROM PAKISTAN

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As part of my D. Phil. research, I investigated the role of social interactions in students' learning of mathematics. The research methodology was qualitative in nature. Over a period of nine months I observed two small groups of students as they did mathematics in their respective classrooms in Karachi, and recorded the social interactions on videotapes. To follow up on issues arising from the classroom observations, students were interviewed under stimulated recall. During these interviews students were also set mathematics tasks similar to the ones used in the classroom. Grounded theory procedures (Strauss & Corbin, 1998) were used in analysis of data.

Preliminary findings from one case, indicate that the socio cultural norms being constituted and stabilised in the classroom, in turn constituted and stabilised mathematical norms where mathematical norms were criteria of values for mathematics activities. For example, the classroom organisation of working at mathematics tasks in small groups, preceded by some introductory work by the teacher and followed by presentations to whole class, was very consistent across the lessons observed. The social norms were that students were expected to explain their solutions to others, give reasons for their thinking, and make sense of other's explanation. As different groups presented their work using varied approaches to the solution, a mathematical norm being stabilised was that there could be more than one solution to a mathematics problem. This had implications for mathematics development in the classroom community.

Moreover, the teacher's effort to ensure individual accountability in group work led to qualitatively different patterns of interaction in the group with further implications for students' learning.

The study also revealed that students' responses were different to mathematics tasks with the same content but given under different social settings that of the classroom and of the interview. Thus raising deep and intangible questions about the processes involved in learning.

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MATHEMATICS ANXIETY IN ACTION

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Anxiety is an unpleasant emotional state of fear. It is directed toward an unwanted possible future outcome, and is typically out of all proportion to the threat. Mathematics anxiety is anxiety in mathematical situations. It is related to general anxiety and its sub-constructs (e.g. test anxiety). (Hembree, 1990)

Experimental psychology has concluded that anxiety biases cognitive processing. Attention is biased towards threatening information, and judgments towards more threatening alternatives. There may be also memory bias towards threatening information. All this leads to impaired performance in cognitive tasks. (MacLeod, 1999)

I shall present a case study of one classroom interaction that illustrates the role of anxiety in the interaction between teacher and student. The student was an exceptionally well-behaved girl and rather quiet in the class. She was a diligent worker, yet she achieved below average in tests. For Helena, mathematics was characterized by strong negative emotions. She did not, however, express these emotions in the classroom. The severity of her negative affect became apparent first in an interview.

Helena: *Mathematics makes me at least so quite anxious ... and agonized and distressed. ... mood goes really down or you start ((to think)), depressed, that 'I don't understand this again'. Then just wait for the next exam horrified ... For me, that ((problem solving)) is exactly what makes me feel anxious, and then sort of, like, in a way, somewhat unpleasant feeling that sort of: 'Am I stupid or what - or feeble-minded?' - 'coz everyone else can solve this. So, why can't I?*

I shall present and analyze in detail a teaching interaction between Helena and her teacher. During one class, the teacher spent 15 minutes helping Helena. When teacher and Helena had finished the task, Helena had not learned to do the task. Instead she was feeling incompetent and frustrated.

The case study is a good example of unsuccessful interaction. Interaction with anxious pupils is more difficult than with normal pupils because of their biased attention and judgments. Anxious pupils need a learning environment where they can feel safe.

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Instruction to use a graphic calculator to solve problems in a context of secondary vocational education

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Mathematics education is gradually changing from teacher oriented instruction towards student self-regulated learning. Co-operative and collaborative learning are increasingly emphasised as effective learning environments. This means that both the learning context and the didactics for teaching mathematics are changing, emphasising that students have to plan, to discuss and construct their own knowledge.

Beside the context of the learning situation the content of mathematics is also changing. In the Netherlands a new mathematics curriculum has been implemented, in which the use of a graphic calculator is more or less integrated in the booklets. The graphic calculator is integrated in the new mathematics curriculum because it is expected that the use of a graphic calculator in a setting of co-operative, small group work will improve students' flexibility in solving mathematical problems in an engineering context.

To investigate this hypothesis teachers of four schools of secondary vocational education are being supported in using the graphic calculator as a problem solving and learning tool and for applying mathematics as well as in guiding small group work. We try to describe the possibilities and obstacles of instruction in this learning environment. Students' discourses are analysed while they are working collaboratively. Videotapes are used to write down verbatim protocols. Preliminary qualitative analyses of verbatim protocols and observations show that students mostly use their graphic calculator as a computational tool. However, during a whole class instruction and discussion the graphic calculator stimulates discussion between students. Especially when the teacher asks challenging questions about different mathematical phenomena.

On the basis of the protocols, the interviews and the field notes conclusions are drawn and recommendations are formulated. Finally we will discuss some implications for the use of the graphic calculator as an integral part of mathematics education.

COMPUTER SIMULATIONS IN MATHEMATICS EDUCATION

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Many educators agree that a more extensive use of computers in school in general, and in mathematical education in particular, is desirable. The NCTM standards (1989) recommended this over 10 years ago.

In this presentation, I will present a teaching module (Hoffmann, 1996), meant for teachers intending to teach in the higher grades of elementary school and/or the upper grades in high school.

To demonstrate the teaching module I will describe **The Monte Carlo simulation for area approximation,** focusing on **the computation of the area of a unit circle** (a topic taught in the 6th grade in Israel), which leads to the approximation of the number π. (Hoffmann, 2000). The term "Monte Carlo method" is general in nature. It refers to numerical methods based on probabilistic or randomized algorithms, which use elementary statistical methods, allowing rapid approximate solutions for problems for which computational solutions are either not known or are inefficient (Harel, 1992). Monte Carlo methods are used in various fields of computational science: economics, statistics, nuclear physics, chemistry, biology, mathematics, and the like.

In my talk I will show how modern day technology enables embedding this topic within the school curriculum. I will review our experience in teaching the topic to a variety of populations in teacher education. The following data will be presented : (a) algorithms written by students; (b) the numerical output received using the Excel spreadsheet; (c) a mathematical graphic presentation of the convergence to the approximate area; (d) a visual presentation clarifying the process as a whole; (e) the "Buffon`s needle" simulation (Breuer & Zwas, 1993); (f) some simulations in basic probability.

The students in all the classes engaged in this module with great enthusiasm and interest. The topic created opportunities to discuss important mathematical concepts such as: area; probability; the use of simulation; approximations. In addition, the students were introduced to yet another facet of computer use in mathematics, and were lead to experience mathematical studies from a new perspective.

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CONSTRUCTING PROCEDURES AND CONCEPTS IN THE CLASSROOM – ADDITION AND SUBTRACTION UP TO 100

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Children learning in mathematics can be taken as a specific activity constructing **cognitive structures** (various relationships between real situations, their models, and mathematical notations etc.) in children's minds (Ku• ina, [2]). Various interpretations of concepts, models and visual aids support the understanding of the concepts and processes in a determined way. For illustration this idea an introduction of addition and subtraction up to 100 will be used. The nature of children strategies will be described in the framework of theory of procept (Gray, Tall [1].

The organisation and framework of the study:

We conducted an experimental education in 4 classes (children aged 7) where different teaching materials were used. Starting points for the study:

- in teaching children the teacher usually come out from the textbook – it means they use the same representations;

- children will be given the opportunity to create and use their own strategies, because they are able to introduce new procedure of addition and subtraction up to 100 without any explanation of the teacher. This promotes understanding and creation of procepts.

Questions of the study: What strategies do children use in mentally adding and subtracting up to 100 in respect to the aids used mainly in the classroom?

What nature have these strategies from the point of view of creation the procept?

Conclusions:

- Pupils are better motivated to learn when teaching builds on their own strategies.
- Paying attention to relation between the learning aid, procedure and concept is necessary. Various tools are essential to carry out activities but they do not lead direct to creation of the processes, concepts or procepts.
- When pupils learn to co-ordinate systems of signs, with their activity, their reasoning system becomes more powerful (more efficient, faster, less subject to distraction).

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PROBABILITY PROBLEM SOLVING IN TRAINING MATHEMATICS TEACHERS. THE CAVE PROBLEM.

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Several years ago Shaughessy (1992) emphasised the close ties between the two areas of research I consider here, stochastics (probability and statistics) and problem solving. He also pointed out the unfortunate situation of teaching probability and statistics in primary and secondary schools, citing one of the possible issue that hinder the effective teaching of stochastics the preparation of the mathematics teachers.

The Cave problem is a problem I used as a context in teaching and learning probability and statistics in training primary and secondary teachers. The problem-situation talks about 27 explorers that are into the cave in a randomise situation. There are three paths into the cave but only one let a explorer to leave the cave in one hour walking, but not the other two paths, coming back to cave inside after 2 days and 3 days walking. Each explorer has meals for 6 days as maximum. The question is how many explorers will leave the cave?

As we know, problem solving can be situated in several worlds (Puig, 1996). One of them considers problems in relation to the mathematical sign systems (MSS) (Filloy, 2001) with which they are solved. In this context, learning probability and statistics is seen as a process to gain competence in continuos stratums of the MSS that solve this problem or others. Teaching probability and statistics, consequently, will organise the step from a stratum of the MSS to a new stratum of the MSS from which the first is seen as more concrete and with which what was previously describe as separated or unconnected is described the same way, and therefore, are produced as new concepts and new signs.

In solving Cave Problem I identify three stratums of the MSS mentioned. The more concrete one uses signs from the play that solves the problem. The intermediate one uses signs from the representation systems and associated rules and, finally, the more abstract one uses signs from the more formal representation in mathematics. The use of each stratum can be associated to learning process at different school levels: primary school, secondary school and college.

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GENDER-RELATED DIFFERENCES IN ATTITUDE TOWARDS MATHEMATICS AT PRIMARY AND SECONDARY SCHOOL LEVELS IN HARARE

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There is a national policy in Zimbabwe to address all the issues of gender equity. However in the education field, national statistics still show that more girls than boys drop mathematics, before completing secondary school (Dhliwayo et al 1996). This study investigated the extent to which the attitudes of girls and boys towards mathematics differ at primary (Grade Seven) and secondary (Form Four) school levels in Harare. In his research on gender-related and achievement-related beliefs in mathematics, Risnes (1998) found that boys displayed more positive motivating and achievement-related beliefs girls.

Two primary and two secondary schools were randomly selected in Harare. In each school, one Grade Seven class (primary school) and one Form Four class (secondary school) participated in the study. Two questionnaires were administered to Grade pupils and to Form Four pupils respectively. The questionnaires were used to survey the students' attitudes towards mathematics. The results showed that more Form Four male and Grade Seven female pupils find mathematics enjoyable and interesting than the Form Four female pupils. Similar results were observed about the level of participation in mathematics lessons and not fearing the subject. It was observed that more Form Four boys and Grade Seven girls indicated that they are good at mathematics and denied being poor at mathematics than did their Form Four female counterparts. In another set of results more Form Four male and Grade Seven female pupils claimed that mathematics was their favourite subject than the Form Four female pupils. In contrast fewer Form Four male and Grade Seven pupils indicated that mathematics was their most difficult subject than the Form Four female pupils. In all the cases Grade Seven male and female pupils gave similar responses.

The study concluded that there was evidence that more Form Four male and Grade Seven female pupils have a more positive attitude towards mathematics than the Form Four female pupils. However this difference was not apparent between Grade Seven female and male pupils. My conjecture is that in the Zimbabwean culture, maturation and gender socialisation may contribute to the decline of positive attitude towards mathematics observed in senior girls. For example, girls do more household chores than boys and this may reduce their preparation time for lessons. Further research should investigate this conjecture since it may not be true in other cultures.

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AN ASPECT OF A DIDACTICAL PATH FOR APPROACHING THE CONCEPT OF FUNCTION: THE QUALITATIVE INTERPRETATION OF A GRAPH

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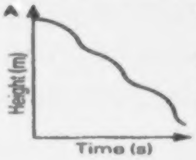
There are many teaching/learning problems surrounding the concept of function. These are bound up in the complexity of its history which are reflected in the partial or distorted views of the concept held by students as well as in the difficulties they meet as shown in various studies on the matter (we cannot give here any reference). We believe that in order to develop a true knowledge of a subject, it is necessary for the student to distinguish between mathematical concepts and their representations. Our hypothesis is that only through the co-ordination of different representative registers (verbal, tabular, algebraic, graphic) of a function the pupils can move flexibly and consistently between variational/qualitative and pointwise approaches to the funtion, even if the graphic level clearly plays a fundamental role.

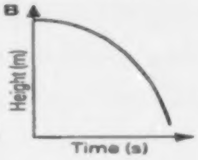
What we shall present is a fragment of a wide research developed in this frame. We shall concentrate on the results of some experiments, carried out mainly in 7th and 8th grade, which concern the recognition and the qualitative interpretation of graphs relating to physical phenomena not associated to formalised rules, for what they express compared to the variability of the magnitudes in question. In particular, we shall look at the main concepts and difficulties encountered by pupils by analysing their work on the co-ordination of the graphic representation of a given phenomenon and the character of the same phenomenon when expressed verbally or which derives from the relative interpretation. An example of these activities is below.

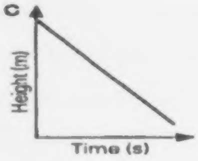
The following table refers to a parachutist's free falling jump from an aeroplane:

time (s)	0	5	10	15	20
height (m)	3000	2875	2500	1875	1000

a) how high is the plane the moment you start the drop?
b) how many meters lower was the parachuter after the first 5 seconds?
c) one of these drops describes the drop. Which? Why?







The results about the pupils' real understanding of the meaning of a given graph, from the observation of its tendency, confirm the foreseen difficulties of reading if not preceded by activities of effective construction. In particular, pupils confuse increasing and decreasing development patterns, especially in the case of non-straight lined graphs. As far as we have been able to observe in the course of their studies, these difficulties persist substantially into the third year of middle school, even when the pupils have been introduced to further graph-drawing activities starting from functions given to them in the form of algebraic formulation.

RECOVERING A POSITIVE ATTITUDE TOWARD MATHEMATICS IN FUTURE ELEMENTARY TEACHERS

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For the first time, in Italy, future elementary teachers follow universitary courses, where, however, the mathematical and scientific formation is confined in a few terms. Moreover their mathematics is usually badly structured and felt as extraneous. So, we believe that the crucial task is to help students to recover a sense in doing mathematics, namely, to radically modify their conception of this discipline and to transform their initial hostility into motivation and interest.

History tells us that mathematical notions were invented and gradually sharpened in order to give sense and structure to reality. There is a continuous link among the observation of phenomena in the physical world, their representation in mathematical form, the development of mathematical structures and of scientific language, and again an outcome in the comprehension of real world (Israel, 1996). Moreover the two processes of mathematical abstraction and formalization and of physical modelization appear as different sides of the same coin.

Our teaching activity utilizes this relationship between mathematics and physics, as a powerful didactical resource, both from the cognitive and the motivational points of view. So, the abstraction process comes from two problematic contexts: the first one concerns mathematical "objects", their properties and behaviours, as for example in (Tall, 1992) and (Arcavi, 1994); the second one starts from everyday experience or simple physical phenomena (Polya, 1954). In both cases special attention is devoted to the development of mathematical language, which expresses and supports the development of scientific knowledge.

This strategy naturally brings to view mathematics as an activity (Freudenthal, 1991), and moreover makes the abstraction process from reality to mathematics resonant with natural cognitive processes (Guidoni, 1985).

A special metacognitive tool is what we call the "logbook": in this personal diary each student records her/his individual understanding percourse (exercises, achievements, personal comments, etc.). The awareness of the growth in knowledge turns out to be a source of intrinsic satisfaction (Bruner, 1996).

In the communication, we will report some examples of our activity and of the achievements of our students (in our opinion, quite satisfactory).

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Flexible Spaces of Language

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'Flexible Spaces of Language' reflects
- dialogical concept of knowledge
- perspective on learning
- reflections on interaction according to the dialogisity.

The background for my study has been earlier work done on the richness and variation in children's use of language when solving problems in school and in their 'daily-life'. By use of Vygotsky's theory I focused on the function of language in the learning processes. The *translation link* was seen as a tool for making "new languages", as mathematical symbolistic language, as a tool for thinking. When I focused on the manifold of children's 'own language' it became clear that it is an educational challenge to facilitate the learning process by making it natural for the children to use 'their languages' in the processes of leaning more mathematics. This was seen as a background into which we could introduce or offer formal and 'authorised' languages, in order to obtain students developing ownership of the formal language.(Johnsen Høines1998, Mellin-Olsen 1987)

The data in my present study emerge from a learning situation where teacher students are working on calculus. Analysing the data and reading Bakhtin widened my concepts of texts, utterances, voices and dialogisity. It helped me to get insight into a field of learning characterised by manifold, complexity and movement. The multiplicity of voices in utterances that are written, drawn, said or uttered in other ways by students or by other, got visible. The different voices are, however, not enough to create meaning in this respect. According to Bakhtin, it is the tension and struggle between them that create understanding. (Dysthe 1999:76). It is about how our understanding is constituted by understandings and how it is a consequence that there would not be an explanation without interactive explanations, without multiplicity of voices.

I got, by focusing on one situation, insight into a Bakhtinian concept of knowledge. I have arguments I would not have had without this insight: It is about dialogisity knowing. It is about intentional and dialogisity learning.

As an outcome of this presentation I would like a focus on:
- How do we se the educational challenge represented by 'flexible spaces of language'?
- How do we see interaction according to the dialogisity?
- And - how to interact with the teachers practice?

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FIRST RESULTS OF A STUDY IN DIFFERENT MATHEMATICAL THINKING STYLES OF SCHOOL CHILDREN

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First results of an empirical Study of different styles of mathematical thinking of teenagers (9th and 10th class) will be represented. In connection with the empirical Studies by Leone Burton (1997) on practising mathematicians by means of a classification attempt by Felix Klein (1892), the following thinking styles are distinguished: • Visual thinking style • Analytic thinking style • Conceptual thinking style. Single mathematicians change between several thinking styles, but nevertheless one style is dominant.

It shall be investigated, how far the recognised styles of mathematical thinking of the practising mathematicians can be reconstructed with school children or, how far other thinking styles can be distinguished. This study is laid out as a quality oriented case study based on the following design: Pairs of female or male pupils will be video-taped while working on mathematical problems. Then the problem solving processes carried out by the pupils and the underlying thinking style will be documented by means of the method of stimulated recall and an interview which will be recorded by audio-tape. In the following the video and audio recordings will be transcribed and analysed by categories which still have to be developed. A central point of this design are the problems itself, originating from different mathematical topics which need different problem-solving strategies (e.g. graphic-visual strategies, numeric strategies, algebraic-algorithmic strategies, trial-and-error methods.). By this the use of different thinking styles shall be stimulated.

Within the framework of school teaching this study is of highly explosive nature: Generally, a teacher has his own dominant style of mathematical thinking, that normally he is not conscious of, and this style is underlying his school lessons and simultaneously structuring it. School children and teenagers, using other thinking styles than a teacher, have – this must be expected – clearly more difficulties with the lessons than those using thinking styles which are similar to those of their teachers.

In order to provide all teenagers the same proportion of participation in the lessons, and to give them the same chances, those different thinking styles must be made conscious, so that one can handle them in a conscious way.

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EFFECTS OF TEACHING FOR DEVELOPMENT OF METACOGNITIVE ABILITY

— CLASSROOM SESSIONS —

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SUMMARY

The purpose of this paper is to investigate the effects of teaching for developing of metacognitive ability. This experiment involved fourth grade 2 classrooms (experimental classroom and control classroom). Classroom sessions were carried out for the both by same teacher used same problems. But it was the differences of both classrooms' situations that were the teacher's activities and the work-sheets used on the sessions. In the experimental classroom the teacher promoted the children to do metacognitive activities through the framework of teaching for development of metacognitive ability (Table 1). Instead, a pretest and posttest were carried out for all children.

The main findings of this investigation are the followings :

- Both metacognitive and cognitive growth of the experimental classroom are higher than those of the control classroom
- In the posttest, some children in the experimental classroom did some metacognitive activities that they had not done in the pretest

METHOD

Classroom Session Table 1 is the framework of the teaching of the experimental classroom. It is based on Schoenfeld (1987) and others.

Pretest and Posttest In the tests, each child was asked to solve the problem on the work-sheet and to answer the stimulated recall questionnaire. The work-sheet was used to analyze his/her problem solving process, and the stimulated recall questionnaire was used to represent his/her metacognitive activities.

Analysis These problems of the pretest and posttest were the same. Then from the tests, children's metacognitive growths and cognitive growths were respectively identified as followings.

Table 1 Framework of teaching for development of metacognitive ability

Ⅰ. Solving the problem by oneself.
Ⅱ. Talking about cognitive activities.
Ⅲ. Talking about metacognitive activities.
Ⅳ. Using the metacognitive activities for a similar problem.

Table 2 Framework of investigation of metacognitive activity

1. Method of Investigation
   (1-1) Solving a problem on the work-sheet
   (1-2) Answering the stimulated recall questionnaire
2. Method of Analysis
   (2-1) Scoring the mathematical problem solving processes
   (2-2) Counting of metacognitive activities

Metacognitive growth is defined as

[the number of his/her metacognitive activities on the posttest]
—* [the number of his/her metacognitive activities on the pretest].

Cognitive growth is defined as

[the marks at his/her work–sheet on the posttest] — [the marks at his/her work–sheet on the pretest].

The results of teaching will be reported in this presentation.

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* This means to subtract.

DIRECT OR INVERSE RATIO PROBLEMS- DOES IT REALLY MATTER?

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In recent PME papers we described a workshop that was specifically designed for enhancing teachers' knowledge of students' ways of thinking. This workshop focused on students' misconceptions, possible incorrect responses and their sources and presented theories and research findings concerning students' ways of thinking (e.g. Klein & Tirosh, 2000).

During the workshop it became obvious that teachers were using and teaching almost only mapping tables as means for solving specific word problems, without considering the appropriateness of such a procedure. They believed that such a technical and automatic way always works well. We challenged the automatic use of mapping tables by presenting participants with both direct and inverse ratio problems. Most teachers did not identify inverse ratio problems as such and consequently arrived at incorrect solutions when using mapping tables. We used the teaching by conflict method (Swan, 1983) in order to raise teachers' awareness of the correct use of the mapping tables procedure.

In the short oral presentation we will describe the problems we used during the workshop, teachers' responses and their beliefs. Our data supports research findings that change in teachers' knowledge without change in teachers' beliefs is not significant. After the workshop teachers beliefs were only slightly changed and they still supported the automatic use of mapping tables. These findings are in line with former research demonstrating inflexibility in teachers' understanding of ratio concepts (Klemer & Peled, 1998).

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COUNTER EXAMPLES AND CONFLICTS AS A REMEDY TO ELIMINATE MISCONCEPTIONS AND MISTAKES: A CASE STUDY

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This case study deals with typical students' misconceptions in first-year university engineering mathematics courses. Unfortunately sometimes those misconceptions were caused by mistakes and mathematical inaccuracies in textbooks. In order to eliminate misconceptions and correct mistakes counter examples were used. The students were given the extracts from their textbook containing mistakes and were asked to give counter examples. They had enough knowledge to do that. However, for most of the students that kind of activity was very challenging and even created psychological discomfort and conflict for a number of reasons. Some of the reasons were consistent with findings from another study of the author (Klymchuk S, 1999).

In this study, practice was selected as the basis for the research framework and, it was decided 'to follow conventional wisdom as understood by the people who are stakeholders in the practice' (Zevenbergen R, Bcgg A, 1999). The theoretical framework was based on Piaget's notion of cognitive conflict (Piaget, 1985).

Below are some examples of bad mistakes from the textbook (Bolton W, 2000) that were used in the study:

- 'With a continuous function, i.e. a function which has values of y which smoothly and continuously change for all values of x, we have derivatives for all values of x' (p.332).
- 'If $\frac{dy}{dx} = 0$ then y is neither increasing nor decreasing' (p.353).
- 'At a maximum $\frac{d^2y}{dx^2}$ is negative and at a minimum positive' (p.353).

Eighty students were questioned regarding their attitudes towards the method of using counter examples to eliminate misconceptions. The majority of the students reported that the method was strong, effective and successful.

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A THEORETICAL FRAMEWORK FOR EXAMINING DISCOURSE IN MATHEMATICS CLASSROOMS

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The purpose of this session is to present a theoretical framework for examining discourse in mathematics classrooms that draws upon work from outside the body of mainstream mathematics education literature. Specifically, we draw upon Mikhail Bakhtin's notions of speech genre and voice (see Knuth & Peressini, in press). Bakhtin (1986) posited that discourse can be characterized—in terms of speech genres—by the nature of its utterances: "A speech genre is not a form of language, but a typical form of utterance.... Genres correspond to typical situations of speech communication, typical themes and, consequently, also to particular contacts between the *meanings* of words" (p. 87). Bakhtin differentiated between different types of speech genres in terms of the degree to which one voice can come into contact with and interanimate another (Wertsch, 1991). According to Bakhtin, true understanding results only when the voice of a listener comes into contact with and confronts the voice of the speaker, that is, through the interanimation of voices. Yet, the degree of interanimation of voices—and thus the understanding developed—may differ depending upon the nature of discourse in which the interlocutors engage.

Using this framework, we explore the theoretical underpinnings of discourse and how discourse actually functions in mathematics classrooms. We then address possible ramifications of these different functions of discourse in mathematics classrooms. In conclusion, we discuss a variety of issues that have arisen as we have used this theoretical framework as well as considerations for future research.

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TEACHING MATHEMATICS METHODS COURSES TO PROSPECTIVE ELEMENTARY SCHOOL TEACHERS: FIVE YEARS OF REFLECTION

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In the last decade, researchers have shown a great deal of interest in the study of learning to teach prospective elementary school teachers. They have reported that there is often a conflict between what the university teacher educators want to teach in their methods courses and what prospective teachers want to learn (Katz & Raths, 1992; Nicol, 1999; Wineburg, 1991). Mathematics teacher educators usually want to engage prospective teachers in theory-based teaching approaches. Prospective teachers, on the other hand, usually want to be told "best practices" of teaching. These conflicts create dilemmas and complexities in classrooms. This presentation describes such a conflict that this author has consistently experienced in the teaching of mathematics methods courses to prospective elementary school teachers.

This study was carried out within a framework of case study research. The data for this study were collected from seven groups of elementary mathematics methods courses that I taught over a period of five years. In order to prepare this presentation I reviewed my course materials and analyzed them using a qualitative approach.

The data indicate that prospective teachers often wanted a collection of activities, lessons, and units in order to use them in their future classrooms. They wanted to make sure that I cover the curriculum by giving them enough activities from each content area. However, my feeling was that the carrying out of activities without proper philosophy is superficial and cannot get prospective teachers into inquiry and analysis. I wanted to cover only a fewer topics with appropriate theories and philosophies while prospective teachers wanted to cover more topics eliminating the time for theories. Basically I wanted to emphasize the depth and the prospective teachers wanted a breadth of activities.

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DETECTING VISUAL CHARACTER OF THE ISOSCELES-TRIANGLE SCHEMA

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The purpose of this research is to investigate the relation between a perceptive and an interpretive treatment of a simple geometric figure. A figure is more than what is initially stated in the building instructions and the surplus is not immediately visible but requires a mental reconfiguration. This heuristic approach permits the connection of different concepts under the cognitive construction of a geometric schema, in our case: the Isosceles-Triangle Schema. (ITS).

A concept-understanding schema combines information about concept-image and concept-usage (Moore 1994, Chinnappan 1998). This information follows from a treatment of aa geometric figure in two levels: The level of gestalt-apprehension and the one of operational apprehension (Duval 1995). Both treatments may be analyzed in the wider context of visual imagery. Our study aims to trace the 10-grade students' ITS on the basis of their responses to recognition and construction activities demanding different types of imagery as these are defined by Owens and Clements (1998).

The participants were 63 students (31 high achievers and 32 low achievers) who, according to curricula, had been taught geometry in a deductive level for a year.

The language used by the students in order to explain the way of acting was used as a tool of analysis of their conceptual schemata. The analysis of students' responses showed that a successful mathematical treatment of a geometric problem does not assure that students are also capable for a successful conceptual treatment. We have observed a tendency for overgeneralization of the application field of some basic mathematical concepts, but also a tendency for limitation of the operational treatment of the geometric figure on the base of quantitative relations at the expense of qualitative ones.

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STRUCTURING GROUP INTERACTION IN MATHEMATICS COOPERATIVE CLASSROOMS: THE EFFECT OF METACOGNITIVE INSTRUCTION VS. WORKED OUT EXAMPLES

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The purpose of the present study is three fold: (a) to investigate the effects of metacognitive instruction versus worked-out examples on students' mathematical reasoning and mathematical communication; (b) to examine the extend to which the two methods exert different effects on group problem-solving behaviors; and (c) to compare the long-term effects of the two methods on students' mathematical achievement. Both the metacognitive instruction as well as the worked-out examples were embedded within cooperative learning settings.

Worked-out examples specified all the steps needed to solve the problem and provided complete explanation regarding the sequence of actions required. The metacognitive instruction was based on the IMPROVE method (Mevarech & Kramarski, 1997) implementing metacognitive questioning in small groups. The metacognitive questions focus on : (a) the nature of the problem/ task (b) the construction of relationships between previous and new knowledge; and (c) the use of strategies appropriate for solving the problem/ task.

The study was conducted in two academic years. Participants for the first year of the study were 122 eighth-grade Israeli students who studied algebra in five heterogeneous classrooms with no tracking. In addition, problem-solving behaviors of eight groups (N=32) were videotaped and analyzed. A year later, when these participants were ninth graders, they were re-examined using the same test as the one administered in eighth grade. Within cooperative settings, students who were exposed to metacognitive instruction outperformed students who were exposed to worked-out examples on both the immediate and delayed posttests. In particular, the differences between the two conditions were observed on students' ability to explain their mathematical reasoning during the discourse and in writing.

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THE SELF-CRITICAL REFLECTION ON EXPERIMENTAL WORK

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Mistakes by pupils and teachers/researchers are often treated as crime in the case of pupils and not even recognised in the case of teachers/researchers. In this research we attempt to show that 'everyone can learn by their mistakes'. If the teacher/researcher undertakes self-reflection critically of the work they have done with students then this will show where mistakes occur, which might hamper the student`s thinking.

Research

We will present examples of self-critical reflections of our own experimental work. The theoretical framework for this study is based on the constructivist view of teaching. Analysing our experiments we have found that the researchers have made various types of mistakes:

1. The researcher does not react to the indirect information given by a student.
2. The researcher is not sensitive on the ambiguity in the interpretation of some statements.
3. The researcher is not sensitive enough for the important moments.
4. The researcher is engaged by his/her own idea and cannot follow the pupil`s way.

We have tried to determine what is fundamental about different mistake. Our incomplete classification of mistakes describes phenomena which show that mistakes do not depend on the types of problems given nor on the level of pupils' development.

We have concentrated on the different ways of interactive mathematical and social communication: questions, sentences, and non-verbal expressions. We have also observed the social aspect of pedagogical interaction. Following this we have tried to evaluate our interpretations and reactions in the directions of the sensitivities on special "sense" and meaning that emerged during the discourse and on our skills in understanding correctly the speakers' intentions.

Every researcher has his/her own mental model of a concept or of a solving process and this model is autonomous for each researcher. Sometimes it is impossible to foresee what kind of obstacles and restrictions this individual model brings into the observed situation. The examples of mistakes presented will show that the researcher occupied by his/her model is not sufficiently sensitive to perceive the student`s model and hence causes communication difficulties between researcher and student.

We will show that the interaction between the researcher and the student could be detrimental to the student`s thinking process. In the theoretical model of research it is the researcher`s role to be objective during his/her work, to create confidence and accept fully everything that a student says and does during the experiment. But reality shows that it is not easy to work according to this rule. The self-critical reflections on experimental work can help avoid the same mistakes in future work. In addition this reflection increases the sensitivities of researchers and develops a wider perspective when analysing research results.

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DYNAMIC MODEL OF MATHEMATICS TEACHER TRAINING

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> "The purpose of education can be seen in the learners' cultivation, particularly in the cultivation of their mental world." (Hejný & Kuřina, *Pedagogika* 2000)

In mathematics teacher training, we put emphasis on the support of constructivist approaches to mathematics education; not only verbally, but mainly in changes in actual teaching. In (Kubínová, Mareš, Novotná, 2000) we showed attempts to present constructivist teaching even in cases where detailed analysis showed that teaching by means of transmissive or instructive methods were used.

Our research

Large scale research done in 1999-2000 showed that:

- In mathematics, teacher training was conducted mainly by frontal and receptive methods. A static approach prevails (Kubínová - Novotná, 2000).
- From their own experience at school, students training to become teachers - our undergraduates - come to us with believing all teaching is transmissive.

If we want to change the nature of education we have to influence, above all, future teachers. We need to develop and cultivate their feelings about the functions and purposes of mathematical education. This implies a total change in the nature of existing teacher training concepts in the following directions:

- To create new schemas of transformation of scientific disciplines forming theoretical basis of pre-graduate mathematics teacher training.
- To move from a static, frontal teaching strategy and faculty based environment to a dynamic one, characterised by constructivist methods of teaching and learning taught both at the faculty and in faculty schools.
- To train future teachers in concrete activities using modern forms of didactical interpretation rather than traditional ones.

We prepared a new model of mathematics teacher training and from Autumn 2000 we have been teaching it. The model is "dynamic" because it makes use of constructivist approaches in teacher training, including breaking the isolation of individual subjects. There are strong links between the faculty and faculty school environments and these together with the creative approach to teaching profession become a permanent part of the teachers' future work.

Concrete examples of the proposed changes will illustrate the results of our research.

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COMBINING DYNAMIC MANIPULATION WITH FORMALISATION: MEANINGS CREATED AROUND THE CONCEPT OF CARVATURE

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Meaning-making processes based on the use of dynamic exploratory mathematical software in the classroom is attracting prolonged research interest (Noss & Hoyles, 1997). It is suggested that the enrichment of representation, functionality and feedback achieved with computer based learning environments could enable more focus on the ways by which medium resources within specially designed activities support the emergence of mathematical meanings. We report research aiming to explore how children construct meanings around the concept of curvature while working with software which combines symbolic notation (through a programming language) to construct geometrical figures with dynamic manipulation of variable values. "Turtleworld" enables constructions with variable Logo procedures and dynamic manipulation of these constructions using the "variation tool" (Kynigos et. al. 1997) to sequentially and continually change variable values. The students were engaged in trying to construct the arcs of bridges of different sizes and shapes. We investigated emerging meanings about the notions of curvature, arcs and angles and how these were constructed with the use of the variation tool. The analysis of pupil's interactions suggests that vivid interplay between symbolic and graphical representation was an integral part of the process by which they expressed "theorems" (Vergnaud, 1987) on length of arc, length of corresponding chord and the tilt of the arc as they were trying to create horizontal bridge arcs. Unexpected levels of abstraction emerged within these situations (Noss and Hoyles, 1996) and formalisation in quasi-algebraic terms became a means of thinking about and expressing relationships in a notational form. The dynamic manipulation and continuity effect of the variation tool facilitated pupil's spontaneous conceptions of arcs as well as their hypotheses drawn from pencil and paper calculations. Further research is suggested on the entanglement of symbolic expression in mathematical activity with dynamic manipulation software.

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THE DEVELOPMENT OF PERFORMANCE INDICES FOR THE ASSESSMENT OF STUDENTS' ABILITIES IN CULTURALLY ORIENTED SPACE AND SHAPE TASKS

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Curriculum change in South Africa requires that students engage with culturally oriented activities related to space and shape. Curriculum guidelines provide for assessment in 6 levels in a space and shape strand over grades 1 to 9. The particular specific outcome being addressed by the research reads as follows: "Analyse natural forms, cultural products and processes as representations of shape, space and time" (National Department of Education, 1997: 4). No guidelines are however provided for judging the developmental level at which the learner is engaging with such activities in terms of the mathematics embedded in the culturally oriented activities suggested.

The RADMASTE Centre has been engaged in developmental research (Gravemeijer, 1994) related to the implementation of an ethnomathematical approach in classrooms in South Africa for a number of years. An educative assessment framework (Wiggins, 1998) has been adopted in the development of tasks arising out of this research. These tasks were given to grade 7 and 8 students in well resourced suburban schools as well as in disadvantaged township (e.g., Soweto) schools.

This paper reports on a preliminary analysis of the responses of a sample of students on three of these tasks: Ndebele designs, house designs and map symbolism. The approach to the analysis was similar to that adopted by Sproule (1999) on culturally-based counting practices. The analysis has resulted in a tentative rubric involving performance indices and related assessment criteria. Learner performance in symmetry, scale, notation, symbolism, was identified in the tasks as a basis for the assessment criteria.

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SMALL GROUP INTERACTION: A STUDENT'S DEVELOPMENT TOWARDS INTELLECTUAL AUTONOMY.

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According to Kuhn (in Wood, 1999), "argument is central to thought and the construction of knowledge". Students could be disadvantaged in a problem-centred classroom if they lack the confidence to explain their own methods or to challenge other methods with which they do not agree with or do not understand. This paper suggests that this can be overcome if small group interaction is sensitively facilitated.

Classroom observations were made as part of a project to investigate the role of argumentation and articulation in students' conceptions of fractions. Students worked in heterogeneous groups. How discourse should be conducted was addressed by the teacher and the researchers in whole-class and small-group discussions. Some 'weaker' students did not appear to participate in heterogeneous group discourse, although the expected kind of discourse was demonstrated by others in the group.

After assessment of the concepts formed, students were regrouped into relatively homogeneous groups for remedial purposes. Two of the apparently disadvantaged students, Riyah and Lindile, were in a group whose assessment showed that the relevant concepts were not yet evident. The two girls were allowed to work together, on their own, in the hope that facilitation and support could help them to make sense of the mathematics and to participate in discourse where methods and solutions were explained, challenged and justified. A deliberate effort was made to illustrate the role of the listener by asking questions like *Why do you say that?* and *Do you agree with what she said?* and by requiring the students to verbalize their ideas - *If you want to [do something], what do you have to do?*

Although the two students did not make major conceptual gains in the first few sessions, they were soon able to participate in discourse that was characterised by reasoning and thinking. Riyah consistently demonstrated that she was willing to grapple with a problem for a length of time and try out various strategies. This type of behaviour is regarded as necessary for the construction of understanding (Hiebert et al., 1996). Lindile still tended to act irrationally, for example, adding the numerator and denominator of a fraction. But Riyah challenged her every time, genuinely interested in the reasons for her actions and trying to convince her to a different view. Assessment at the end of the year indicated an improvement in Riyah's conceptions and her ability to work abstractly.

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Mathematics Pedagogical Value System centering on mathematics knowledge acquisition in elementary school

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The paper investigated the pedagogical values presented in the mathematics teaching of a fifth grader elementary school teacher. Based on the valuing theory of Raths, Harmin and Simon (1987), the study defined a value as any belief or attitude or other similarity which meets the three criteria of choosing, prizing and acting. The criteria have different methods of examination.

The methodology of the research was classroom observation and interviews. The time for research was one year. Ten lessons were observed and 18 interviewed were conducted. The research subject was a teacher, Ms. Lin, who has been teaching in elementary school for nine years.

The purpose of classroom observation was to look for the repeated behavioral patterns of Ms. Lin, such as the demand for previews, the raise of testing questions, the neglect of student's wrong solution, and the repetition of reviews.

Furthermore, interviews were used to examine if Ms. Lin's value indicator met the criteria of choosing and prizing. For example, after considering the pros and cons of teacher-centered and student-centered teaching, Ms. Lin still chose to teach in former teaching style. Ms. Lin said, "going over them three times is better than going over them twice; twice is better than once." These words revealed that she stressed that students must learn the knowledge from the textbooks well.

Therefore it was concluded that the purpose of mathematics teaching is to teach student to learn the knowledge in the textbooks is a mathematics pedagogical value of Ms. Lin.

In addition to the mathematics pedagogical value stated, two more values of Ms. Lin were defined. But these values were not equally important. In the educating of teachers, the relationship among the values must be clarified and the core values must be found first. Then we further examine whether the core values accord with the current education policies. If not, in modifying teachers' mathematics pedagogical values, the change of core values will be the underlying solution. Of course, how to change teachers' mathematics pedagogical values is a new subject for research.

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ANALYZING TEACHERS' CONCEPTUAL CHANGE THROUGH THEIR REFLECTIVE JOURNALS

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The importance of developing reflective skills has gained acceptance in mathematics teacher education in recent years. To gain insights into the teaching, critical reflection is an essential skill for teachers to develop (Krainer, 1999). Cobb and his colleagues (1997) suggest that if reflection becomes a regular part of the process, teaching and learning will improve. Therefore, this study was intended to provide teachers with opportunities for reflecting on their teaching and learning mathematics experience.

Eight teachers with different teaching experience were asked to share their knowledge and experience in weekly meetings. They piloted journal-writing efforts by reflecting on what was discussed in the meeting. The researcher also responded with written comments to the questions teachers posed in their journals. The teachers were asked to reread their own journal entries in order to promote in-depth reflection rather than simply replicate others' ideas.

The reflective journals of the teachers were analyzed according to the parameters of a three-dimensional diagram that emerged in the course of the present study. "Perspective" describes the ways teachers organized and reviewed their teaching and learning experiences; focusing on mathematics content (M), pedagogy (N), or student learning (P). "View" describes whether participants reflected on themselves (E) or others (S). "Level" indicating the degree of reflection was separated into five levels: descriptive (1), illustrative (2), reflective (3), critical (4), and meta-grow (5). These three dimensions are represented as a 2x3x5 matrix by means of a three dimensional solid. This matrix is illustrated in Figure 1.

It is found that perspective of reflection was determined by concerns that depend on teachers' professional background. The pedagogical aspects of teaching were reflected on more frequently than mathematical content or student learning.

This study found that reflective writing was an effective tool in promoting conceptual change. The longer the participants took part on the development program, which emphasized a learner-centered approach, the better they understood how their students learned. However, the novice teachers focused more on their own teaching skills than students' cognitive processes in their reflections.

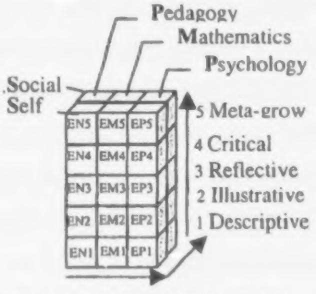

Figure 1 Schema of Analyzing Reflective Journals

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TOWARDS AN ANTI-ESSENTIALIST VIEW OF TECHNOLOGY IN MATHEMATICS EDUCATION: THE CASE OF CABRI-GÉOMÈTRE

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In this paper both theoretical and empirical support to a view on secondary mathematics teachers' use of Cabri-Géomètre is presented. In particular, I argue that the use of a software package for teaching is not only linked to the school curriculum but also strongly linked to what a teacher *sees* in such a software package. By treating software packages as *texts* and secondary mathematics teachers as *readers* of such texts from an *anti-essentialist* viewpoint of technology (Grint and Woolgar, 1997), this paper discusses a preliminary analysis of two of the case studies -*The Cabri of Anthony* and *The Cabri of Camilla* - of my ongoing PhD.

Anthony and Camilla, two secondary mathematics teachers from a state school in Bristol (UK), were interviewed both in front of and away from a computer, talking about and describing *her/his* Cabri. They also had two of their lessons within a Cabri environment observed. Methodological issues on how the research had been designed will be given in the talk.

The research project aims to look at *what is actually being said* by secondary mathematics teachers about Cabri, and to investigate to what extent this is linked to the teachers' use of Cabri in the classroom, in their teaching.

Here, to look at 'what is actually being said' means to look at what meanings are being produced by teachers for Cabri. One of my assumptions is that the software package which reaches the classroom environment is not *the* software that once had been designed but rather *a* software: the one that the teacher has constituted. The Cabri presented in a classroom is *a* Cabri: *the Cabri of the teacher*.

One of the said powerful features of Cabri-Géomètre is *drag-mode* that allows deformation of figures, which brings dynamism, where ideas of dependence and independence can be explored by establishing relationships among points on geometrical figures. From the two case studies, *seeing* and *treating* Cabri as such has shown not to be the case. The drag-mode has nothing to do with *the Cabri of Anthony* and *the Cabri of Camilla* at the time they were interviewed. This does not imply that it will never be. New meanings can be or will be produced by each teacher for Cabri, as meaning production is to be viewed and understood as a process rather than something static and fixed. The point is the importance of such *awareness* of *the Cabri of the teacher* in order to understand *how* and *why* Cabri is being taken and used in a classroom in a certain way.

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THE ROLE OF TECHNOLOGY IN THE STUDY OF SEMIOTIC REPRESENTATIONS IN MATHEMATICS

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We assume that, in the realm of mathematics, a *representation* is a symbolic, graphic or verbal notation to express concepts and procedures, as well as their more relevant characteristics and properties. According to these properties, representations can be classified in *semiotic registers of representation* (Duval, 1999). In this work, we reflect on the role that technology plays into cognitive processes, and its relation with the semiotic systems of representation, which constitute the key point to understand the way in which students construct mathematical knowledge. We will consider the calculator as a mediating tool in that knowledge construction process.

A repeated argument against the use of technology in mathematics teaching is that student forget and drop what they do with paper and pencil, and that this fact is considered as a damage for the quality of their education. But in the same way that the writing is not an obstacle for mental calculus, neither the calculator is. Instead, the representations supported by this tool have features which makes them especially productive for the learning of mathematics. They are *executable representations*, that is, they can simulate cognitive actions independently of user of calculator.

The power of technology is mainly epistemological, as its impact is based in a reification of mathematical objects and relations (Balacheff & Kaput, 1996, p.469): the calculator allows them to be regarded as "manipulable" objects, and gives us the possibility of working "about" them. A calculator as the TI-89 or TI-92, supply a big number of representations of mathematical objects in different registers, and they allow making conversions between registers, and this is a valuable tool in mathematics education. In the environment supplied by this technology, we can obtain mathematical properties very different of those possible with paper and pencil.

This comes to emphasise the notion of executable representations. The abstract ideas and concepts of mathematics become to be real with use of calculator: they can are mathematically manipulated and transformed, as we will whit many examples.

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RATIO COMPARISON: PERFORMANCE ON RATIO IN SIMILARITY TASKS

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Since 1996 a study about ratio, proportion and proportionality learning and teaching has been carried out at Valencia (see for example, Fernández and Figueras, 1999; Fernández et al, 1998). As part of this research an investigation to identify the ways in which secondary pupils —13 to 16 years old— solve Σ-construct tasks (Freudenthal, 1983) was made. These tasks comprise problem situations related to similarity, scale, the use of Thales theorem and the graphic of linear functions.

Students' answers to ratio comparison problems of a paper and pencil test in which similarity criteria play an important role were analysed using a classification scheme built up for other parts of the aforementioned global research. In search of elements which could serve as a framework to further understand pupils' answers a study with mathematics teachers was done. Interviews were made individually in a group setting, where teachers were asked to solve the pupils tasks, to describe their strategies and to compare them with those of their colleagues, to assess students answers and to discuss their characterisation.

The purpose of this communication is to describe cognitive tendencies in students' and teachers' ways of reasoning. Characterisation of pupils and teachers performance took into account criteria for preservation of ratio in similarities such as: preservation of equality of lengths, of angles, of congruence, of internal ratios and constancy of external ratios. Two differentiated types of reasoning were identified. One called "formal" was characterised by the use individuals do of necessary and sufficient criteria to sustain their answers. The other type named "qualitative performance" due to the use of visual reasoning which considers aspects that are necessary but not sufficient. These aspects are based fundamentally on the perception of form and relations that can be identify in a visual inspection of figures. Among the relevant results it can be mentioned the identification of similar behaviours between students and teachers. The qualitative performance of some of the teachers shows that it is necessary to design a teaching model (in the sense of Filly, 1999) based on visual reasoning which favours the construction of mental objets closer related to the ratio in similarity mathematical concept.

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RESEARCH ON EMOTION IN MATHEMATICS EDUCATION — THEORIES AND METHODOLOGIES

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Jeff Evans, Middlesex University

Within the Project Mathematical Thinking the research team has developed ways of looking at key issues in mathematics education research, namely emotion, transfer and assessment, through the lenses of four theoretical *perspectives* current in mathematics education and in social research more broadly: theories of symbolic control; situated learning; discursive practices; and semiotic mediation.

In our work on emotion, as with much recent work in mathematics education, we tried to move beyond a narrowly psychological approach and, influenced by cultural, sociological and psychoanalytic approaches, developed new ways of addressing how emotion permeates mathematical activity in the classroom.

Drawing from earlier work of Evans (2000) who used textual analysis of interview transcripts based on a discursive practice framework and a psychoanalytic basis for the methodology we then applied a range of methodological tools, including emotion metaphors (Kovecses, 2000; Lakoff, 1987), cognitive scenarios and scripts (Wierzbicka, 1999) to analyse classroom episodes. Through the analysis of transcripts of problem-solving interviews and classroom episodes it was possible to notice how emotions apparently attached to mathematical activity nevertheless draw on the experience and beliefs related to schooling more broadly and to non-school contexts rather than to narrowly school mathematical issues. Emotion is seen as integral with cognition and, whether it manifests itself as enjoyment or aversion, a relevant issue in the development and sustaining of mathematical activity in the classroom.

In this presentation we will discuss the methodologies used to address the corpus of data.

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The Construction of Objects as a Crucial Phase for the Articulation of Registers in Tridimensional Situations [*]

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The study presented here concerns a longitudinal project which privileges space and action in the learning of geometry at school; the project is being implemented in a primary school of the northern of France, since 1997/1998, with about fifty pupils, from six to eleven years old, during their school-attendance.

Activities of modelization of space are decisive to our approach (Mesquita, 2000). These activities are linked to the construction of tridimensional objects and other actions of manipulation (Berthoz, 1997). Along these activities, we give a special attention to the articulation of the different registers (the semiotical systems of presentation of knowledge considered by Duval, 2000) which can be used in tridimensional situations with children.

In this presentation we will focus on a particular moment of the study, where 8 to 9 years-old pupils were asked to construct a relief map of a part of the school quarter (including buildings, streets and hills). We will show how the construction of a maquette enables the coordination between the real objects and their tridimensional representation. This coordination is essential to the articulation among the representations used in this phase of the study: the tridimensional representation (the relief map, a non-semiotical representation, in the sense of Duval, 1999) and the different forms of bidimensional representations used by pupils: orthogonal projections (a land-register map, a semiotical representation) and perspective views (air views, non-semiotical representations). Actions of manipulation have a major role on this coordination.

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EXPLORATORY STUDY ON INFERENTIALS CONCEPTS 'S LEARNING IN SECONDARY LEVEL IN SPAIN

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SUMMARY

In this paper we in summary communicate the first results obtained in an exploratory study on the learning of statistical inference basic concepts for Spanish students in secondary level. This is a part of an ongoing research project, with the aim to determine problematic areas, the students' difficulties, favourable learning environments or methodological questions that can serve like base to improve the teaching of the statistical inference in the secondary level. The taking of data has been carried out in two courses of secondary level, in total 49 students of different age and previous statistical formation. The students have answered a written questionnaire whose content refers to inferential basic concepts as those of population and samples, the influence of the type and size of the sample in the realisation of inferences, the different sampling types role in the inferential process as well as the implications of the possible biases in the obtaining of data. The enunciates are posed in three different contexts, concrete, narrative and numeric. The analysis of the answers has been, fundamentally, of qualitative type. First, each one of the items separately; later, each researched conceptual nucleus, grouping for it all the results of the items that refer to the same one. As it is an exploratory study we intend especially to determine the open questions and to formulate queries to research later more than to reach accelerated conclusions. Nevertheless, we have obtained some first interesting results: a categorisation of answers about concepts, only seemingly easy, as those of population and samples; differences among the students' conceptions about the sampling process in connection with the age and the context in that the question it posed; appreciation lack about the importance of the randomness in the taking of data or the influence of the sampling type. All these questions have a certain social importance, given their incidence in the taking of decisions in situation of uncertainty for all the citizens, therefore, the improvement in the basic education in these aspects is so much an obligation of the educational system as of the teachers and the educational research.

Acknowledgement: To the Research Project PB97-0827, financed by the Spanish Ministry of Education, Madrid, Spain.

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CONCEPTIONS AND BELIEFS OF MATHEMATICS UNIVERSITY PROFESSORS ABOUT THE TEACHING OF DIFFERENTIAL EQUATIONS

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The present research is an approach to the conceptions and beliefs of mathematics university professors related to the teaching of differential equations within the first years of scientific and experimental studies (Moreno, 2001). The teaching objectives are twofold: on the one side, to discern the most relevant characteristics of present way of teaching differential equations ant to explain the persistence of traditional teaching methods; an secondly, categorise each professor under study in terms of his conceptions and beliefs about the subject matter itself, about the teaching and learning of it, and also to establish the level of coherence and consistency or permeability of such conceptions and beliefs.

The theoretical elements on which the conceptual framework is built on comprehend aspects related to the knowledge each professor has of the subject matter and his relations with the elements of the teaching practices and with his conceptions and beliefs (Brophy, 1991). Furthermore, we resort to epistemology as a bridge to link the subject matter and the teaching and learning process with the conceptions and beliefs. The study is a qualitative one, involving six mathematics university professors, all specialised in practical mathematics. An *ad hoc* designed questionnaire and a recorded interview are used for data collection. The data analysis includes eight phases, during which the corresponding tables, layouts, descriptor lists and the like are produced.

The analysis results corroborate the predominance of teaching approaches encouraging algorithmic and algebraic practices towards differential equations (Artigue, 1995; Yusof and Tall, 1999), also signaling some asymmetry and a certain lack of transparency as far as the teaching objectives are concerned. In addition, we explain: the professors persistence in their conception and belief of the conceptual difficulty of this mathematical subject, the personal conception each professor has about this equations, within mathematics in general, their fear to lose the mathematical contents considered as "real mathematics" and the ease and simplicity implied in a way of teachings based on mechanical solving.

The differences when categorising each professor arise from coherence between his conceptions and his beliefs, and they allow us to set up three groups of professors (I, II and III). Group I consist of the most incoherent holding the most permeable beliefs; group II includes the most coherent and consistent; and finally, professor of group III is fairly incoherent though consistent in his beliefs about teaching differential equations.

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ANALYZING "MORE THAN" AND "n TIMES AS MANY AS" IN MOZAMBICAN BANTU LANGUAGES

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In English, comparative relations are often described using *more than* and *n times as many as* (MacGregor, 1991). The former is used to describe a linear difference (e.g. 5 *more apples than*) and the latter, a proportional comparison (e.g. 3 times as *many apples as*). MacGregor analyzing these expressions in English and in other languages found that difficulties pupils have in understanding them may vary from language to language. "In some languages, the logical form of a relation matches the grammatical form used to express it. In others, including English, grammatical form obscures logical form. Some languages offer no way of translating certain English expressions until their logical form is revealed by paraphrasing (MacGregor, 1991)". Misinterpretations of *more than* and *n times as many as* in mathematics are also reported by Zepp (1989), Dickson, Brown and Gibson (1984).

An ongoing study carried out among Mozambican Bantu language speakers, representing three different communities (*Nyungwe*: 8 participants, *Changana*:12 participants and *Makhuwa*:17 participants), showed that in local languages people may better state and understand simple linear difference problems and proportional comparison problems than in Portuguese, the language of instruction in schools.

For the research, the following verbal problem was used: *Paulo has 7 apples. Maria has 5 more apples than Paulo. Rita has 3 times as many apples as Paulo. How many apples does Maria have? How many apples does Rita have?* The problem was given to participants (unschooled adults and children) in Portuguese. The statement *Rita has 3 times as many apples as Paul* is translated into Portuguese as *A Rita tem três vezes mais maçãs do que o Paulo (Rita has 3 times more apples than Paulo).* MacGregor observes this statement as being seen by some pupils as a complex relation that combines difference and proportion.

The participants were asked to solve the problems in Portuguese and then, to translate into their Bantu language as clear as possible. Their statements were collected and confronted with other speakers of the same language. Preliminary results showed that people understand these relations better than in Portuguese.

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The German addition to the OECD-PISA mathematics assessment: Framework for the supplementary test and its connection to the international framework.

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The OECD-PISA assessment in 2000 (PISA = Programme for International Student Assessment) included two parts in Germany: the international test and national supplementary tests. These two parts are intended to complement each other. In this presentation, the reasons for the German national option are described and the framework for the test is explained. We will demonstrate how the national framework relates to, refines, and supplements the international framework for PISA-mathematics.

The basis the national framework is based upon is the international PISA framework (OECD 1999). There are essential differencies between former international studies like TIMSS and the general aims of PISA. The most valuable point from a mathematics education viewpoint is that PISA aims to evaluate „mathematical literacy", mathematization being the central instance. One is therefore forced to define this conception, and the international framweork does it while pointing to the competencies necessary for doing mathematics. However, those competencies never occur as single capabilities. They occur in bundles when specific tasks are worked out. So, items should be classified in competency classes which indicate some qualitative distinction between different modes of mathematical thinking:

Class 1: reproduction, definitions, and computations;
Class 2: connections and integration for problem solving;
Class 3: mathematical thinking, generalisation and insight. (OECD 1999)

The German national framework takes the three classes as a starting point, but differentiates it for the reason to describe in more detail the achievements of German students. There were formed five classes, two for each, Class 1 and 2, and Class 3 leaving unchanged. The classification follows what in the German discussion was called „mathematische Grundbildung" (Winter 1995). The differentiated classes allow to distinct between the mastering of purely technical tasks and the various steps towards mathematization, which can be both, intra- and extra-mathematical modelling.

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PROBLEM: CONCEPTUAL INCOMPETENCE IN GRADE 7 GEOMETRY! SOLUTION: TEACHER TRAINING?

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In South Africa consensus is growing among researchers, as well as teachers that mathematics teachers' preparation to fulfil their role as learning facilitators needs to be improved (Taylor & Vinjevold, 1999). The results of the TIMSS and other surveys indicate that a disturbing number of school teachers are incompetent, especially with respect to domain specific conceptual and pedagogical knowledge in critical areas of mathematics, like geometry (Howie, 1997; Strauss, 1999; Van der Kooy, 1996). Taylor and Vinjevold (1999) found that South African mathematics learners, as well as their teachers performed very poorly with regards to conceptual knowledge, while a substantial number of the teachers did not outperform their learners by much, if at all, when completing exactly the same test. As teaching behaviour is fundamentally influenced by teachers' domain-specific conceptual knowledge (Koehler & Grouws, 1992), Kennedy (1998) rightfully appeals for concrete evidence of what prospective and practising teachers do in fact know and understand about the content they learn or teach.

The current project primarily aims to reveal further evidence to this effect, and to explore some ways of tackling the problematic situation. The project, undertaken in the Northwest Province of South Africa, involves selected Grade 7 mathematics teachers, their learners and the final-year mathematics student teachers (primary school) at all training institutions. Teachers teaching at schools with supposedly good track records regarding mathematics performance were specifically selected for the purpose of the study. Initially five selected "good" teachers and their learners (n=142) were subjected to the same Mayberry-type Van Hiele Test, set on the topics in the Grade 7 geometry syllabus, as to determine and compare (relate) their conceptual competence. The teachers also participated in a survey about their beliefs, attitudes and practices regarding the teaching and learning of mathematics, particularly geometry. Afterwards an additional twenty "good" teachers and their learners were involved. Results suggest that even "good" teachers are not much, if at all, above the level of geometry acquisition of their learners. They also suggest the existence of a seemingly intricate relationship between teaching success, and teachers' beliefs, conceptual competence and relevant own school experiences.

The results of the student teachers (n=102) do not seem to reveal a clear pattern, suggesting that even "good" hands-on geometry training programmes are not yet specific or focused enough to make a marked difference. However, as with teachers, own school experiences, and positive beliefs and attitudes about teaching and learning seem to be significant factors to be accounted for in the relevant teacher training.

LEARNING SCHOOL MATHEMATICS IN THE ABSENCE OF A TEACHER IN CLASS: REALLY?

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In South Africa, as in many developing countries, poorly trained teachers, absenteeism and lack of financial resources pose serious problems for school education, particularly in mathematics. In this time and age, it seems logical to ask whether relatively accessible and affordable technologies, such as TV and video, can contribute towards addressing problems like these. Huge (1990) reports on the use of video recordings of mathematics lessons to successfully compensate for the absence of teachers. In addition, there is evidence that the use of a "video class system" (VCS) does not impair the learning and teaching of school mathematics (Lowry & Thorkildsen, 1991). In the current project the influence of a VCS on learning variables and achievement in mathematics was investigated in a pretest-posttest experimental setting; qualitative analyses of observations were noted too.

Three groups of classes participated in this project over a period of six months. The mathematics lessons of experimental group 1 (E1) were videotaped, which were then played back to experimental group 2 (E2) in the absence of a teacher. A third group of classes acted as a control group (C), taught by the same teacher in the "conventional" way. Although the experiment was conducted in two suburban state schools, this presentation focuses mainly on the observations and results obtained in a multicultural high school for girls, involving grade 9 learners (n=100). The instrumentation used comprised a version of the LASSI-HS (Weinstein & Palmer, 1990), adapted for mathematics, which was used to measure variables influencing classroom learning (e.g. attention, motivation, etc.), as well as a series of self-constructed mathematics tests, which were used to measure achievement.

Overall, the results seem to support the claim that the use of a VCS does not impair learning performance in classes E1 and E2, as no significant differences occurred between the achievement of the learners in the respective groups. Surprisingly significant differences ($p<0.05$) occurred concerning the learning variables: Compared to E1, E2-learners' attention, use of main ideas and use of test strategies improved with medium effect ($d \geq 0.5$), however, their anxiety also raised markedly ($d=0.6$). Compared to E1 and C, E2-learners' learning motivation differed with medium effect to almost practically significance ($d=0.7$; $d=0.5$), while their concentration ($d=0.5$) improved with medium effect. Finally, these results can, and will be connected to mostly "positive" qualitative changes observed in learning behaviour, especially in E2, as well as with marked mostly "positive" adaptations of teacher behaviour.

SOME NOTES ON REFLEXIVE WRITING ACTIVITIES

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There have been a lot of discussions on the importance of either metacognition, reflective thinking, or affective issues in mathematics learning. On the other hand, there have been lots of reports on the writing activities in mathematics education, such as Journal writing, Reflective writing, etc. In this paper, a kind of writing activity which refers to the reflective and reflexive nature of mathematics learning is defined, and the example of "Reflexive Writing" is presented.

A kind of statement which is written in mathematical notation, such as numbers, expressions, proofs, is one kind of writing. In this paper, such kind of writing is called "Objective Writing", which means such writing is *the object* of mathematics learning. On the other hand, a kind of writing which express the students' metacognition, reflective thinking, or affective issues should be in another category of writing in mathematics education. Considering the importance of these mental activities in students' learning, such writing is also crucial in mathematics education. In this paper, this is called "Reflexive Writing", because there might be reflections or reflexions between the object of learning (Objective Writing) and Reflexive Writing.

Finally, the Fig.1 is an example of Reflexive Writing

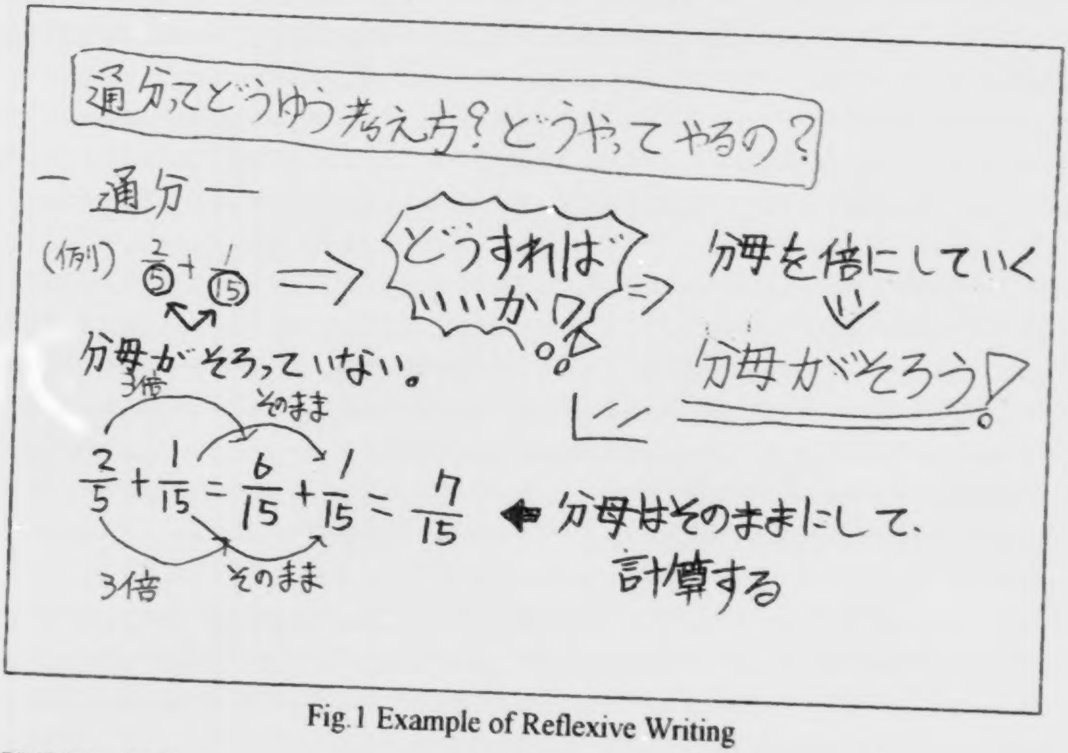
通分ってどうゆう考え方？どうやってやるの？
ー通分ー
(例) 2/5 + 1/15
どうすればいいか？！
分母を倍にしていく
分母がそろう！
分母がそろっていない。
3倍
そのまま
2/5 + 1/15 = 6/15 + 1/15 = 7/15
分母はそのままにして、計算する
3倍
そのまま

Fig.1 Example of Reflexive Writing

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Development of Naïve Algebraic Ideas during Solving Problems and Explaining the Solution Processes

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Our concern is to rethink mathematical learning from a holistic perspective, especially Japanese 7th grade's unit "letters and algebraic expressions" in which the teaching is likely to be atomistic, since most students don't feel the need to learn algebraic methods.

Bednarz & Janvier (1996) discussed students' approaches in solving the algebraic type of problems in their research of the transition from arithmetic to algebra, and suggested some gaps between their approaches and the algebraic method. We think their problems are good situations as a starting point for learning algebra in that they offer a picture of linear equation to students, and for our study it is important to understand how they overcome the gaps and develop their naïve ideas towards algebra.

The aim of this paper is to analyze several case studies and to suggest how students who haven't learned the contents of letters and algebraic expressions can construct and develop some algebraic ideas through solving problems and explaining the solutions. The following are suggestions resulting from the analysis.

1. The interplay between using trial-and-error methods and setting up expressions could help them to understand local relationships and to create algebraic ideas.
2. The naïve idea of uniting several terms emerged in two situations. One was the situation of simplifying the numerical calculation and the other was that of understanding the structure of the parts in the expression that they set up.
3. It was effective for the emergence of the naïve idea of distributive law that they conceived the problem situation and the numerical computations structurally. The idea became explicit when the explanation was shifted from numerical calculation to algebraic expression using letters and when they found the number of uses of the letter in the expression.
4. Students could create a feeling of necessity and appreciation of using letters through solving problems in the context of linear equation before they learned algebraic expressions.

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Attitudes of Preservice Mathematics Teachers towards Modeling and the Graphic Calculator

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In this paper we study the attitudes of a group of eight mathematics students following a preservice teacher training Program, based on the use of the graphic calculator in the process of teaching and learning mathematics. This Program was implemented by means of a course-workshop. We considered it important to elicit the attitudes of the participants involved in the Program both at the beginning and the end of the Program. In order to do so, a Likert-style attitude questionnaire was designed, the aim of which was to record the attitudinal changes in the mathematics students arising from the development of the Program which incorporates modeling and the graphic calculator in the context of linear algebra. This scale was set up from a matrix structure supported on a system of categories taking two variables into account. The first variable was defined by means of the specific objectives stated for the Program: i.e. mastery of the graphic calculator (O_1), knowledge of linear algebra and problem-solving strategies (O_2), modeling of real-world problems (O_3) and the design of teaching units (O_4). The second variable was defined by means of the components of the curriculum concept: i.e. student (C_1), teacher (C_2), mathematical content (C_3) and social use (C_4). From these two variables, a bidimensional variable O_iC_j was built with 16 values, from the combinations of the previous variables. The reliability of the instrument was measured, by applying the Spearman rho (r_s) coefficient.

The scale of attitudes allows favorable changes to be appreciated in the student's attitude towards modeling (O_3C_1), the teacher's attitude towards modeling (O_3C_2), the attitude towards solving algebraic problems concerning evaluation (O_2C_4), and the student's attitude towards the graphic calculator (O_1C_1). However, in view of the log-linear analysis, these were not statistically significant. Similarly, attitudes were detected which did not increase after the application of the Program, as is the case in O_4C_1, O_4C_4, O_3C_4, and O_4C_2, which suggests that some aspects should be revised. These aspects include practical and theoretical activities, teaching tasks and activities which involve evaluation with a calculator and modeling.

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SHORT ORAL COMMUNICATION

TITLE: Initial Mathematics teachers training at the Ecole Normale Supérieure of Nouakchott

AUTHOR: Mohamed Ahmed Ould Sidaty

ABSTRACT

The mauritanian educational system reform, recently decided by the government, aims first of all to improve the quality of scientific teaching, especially teaching Mathematics.

The Ecole Normale Supérieure (Teacher Training College) of Nouakchott, the only institution in the country in charge of training secondary school teachers, is aware of the fact that the success of such a reform depends mainly on the teachers responsible for putting it into practice. Therefore the E.N.S. took a certain number of measures aiming to improve the quality of mathematics teachers' training.

These measures consist of:

Deepening vocational guidance of training programs.

Making trainees more responsile to be able to monitor their own training.

The objective of this communication is

- to give a short overview of the new training programs and their modes of application.
- to precise the impact of these measures on
  the course of Mathematics,
  the course of didactic of Mathematics,
  the supervision (follow up) of teaching practice in high schools.
  the attitude of trainees when they become resposiblefor their own training.

Finally, we shall talk about obstaclesmet when applying these measures.

Exploring the Link between Preservice Teachers' Conception of Proof and the Use of Dynamic Geometry Software

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This study seeks to connect two well-established research strands. One is the role of proof in geometry, and the other is the link between teachers' subject matter knowledge and pedagogical content knowledge. Of particular interest is how secondary preservice teachers perceive the need for, and benefits of, formal proof when given geometric tasks in the context of dynamic geometry software.

Building on the work of Eves (1972), who conjectured that mathematicians determine truth by methods that are originally intuitive and empirical, and Lakatos (1976), who suggested the deductive format in which proofs are presented is misleading, Battista and Clements (1995) suggested that students should learn the meaningful use of proof by avoiding formal proof and focusing instead on justifying ideas and building visual and empirical foundations for later work. One way to do this is with dynamic geometry software, such as Cabri and Geometer's Sketchpad, which facilitate the making and testing of conjectures. Bershadsky and Zaslavsky (1999) also investigated how such dynamic environments impacted students' awareness of the intuitive, visual aspects of geometric situations, and described how this in turn was reflected in students' understanding of the ideas under study. Crisan (1999) suggests that the use of mathematical software both challenges and enriches teachers' subject matter knowledge as well as pedagogical content knowledge.

This report presents results of a case study of four preservice teachers as they solve two geometric problems posed in the form of questions, and then attempt to create formal proofs that generalize their results. The problems were chosen to be unfamiliar in their specifics, yet based on traditional Euclidean concepts. Thus, the preservice teachers were faced with issues of proof in areas that challenged their subject matter knowledge, and were asked to do so in a software environment that also challenged their pedagogical content knowledge. Preliminary results indicate that students see dynamic software as tool to make sense of proofs, but not necessarily helpful as a tool to create proofs.

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SYMBOLIC EXPRESSIONS ADOPTED BY PUPILS ANSWERING TO A WORD PROBLEM

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In this article we study the answers to a word problem given to 69 pupils of 7th grade of a "multicultural" school in Greece. This problem was an adapted version of a problem proposed, among others, in several research projects by Sowder (1988), and by Lester et al (1989). The adapted version reads as follows: *"A teacher plays the following game with her pupils. Each pupil would exchange a teacher's one hundred drachmas coin with lower value coins, excluding the one drachma coin. How many pupils could play this game if each exchange should be made in a unique way?"*

Greek language was not the mother tongue for 12 of the pupils. Although there was, partly, some misunderstanding of the problem (some pupils constructed questions of their own in a way similar to that described in Lean et al (1990)), it seems that being part of a linguistic or ethnic minority (e.g. albanian, russian, gipsy etc.) does not seem to be a significant factor, that affected the understanding of the problem.

Of particular interest are the symbolic-arithmetic expressions (or forms) adopted by pupils in their answers, in order to represent the various ways of exchanging a 100 drachmas coin. A large part of the symbolic-arithmetic forms we refer to sometimes resemble arithmetic «mononyms» and «polynomials», where the structural units represent the «basis» elements (50, 20, 10, 5 and 2 drachmas). For example: «5(2) + 2(5) + 1(10) + 1(20) + 1(50) = 100» or «100 = [10+ 10 + 10 + 10 + 10] · 2» or even in «exponential»(!) form «$5^{10} + 2^{20} + 2^{5} = 100$». We would like to emphasize that a variety of symbolic forms as above could not probably be achieved without the particular verbal context of the problem. It has been argued that only the development of non-verbal mathematical activities can support cognitive activation and "integration" of pupils who are socially isolated due to linguistic problems. Our research provides some evidence that this claim does not seem to be fully justified.

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TOWARDS A COMMON CHARACTERIZATION OF BELIEFS AND CONCEPTIONS

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The purpose of this paper is to draw attention to theoretical deficiencies of belief research (cf. Furinghetti & Pehkonen 1999). The concept of belief (and other related concepts) are often left undefined or researchers give their own definitions which might be even in contradiction with each.

On the ground of the previous considerations we have worked out a questionnaire in which we listed from the recent literature (1987–98) nine belief characterizations which focus on one or more terms of the triad in question (beliefs-conceptions-knowledge). In March 1999 we sent via e-mail our questionnaire to the 22 specialists invited to the international meeting "Mathematical Beliefs and their Impact on Teaching and Learning of Mathematics" (see Pehkonen & Törner 1999).

Our first observation was that in the responses of the specialists, there was no clear pattern to be observed. But in some points, one can find some regularity. The answers were most unified in one characterization ("*Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component.*") in which 15 specialists (83 %) disagreed with the statement. Then we have singled out quite clearly two central features determining the disagreement: the adjective *incontrovertible* and the *relation between beliefs and knowledge*. In gathering the criticisms and the constructive parts of the answers that we had at disposal we realized that there are points on which future research may be based.

In summarizing the results, we propose for studying beliefs and the related terms a list of basic recommendations, which should be used flexibly according to the situation, analyzed. They are, as follows: to consider two types of knowledge (objective and subjective); to consider that beliefs belong to subjective knowledge; to include affective factors in the belief systems, and distinguish affective and cognitive beliefs, if needed; to consider degrees of stability, and to leave beliefs open to change; to take care of the context (e.g. population, subject, etc.) and the research goal in which beliefs are considered.

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THE MATTER OF CHANGE - BEEING RESPECTABLE

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In the last decade, studies on teacher change (e.g., Clark and Peterson 1986) have presented many aspects of the change process. One of the main interests has been to find out the conditions for change. Recent studies have suggested the complex nature of teacher change (e.g., Senger 1999). In Finland all teachers are educated at universities. Anyway, also to these well educated teachers, it seems hard to meet changes in curriculum and education (Pehkonen, E. 1993). The aim of this paper is to understand the difficult question of change from the perspective of stability. What constitutes the good and stable elements in school mathematics teaching and learning in the minds of teachers?

The data consists of theme interviews of nine competent elementary school teachers. They had teaching experiences from one year to seventeen years. The themes in interviews were: 1) The teacher's personal conception about the state of mathematics teaching; 2) The good and proper elements in mathematics teaching; 3) The elements in mathematics teaching which should not be changed in the case of any change. The data was analyzed qualitatively. The aim was to find out the stable key elements most important to teachers and to interprete the prerequisites of change. The interpretation was later negotiated with the involved teachers.

Three key elements found in maths teaching were: the math book, basic counting skills and teacher-centred classroom teaching. These were the elements most frequently mentioned in every theme. They were also seen as the main elements not be changed in case of any change.Teachers do not object the changes, but they want to do their job properly. Teachers know the curriculum, text books, learning materials, teaching methods. They argumentate their action by explaining that they use best possible text books. Pupils seem to achieve what they are expected to achieve according to official standards and tests. Teachers could be satisfied.

But - however all teachers express their feelings of guiltiness. They know the trends of modern mathematics education, but they are not sure how to be able meet them. They are afraid that they have given up a part of their professional competence to the text books. Teachers are busy and teaching groups are very heterogenous.

Teachers want be respectable. They do not want to take too serious risks. They have experience-based evidence that their teaching meets well enough the set standards. But they do not have any evidence what happens, if they will spend their time on something else - for example on problem-solving.

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A COMPUTERGAME FOR MATHEMATICAL LEVEL RAISING

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Aim of this research project is to use ICT to attain level raising in the learning of mathematics. We define level raising as the transition from a visual or perceptual approach of a problem towards a conceptual approach. The assumption is that students attain level raising by executing the so-called key activities from the process model of Dekker and Elshout-Mohr (Dekker & Elshout-Mohr, 1998): to show, to explain, to justify and to reconstruct one's work. The expectation is that the use of computer simulations and working in pairs on investigations tasks will promote the occurrence of key activities.

A computer game is the starting-point for a couple of investigation tasks on the subject of probability theory in upper secondary education with 16-year-old students. Students are working in pairs on these tasks and the teacher is giving minimal help. The mathematical model underlying this game is the model for binomial chances. The research question of the first experimental study in this project is the question on which moment in the learning process working with investigation tasks is most effective. Executing mathematical investigation tasks can have different functions in the learning process. At first, they can activate prior knowledge and make students curious. Secondly, their function can be to give students the opportunity to apply and to process what they have learned. Thirdly, the investigations tasks can support the process of reinvention of the model for binomial chances. These three functions occur in this research project in three conditions. The question that we want to answer is in which condition the students attain most level raising. Besides, we want to know if there is an difference in the occurrence of key activities.

In January 2001 an experiment was executed in which 68 students were working on the investigation tasks in the three conditions. The data set consists of a pre- and posttest, audiotapes of 9 pairs, log files of all students and written tasks. These data will be analyzed on the occurrence of key activities and indications for level raising.

In the presentation we will make explicit the concept level raising and explain how in the process model of Dekker and Elshout-Mohr the occurrence of key activities involves level raising. We will show how this model is applied and the preliminary results of the experimental study will be presented.

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IMAGERY AND VISUALIZATION WITHIN PRE-SCHOOL AND ELEMENTARY NUMBER: A 2-YEAR CASE STUDY OF 11 SCHOOLS

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For two school years from Autumn 1998, 11 schools teaching 4-9 year old students adopted and pursued a number curriculum with a strong emphasis on progressively building imagery and visualization. Both authors studied this by visiting the schools, discussing developments with teachers, and observing lessons.

Research questions identified were:

- will the initial commitment of teachers to the more visual curriculum be sustained? (1. we assert that there is not a culture of persistence in English elementary education; 2. the 2-year period included, from Autumn 1999, the introduction of the English *National Numeracy Strategy*);
- will the students show themselves as in a different relationship to numbers – e.g. as having a greater 'at-homeness' with numbers – after 1 year, after 2 years of the visual curriculum;
- will the students' achievements with numbers be enhanced: *specifically* their number location, ordering, counting, complement-finding, calculating and calculation-adjustment abilities.

The oral presentation will:

- identify key features of the responses of teachers and their students, including gains in 'at-homeness' with numbers;
- illustrate these with respect to *one* of the teachers and her class;
- present evidence of increased abilities to use imagery and to visualize when dealing with number problems;
- present evidence of increased achievement levels within the aspects specified above;
- outline how these schools reacted to the English *National Numeracy Strategy*;
- relate this study to the ideas of Freudenthal, Gattegno, Wittmann and others.

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ACCELERATED LEARNING IN SECONDARY SCHOOL MATHEMATICS: A NEW ZEALAND PERSPECTIVE

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Within New Zealand there is a "growing awareness of the special needs of gifted and talented students and of the importance of providing them with an educational environment that offers maximum opportunities to develop their special abilities" (Ministry of Education, 2000, p. 6). However, despite National Education Goals that charge schools with the task of providing programmes that enable all students to realise their full potential, many (32%) secondary schools have no school-wide policy on gifted education.

This paper combines the findings of two related research projects in order to provide an overview of the types and frequency of programmes offered to secondary gifted and talented mathematics students and examines both teacher and student perceptions of accelerated programmes in mathematics.

While mathematics is the subject that is most commonly targeted for special attention for gifted and talented secondary school students, there exists a great variation in the types of programmes available. Factors such as school size, decile (SES) rating, proximity to competing schools, and school culture appear to influence both the availability and the scope and nature of programmes. Many schools that offer accelerated programmes reported problems in timetabling, parental pressure, and student achievement in subsequent years. However, 25% of respondent schools reported no problems.

In contrast, students' perceptions of accelerated programmes (4 case study schools) were very favourable. Contrary to fears identified in the literature the students perceived that inclusion in the accelerated programme had not affected their friendship base; they believed that they had a normal adolescent social and emotional development. Participant students felt that the accelerate programmes enhanced their learning and they identified no significant problems with compaction of the curriculum or gaps in knowledge. Issues of selection into accelerate programmes and reasons for students wanting to be accelerated in mathematics were often influenced by school culture.

While acceleration is currently one way to provide for gifted and talent mathematics students in New Zealand, this research suggests the need for schools to develop more cohesive and flexible programmes, with clearly identified goals, in order to best meet the needs of talented and gifted mathematics students.

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PROOF, PROOFS, PROVING AND PROBING: RESEARCH RELATED TO PROOF

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Research on proof and proving in mathematics education makes use of several different meanings for the words "proof" and "proving". In some cases this can lead to seeming contradictions in research findings. This paper identifies four current usages.

The concept of proof: Most professional mathematicians would say a proof of a statement confers absolute certainty on it. Some research has concluded that many secondary students do not understand proof in this way (see, e.g., Healy & Hoyles 2000). On the other hand, unpublished data (Zack 2000) suggests that some younger students may understand this concept of proof. For example, one Grade 5 student stated: "I think proving means showing that your answer is correct and it can't be wrong."

Proofs: There are sections of writing in mathematics textbooks and journals, which are called "proofs". They are characterized by a particular form and style. Writing proofs is a goal of recent reform documents but further research in this area is needed.

Proving: "Proving" is usually connected with deductive reasoning. Quite young children have been observed to be reasoning deductively (see, e.g., Zack 1997) but their reasoning seems to depend strongly on context. Additional research on the contexts in which children find deductive reasoning useful is needed.

Probing: Lakatos (1976) describes a process of "proof-analysis" in which "proving" is probing, testing the truth of a statement. Lakatos claims that his version of the nature of proving is incompatible with the concept of proof described above, which implies that researchers working from either perspective need to be careful they are not misunderstood as working from the other.

Note: A longer version of this paper is available at
http://plato.acadiau.ca/courses/educ/reid/papers/Pme-paper-2001.htm

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A TEACHING PROPOSAL ABOUT RATIO AND PROPORTION WORKED WITH STUDENTS OF ELEMENTARY SCHOOL

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The present document is the report of a research that deals with the topics of ratio and proportion whose great importance nowadays has been shown on the studies that the researchers of different countries have made during various decades. The problem of research consists in reviewing the strategies used by the Mexican students of 6th grade of Elementary Education when they solve problems involving ratio and simple and direct proportions, in order to identify qualitative and quantitative components of though linked to these topics and their diverse modes of representation; which is the base of the design and the application of a teaching proposal about, that was adapted to the school's program.

The theoretical and empirical antecedents include a revision the cognitive aspect as a part of a disciplinary development, as well as didactic and psychopedagogic aspects, which was taking from Freudenthal (1983), Streefland (1993) and Coll (1998); respectively. It was pretended to do a conjunction between the mathematical reflection about ratio and proportion and the didactic aspects. Teaching proposal was conformed by six teaching models, considering the definition that Figueras, O.; Filloy, E. and Valdemoros, M. (1987) give of teaching model, which were retaken in different sessions as they were required by the advance of the teaching process, which is similar to what Streefland (1990) points out in his Realistic Theory, referring to the strategy of change in perspective. The elaboration of different tasks shows a progression, starting with the review of qualitative aspects for reaching the quantitative because the recognition of ratio as a relationship between quantities and proportion as the relationship between ratios.

Two phases manifested, interconnected, of the progression of the methodological instruments. The first one was the exploration phase, this was done through the *observations and the initial questionnaire*, the second phase, that was the one of enhancement, it took place with the *teaching proposal and the interviews*. The observation as well as the Questionnaires gave a light for designing the teaching proposal which was constructivism-didactic; it allowed the students to give sense and meaning to the concepts of ratio and proportion. The validation of the methodological instruments was carried out through a process of piloting by a triangulation of different sources of register and of crossed controls. As a way of evaluating this proposal, a *final questionnaire* was used and after this one the *interviews* were designed. The interviews had two intentions: evaluation and feedback.

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SURPRISE AND HUMOUR IN TEACHING UNDERGRADUATE MATHEMATICS

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Nobody doubts that undergraduate mathematics teaching should be interesting. The elements of surprise and humour can help to solve this problem.

In our view, the most effective and useful surprises are of three types: 1) when the concepts and elements, familiar to students from one mathematical context, appear in another mathematical context, resulting in new contents - new properties of these elements, or other mathematical results; 2) unexpected interpretation (geometrical, graphic, or, on the contrary, algebraic interpretation of the geometrical facts), allowing to look at things from a new viewpoint and to receive, unexpectedly and effectively, new mathematical contents; 3) unexpected applications of mathematical concepts and results in new areas outside mathematics.

As well as in art, the surprises are more effective when they are well prepared. Any concept intended to be considered in a new, unexpected context, earlier should be imprinted on the minds of the students, so that they really could recall it in a new situation. Lecturers might imitate authors of detective stories: the keys to the disclosure of a crime or mystery usually are distributed in different parts of a story, so that the reader, even after overlooking these moments, at once recollects them in a final scene. In mathematics it means that the lecturer, introducing a new concept, should make it unusual, connect it with an interesting example, method or application.

The humour is connected to surprises: to a paradoxicality of the phenomena, unexpected connections of serious with ridiculous. If the purpose of elements of unexpected is to make learning interesting, the purpose of humour is to give ease to the process of learning.

The humour is proper both for oral teaching and for textbooks. Certainly, the elements of humour in oral teaching have features, quite different from those in the textbooks. In oral teaching there is a place to the improvisation, it is possible to use peculiarities of an audience, moment, external circumstances, of a lecturer's own mood. However, in oral teaching the humour frequently is of the external (w.r. to mathematical contents) character, being not related to mathematics and aiming only to create a respite or pause for switching or refreshing attention.

In the textbooks the humour, as a rule, is in greater extent related to elements of the mathematical contents, connecting this contents with non-mathematical (e.g., everyday) things or pushing it together with other intellectual contents (e.g., humanities or other manifestations of the spiritual life), giving birth to paradoxes and amusing contradictions.

An Approach to Teachers' Knowledge about the Mathematical Concept of Volume

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In Mexico, the study of the concept of volume forms part of the program of studies for primary education. In accordance with Freudenthal (1983) the objective of mathematical education should be the forming of rich, well founded mental objects, related with the concepts of mathematics, nevertheless, experience in workshops with teachers of elementary school indicate that on occasion these "mental objects" of teachers are very limited. On the other hand, the participants in these workshops have pointed out volume as one of the thematic contents in which they found more difficulties for the learning of the children.

These considerations motivated an investigation project whose principal objectives to characterize beliefs and knowledge of the teachers related with the concept of volume and its teaching. The theoretical and methodological frameworks, as well as some preliminary results, have been commented on Saiz & Figueras (1999 and 2000).

Evidences that teacher beliefs and knowledge affect the way they teach have been obtained in different studies (Thompson, 1992). The work reported herein fits in this line of investigation, now that it focuses attention on the beliefs with respect to a mathematical concept and its teaching. A group of elementary school teachers were video and audio taped during a workshop designed to permit teachers spontaneously comment and share their ideas with their peers when working with tasks related to the concept of volume.

Some segments of these tapes' transcriptions have been analyzed. Until now, some results related to the teachers' subject-matter knowledge have been obtained: a) Teachers define volume as the place occupied by a body in space; nevertheless they themselves use this term, at times, as a synonym of capacity. b) They relate the raising in the level of liquid upon submerging an object to the weight, not to the volume. c) They believe that the lateral area is directly related to volume, that is, they believe that the greater the lateral area, greater volume and vice-versa. d) They have problems with the conversions between distinct units. e) They believe that enlarging the linear dimensions of a solid in k times gives place to k times enlargement in volume.

As collateral results of this investigation some are found that point toward the formation and actual circumstances of the teachers. When they work in groups and confront designed didactic situations –with the purpose that they themselves discover misconceptions and erroneous beliefs– they reconsider their own knowledge. They think over the difficulties that the children can confront, which will cause them to reflect about their role in the classroom, as it has been the situation in the workshop herein commented.

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INSTITUTIONAL RELATIONSHIP VERSUS PERSONAL RELATIONSHIP OF MOZAMBICAN MATHEMATICS TEACHERS TO INEQUALITIES

By Luis San, Ribas Guambe, Ida Alvarinho and Torgeir Onstad

The understanding and handling of inequalities by pupils is analysed into the general framework of algebra initiation. The transition from arithmetic to algebra is a deep qualitative step, which is, often assumed by pupils as an introduction of a set of rules that constitute an aim in them selves (Sfard & Linchevski, 1999)[1]. It is within this theoretical framework that this present study intends to identify the related factors that influence, in secondary schools in Mozambique, the pupils' skills and understanding of the concept of inequality and its solution. It is a correlated research where we intend to explain a phenomenon(a), relating some variables such as the institution, the teachers, and the pupils.

In terms of institution we analysed the syllabi, textbooks, and final exams. Moreover, 20 teachers of grade 10 answered a questionnnaire which aimed to collect data from the teachers them selves about inequalities so as their opinions about the performance of pupils in this topic.

These activities had as the start point the following hypotheses: (1) It seems that there is some confusion between the concepts of equation and inequality; (2) in solving inequalities, the relation between the numerical, algebraic and graphical methods, if it is done, is very incipient and (3) the resolution of inequalities seems to be assumed as a "game of mathematical symbols" without any relation to reality, loosing the sense of what one is really looking for.

The findings corroborate the hypotheses assumed beforehand. At the institutional level it seems that the inequalities are handled in a disorganized and discontinuous way as follows: (1) Different types of inequalities are treated in different ways. Some are treated as algebraic and others as function inequalities; (2) some grades inbetween do not handle inequalities; (3) Questions formulation in most of final exams allows rigidity instead of flexibility of resolution methods and (4) the syllabi and textbooks seem to reinforce the similarities between equations and inequalities. According to the results of the questionnaire, teachers are faithful implementers of what the institution sets up and pupils face a lot of difficulties when solving inequalities. This study recommends curriculum changes that help a (smooth) transition from arithmetic to algebra (Gimenez,)[2] taking into account what Banzini[3] says: "the passage from natural to symbolic language is a key point in the development of algebraic thinking and asks for special attention" (Banzini, 1999).

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TEACHERS' PRACTICE AS A PROBLEM SOLVING ACTIVITY

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This presentation, based in a larger investigation (Santos, 2000), focus on the professional problems posed by mathematics teachers in a process of curricular change. Many authors have proposed to regard professional practice as a problem solving activity. Dewey (1910/1997), associated thinking to problem solving, Schön (1991), spoke about professional problems and their ill-structured nature. In this study, I tried to understand more deeply the characteristics of the problems that teachers daily face and solve in individual and collaborative contexts of practice.

The study methodology stands on an interpretative approach, using case studies. A secondary school was selected where three teachers planned to work collaboratively, applying for the first time a new mathematics program in their 11h grade classes. Data was gathered by observing collaborative work and each teacher's classes, carrying out reflective sessions concerning these classes and in interviews.

Results strongly suggest that professional problems are ill-formulated and ill-defined. They are progressively reconstructed and understood by teachers along their work. In the collaborative context, the problems are range over a great variety of contents (mathematics; didactics, organizational) and are marked by their global character — normalization is valued. Their starting point is the curriculum and the end point is the practice; some of them enter in the public domain. In the individual context we find that problems are centered on the didactical knowledge. They are very specific and particular — singularity is valued —, starting and ending in practice, while curriculum has a guiding role, and they belong to the private domain. The main processes used are analysis, consultation and "living with the problem". The extent to which problems identified in the collective working context are solved much greater than that found regarding the individual context.

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AFFECT AND COGNITION - TWO INSEPERABLE COMPONENTS OF MATHEMATICS LEARNING

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McLeod (1992) has emphasised the central role of affects for the learning of mathematics and characterised the affective domain in mathematics education by three types of affect "beliefs", "attitudes" and "emotions". The latter are ranked in order of increasing intensity and decreasing stability. There are many approachs to linking affect and cognition and these were characterised by Evans (2000) according to four types of models. The aim of this paper is to analyse the relationship between affect and cognition on the level of the individual learner. The concept of "affect logic" (due to Ciompi (1982)), which combines the psychoanalysis of Freud with the genetic epistemology of Piaget, were employed. Within this concept, the psyche is understood as a complex hierarchical structure consisting of affective-cognitive schemata. These are the result of maturation and learning processes which are based on assimilatory/accomodatory interactions with reality. The entirety of affective-cognitive schemata form at each moment the "world view" of an individual and control further learning processes. If we regard affect and cognition as two inseparable componets of a mental unit, we are in a position to understand why we can know anything about our feelings. This type of knowlede is used in the research of beliefs and attitudes.

Roth (1996) sees the role of affects as an assessment system for thinking and acting. This aspect allows us to understand the importance of social factors for the learning process.

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THE QUALITY OF INSTRUCTION IN THE CONTEXT OF REFORM

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Research has identified elements of U.S. school capacity that directly affect instruction, sociomathematical norms that affect the learning environment, factors that support students' reasoning, ways to judge the intellectual quality of class work, and ways to examine cross-national differences in instruction. Remaining to be developed, however, are research methodologies that characterize instruction in ways that examine the impact of standards-based curricula on student achievement. This report focuses on the results of the development and use of a composite index that characterized and described the variation in instruction among teachers in a longitudinal/cross-sectional study using the reform curriculum *Mathematics in Context* (MiC; NCRMSE & Freudenthal Institute, 1997–1998).

Standards-based curricula present mathematics in ways often unfamiliar to U.S. teachers and require more planning for successful instruction than generally perceived necessary for teaching via traditional methods. In MiC, topics traditionally reserved for high school are introduced in middle school using real-world contexts, with an emphasis on student reasoning rather than on memorization of procedures, providing challenges to U.S. teachers, who are often accustomed to teaching mathematics as isolated pieces of knowledge (Romberg, 1997).

The index used to examine instruction discriminated differences in instruction among study teachers in two districts who used either standards-based or conventional curricula. Variation was found by grade level, curriculum, and district. These results confirmed (a) the significance of integrating assessment practice into instruction, with the content corresponding to content standards and emulating what it means to know and do mathematics; and (b) the need to develop instruction that emphasizes teaching mathematics for understanding, that is, developing classroom norms in which students value and participate in discussion, providing meaningful tasks, creating opportunities for students to articulate their thinking orally and in writing, and seeking multiple forms of evidence of understanding from each student in class. The results also suggest that even in classrooms using reform materials, students are rarely given opportunities to reflect on and express their own mathematical ideas or to listen to the reasoning of other students. Teachers' own understanding of the mathematics, the ways the mathematics is presented, and their developing pedagogical content knowledge related to the units have a significant impact on classroom instruction and on student achievement.

The findings in this study underline the need to take into account such variation in instruction in any interpretation of the impact of reform curricula on student achievement.

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Argumentation Reasoning and Mathematics proof

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This study reports a way to help students to organize their strategies to write mathematics proofs. The purpose of this is to justify why argumentation is important as a pedagogic strategy to facilitate the study of mathematics proof. In Brazil the taught of proofs follows a traditional deductive approach, where students try to repeat what their teachers wrote before. These students are engaged in a course with emphasis on deduction and proof but they are, in general, unable to understand logical deductions and consequently cannot write their own demonstrations. The researcher concluded that there is a necessity of develop a rhetoric argumentative thinking as mentioned by Perelman at the theory of the Argument or New Rhetoric. For him the New Rhetoric can be involved in the context of discussions about relationship between formal language and natural language. For that students are asked to talk aloud about their reflections. The purpose is for them organize their own reflections about their study and understanding deeply their strategies. They try to convince others about their thinking wishing to have others to affirm their statements. They make observations and conjectures, keep a record off all decisions made by the class, while use prior statements to validate or support new conjectures. After this students are asked to write logically their conclusions based on their according and register their observations producing a text to explain these conclusions. On this way the teacher is able to identify some of the organizing strategies students used for constructing their proof while students write informal proof naturally and with sense for them.

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TEACHERS' BELIEF STRUCTURE IN GENDER AND MATHEMATICS

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The focus of this paper is to examine teachers' beliefs about the differences of boys and girls (aged 13-15 years) as learners of mathematics. For this purpose we developed a questionnaire with a new answering scale. A sample of Finnish teachers of mathematics, 110 female and 94 male, answered to this questionnaire in February 2000. They classified a list of characteristics as being more frequent among girls or among boys in their mathematics classes.

Research on affect and mathematics has focused on the affective responses of students rather than those of teachers (McLeod, 1994; Li, 1999). Teachers' knowledge of, and beliefs about, mathematics have been studied from the perspective of cognitive science, but this perspective is less used in studies concerned with gender (Fennema & Hart, 1994). Studies that deal with mental processes of teachers might give insight into why teachers interact with boys and girls the way they do.

G	usually a girl	-2
g	a girl more often than a boy	-1
±	a girl as often as a boy	0
b	a boy more often than a girl	1
B	usually a boy	2

Table 1. Alternatives and scores for X.

Our questionnaire consisted of 55 items of the type: "X finds mathematics difficult." For each statement, teachers had to select the subject X out of the five alternatives: G, g, ±, b, and B. In the analysis the neutral alternative was scored as 0, the direction "girls more often" was scored negative and the direction "boys more often" positive as presented in Table 1.

The results indicated that a great majority of teachers held different beliefs about girls and boys as mathematics learners. Factor analysis revealed six main factors in the structure of beliefs about gender differences: *Avoid using intelligence*, *Talent*, *Lack of equity*, *Work-orientation*, *Expectations of success*, and *Teacher attention*. The two most common beliefs stated girls avoid using intelligence and boys attain most of teacher attention. Beliefs are gathered as parts of a belief system (Green 1971, p.42). Correlations between the six belief dimensions will be discussed in the presentation.

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The atomic analysis of the conceptual field: *similarity*

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My work was related to the very large research problem: structuralisation.

Filip, a junior-secondary student, was subjected to a series of tests and observations. As the 10 years old boy, at the end of 1996 he took part in the experiment in which he was taught how to understand the statement: "figures have the same shape". Three years later I met him again. During the investigation I tried to describe the actual level of his competence by using an informal idea of similarity.

The first part of the investigation was related to the former experiences of Filip. His reaction to the task which he solved in 1996, and tools used, gave me a possibility to describe the actual structure of his field concerned transformations. It contained three almost separate conceptual fields: deformation, isometries, and similar figures. The differences between concept derived mainly from their epistemological nature.

During the second part of the investigation Filip solved a new series of tasks related to similar figures. After that he was asked to describe that concept. The atomic analysis showed how the solving process changed the scheme of similarity. I isolated 14 phenomena which described the mathematical activities connected to his work on the task. The analysis showed also the new linkage created between the concept "similar figure" and the concept "isometry".

New capacity of the concept treated gave Filip a chance to build the new meaning for the statement: "figures have the same shape".

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EFFECTS OF PRE-SERVICE TEACHING PRACTICE ON TEACHER-STUDENTS' EVALUATION OF PROBLEM SOLVING STRATEGIES

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In Hungary primary teachers' preparation includes about 500 hours of teaching practice, with more than 400 hours of teamwork concerning classroom experiences. My presentation will deal with the following question: How does the teaching practice influence the teacher-students' evaluation of pupils' problem solving strategies? In order to tackle this question, an experimental study was planned. Two samples of 25 second year (before teaching practice) and fourth year (after t. p.) students were chosen. Ten primary school pupils' solutions to the same problem were analysed by the teacher-students, according to the following guidelines:

A) Try to describe the ways of thinking of this pupil.
B) Evaluate the solution, if the maximum score is 7.
C) Evaluate what this particular pupil knows, and what are the necessary elements in his or her knowledge to be acquired in the direction of solving this problem.

The problem was:

With 32 forints for stamps one can mail a letter weighing no more than 250 grams. Eva has an envelop weighing 14 grams. How many drawing sheets, weighing 16 grams each, may she put in the envelop in order not to exceed (with the envelop) the weight of 250 grams?

It was chosen because it is a true 'word problem' (not an usual 'exercise' in Hungary!) and, according to precedent studies (see Boero and Shapiro, 1992) it allows different strategies (in particular, trial and error strategies and pre-algebraic strategies). The ten solutions were chosen amongst 300 solutions produced by fourth grade Hungarian pupils. The chosen solutions contained: different strategies; different mistakes (in particular, ineffective calculation or notation mistakes); correct numerical answers with evident loss of meaning, or wrong but reasonable numerical answers; etc.

Some preliminary analyses of student-teachers' protocols show: an overall stability of evaluation (mainly centred on formal correctness) across two years of teaching practice (and attended courses), especially as concerns item B; and (especially for items A and C) some changes as concerns the depth of analyses, the mastery of professional-technical expressions and the security in perfoming the task. This study raises interesting questions about the reasons for stability, and how to change the student-teachers' deep attitudes towards pupils' ways of thinking.

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SOME CHARACTERISTICS OF STUDENTS' INNER WORLD IN LEARNING SCHOOL MATHEMATICS

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Takahashi(2000) proposed that students had internal frames of reference which were their own grounds to view something, to perceive something and to reason in learning school mathematics. Four participants were interviewed two times in grade six and two times in grade seven. Eight modalities of the participants' inner worlds, which make up their internal frames of reference, were found. They were their view of mathematics, attitude/affection towards mathematics, objectifying knowledge, influence of significant others, ways of learning, relating mathematics with daily life, relating mathematics with other subjects, and formalized mathematical knowledge.

This research was extension of Takahashi(2000). The purpose of this research was to explore some of the characteristics of modalities of internal frames of reference in students' learning mathematics through additional two interviews.

This research was based on Polanyi's theory of knowledge. Polanyi (1958) proposed that scientific knowledge doesn't exist as an impersonal universally established. Mathematical knowledge of a society also consists of relationships between personal knowledge and includes a tacit dimension. Owing to the tacit dimension we can try to know actively through intellectual passion, images, and the belief in the existence of mathematical answers to problems.

Knowing depends on a conceptual framework which either assimilates new experiences or ideas, or adapts to them. The framework includes a tacit dimension which supports and fosters the activity of the framework. The tacit dimension is key domain to form internal frames of reference. Polanyi's theory proposes that knowing constructs metaphorical relationships in our mind. The theory has the same standpoints as Lakoff, Johnson and Núñez'.

Three of four participants were interviewed six times and one was interviewed five times. At fifth and sixth interviews the participants were eighth graders.

As results there were remarkable changes in two participants' modalities. One student changed her way of learning based on the change in her view of mathematics. Another student changed his way of learning based on the relationship with his friends. There weren't remarkable change in the other two participants.

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STUDENT'S TRANSLATIONS BETWEEN MODES OF REPRESENTATIONS AND ITS UTILISATION IN DIDACTICAL RESEARCH

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Results of many investigations confirm great significance of the development of representations for the process of education. Halford (1993) noted agreement between theorists „... that understanding means having an internal representation or mental model that corresponds to a concept, task or phenomena (p. 9)." Bills and Gray (1999) pointed out the relationship and connections between internal and external representations. Lesh, Post and Behr (1987) identified five types of representation relevant for the teaching of mathematics. Janvier (1987) stressed that the level of understanding is related to the continuous enrichment of a set of various modes of representation and Bönig (1994) emphasised the development of student's capability of translation between modes of representation.

These works inspired the present research that follows the previous investigation (Tichá, 2000). This contribution focuses on the study of possibilities for utilising the students' translations between modes of representations (a) as a diagnostic means in the course of study of images and understanding, misunderstandings and obstacles for understanding of the notion of fraction, and (b) as a re-educational method that helps to eliminate students' difficulties, insufficiency and obstacles for understanding.

Sample: 10 - 13 years old students from 15 classes at various places in the Czech Republic. Classes were in no manner specialised. In one of classes the students worked in groups since we wanted to observe communications.

Background: Students' formulations of word problems with fractions to the given calculation or to the given pictorial representation.

Method: Analysis of written work. Students were asked to write only on the sheet with printed problem, without erasing anything. With some of them, we discussed "their problems" after they submitted the test.

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ELEMENTARY SCHOOL PROSPECTIVE TEACHERS' SOLUTIONS OF $200 \div 50 \div 10$

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It is widely agreed that students' ways of thinking, their correct and incorrect ideas, should play a role in teaching (e.g., NCTM, 2000). This paper illustrates a way to base instruction on students' reactions to mathematical tasks.

A class of elementary school mathematics prospective teachers was asked to solve $200 \div 50 \div 10$ as part of a weekly home assignment. An examination of their solutions revealed a variety of correct as well as incorrect methods that led to different results. The correct solutions included, $200 \div 50 \div 10 = (200 \div 50) \div 10 = 4 \cdot \frac{1}{10} = 0.4$, and $200 \div 50 \div 10 = 200 \div (50 \cdot 10) = 200 \div 500 = 0.4$. The incorrect solutions included, for example, $200 \div 50 \div 10 = 200 \div 50 \div 10 = 20{:}5{:}1 = 4$ and $200 \div 50 \div 10 = 200 \div (50 \div 10) = 200 \div 5 = 40$.

In order to trigger the prospective teachers to reflect on their own and on their peers' solutions, a class assignment was designed, consisting of their correct and incorrect solutions for the above task. In a 90 minutes class session, the prospective teachers were first asked to work individually and decide whether each suggested solution was correct or not, and why. Later in this session they formed small working groups, and had to discuss their ideas and come up with an agreed response to each suggested solution. Finally, each group presented its decisions to the entire class, followed by further discussions.

In the written responses and in the class discussion the prospective teachers exhibited a rule-based mathematical approach. For instance, several students claimed that "In division one can reduce the elements by cancellation" as a justification for $200 \div 50 \div 10 = 200 \div 50 \div 10$. Some others stated that "there is no commutative law for division, hence $200 \div 50 \div 10$ cannot be computed as $200 \div 10 \div 50$ (see also Tirosh, Hadass & Movshovitz-Hadar, 1991).

Additional findings regarding the prospective teachers' criteria for accepting or rejecting the suggested solutions, and regarding the consistency of their solutions will be provided in the oral presentation.

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Teacher behaviours in computer based mathematics – gender implications

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Mathematics curriculum policy in Australia requires teachers to use information technology as a focus and aid to students' learning and understanding of mathematics. However it is unclear how the use of technology will impact on the participation, engagement and outcomes of girls and boys (Burton & Jaworski, 1995).

In the study reported in this short oral communication the behaviours of two secondary mathematics teachers, who used computers as a resource in their mathematics lessons, were explored. The study was part of a larger ethnographic study concerning the culture of computer based mathematics in secondary coeducational classrooms. Two mathematics teachers, a year 8 and year 9 mathematics class from an urban school participated in the study. The year 8 class used computers in a laboratory for two of their five mathematics lessons each week. During the period of the study they were learning to solve multi-step equations and used *Microsoft PowerPoint* to display their understanding. The students in the year 9 classed owned laptop computers and used *Geometers' Sketchpad* to investigate and demonstrate properties of particular shapes during the period of observation.

Six computer based mathematics lessons for each class were video taped, documents collected and four students from each class and the two teachers were interviewed. Data were analysed qualitatively. Teacher behaviours were coded for the content and cognitive nature of interactions (Geiger & Goos, 1996), the attitudes and feelings about people conveyed in their behaviour (Lee, 1993), the methods used to teach with computers and ways of solving problems. Teacher behaviours were compared for interactions with girls and boys and with high and low achieving mathematics students.

Data that illustrates the way that teachers' behaviours contributed to the culture of these classrooms will be presented. Whilst each classroom was culturally different, and influenced by other factors in addition to teacher behaviour, they advantaged the learning of high achievers and boys.

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THE ROLE OF COMMUNITY IN LEARNING TO TEACH MATHEMATICS

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The current environment of reform in the United States, prompted in great part by calls published by the National Council of Teachers of Mathematics, has created the opportunity for those who are learning to become teachers for the first time to partner with experienced teachers who are learning to teach in a new way. For the past four years, the Learning to Teach in a Reform Environment (LTRE) project has been investigating this learning process for six secondary school mathematics teacher education students from Western Michigan University. During their two-semester pre-intern and one-semester intern teaching experiences, these students partnered with teachers from a school that was implementing the Core-Plus Mathematics Project (CPMP) curricular materials. They also attended two week-long summer workshops with teachers from the school to learn more about teaching the CPMP curriculum effectively. The partnership was designed to provide three key benefits beyond a traditional field experience: 1) including the preservice teachers in a community of learners that had made a commitment to continually improve their teaching to meet the challenge of current calls for reform; 2) putting the preservice teachers in classrooms with teachers who were learning to teach an innovative mathematics curriculum; and 3) ensuring that the field experience was consistent with the teacher education coursework at the university.

Extensive interview and observation data was collected from the participants during their pre-intern and intern teaching experiences and follow-up data was collected through their second year as full-time teachers. In this short oral presentation, results of the analysis around the issues of community will be discussed. These issues include: the effect of participating in a community that is in the process of redefining itself, the different nature of discussions depending on which members of the community are present, and aspects of the community that the interns sought out or tried to recreate in their permanent teaching positions. Looking at these issues leads to a better understanding of the role of community in optimizing the benefits of field experiences.

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TEACHING ANGLE CONCEPTS BY ABSTRACTION AND GENERALISATION: AN EXPLORATORY STUDY

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Mitchelmore & White (2000) have recently advanced a developmental theory, based on the theory of abstraction, which attempts to explain the formation of a single concept for angle which results from recognising the underlying similarity between a variety of angle contexts which all involve two lines meeting at a point with some meaning (often "turning") attached to the opening between them. An angle diagram represents the common features of all the different angle contexts.

The authors have shown conclusively that children rarely recognise the dynamic similarity (turning) between two angle situations. and more easily recognise the static similarity (two lines meeting at a point), the facility depending on the salience of the two lines which form the angle (over 90% of Grade 2 children where the two lines are obvious in situations such as tile corners and scissors; but only about 40% of Grade 6 children where one or both lines has to be imagined in situations such as hills, doors, and wheels). Based on their findings the authors tested a method for teaching angles which focused on the lines (in particular), vertex and opening in paired situations where one situation had both arms of the angle visible and one situation did not.

The data showed that focusing students' attention to the individual attributes of an angle helped them to make appropriate matches, to identify the critical features of an angle, to recognise irrelevant matches, and to draw abstract angle diagrams. However, it was not as effective as hoped because it sometimes seems to have led to an angle being viewed just as a line, a vertex, or an opening, and not as a single mental object integrating all three. The data also showed that focusing on these features in a 2-line situation often did little to help students recognise angles in 1-line situations or 0-line situations. The data on drawing abstract angle diagrams at first seems paradoxical because even though students could not match the angles in a pairs of situation, they easily learnt to make an abstract drawing of an angle. This finding can be regarded as a *non-transitivity* effect: Being able to match a drawing to two angle situations does not imply being able to match the two situations.

In hindsight, we realise that we failed to correctly interpret our own research findings and in fact used an approach which is the exact opposite to that suggested by the theory of abstraction. Hence, taken overall, the teaching sequence suggested is supported, but with a specific focus on the three critical features of an angle occurring *after* the appropriate similarity has been recognised, not before. Further teaching experiments are currently being undertaken to test the revised approach.

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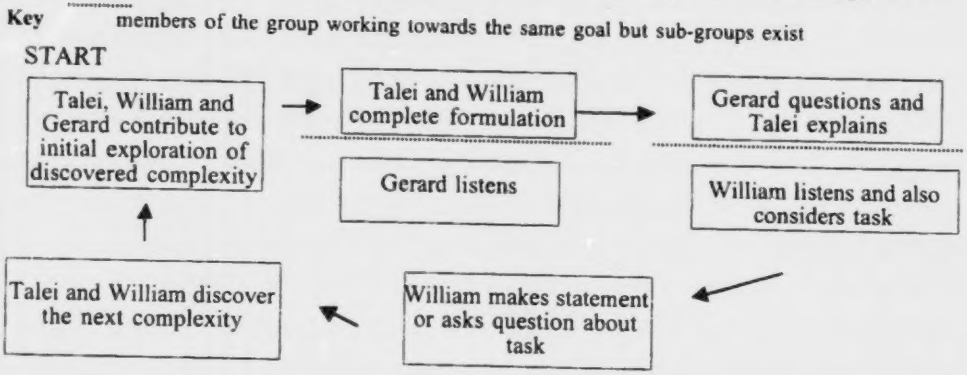
SUSTAINED OVERLAP IN ZONES OF PROXIMAL DEVELOPMENT? IF SO, HOW DID THAT HAPPEN?

Gaye Williams; University of Melbourne; email: *g.williams@pgrad.unimelb.edu.au*

A collaborative group of senior secondary calculus students sustained a high level of engagement with an unfamiliar challenging problem and progressively developed concepts new to all group members. This is explained using a schematic representation that connects aspects of the *zone of proximal development* (Vygotsky, 1978) with *flow* (Csikszentmihalyi & Csikszentmihalyi, 1992) and *discovered complexity* (Williams, 2000). Discovered complexity occurs during task completion if a problem solver or a group of problem solvers perceive *intellectual* and *conceptual complexities* (Williams & Clarke, 1997) not evident at the commencement of the task. This altered perception arises when the group spontaneously formulate a question (intellectual challenge) that is resolved as they all work with unfamiliar mathematical ideas. Discovered complexity meets the conditions for flow; students work just above their present skill level to meet a challenge almost out of reach. Even though the group members are progressing through their individual ZPDs, the *expert other* (Vygotsky, 1978) is not apparent. Evidence of the group's interaction pattern (Figure 1) and a profile of each student's individual ability to solve unfamiliar challenging problems (Krutetskii, 1976) are used to explain how the composition of the group facilitated concept creation.

Figure 1 Usual interaction pattern as the group discover complexities


Key .... members of the group working towards the same goal but sub-groups exist
START
Talei, William and Gerard contribute to initial exploration of discovered complexity
Talei and William complete formulation
Gerard questions and Talei explains
Gerard listens
William listens and also considers task
William makes statement or asks question about task
Talei and William discover the next complexity

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A Community of Practice associated with Graphics Calculators

Peter Winbourne, South Bank University, UK; Bill Barton, Auckland University, NZ;
Megan Clark, Victoria University, NZ; Gerri Shorter, St Cuthberts College, NZ

This report will present some results of a collaborative research project exploring the way secondary school students use graphics calculators. It builds on and complements the work of other colleagues involved in the project (Graham and Thomas, 2000). Here we will discuss how graphics calculators have affected the lives of some senior students: how the technology has helped or hindered their learning; how it has affected the mathematics they learn in mathematics classrooms and how they see the mathematics they are required to learn in other classroom contexts.

The report will be framed by theories of situated cognition (Lave 1996, Lave and Wenger, 1991, Wenger 1998) and adopt the perspective derived from this (Winbourne and Watson, 1998, Winbourne, 1999) where mathematics classrooms are seen as multiple intersections of practices and developing identities.

Students in the study attended schools in England (London) and New Zealand (Auckland). The schools were chosen for their support for graphics calculators in senior classes and for the general and systematic use of ICT in teaching and learning. Student interviews revealed a wide range of student use of graphics calculators. In particular, they revealed students' predisposition to 'see' uses for the technology in their non-mathematics lessons. This predisposition, varying slightly between the schools, was seen to be practically significant. For example, it allowed students to engage in spontaneous graphics calculator-based mathematical discussion in 'non-mathematical' contexts. It was also seen to be theoretically significant; from our perspective, we see this kind of spontaneous discussion as signifying the growing and developing mathematical practices participation in which is central to the success of these students. Such 'predispositions' then, are taken to indicate a community of practice associated with graphics calculators. We will suggest that they are thus desirable as outcomes of teaching and discuss how teachers might aim to achieve them.

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WHY AND HOW TO PROVE:
THE PYTHAGORAS' THEOREM IN TWO CLASSROOMS

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We have long recognised the pedagogical significance of proofs, even informal proofs, in the mathematics classroom. The teaching of proof concerns itself more with meaning and understanding than validity. Wittmann (1996), in his case with the teaching of the Pythagoras' Theorem, has suggested that those diagrammatic proofs are conducive to students' construction of conceptual relationships.

Despite such theoretical concerns, however, we have been increasingly aware that for mathematics learning in the classroom context, a wide range of factors, intertwining with each other, are impinging on students' understanding of what purposes a 'proof' serves and how it constitutes a mathematical proof (e.g. Sekiguchi (1992)). Taking on board the significant shaping effect of everyday classroom practice, analyses of teacher-student interaction (mainly found in recent German mathematics-education research) from the sociological interactionist position look closely into the communication and negotiation process occurring in the mathematics classroom.

Against this general background and following the method of analysis proposed by Steinbring (1998), teaching episodes are taken from two classrooms (at Secondary 2 level in Hong Kong) during the teaching and learning of the Pythagoras' Theorem. Attempts in using Steinbring's conceptual tool, the epistemological triangle, open a window on our understanding of how students conceptualise the Pythagorean formula (i.e. the symbolic representation $a^2 + b^2 = c^2$) in relation to certain reference contexts. Although the teachers took more or less the same approach (i.e. started from a few examples and then went through the same 'dissection proof' to reach the general theorem), results of analysis reveal that the development of the mathematical arguments and explanations in the classrooms unfolded in their own self-referential ways, qualitatively different from each other. Among others, the difference lies in why the proof was perceived as needed, and also in how the proof was done. These two aspects are particularly considered as regards the possible meaning of proof conjointly constructed and emerged amidst the teacher-student interaction.

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POSTER PRESENTATIONS

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STATISTICAL MINITOOLS IN A LEARNING TRAJECTORY

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The three demonstrated statistical minitools are Java applets that can run via the Internet (e.g. www.fi.uu.nl/~arthur), but are also available as stand-alone applications. They have been designed for teaching experiments in grade 7 and 8 for Cobb, McClain and Gravemeijer of the Vanderbilt University (USA), and are now used for Dutch experiments of Bakker in grade 7.

At first sight, user-friendly data-analysis software packages seem to be the self-evident accessories for exploratory data analysis. However, working with such packages rather signifies an end point of the intended instructional sequence than a means of supporting it. Instead of using ready-made statistical tools, the present minitool sequence incorporates software tools that can be used for *learning* exploratory data analysis.

The point of departure is a bottom-up approach in which the minitools are perceived by the students as sensible tools that are compatible with their actual conception of analyzing data. For the students, the primary function of the minitools is to help them structure and describe data sets in order to make a decision or judgement. In this process, notions such as mean, mode, median, skewness, spreadoutness, and frequency may emerge as ways of describing how specific data are distributed within this space of values. Further in this approach, various statistical representations or inscriptions like histogram and box plot may emerge as means of structuring or describing distributions. In fact, the minitools are so designed, that they can support a process of progressive mathematization by which conventional statistical concepts and representations are reinvented.

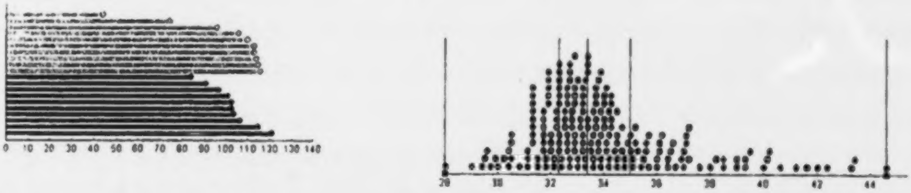

Figure. Minitool 1 and 2: value bar graph and stacked dot plot.

In minitool 1 every measurement is displayed as a bar. The data can be sorted and investigated with several tools. While working on problems with the simple bar representations, the students learn to relate characteristics of the graphs to the meaning within the context of the problem. If this relationship is consolidated, a more advanced representation is used: not bars but dots in a stacked dot plot are used, the representation of minitool 2. The data in minitool 2 can be organized in several ways, e.g. make your own groups, fixed interval width, two and four equal groups. The last three are useful precursors to histogram, median and box plot. Later, the students can go on with minitool 3, which offers a multi-dimensional scatter plot with several grouping options.

THE GRANDPA'S MATH BOOKS

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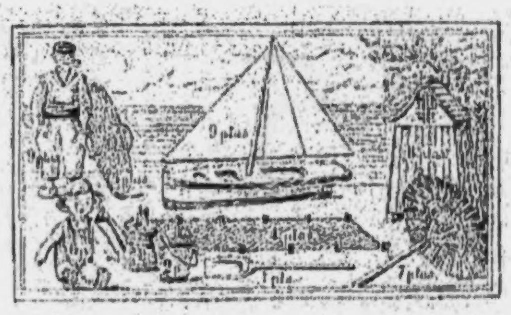
The years from 1936 to 1939 were a time of Civil War in Spain. The conflict brought to an end the progressist ideas of the Second Republic, and cut off any impulse for innovation in the different regions of the country.

Concerning mathematics and education, proposals coming from foreign authors such as Decroly and Montessory had profoundly influenced teacher training during the years before the war. English textbooks were translated into Spanish to *modernize* the learning of mathematics, and interesting proposals written by Spanish authors became broadly diffused in the form of textbooks. It was a moment of change and renovation for the whole of Europe, but abruptly stopped in Spain because of the war.

New perspectives opened up during the sixties when *old* teachers -those plenty of enthusiasm before the war- worked, hand with hand, with young recently graduated teachers. It was again a moment of change and eagerness for recuperating their history, which we also wanted to recuperate by consulting their textbooks and their class notes; by chatting with those *young teachers*, the grandpas and grandmas of today. The result of this experience permits us to value many things we do today in our school with a different perspective: mental calculations; manipulation of different materials and use of drawing tools; sequence for presenting contents, etc. And the most beautiful thing that this experience permits us is to gather so many grandpa's hopes about teaching mathematics, so many histories of life!



Juego de compras

En la playa:

Una barca....................	9	pesetas
Un pescador..................	0	»
Una red......................	4	»
Una panera...................	5	»
Un niño......................	6	»
Un cubo......................	2	»
Una regadera.................	3	»
Una pala.....................	1	»
Una sombrilla................	7	»
Una casita...................	8	»

The presentation shows some examples extracted from the books we consulted and from the interviews we made.

PHYSICS AND MATHEMATICS AS INTERRELATED FIELDS OF THOUGHT DEVELOPMENT

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Students of all ages struggle with Physics not only due to the complexities of the subject, but also due to inadequacies with their skills and knowledge of Mathematics. Mathematics is the "language" of Physics and it is clear that the learning problems of students in Mathematics are transferred to the learning environment in Physics. Lecturers and teachers in Physics consume hours of teaching/learning time to "redo" the Mathematics required to describe physical phenomena. Internationally it is common to find courses such as "Mathematics for Scientists and Engineers" as part of curricula in Physics, since Physics teachers feel that they are in a better position to teach the Mathematics required for their courses. It is proposed that the learning environment in Physics in particular and Science in general, could be much more effectively structured when Physics/Science and Mathematics are considered as interrelated fields of thought development.

Concept formation in the physical sciences depend heavily on two aspects namely a (1) suitable context in which the physical phenomenon takes place and (2) an accessible language in which to comprehend and express these physical phenomenon. One of the criteria for a suitable context would have to be that it is accessible from the real world in which the child/student finds him or herself at that moment as well as that the concepts (e.g. length) within the context, adhere to the psychological (cognitive) structures which the child/student has available at that moment. The context must also be utilised in such a way so that the child/student is exposed to a multitude of interactions in order to prevent him/her from forming so called limiting constructions. The teaching of the physical sciences at school level (secondary or primary) presupposes that some basic building blocks (e.g. position, length, direction, etc.) are in place. By the time the child reaches high school the endeavour is to develop scientific concepts that are already a combination (e.g. velocity, acceleration, etc) of the basic building blocks. The majority of these scientific concepts have their foundation in mathematical sub-structures (magnitude, space, time etc) that the child encounters from his first day of formal schooling (and long before that from birth).

We would like to propose a model that reflects the complexity of this interaction between mathematics and physics with the endeavour to assist in the design of appropriate materials and activities for the development of specific scientific concepts using acceleration as an example.

PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS: TWO RESEARCH PROJECTS

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Fátima Alonso Guimarães, Escola 2,3 do E. B. de Telheiras

Today's world, characterised by sudden, complex, diverse and uncertain changes that profoundly influence the social practices, is obliged to rethink the concept of professional development in the educational context by associating it to a personal, continuous, dynamic, unlimited and situated process.

This poster will present, schematically, two ongoing research projects focused on teacher professional development. These projects are integrated into DIF, a research group of the University of Lisbon, whose study object is the mathematics teacher.

The poster will include a diagram organised in three parts (A, B, C): A will be a DIF's characterising schema, while B e C will present, with more detail, the research projects **P1**, *Professional development of mathematics teachers: two life stories* and **P2**, *Professional development of mathematics teachers and the teaching of mathematical argumentation,* showing the relationships between them and DIF's work.

Both P1 and P2 have in common the fact that they aim to describe and understand professional development processes of elementary mathematics teachers, adopting, in methodological terms, an interpretative approach of phenomenogical inspiration. P1 is included as a biographical approach, using life stories (Pineau, G. & Jorbert, G., 1989) and P2 frames in the paradigm of the co-operative research proposed by Reason (1988). In order to understand the processes and dimensions of the professional development – both in its more broad aspects and in the ones more connected to mathematics teaching – the perspective transformation theory (Mezirow, 1991) and the concept of project (Boutinet, 1996) will be applied in P1. P2, which focus on teachers' knowledge, competences and dilemmas, will be developed in a context of a collaborative work organised around the teaching of mathematical argumentation. Concerning this last aspect, the concepts of argumentation (Toulmin, 1993; Perelman, 1993) and speech genre (Bakthin, 1986) will be applied, as well as the relationships between proof and argumentation in mathematical activity.

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READING SKILLS AND MATHEMATICS

Carol Bohlmann, University of South Africa

The Departments of Mathematics and Linguistics at Unisa have been investigating the relationship between reading difficulties and mathematics learning, with a group of foundation level students. Some issues involved are the following.

- Potential problems when the primary language is oral rather than written.
- Relationship between early language acquisition and future mathematical ability. (Ellerton and Clements, 1988; Souviney, 1981, in Garaway, 1994)
- Relationship between bilingualism and the ability to learn mathematics. (Bain & Yu, 1980; Ben-Zeev, 1977; Clarkson, 1991; Cummins and Gulutsan, 1974; Dawe, 1983; Duran, 1988; Hakuta & Diaz ,1984; Prins, 1997; Secada, 1988; Zepp, 1989)
- Specialised academic and mathematical vocabulary and skills; technical terms.
- Specialised semantic and syntactic structure of mathematical discourse.
- Special symbols which denote processes and concepts.
- A variety of ways in which the same operation can be indicated.
- The use of passive voice.
- Visualisation of problems or concepts.
- Lack of equivalent words in the mother tongue.
- The redundancy of words in ordinary language
- Reading rate adjustment; eye movement patterns.

The Unisa team postulated that a strategy for effective reading and learning of mathematics, initially using a volunteer group of foundation level students, and applying the SLAMS (Second Language Approach to Mathematics Skills) (Dale & Cuevas, 1987) method, should take into account

- *level of difficulty of text*, in particular: general vocabulary and language proficiency; academic vocabulary and mathematical vocabulary; identification of ordinary, mathematical or multiple meanings of words
- *syntactic structures* specific to mathematics
- *semantic/logical relations* specific to mathematics
- *reading rates*
- *trialing the PQ4R technique*: (P)preview, (Q)question, (R) read detail, (R)reflect, (R)rewrite, (R)review
- *verbalisation of mathematical notation*
- *anaphoric references within the text.*

We plan to generalise the results into a strategy we can use with an entire cohort of similar students in 2002. We hope to determine whether the reading enhancement programme has any significant impact on the extent to which these students can meaningfully read, and hence learn, mathematics.

The above information will be more comprehensively available at the conference, as well as details of the references on which this paper is based.

WHAT STRATEGIES DO PRIMARY SCHOOL CHILDREN USE IN MEASUREMENT ESTIMATION PROBLEMS?

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In mathematics instruction at school frequently the search for a precise answer is dominant. In every-day life however, we often base our decisions on estimations, sometimes because it takes less time or effort, sometimes because it is the only suitable method to solve a problem. In addition, (measurement) estimation competencies play an essential role in the development of number sense (Dehaene 1997; Greeno 1991). Nevertheless, currently little attention is given to this topic both in education research and classroom instruction (e.g. Sowder 1992).

The poster presents the estimation problems and findings of a large scale empirical study involving over 100 primary school children (grade 3 and 4), who were asked to solve a variety of estimation problems in an interview setting. Since the main research interest was concerned with the qualitative analysis of the estimation process, the key findings relate to the estimation strategies the children used in these contexts. The poster will display photographs of several problems used in the clinical interviews and illustrate the observed student strategies. In conclusion, a system of major categories which has been developed on the basis of these results as well as under consideration of models dicussed in the research literature (Forrester et al. 1990, Hildreth 1983, Siegel et al. 1982) will be proposed. In this connection, special attention is given to the relationship between task demands and the observed strategies as well as to the role of spatial structuring in measurement estimation situations (Battista et al. 1998).

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A NEW METHOD FOR RESEARCH IN COMPUTER-BASED LEARNING ENVIRONMENTS

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Instructional designers are becoming much more attuned to the importance of user-centred design, the wide variety of experiential modalities associated with human-computer interfacing, and various relations between instructional design and cognition. As a result, computer-based learning environments (CBLEs) for mathematics education now typically involve multiple representations of subject matter content in ways that attempt to accommodate different learning styles. Typically, human-system interaction research has relied on monitoring user activity via computer tabulation of keystrokes and mouse clicks. Advances in digital video and telecommunications technology have now provided the means to develop a new method for dynamically tracking teaching and learning in these environments. Some empirical results demonstrating this method of dynamic tracking are presented.

Dynamic tracking presents teachers, learners, researchers and instructional designers alike with more intuitively accessible opportunities for investigating symbiotic cognitive development in CBLEs. Dynamic tracking allows teachers to conduct action research on the effectiveness of their teaching with CBLEs, and learners with the opportunity to reflect on their own learning. What researchers can discover about various factors that promote or inhibit learning in CBLEs using this method can also serve to inform instructional design. On the other hand, instructional designers, through the CBLEs they have developed, provide researchers with a variety of different constrained conditions in which to investigate factors that promote or inhibit learning with respect to a variety of different learning styles.

This presentation will demonstrate dynamic tracking using familiar protocols (such as "talking aloud") as well as new ones that were simply not possible using previous methods (such as "keeping your mouse where your mind is"). The method of dynamic tracking will be demonstrated in both teaching and learning situations, as well as in classroom and clinical contexts. Once data have been collected, they can also be analysed using digital video processing software. Thus, various action sequences and meta-cognitive factors can be identified and used for data interpretation. Using this method, CBLEs can be analysed to help identify potential systemic constraints placed on the learner, as well as the kinds and effectiveness of the scaffolds for learning that they offer. Dynamic tracking thus provides a variety of avenues for investigating various modalities of mathematical cognition and how they are interconnected, for ways in which CBLEs can be improved, and for ways in which teaching and learning with them can be more effective.

UNESCO PRINCIPLES AS A BASIS FOR MATHEMATIC EDUCATION

SEP- DGENAMDF- BENM
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Purposes and basis for the proposal: Our intention is to design environments for children to use the math they learn in the classroom for their lives, to learn it through their lives and during their whole lives; a mathematic knowledge that supports the upbringing/raising of integer human beings and improves the society's life quality.

For this reason, we take UNESCO principles as a basis, and we propose that the process to construct mathematic notions, to practice and apply them should be the basis for:

- Learn to be: Development of creativity, attitudes, values, conscious decision taking.
- Learn "to live together": Enhance respect for learning styles diversity, intelligence, cultures, preferences, sex, perceptions,....as a guide to find common elements to make grow the unity.
- Learn to do and to learn: Development of skills of thinking, mathematics, designing plans, project participation, competitions.
- Learn to know: Construction of math notions departing from dialectic processes which suppose that learning is mediated, among other things, by the individual maturity level, social historic experience and technology; given through approximations; which is the product of individual interaction and social collaboration, and that is meaningful when gradually, the consciousness levels of reality increase while participating in environments with projects related to the knowledge of oneself, the family, the school, the place where one lives, the country, the world, the universe, the present, the past, the future.....combined with actions on didactic situations requiring descriptions of surroundings, solving problems, games, construction, organisation, research....(Vigotsky, Piaget, Freire, Papert).

The experience: Departing from the observations made with 20 children for 10 years where they were building the mathematic notions during their daily activities, related to their family-life and outside school; later, a test phase was carried out with 10 children for 3 years (1997- 1999) in which, the practice of values, attitudes and decisions-making were proposed, together with learning maths and using computers. Beside this we worked with 200 in service primary teachers (1998 – 2000) and 20 pre-service teachers from the Benemérita Escuela Nacional de Maestros (2000) to promote the study about UNESCO Principles as a synthesis of different theories of education philosophy, and important data was got on the human being's potentials, his creativity, attitude, will and responsibility in making decisions, the diversity of learning styles and the diversity of reactions over difficult situations, the skills and the learning foundations, based on which we designed didactic strategies as well as the creation of environments in which mathematic knowledge is integrated with learning to be, to "live together ", to do and to learn.

Produced materials and applications: To make concrete this proposal we have elaborated text books for basic school children with the sections Monthly Game (recreational exercise), For Thinking (skill development), Giving Opinion (respecting the diversity and recognising the unity), Deciding (decision making and attitude recognition); Suggestion on Didactic Planning in which the activities gather the concepts and procedures with skills, attention to diversity – unity, creativity, decision making, values and attitudes; CD with math games, courses and workshops designed for teachers and future teachers. A Web site has been set with the didactic planning.

Future actions: During the year 2001 we will keep working on didactic planning with in service and pre-service teachers in order to let them learn about the proposal and collect relevant feedback for evaluation.

MATHEMATICS TEACHERS' DIDACTICAL DEVELOPMENT AND THE QUADRATIC FUNCTION

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In this study, mathematics teachers' didactical development was explored on the basis of a series of conceptual maps that they produced while working on small groups. The teachers participated in a one-year long in-service training program in which they were asked to produce content, instruction and cognitive analysis of the quadratic function. The training program methodology was based on the presentation and discussion of the groups progressive productions for each analysis. The notion of representation system was the organizing idea for the teachers' conceptual maps. These maps were codified using a series of attributes for identifying what the teachers saw as the essential characteristics of the mathematical object and the type of representations they used to describe it. Using these attributes, a characterization of all possible maps based was produced and used to analyze the results of the codification process. The results show that this in-service training program, designed using the notions of didactical analysis and representation systems as conceptual structure, using conceptual maps as communicating tool for the teachers, and promoting small group work and whole class discussion, induced a didactical development of the participants that expressed itself in the increasing complexity with which the teachers represented the mathematical object at hand.

We describe briefly the methodology used to codify and analyze the nine conceptual maps produced by each of the five groups of teachers. The attributes considered were the following: whether the map is based on representation systems or not; number of structured representation systems present; number of connections between representation systems; number symbolic forms of the quadratic function present; whether there are connections among these symbolic forms; whether the map is centered on the quadratic equation; and whether the symbolic manipulation techniques were presented as objects or as relationships. On the basis of these attributes some conceptual maps are possible and others are not (i.e., one cannot talk of connections between representation systems if the map is not organized on them). These produced four different structures of the evolution of the possible maps and the paths the maps of a group of teachers can follow while improving them. In this way, we were able to characterize the development of the productions of each group of teachers.

All groups showed significant differences between the first and last conceptual maps produced. However, while some groups were able to make steady progress, other groups showed that there are obstacles to this development, related mainly to the use of structured representation systems and the connections among them. Furthermore, while some groups productions evolved using always the same basic structure, others groups changed that structure at least once along the program.

A Model of Teaching Integrated Theories of Teaching and Learning

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In 1992 the Taiwanese National Science Council began funding a two-year project which had the goals of: (a) investigating the practicality of implementing problem-centered instruction (Wheatley, 1991) in Taiwan, (b) increasing the possibilities for effective mathematics teaching through teaching strategies change, (c) developing a new model of teaching for Taiwanese teachers. The research results of (a) and (b) have been reported (Chang, 1995, 2000). This paper reports the research results of (c), a new model of teaching (Figure 1).

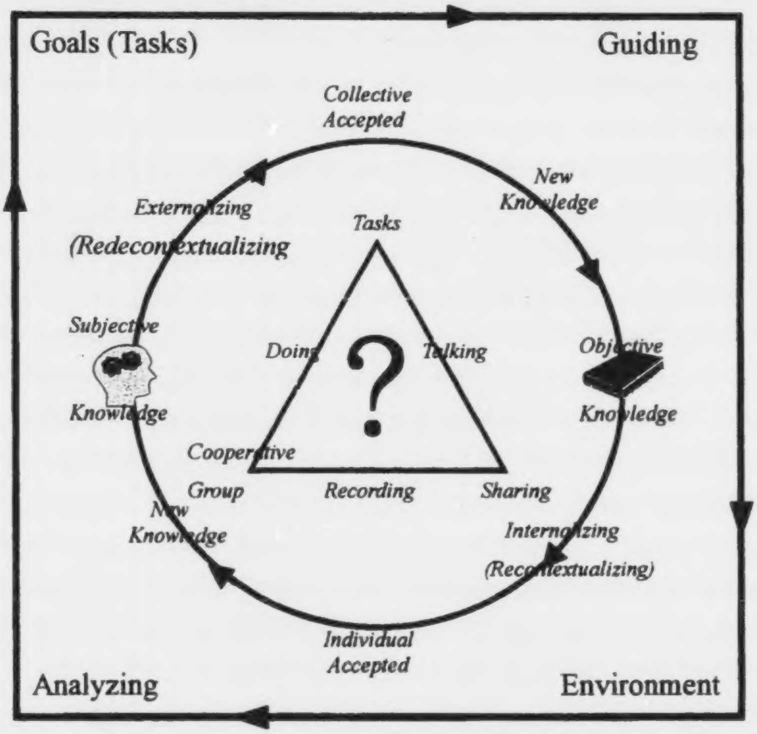
Goals (Tasks)
Guiding
Collective
Accepted
New
Knowledge
Externalizing
(Redecontextualizing
Tasks
Subjective
Knowledge
Doing
Talking
?
Objective
Knowledge
Cooperative
Group
Recording
Sharing
New
Knowledge
Internalizing
(Recontextualizing)
Individual
Accepted
Analyzing
Environment

Figure 1. Problem-Centered with Double Cycles (PCDC) Teaching Model

The PCDC model integrated the problem centered learning model of Wheatley (1991), the DO-TALK-RECORD framework of Open University (Brissenden, 1988), the social constructivism of Ernest (1991), and the recent recommendations of NCTM Standards (1989, 1991).

SPACE: AN INTERDISCIPLINARY PROGRAM FOR TEACHERS

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1. The difficulties and questions arising from the implementation of single-discipline programs in educational institutions have lead to the search for a different approach to teaching. In France, the Ministry of National Education has set up interdisciplinary studies at various levels of secondary education in order to bring about this new approach.

2. MAG7 (Paris 7 IREM), a multidisciplinary group consisting of seven teachers and researchers in education, directs its efforts toward elucidating the relationship with space as it appears both specifically within the three disciplines taught (maths, plastic arts, and geography), and in ordinary communication. MAG7 uses the results of its work to construct a common training program on the subject of the relationship with space for teachers in these disciplines at the middle-school level (years six through nine of compulsory education).

3. The objectives of the program are the following:

- to promote awareness of the ways each discipline approaches the concept of space
- to compare and articulate the various points of view, and attempt to define an axis of communication and collaboration among the disciplines.

4. This training program consists of eight three-hour sessions.

• The goals of the course are presented, followed by an interactive introduction to ordinary communication and the analysis of images (diagrams, photos, multimedia) that are considered to be inter- or trans-disciplinary. • Each of the disciplines is introduced by means of a case study concerning learning about space. Although placed in a specific context, it can prompt an exchange of ideas from the points of view of the various disciplines. • Activities involving the collaboration of teachers from at least two disciplines are proposed, carried out, analyzed, and evaluated.

5. Is it possible to join different sorts of knowledge together, and to build bridges between the disciplines involved in learning about space? If so, how? Herein lies the heart of the debate on interdisciplinarity in education.

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TRANSLATING BETWEEN REPRESENTATIONS OF FUNCTIONS

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This poster presents some results of Project AMECC (Learning in Mathematics: A study on the construction of concepts[1]), focused on the topic of functions taught in the 10th grade. This study intended to characterize the concept of function and of graph of function developed by pupils in a situation of formal learning. More specifically we intended:

- to identify the nature and the use of pupils' own conceptual models
- to understand how these models are changed through education.

The methodology, of qualitative nature, was based on semi-structured interviews of 6 pupils at 3 different moments: before the beginning of the formal study of the topic, immediately after this study, and about 3 months later.

The results presented are centred on the performance of pupils in the accomplishment of two tasks that involved the translation between different representations of the concept of function, namely, a two-way translation between a written representation of a real-life situation and a graphical representation. This type of translation posed some difficulties to the pupils who engaged in the tasks using their intuitive knowledge and who gave less relevance to the variables involved and to the functional relationship among them. Some graphics constructed by the pupils showing these types of difficulties and usually related to special kinds of pictorial representations will be shown. Although in general accordance to previous works, (Janvier,Dreyfus, Kaput, Vinner), some variations will be presented.

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[1] AMECC (Learning in Mathematics: A study on the construction of concepts) — JNICT-PCSH/C/CED/571/93; IIE-PI/12/93

Promoting children's number sense

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Number sense has been a focus of the mathematics education in about ten years. And, in recent years, instructions have focused on promoting number sense in the domain of numbers and calculations at elementary schools in Japan. These instructions have reflected on old instructions for the acquisition of formal calculation skills. Performing calculations quickly and correctly is completely different from performing calculations using many kinds of methods flexibly or seeing numbers in various ways. If the instructional aim is one of these, the other one can not be achieved automatically. In other words, we have to aim at qualitative improvements in such capabilities so that children can utilize meaningfully numbers and operations and can solve problems effectively.

Children come to treat numbers in various contexts before entering school, and have naive knowledge concerning numbers. In school, they are expected to develop rich knowledge about numbers, to perform well with numbers or calculations in mathematics. A rich number sense can not develop only within a certain one domain or in a fixed time period. Such a sense is developed gradually and refined in various activities involving numbers and calculations. Furthermore, a rich number sense cannot be directly taught as a certain fixed procedure. The number sense of children can only be developed after they recognize the merits of calculation methods or the necessity of using them. For example, the following important points can be mentioned in calculation instructions.

- Do not focus only on performing calculations quickly and correctly.
- Do not focus only on whether calculation results are correct or not.
- Have children develop their own calculation methods.
- Put a child's original way of thinking as the basis of the instructions.
- Provide a lot of opportunities for children to experience and compare various calculation methods.
- Provide a lot of opportunities for children to apply trial-and-error methods using various calculation methods.
- Provide the opportunity for children to recognize the necessity of examining calculation results.

Each of the above-mentioned important points is apparently very common. In spite of this, it is very difficult to implement these points in daily classrooms or everyday life. The amount of daily activities has a serious influence, making it difficult to promote such a number sense.

SPATIAL ABILITY & GEOMETRY LEARNING

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Spatial ability is regarded to be an important part of mathematical competence. In particular, research studies have shown that spatial ability is positively related to achievement in mathematics (e.g. Fennema & Sherman 1977). Other studies gave evidence of transfer effects between a training of spatial ability and problem solving competence (e.g. Souvignier 2000). Moreover, transfer effects between spatial ability training and the enhancement of mathematical skills is discussed.

Our research contributes to the role of spatial ability in geometry problem solving processes. We present two studies concerning
(1) the effects of computer presented spatial geometry problems on spatial ability,
(2) the influence of spatial ability on geometry performance, and
(3) the differences between low-achievement and high-achievement students with respect to the training effects.

The sample comprised 63 elementary school children (study I) and 110 students in special education (study II). The students were assigned to one of two groups. Students of the experimental group took part in a computer-based training of spatial ability. All students took part in a series of regular geometry lessons. Both groups were presented questionnaires with items concerning their spatial abilities and their geometry performance as pre-test and as post-test.

Our results show that spatial ability can be enhanced in particular with respect to specific items closely related to the training environment (study I: Mann-Whitney-U-Test $p= .000$, Cohen`s effect size $d= 1.1$; study II: $p= .014$, $d= .404$). We did not identify a significant effect concerning the influence of spatial ability on geometry performance. These results confirm results of former studies with secondary school students (Hartmann & Reiss 1999).

Our study-in-progress aims at refining the results of previous studies: In particular, we will consider content, environment and duration of training as important components influencing the effectiveness of training.

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A NEW COURSE FOR PRE-SERVICE MATHEMATICS TEACHERS

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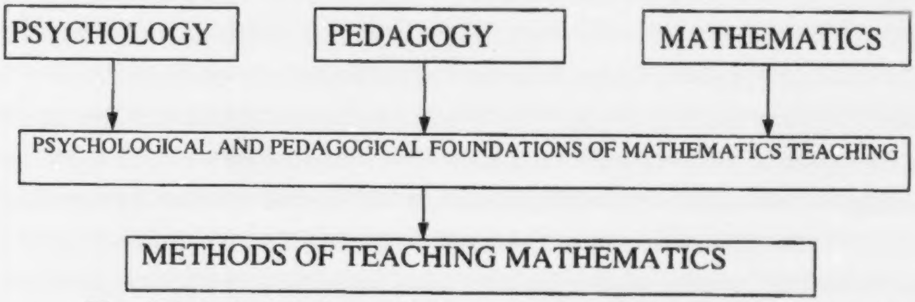
We will consider the new approaches to pre-service teacher education in a leading pedagogical university in Russia – in the Moscow State Pedagogical University. For the last few years here the problems connected to the realization of the complex approach to mathematical, psychological, pedagogical and methodical preparation of the mathematics teacher have been studied.

In particular, the program of a course "Psychological and pedagogical foundations of mathematics teaching" is elaborated. In the following we describe some features of the structure and contents of the course.

Essentially, the traditional general methods duplicate didactics, not concerning at all psychology of teaching. On the other hand, the particular methods consist of exact prescriptions for teaching certain themes of school mathematics, sometimes simply describing the school course. Necessity of the intermediate course, that would serve for a bridge between psychology, pedagogics and mathematics, on the one hand, and methods of teaching of mathematics, on the other hand, is obvious. This necessity is caused by the impossibility to effectively refer to general psychology and pedagogy. It is assumed that the course "Psychological and pedagogical foundations of mathematics teaching" will be studied by the students, that have already learnt psychology, pedagogics and some part of mathematics, before the course of methods of teaching mathematics (see the scheme below):


PSYCHOLOGY
PEDAGOGY
MATHEMATICS
PSYCHOLOGICAL AND PEDAGOGICAL FOUNDATIONS OF MATHEMATICS TEACHING
METHODS OF TEACHING MATHEMATICS

The brief description of the contents of the course "Psychological and pedagogical foundations of mathematics teaching" follows:

Mathematics as a science and as an educational subject. Mathematical thinking. Process of teaching/learning. Methods of teaching. Activity: the teaching activity of the teacher, the learning activity of the pupil. The activity approach to learning mathematics. Mathematical knowledge and skills. Scientific and educational knowledge. Transformation of scientific knowledge into educational (didactical transposition). Basic results of teaching mathematics: mathematical knowledge and mathematical development. Motivation of learning.

MATHEMATICAL UNDERSTANDING OF GRADE 8 STUDENTS

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We will report on a survey in four classes of 8th grade at a secondary modern school (German Realschule). Overall 106 pupils were asked to answer ten exercises which mainly dealt with school geometry. The survey aims to investigate the pupils' knowledge on five general mathematical concepts, which are not taught explicitly in mathematics classroom. We want to find out: (1) Are grade 8 students able to give definitions; (2) Are they able to identify equivalent descriptions of a simple geometrical object; (3) Do they have an understanding of a mathematical proof; (4) Do they know that in general the converse of a logical implication is not true; (5) Are they familiar with the concept "proof by a counterexample".

First results indicates the pupils understanding of a mathematical proof. We gave the students three "proofs" for the statement that the vertical angles are equal. We asked the students a) to select the "proof" which they would choose as own approach and b) to select the "proof(s)" which are correct. It turns out that more than 50% of the pupils preferred the proof with the empirical arguments as own approach but only 39% chose this approach as right answer. This result corresponds to those of Healy and Hoyles (1998). In contrast to the results of Healy and Hoyles in our study more students preferred empirical arguments. This difference seems to base on different age groups and different mathematical skills of the populations.

With respect to logical implication, we presented a daily life problem and a mathematical problem. Most of the pupils answered correctly to the problem of daily life: More than 65% stated that the converse of the implication "If it rains, then the street is wet." is not true. Around 45% of this correct answers were given by a counterexample. However, in the case of the converse of the mathematical implication "If a square is a rectangle, then the opposite lines are parallel.", only 10% offered a correct answer. These results show similar tendency as those of Douek (1999) and Duval (1991).

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Description of Doctoral Project

I have been working with teacher education in mathematics for 25 years. Through this work I have gained some experience and have made a number of reflections about the learning of mathematics.

My doctoral project will be based on my experiences and thoughts connected with my work as a teacher educator. This will cause me to choose a philosophic analysis of mathematical content and/or of the process of learning. In order to restrict myself to something accessible, but still relevant to all learning of mathematics, I will concentrate on the *formation of concepts*. The problem I will pose is therefore this:

How do human beings form mathematical concepts?

This problem leads straight to the core of the epistemology of mathematics. The most important sub-problem here is

- What is a mathematical concept?

To answer this, we must answer something even more fundamental:

- What is a concept?

Then we must consider the process of acquiring knowledge:

- How do human beings form concepts?

If we combine these points, we will closer to answering the question I have posed..

In order to say what mathematical concepts are, one has to consider the philosophy of mathematics. As a result, one has to pose the didactical question:

- How does the teacher's philosophical view on mathematics influence the teaching?

Then I have to cope with the very difficult task of separating mathematical concepts from all other concepts. This is about finding criteria describing a clearly defined hierarchy of concepts. These criteria shall fit concepts that are so different as circle, addition and topological space. I am not sure what the result of this analysis will be, but a doctoral dissertation must find some new paths. I think part of this is definable, but I imagine some concepts must be considered as mathematical by cultural tradition.

Constructivism as a philosophy of learning holds a strong position today. Recently, it has been claimed that this philosophy has no didactical influence, because the theory does not incorporate guidelines for good teaching. I will thoroughly investigate the importance of discriminating between on the one hand the philosophy of how mathematics is learned and on the other hand methodological recommendations. At the same time I want to show how philosophical insight combined with didactical goals may influence teaching.

The real core in my project is ***to use philosophy not only to understand, but as guidance for practice.***

Einar Jahr

CITIZENSHIP BY WATERS

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Instituto de Educação Matemática – Universidade Santa Úrsula

The understanding of the environmental questions supposes an interdisciplinary work in which Mathematics in inserted. The quantification of involved aspects in environmental problems favors its clearer vision, helping in making the decisions and permitting the needed interventions (recycling and reutilization of materials, for example). This project is part of an investigation in which the line of research is Mathematics and Society, of the Master's Program in Mathematics Education of the Institute of Mathematics Education of Santa Úrsula University. This project was elaborated and implanted during the year of 1999 in three classes of he Sixth year of the Fundamental Course, totalizing 80 students of a private school in the south zone of Rio de Janeiro Country. The investigated problem was how to avoid the waste of water, since the missing of hydric resources in one of the most present day concerns of humanity, The project had as its targets – development of peoples' capacity to approach questions in the environment – environmental and ethical consciousness. The study implied the disciplines Geography, Portuguese, Mathematics and Sciences.At the beginning,the students and the teachers of the citaded disciplines visited the State Company for Waters and Sewers –CEDAE- where they received an explanatory video and participated of activities that gave information about the origin, supplying, treatment and reutilization of the water in the city of Rio de Janeiro. The classes were divided in groups of three or four students; each group was held responsible for a specific theme related to one of the disciplines implied in the project. The groups in charge of the mathematical part chose as their theme: the quantity of water wasted in homes. It was asked to the students and each group was held responsible for the data collection in the following actions: dishwashing, teethbrushing, taking a bath, washing the sidewalk and washing a car. The students made tables that indicated the quantity of water spent in the action leaving the tap always open or closing it when flowing water was not needed. They calculated arithmetic means, ratios, percentages and, using EXCEL, made a graphic of sections where there appeared intuitively an idea of angle an its measures.Each group showed a synthesis of its investigation to all the students.The students took conscience of the environmental and ethical question of water economy with the aim of having a sustainable development. They influenced their families in this respectful attitude of preservation of the ecosystems.

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CONTEXTUALIZED MODELS FOR SLOPE AND LINEAR EQUATIONS

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This poster presentation will describe and illustrate photos and products of student work from the results of a classroom-based case study involving 10 eighth grade students' use of visual and tactile models during instruction of a unit involving the topics of slope, graphing, and equations. The models were derived from the *Mathematics in Context* curriculum unit *Graphing Equations* (Kindt, et al., 1998). Samples of student work from pre- and post-unit interviews and classroom work will be used to illustrate the impact of various models on their ability to determine the slope of a line and solve a linear equation for an unknown.

Pre-unit tasks were used to assess students' informal and prior knowledge of slope and solving linear equations. Most of the informal/prior knowledge consisted of recognizing slope as "something slanted". Several students related the word to skiing and drew a diagonal line to represent a ski slope. In addition, four of the ten students were able to use the slope of a line within a context to determine an unknown quantity. None of the students, prior to the study, used any formulas or symbols to describe slope when asked to do so in their own words. During the study, compass cards and graphing calculators were used to locate points, draw lines, and determine the slope of a line. The results of post-unit interviews revealed that these activities enhanced eight out of the ten students' abilities to conceptually describe slope in their own words and determine the slope of a line using a variety of informal and symbolic strategies.

Prior to instruction, none of the students were able to solve linear equations for an unknown. A frog jump context and linear (visual and tactile) models were used during instruction to connect their knowledge of modifying "equal lengths" to the steps involved in solving a linear equation for an unknown. The results on post-unit tasks showed that eight out of the ten students were able to solve multi-step linear equations for an unknown.

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Investigating the Mathematical Components of Engineering Expertise

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We will present the design, and discuss the mid-project findings, of a one-year project that is examining how mathematical ideas and techniques are used in the practice of engineering, based on case studies of professional engineers working on design projects in several large civil engineering consulting firms. We are especially concerned with the observation of mathematics as used in workplace activities, rather than merely asking people to describe what they think they do. This is because we want to look beyond those elements of work conventionally described as mathematical (such as techniques learnt in school or university) to try to observe situations in which a more general, and harder to specify, "mathematical literacy" is involved. These situations are made complex by the fact that engineering design is done by teams of people, made up of different kinds of specialists, where mathematical work may be the principle responsibility of only a small part of the team. We have a particular interest in the role of digital technologies in this collective design process: is mathematics becoming an ever more specialist realm, where non-specialists are increasingly the consumers of "pre-cooked" mathematics hidden behind the interfaces of software packages? We believe that this question is not as clear cut as it seems, if we admit a broader definition of mathematisation than the visible application of mathematical techniques.

The methodology and theoretical basis of this project are derived from a previous project carried out several years ago at the Institute of Education, which examined the mathematical practices involved in nursing, commercial aeroplane piloting, and investment banking (Hoyles, Noss & Pozzi 2001). We found that the various professional groups employed a variety of mathematical strategies which were finely tuned to their practice, yet simultaneously retained the notion of invariance that typifies abstract knowledge. We have begun to explain this phenomenon in terms of a theory of "situated abstraction" (Noss & Hoyles 1996), and the current project is developing this theory by carrying out research in a domain of practice where we can see sophisticated kinds of mathematics (geometry, algebra, calculus) that entail complex structures of abstractions in their practical application.

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ELEMENTARY SCHOOL PRESERVICE TEACHERS' UNDERSTANDING OF ZERO

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Students often find the concept of zero difficult to understand. The linguistic of zero is confusing even for university students, including many elementary school preservice teachers (Ball, 1990; Blake & Verhelle, 1985). The purpose of this poster is to highlight some of the difficulties that the elementary school preservice teachers face in understanding zero and its concepts. This poster will depict some dilemmas of teaching these concepts.

The current study primarily focused on two questions. First, what do prospective teachers understand about zero in general? For example, do they consider zero a number? What do they know about the development of zero in various civilizations? Second, what is preservice teachers' understanding of division by zero? For example what is 5/0? 0/0? These questions lead to a discussion on their ability to convey the meaning of zero to their future students.

The data for this study were collected from a course entitled "Mathematics for Elementary School Teachers" for the last five years. Approximately, 200 students have participated in this study. This poster will draw data related to zero from their quizzes, exams, and personal interviews conducted by this researcher at the end of each course.

In all the courses, prospective teachers had difficulty understanding zero and its concepts. Before introducing zero in the class, more than 90% of the preservice teachers thought that a number divided by zero is zero. Even after the concepts of zero were taught for an entire session of one hour, only about 60% of the preservice teachers stated that 5/0 cannot be determined. The other 40% stated that 5/0 was zero. The percentage of correct response did not increase substantially in the final examination when basically the same question was asked. In the personal interviews, many students were confused and even indicated that they would avoid teaching the concept of zero to their future students. The author of this proposal finds it alarming and hopes that it will generate some discussion in the PME poster session.

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A STUDY OF STUDENTS' SPATIAL ABILITIES WITH THE USE OF VIRTUAL ENVIRONMENTS

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Children move, interact, perceive, recall and represent the natural world space and its attributes. (Osberg 1997) The geometrical study of space leads to its reference systems, namely the Euclidean, topological and projective. An interesting research question is the way primary school children perceive and exploit these, intentionally or unintentionally.

The understanding of space and its attributes, the relations and rules govern them, allow children not only to act, but also to conceptualize complicate concepts allowing them to adapt and face similar spaces and learning situations, to develop spatial skills (Wilson 1997,. Doorman, L.M., Kooij, H v.d. :2001)

The present research concerns the study of spatial skills of children of ages 10 –12 years, with the contribution of three-dimensional synthetic environments, implemented by Virtual Reality (VR) technologies.

The virtual environment was designed in order to investigate:

- The nature of internal representations of students concerning space orientation, recognition and handling of two and three dimensional geometrical objects viewed from different viewpoints, following specific verbal instructions
- Language used by students to describe spatial relations in the virtual environment
- The eventual distance between spatial information acquired by exploring a geometric problem in the virtual environment and the one acquired by exploring a paper / pencil environment.

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STRUCTURING MATHEMATICAL KNOWLEDGE DURING A SPECIFIC EDUCATIONAL TASK

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The set of all triads $(a,b,a+b) \in N^3$, $(0 \notin N)$, $a \leq b$ with the mappings Lt: $(a,b,a+b) \rightarrow (a,a+b,2a+b)$, and Rt: $(a,b,a+b) \rightarrow (b,a+b,a+2b)$ creates the structure. This was used as a tool in the research aimed at investigating the building of an infinite arithmetic structure.

Research

The set of triads that is equipped with left and right mappings serves as a good tool for research, diagnosing pupils' abilities to build structure and is an educational field within recreational mathematics. The structure of triads as a research tool enables us to observe a whole process of creating an arithmetical structure because the graphical model of structure evidences many thinking structural processes of the pupils. The research has illustrated the following four facts:

1. Creating a global structure presumes necessarily previous insight into local structures (involving from two to five elements – the triads joined by the mappings).
2. The ability to create a concept of the structure of triads is profoundly individual. Pupils showing the same level of understanding of the structure of natural numbers get insights into the structure of triads at different rates.
3. A powerful tool is the ability to grasp the structure graphically and leads to the understanding of structure which does not depend whether a tree, representing the structure, is orientated up or down.
4. As the pupils solved the problems step by step they created other elements of the structure. During these activities the pupils had misconceptions. By realising this and by understanding the reasons for them, contributed to getting an insight into the structure. The following four cases of misconceptions were found:

A. The notation of triads can be reduced, e.g. triads coded as dyads or monads.

B. The triads on a line higher than line 1 can be written down automatically without thinking about mappings.

C. The generated pattern from the left 'branch' can be applied to the right 'branch'.

D. The number of line can be used as an operator for generating an appropriate triad on the line (the triad on line 10 was made by doubling the numbers in the triad on line 5).

I will give specific examples of pupils' solutions to illustrate the four cases of misconceptions mentioned above in my presentation.

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Changing classroom environment and culture - Case study

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The contribution continues the research presented in (Kubínová, Mareš & Novotná, 2000). It focuses on the analysis of concrete situations in two classes taught in different ways by different teachers in the past but taught by one teacher at present. The differences in students' behaviour, teacher's approaches and results achieved by students are diagnosed. It is shown that even if the teacher who wants to implement the change from instructive to constructive teaching, is sufficiently qualified, has long-term experiences with constructive teaching strategies and has no obvious external obstacles for implementing their plans, has to be open-minded and respect students and their prior experience.

When studying questions related to using constructivist approach to teaching (mathematics) we use variety of methods: longitudinal evaluation of teaching effectiveness by comparison of periodic testing of parallel classes, direct observation of the milieu of the classroom and analysis of teaching strategies, the teachers accounts of their own classroom experience, analysis of audio/video recordings of lessons and children's written work. In the school year 2000-2001 we face a singular opportunity. One of the authors teaches mathematics in two parallel classes of the ninth grade (students are age 14). In one class she has been teaching for five years and using constructivist teaching methods (the SC type class in Kubínová, Mareš & Novotná, 2000). She has never taught the other class before. It is well known that previous teachers taught in an instrumental way.

Our research is in accordance with the ideas about powerful learning environments from (De Corte, 2000). "... the teacher becomes a 'privileged' member of the knowledge-building community, who creates an intellectually stimulating climate, models learning and problem-solving activities, asks provoking questions, provides support to learners through coaching and guidance, and fosters students' agency over and responsibility for their own learning." From our experiments it is clear that the teacher's role is crucial, the teacher has to understand and respect the situation in each individual group of students, it is not possible to transmit the methods and forms of work which were successful with one group of students to another without any modifications, however it is possible to use experiences gained with one group of students to organise work in another group.

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A FRAMEWORK FOR ANALYZING STUDENTS' MATHEMATICAL WRITING

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This study proposes a framework for analyzing students' mathematical writing to promote students' thinking in mathematics classroom.

The educational value of engaging students in mathematical writing has been explored in a variety of research. It helps students, for example, learn and externalize their thinking and feeling about mathematics. This study focuses on promotion of students' thinking through mathematical writing.

There has been much research focusing on writing activities, in particular, journal writing. It is necessary for students, however, to write mathematically in the process of classroom activity, as well as to look back their own thinking at the end of classroom activity. Writing activities should be incorporated into a whole classroom activity.

In this study, two dimensions of mathematical writing are identified. The first dimension is "aspect of mathematical thinking" (i.e. writing in the context of learning mathematics). The second dimension is "aspect of communication" (i.e. writing for and about someone's thought). Each dimension has some "modes" and "levels". The "modes" are used for describing what students write. The "levels" are based on the prescriptions of what teacher wants students to write.

Students' writing is expected to progress along with each of the two dimensions. With respect to the first dimension, students' writing progresses to sophisticated one. On the other hand, with the second dimension, students become more reflectively through mathematical writing. In other words, students' mathematical thinking can be promoted through mathematical writing.

Implications of the framework for teaching are also discussed. The framework can be used for developing worksheets and for planning instruction in the classroom. Classroom activity can be enriched by using mathematical writing.

Some examples of students' writing will be shown in the presentation.

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CHARACTERISTICS OF COGNITIVE ACTIVITY OF STUDENTS WITH LEARNING DIFFICULTIES

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Observation of interactions between teachers and their students with learning difficulties emphasises the links between didactic and cognitive phenomena. More obviously than in any other environment, the status of interactions and knowledge seems to be two-faced. Students and teachers have to play both the didactic game and the apprenticeship game.

- One usual conception of teachers could be "when teaching to students with learning difficulties, one is not to let them inactive". Thus, maintaining and showing off external activity under the pretence of an effective cognitive activity can become the major rule structuring the didactic game. Vannier-Benmostapha and Merri explain how teachers may support this phenomenon in scaffolding interactions. In order to understand how students may face this mechanism, Pichat focuses on the students' pragmatic knowledge : a low level knowledge that allows them to maximise efficiency despite lack of real competency (Pichat and Ricco, in press). Therefore, transfer of responsibility concerning knowledge and skills may not develop in a proper way.
- However, we observe a few students taking advantage of such interactions. Two issues are presented. At a control level, they internalise the structure and the content of former conversational interactions ; by this means, students develop self-control procedures which are isomorphic to the injunctions of the teacher. In this case, the teacher has a chance to be relevantly imitated. Moreover, a second benefit comes from experiencing functional limitations of their pragmatic knowledge ; in such a case, pragmatic knowledge may be re-elaborated so as to be more efficient.

Getting teachers to be aware of these two phenomena could be useful for a better control of the effective cognitive benefit of their students.

Pichat et Ricco. *What is the nature of mathematical conceptualisation in didactic institutions ?* , Cognitive systems, in press.

Vannier-Benmostapha, M.P. *Analyse des tâches et des schèmes de tutelle d'enseignants de C.M.2, $4^{ème}$ technologique et CLIPA.*, doctoral dissertation (director : Gérard Vergnaud), Université Paris V-René Descartes, in preparation.

PEER-ASSESSMENT OF DATA HANDLING (GR 5'S)

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The South African authorities have realised the need for an education system which equip learners with the necessary knowledge, skills and disposition to cope successfully in the world when they leave school. As part of this strategy in mathematics, data handling was recently introduced in the curriculum of the intermediate phase (gr 4 – 6) and presently gr 5's for the first time encounter data handling activities. Facilitating and assessing these activities according to the problem-centered approach gave rise to this experiment.

Peer-assessment is a powerful but lesser known assessment strategy in the new dispensation, and presented a challenge to the facilitator. The linking of classroom activities with real life situations in Outcomes-based Education and especially the assessment of learners' knowledge and skills in data handling has been the focus of this experiment.

Three classes of 30 learners each participated. The classes were divided into five groups of six each. Learners had to solve the problem of determining the most popular chocolate bar in the class and representing the information graphically, in an open socio-constructivistic learning environment. Activities include: collection of information, design of a data sheet to organise information, drawing of a graph, group presentation to explain the whole process, writing of recommendations to the school tuck shop owner based on the conclusions of the survey and assessment of the whole process. Materials: data sheets and graphs from newspapers and other sources, grids for peer-assessment, self-assessment and facilitator assessment of the groups.

Despite a lack of systematised procedural skills and a tendency to over decorate, learners enthusiastically joined in all the activities. Interesting information gathering and data organising techniques were used and different graphs drawn; group co-operation and presentation skills were also developed. The use of peer-assessment techniques proved a useful tool to the facilitator and the learners in the realising of the outcomes for the mathematics strand data handling.

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The role of problems in the mathematics lessons of the TIMSS-Video-Study

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The TIMSS- Video-Study revealed great differences in the teaching of eigth-Grade mathematics classes in the three countries involved, Japan, US, and Germany. Stigler & al. (1999) showed the existence of different cultural scripts in the teaching of mathematics. The reanalysis in this paper is based on the mathematical problems themselves and focusses therefore directly on the content related issues in the videotaped lessons.

Method: From the sample of the lessons in the TIMSS-Video-Study, 22 lessons in each country were selected. So, a sample of 1153 problems, some further divided into related subproblems, were analysed. The instrument used for this analysis is a classification system for mathematical problems, developed for this purpose.

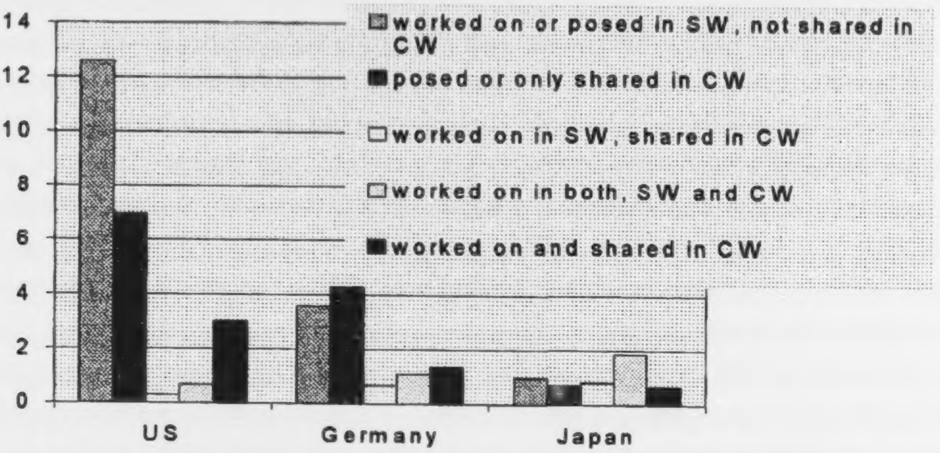

Figure: Number of problems per lesson (means) in different situations (SW=Seatwork, CW=Classwork)

Results and discussion: The most important results from this reanalysis will be presented graphically. The results capitalize the assertion „Teaching is a cultural activity“ (Stigler & al. 1999) from an additional point of view. „Cultural“ does already apply to the content itself. Mathematical problems play - as means to organize the construction of mathematical knowledge - different roles in the three nations‘ classrooms, not only in their number but also in the selection of the problem characteristics and their implementation in the lesson.

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Mathematical teaching and learning by qualitative evaluation using 'Open-Approach Method' in school activities

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Summary :

We will mention some findings concerning the use of both the 'open-approach method' and the 'open-ended problem' to make clear the meaning of 'both activities by students and mathematics are open'. The aims of instruction using the 'open-approach method' are to foster both creative and mathematics thinking in problem solving by students simultaneously. From quantitative to qualitative evaluation of mathematical activities, which are the expression of both mathematical ideas and processes of the ways of thinking used to solve problems, are very useful. In order to develop this usefulness, we make the model for examining these activities which are convenient enough for us to deal with the students' many kinds of processes of solving the problems.

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SAME RULES, DIFFERENT CHALLENGES:
A SUITE OF GAMES FOR PRACTISING NUMBER EFFECTIVELY

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The card games to be presented through this poster were invented by the author in 1978. They can provide specifically differentiated individual and group based practice of mental recall and mental strategic thinking in number contexts. They progress from self-checked working out by one or two students of each set of 6 to 9 cards, through to speed practice in a game format by groups of 4 to 6 players, using packs of 24 to 72 cards. Discussions arising from playing the games can provide diagnostic and formative assessment, while enhancing the insights of students. Using the same rules, around 50 different sets have been designed to meet identified needs, so that students do not have to learn new rules to play gradually more advanced games.

The poster will:

- give full details of a variety of uses of an example game set;
- illustrate key features and how they differ from other superficially similar games;
- relate the use of these games to the ideas of *productive practising*;
- indicate responses of students in pre-school and elementary classes, and views of their teachers, over 2 years of use as a central part of their number curriculum;
- outline how these games relate to the English *National Numeracy Strategy*;
- relate the educational value of these games to the ideas of Vygotsky, Gattegno, Freudenthal, Wittmann and others.

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INVESTIGATING AND LEARNING: A WEB BASED PROJECT

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Aims and context. This project aims to promote innovative perspectives for mathematics teaching and learning through the development of a Web site. It emphasizes the idea of investigation as a process of knowledge construction and provides a link for the community interested in this idea in different parts of the country. The project is developed from 2000 to 2002, as one initiative of Research Group DIF with the support of the Ministry of Education (DAPP). The target audience is all those directly interested in curriculum change in mathematics including teachers, prospective teachers, teacher educators, and pupils of different age groups.

Main assumptions:

(i) Investigations constitute an important form of construction of knowledge. Pupils and teachers may undertake mathematical investigations and, in addition, teachers may undertake professional investigations.

(ii) The Internet is a powerful medium, promoting cultural change. It enables the search for information. It also allows people to publicize their work. It is a medium that promotes interaction among people, enabling the development of new meanings and new identities.

(iii) Information and Communication Technologies (TIC) are a significant resource for teacher education. They also provide an important link between research and professional practice.

Activities carried out in 2000. These include, for example, the 1) set up of a search engine focused in mathematical investigations; 2) actualization of a data base of publications concerning mathematical investigations; 3) creation of a text bank (with documents to download); 4) update of short texts covering issues involved in carrying out mathematical investigations in the classroom; 5) exchange of teaching experiences; 6) update of a links page for other sites; 7) creation of a Web-based forum; 8) divulgation of the project and interchange; and 9) theoretical discussions.

Project interim evaluation. By the end of the first year of the project, we strengthened our perspective about the value of investigative activity as a fundamental strategy for knowledge construction. However, we note some difficulty in getting material, especially descriptions of classroom experiences, to include in the site. We also have some difficulty in promoting interactions with the target audience, as the participation in the forum is below our expectations. These are issues that we want to address in the future development of the project.

Initiatives for 2001. New material will be provided in the Web site, such as the analysis of new curriculum documents regarding the place given to investigative ideas. A closer connection to schools will be attempted, through sessions with features such as demonstrations of the site to teachers and students and research about the place of investigations in teachers' professional practice. We are also planning a survey regarding the how teachers use the Internet.

Graphical form. The poster will include images from several pages of the Web site as well as a global map of the site. A scheme of the project aims, activities, and evaluation will also be provided.

STUDIES ON THE QUALITY OF SCHOOL: ACQUISITION OF CONTENT SPECIFIC AND CROSS-CURRICULAR COMPETENCIES IN MATHEMATICS AND SCIENCE DEPENDING ON IN-SCHOOL AND OUT-OF-SCHOOL CONTEXTS

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Research on teaching and learning as well as the results of TIMSS have revealed that content specific and cross curricular competencies students acquire in German mathematics and science instruction are rather insufficient. Among the most prevalent deficits are 1) a lack of flexible application of knowledge in new situations, 2) difficulties in solving complex problems, and 3) the decline of interest in mathematics and science during high school. It is obvious that these deficits not only concern subject specific achievement but also motivational variables and key qualifications relevant for the efficient acquisition of new knowledge in general (e.g. problem solving strategies, the competence for self-regulatory learning). As a response to these findings a priority program *"Studies on the Quality of School..."* was launched by the Deutsche Forschungsgemeinschaft (DFG) in 2000. The program aims at systematically identifying reasons for this unsatisfactory performance and at suggesting theoretically and empirically well grounded interventions to improve the quality of education in German schools with a particular focus on mathematics performance as well as on science, motivational variables and cross-curricular competencies.

One important but often neglected factor influencing the quality of instruction is the general pattern of the lessons, called scripts. It is eminent to find out, whether there exist consistent, presumably culture specific, patterns of instruction, which patterns dominate, and how they interact with and influence learning processes. Another focus concerns certain components of instruction, like the use of representations or instructional methods and media and their interplay. Apart from the actual learning context (the learning situation), also the wider context of learning has to be considered. Teaching and learning science and mathematics are embedded in a particular school context and are influenced by it. And finally, there is the broad context of the society schools are part of. Students' views of what mathematics and science learning is about and whether it is worth the effort rests also on the attitudes and beliefs of teachers, parents and the peers.

Currently, there are 23 projects working in this program. In most of them mathematics and science educators and psychologists co-operate closely. Co-operation between the individual projects is a further key feature of the program. The projects are problem oriented, i.e., educationally relevant basic research and applied research are combined. The poster presented will provide an overview of the philosophy of the program and the 23 projects funded during the first two year period.

FINDING LINKS BETWEEN EARLY WORK IN PROBABILITY AND THE TEACHING OF FRACTIONS

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Developing an understanding of probability depends on a wide range of activities but it is also dependent on an understanding of part-whole relationships, even before the stage of quantification is reached. This is particularly apparent in such probability tasks as sampling from collections of similar elements and using spinners with differently marked sectors. Since the work of Piaget & Inhelder (1951) much of the research in probability has centred around how children think about chance (Fischbein, 1975; Truran 1994, and many others). In most of the mathematics curicula probability stands apart from other areas of mathematics and how this is to change is a matter to be addressed. The focus of this presentation is on some aspects of a pilot study to explore ways in which early links can be made between foundational activities designed for developing an understanding of probability and for learning about fractions. This study was carried out with children in two age-groups, 7-8 years and 9-10 years, in three schools and consisted of several series of lessons for each age-group. What is presented in this poster outlines the experiences and activities used in one series of the lessons with the 7-8 year-old children.

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DESIGN OF THE SYSTEM OF GENETIC MATHEMATICS TEACHING AT UNIVERSITIES

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First, one should accomplish the analysis consisting of two stages: 1) genetic elaborating of a subject matter and 2) analyses of arrangement of a material and possibilities of using various ways of representation and effect on students. The genetic elaborating of a subject matter, in turn, consists of the analysis of the subject from four points of view: a) historical; b) logical; c) psychological; d) socio-cultural. In designing of the system of genetic teaching very important is to develop problem situations on the basis of historical and epistemological analysis of a theme.

The major aspect of logical organisation of an educational material consists in organising a material so that to reveal the necessity of the construction and development of concepts and ideas. It is necessary to arrange problem situations or tasks, for which the important concepts or ideas, which should be studied, would serve as the best solutions. It is necessary to analyse those problems of knowledge, for which the considered concepts and ideas serve as the necessary solutions. For this purpose, both historical analysis and epistemological considerations, and special search for appropriate problem situations and tasks can help.

When studying university algebra courses, the students usually are encountered with sequentially growing steps of abstraction - with a "ladder of abstractions".

According to the theory of A. N. Leontyev, actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially "contracted" actions with the concepts of the previous level of abstraction. As M. A. Kholodnaya (1997) noted, "a contraction is immediate reorganisation of the complete set of all available … knowledge about the given concept and transformation of that set into a generalised cognitive structure".

In our view, for reaching a contraction of an action with algebraic objects into (automatic) intellectual operation it is necessary, after sufficient training with this action, to include it in another action, connected with the construction of objects of the next step of abstraction.

After two stages of analysis, it is necessary to implement the project of the process of study of an educational material. We divide the process of study into four stages: 1) Construction of a problem situation. 2) Statement of new naturally arising questions). 3) Logical organisation of an educational material. 4) Development of applications and algorithms.

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TRAINING MENTAL CAPACITIES – A STRATEGY TO MAKE LEARNING EFFICIENT AND EFFECTIVE

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National Curriculum Council

The methodology developed during the research covers four stages:

1. **Structuring the information** to be learned, according to an epistemological model.

2. **Systematical training of the mental capacities** focusing in different manners on each of the following: *understanding the concepts, computing procedures, and problem solving.*

3. **Random training of the developed capacities**. This type of training plays an important role in consolidating the mental structures acquired by the child. The technique employed is the mental game starting from isolated information. The start could be: *isolated numbers, groups of numbers, computing exercises, problems, and symbolic schemata*. Examples could be given for different grades, but the most spectacular results could be seen in primary education. For example, in Grade 1, the teacher proposes the number 4 and asks for creating the sequence of natural numbers by fours till 20. Another riddle-game requires the composition of a number in which 4 is a compulsory component, or it is not at all a component. Moreover, the students are stimulated to create problems starting with the number 4. The problems could be connected to practical situations, but also to theoretical mental problems, for example: "Use 4 coins to compose a given amount of money", "We know that we start with number 4 and we add its double. What is the number we get?", etc.; it is important to practice both types and to pass from one to another. These procedures are then carried out starting from other numbers and practicing serial arrangements, comparisons, estimations, creating problems, problem-solving, composing and decomposing numbers. These are practiced orally, mentally, in written forms.

4. **Structured training**, which is targeted at the assimilation of invariants. This is done by constantly resorting to diagrams and models. The exercises become gradually complicated, following the initial model in various ways: with direct support from objects or concrete schemata, or without this support, by operating in an internal language, and later by operating with literal symbols for numbers. The mental structure created in this way is confronted with itself by "shifting" its elements, by applying different thinking operations, by passing from one level of abstraction to another, etc.

Instead of a "drill and practice" strategy, a "structure and practice" strategy is developed. This training leads to the creation of a dynamic mental structure, able to mobilize in various situations and to find creative solutions for complex problems.

THE DEVELOPMENT OF RATIO CONCEPTS IN EARLY ELEMENTARY SCHOOL THROUGH PROBLEM SOLVING

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The concepts of ratio and proportion are considered to be very difficult in elementary school and throughout high school. In fact, Piaget (1966) viewed proportional thinking as a key ability developed in the formal operational stage. Many adults, including student teachers and teachers and people in different occupations (nursing, fishermen, carpenters, etc.), do not understand these concepts well or used localized strategies to solve problems (Fisher, 1988; Keret, 1999 (PME); Klemer & Peled, 1998 (PME); Tourniaire & Pulos, 1985).

Although some ratio problems are very complex and demand abstract thinking, it is important to help children construct ratio concepts while they are young. At this age the children can develop intuitive understanding of the ratio concepts and they can have opportunities to model the problems with concrete objects or drawings. Hopefully, when these children grow up, they will have a sound conceptual base to build on.

This study documents solution strategies for ratio and proportion problems in young children (grades one to three). This data is part of a larger project of learning mathematics through problem solving in a constructivist way by a branch of Cognitively Guided Instruction (CGI) (Carpenter & Fennema, 1996). In this project there is an emphasis on children developing their own unique methods of solution. The children reflect on their solutions and ideas and communicate them to others. The teachers learn about the children's thinking and build the instruction on it.

The children solved the ratio problems by using mainulatives and drawings and by seeing mathematical relationships. Almost all the children could solve the problems correctly, including sophisticated strategies (some needed direction from the teacher such as a request for drawings). Examples of problems: "For the price of 2 thick notebooks you can buy 5 thin ones. How many thin notebooks can you buy for the price of 20 thick ones?" "In a class there are 30 children. For every 3 boys there are 2 girls. How many boys and girls are in the class?" Other problems scale the amounts of ingredients in a recipe. Example: to make 500 grams of beans we used half a cup oil, a quarter teaspoon of pepper, 4 tomatoes etc. How much would we use to make 750 grams? An example of a third grader solution to the last problem: 250 grams goes twice into 500 grams and 3 times into 750 grams. So you divide all the amounts by 2 and multiply by 3. So half a cup of oil corresponds to the 500 grams. 250 grams corresponds to ¼ cup. 750 grams corresponds to 3 times ¼ and this is ¾ cup oil.

Mathematics Curriculum Standards in China
— Present state and prospects for the future

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China has planed to make a change for education. The first action was happened in the field of mathematics education. *The National Mathematics Curriculum Standard* (NMCS 6—15) is the first new published "*Standard*" of the national educational development program in China. There are some new determinants in it. The main ideas were summarized in the fields of targets, curriculum structure, knowledge, learning and teaching, assessment and technology. These new determinants showed that mathematics curriculum should help children to build a practical basis for their continuing development in the future. As a bit soft approach than the former hard one, *NMCS 6—15* can leave more rooms for processes, strategies, constructions, investigations and discourse to the students. *NMCS 6—15* is a symbol and a fundamental step for achieving a changing and development of mathematics education in China. And there are also some new research characteristics for designing *NMCS 6—15*. It proved that *NMCS 6—15* is a production that fully considered the nowadays-international developments and reflected the advantages of Chinese own.

The revising is necessary. *NMCS 6—15* should be more scientifically and practically. Whatever, *NMCS 6—15* has illustrated a good idea. I hope our nation's youth may both excel and hearts in mathematics under the guide of it.

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An Analysis of the Experimental Mathematics Curriculum in Elementary School

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The experimental curriculum for elementary mathematics had been implemented in Taiwan during the period between 1992 and 1997. Researchers studied students' learning processes and scholastic achievements to assess the effectiveness of the experimental mathematics curriculum. The main objectives of this study contain:

1. Investigate the differences in the effectiveness of mathematics learning between students following the experimental curriculum and those following the traditional curriculum in elementary school.
2. Conduct a follow-up study on the continuation of mathematics learning for those experimental students in middle school.
3. Compare the differences in achievement of mathematics learning between the experimental students and non-experimental students in middle school.

Five phases were carried out during this research. Phases I and II were conducted for students in their second semester of sixth grade. Three surveys were collected and analyzed, including Mathematical Learning and Experience Scale (MLES), Mathematics Attitude Scale (MAS), and Mathematical Problem-Solving Test (MPST). In phases III to V, the follow-up study was conducted for students in the first semester of the seventh grade and two mathematics classes were observed. During the period, open questionnaire about elementary and middle school mathematics learning experience was designed and math grades of the students were collected from three periodical test scores in math.

After one year follow-up study, findings were concluded as follows:

1. The experiment students had more positive attitude but less metacognition abilities than the control group students. However, habits of peer discussion, communication and cooperation learning were popular in experimental classes.
2. The students' scores in the problem-solving test were much better in the experimental group than in the control group. The study also indicated that learning effectiveness was dramatically different among experimental classes.
3. Observations in middle school showed that focus of the process-based experimental curriculum is substituted by focus of outcome-based teaching in middle school. The researchers found that a change in teaching style in the math teaching community is difficult and that requires teachers to change their perceptions of mathematical learning.
4. Comparing math achievement in middle school between experimental and control groups, it was found that the achievement of city students is well above the average while the achievement of students in the suburbs is below average.

In view of the above-mentioned conclusions, some suggestions are provided:

1. Establishing professional development programs are essential and should be considered at the earliest possible time.
2. Teachers should increase their flexibility to meet the systemic change in math-teaching classes.
3. A concrete research and development system should be developed to serve as a resource center for math teachers to produce a set of information on structural math concepts.

SEARCHING FOR THE ILLUSION OF LINEARITY IN PROBABILISTIC MISCONCEPTIONS: A LITERATURE REVIEW

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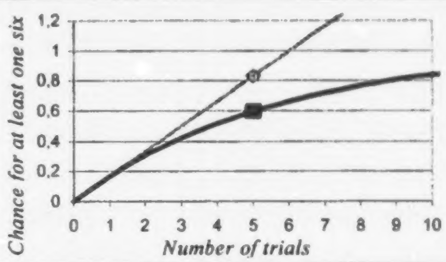
Recent studies by De Bock, Verschaffel and Janssens (1998) have shown strong empirical evidence for the strength of the tendency among secondary school students to apply the linear (or proportional) model on applied geometrical problems about the relationship between the length and the area/volume of similar geometrical shapes, for which this linear model is not suited. This overgeneralisation of the linear model (the so called 'illusion of linearity') has also been exemplarily described in several other mathematical domains, like elementary arithmetic, algebra and probability, however, without providing much empirical support.

This poster will report the results of a literature survey about the linearity illusion in students' probabilistic thinking. Taking Shaughnessy's (1992) review of the existing research on students' misconceptions about probability as the starting point, the poster represents an overview of those misconceptions that can be conceptually linked to the illusion of linearity.

With respect to each of the six identified probabilistic misconceptions for which such a conceptual link could be established, the poster will show (a) an example of a problem that is expected to elicit the incorrect answer resulting from an overgeneralisation of the linear model, and (b) a graph that shows the wrongly supposed linear relationship between the crucial variables in the word problem contrasted with the probabilistically correct nonlinear relationship. One example is given in the figure below.



If a fair die is rolled one time, there is a chance of 1/6 to throw a six. What is the chance of getting at least one six in five trials?

——— ***Misconception: 5/6 = 0.833***
P(at least one six in *n* trials)
= *n* × P(a six in one trial)

━━━ ***Probabilistically correct: 0.589***
P(at least one six in *n* trials)
$= 1 - (5/6)^n$

In our future research we will empirically investigate to what extent these probabilistic misconceptions can indeed be attributed to students' tendency to overgeneralise the linear model. This investigation will help us to further unravel some widely known probabilistic misconceptions, on the one hand, and to provide further insight into the range of the linearity illusion, on the other hand.

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UTILIZE CAS TO CLOSE THE GAP BETWEEN TWO VIEWPOINTS ON PARABOLAS

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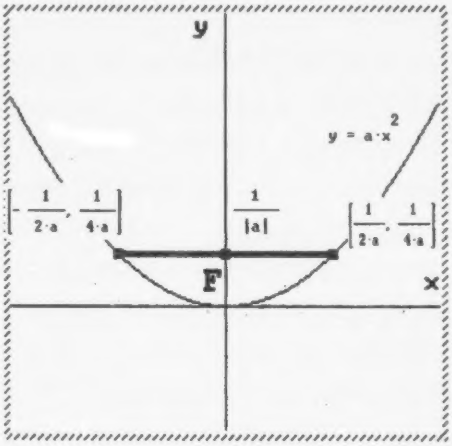
Our aim was to develop a unit for Grade 10 that deals with parabolas. We started by identifying, with the help of teachers, two main limitations of the traditional presentation of parabolas in the high school curriculum:

1. The difficulties students have in expressing the relationship between the parameter 'a ' of the quadratic function $y = ax^2$ and the shape of its graph. Most of the students can determine when the parabola has a cup/hat shape. Students also say something about 'stretching/shrinking', or 'opening/closing' of parabolas, while the teachers are not sure which description is the right one. A common misconception is that 'a ' is the 'slope' of the parabola (analogous to a line).
2. The gap between the two viewpoints on the parabola in the traditional curriculum: the algebraic view of the graph of a quadratic equation, and the analytic-geometry view of loci. This gap causes students to think that there are two different kinds of parabolas.

Our plan was to use the symbolic power and rich graphics of CAS to make connections between analyzing the shape of the parabola and viewing it as a locus of points. Naturally, we concentrated on the *Focus point* of the parabola. But, we did not find a constructivist approach that could direct students at this age (15-16). Consequently, we came with another idea - to develop the notion of "a special chord in a parabola" whose mid-point is the focal point. The chord, which passes through the Focus F$(0,\frac{1}{4a})$, has special properties. The length of this chord is $\frac{1}{|a|}$, which agrees visually with the fact that as *'a' increases*, the parabola 'shrinks'. When we calculate the distance between F and any point on the parabola $P(m,am^2)$, using our 'talented mathematical assistant', we get the expression $|P - F| = m^2\,|a| + \dfrac{1}{4|a|}$. In addition, the slopes at the end-points of the chord are 1 or –1.


y
x
$y = a\cdot x^2$
$\left[-\frac{1}{2\cdot a}, \frac{1}{4\cdot a}\right]$
$\frac{1}{|a|}$
$\left[\frac{1}{2\cdot a}, \frac{1}{4\cdot a}\right]$
F

Our goal was then to develop a sequence of tasks that will help students to open a geometric window on the parabola. The poster will demonstrate critical stages in the formative development of the learning unit.

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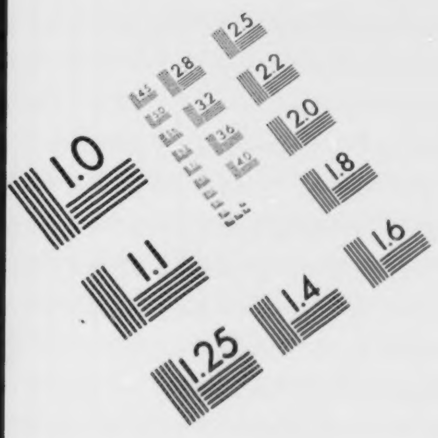

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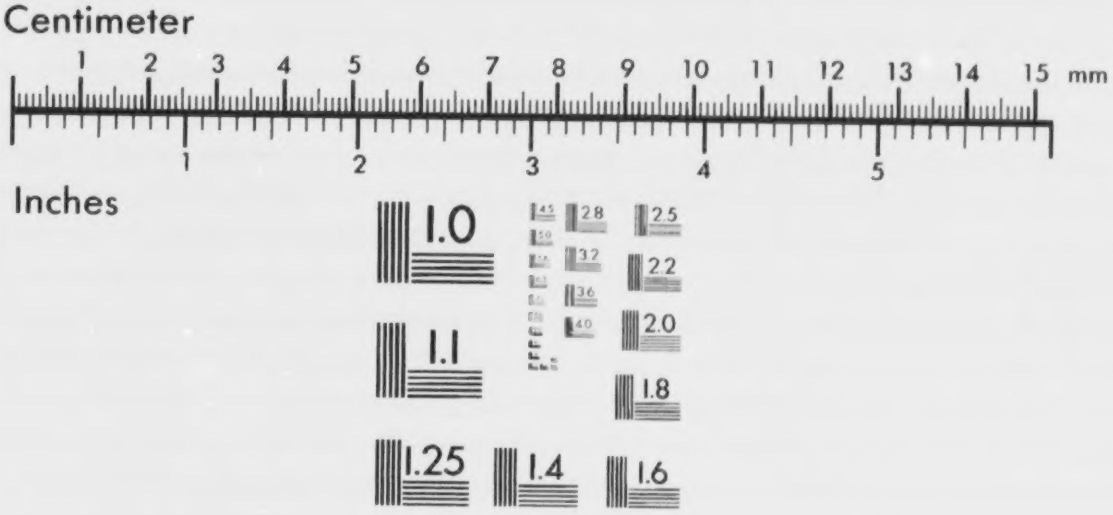
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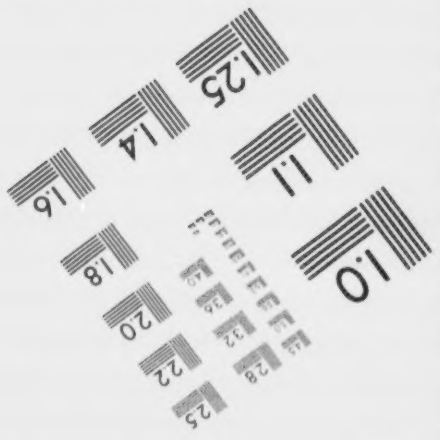

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PROCEEDINGS
OF THE
25th CONFERENCE
OF THE
INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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25@NL

Utrecht - The Netherlands
July 12 - 17, 2001

Editor:
Marja van den Heuvel-Panhuizen

Freudenthal Institute
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RESEARCH REPORTS

Highschool students' conceptions of graphic representations associated to the construction of a straight line of positive abscissas.

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Abstract: *The construction of a straight line of positive abscissas is a task that require to know how make a graph, you can use the point-by-point strategy, but if you don't have any equation is necessary to know where the graph is. We ask to tree different samples of students (16 to 19 years old) where a straight line of positive abscissas have to build. We use a semiotic classification of variables visual and categorial for to find no relevant aspects in their answers, our analysis suggest that some of the students have an "one dimensional conception" about the Cartesian plane, they consider variations only in one coordinate and this conception seems coexist with the point-by-point graphics. They take the y-axis like an anchorage in their attention, it cut the x-axis leaving in one side negative numbers and the positive in the other side. And they use often one part of the graph y=x although they think a complete straight line*

Resumen: *La construcción de una recta de abscisas positivas es una tarea que requiere de conocer cómo se hace una gráfica, pero si además no se tiene la ecuación entonces es necesario saber donde debe estar la gráfica. Proposumos a tres muestras distintas de estudiantes, de 16 a 19 años, la construcción de una recta de abscisas positivas. Utilizamos una clasificación semiótica de variables visuales y categoriales para detectar los aspectos relevantes de la tarea así como los que no lo eran en las respuestas, nuestro análisis sugiere que algunos de nuestros estudiantes tienen una "concepción unidimensional" sobre le plano Euclídeo, es decir, consideran solamente variación en una de las dos coordenadas y esta concepción parece coexitir con la graficación punto a punto. Toman al eje y como un elemento que ancla su atención y corta al eje x dejando de un lado a los números negativos y por otro a los positivos. Utilizan frecuentemente una parte de la gráfica de la recta y= x pensando en la recta completa.*

Introduction

When studying Analytic Geometry, in which objects and conceptions may be expressed through different kinds of representations, Defining and Definiens must be clearly distinguished in accordance to current education, based on the construction of relevant concepts which cannot take place on the basis of only some of their representations.

For to build graphics is neccesary to pass from equation to the graphic and converse, this articulation is a *one to one* relation and always it is possible have one equation for each graphic although you don´t know it.

The idea that *there is a correspondence between the graph and the equation* was used by Pierre de Fermat and Rene Descartes in the creation of Analytic Geometry, according to C.Boyer (1956) which -for teaching purposes- should be understood, as we mentioned before, in the sense that there is *a one to one* correspondance between each algebraic representations of straight lines and their graphic representations and converse.

In fact, the possibility of establishing the relationship between the graphication of the straight line and its equation can see since a semiotic point of view. In this case graph and equation are linking in a univocal way through visual elements (visual variables) in the one case and algebraics (categorical variables) [1] in the other (Duval 1999).

In this paper we make a suggestion about the linking between categorical and visual variables in a construction task: the straight line of positive abscisas. We have been working with tree different samples to see if the typical graphic answers appear on them. We describe the relevant aspects against no relevant aspects about the answers of the students too.

Theoretical framework

The Raymond Duval's point of view is that the conceptualization process involved in mathematical concepts or objects (noesis) through conversion of different semiotic representations (semiosis). Such point of view enables us to observe the conceptualization of mathematical objects or concepts in which all the meaningful variables between the graph and the equation are displayed, thus showing meaningfulness and allowing for the learning of the concept from a sign/significance perspective.

R. Duval (1999) develops a semiotic viewpoint based on graphic and algebraic representations that belong to different semiotic registers and articulation between them, they must be carried out through *conversion*[2].

As to the conversion between registers -which is necessary to make a conversion the graphic and the algebraic representations- one would have to establish the relationships between the visual and the categorical variables on the basis of general variables defined by Duval (1988), as follows, the general variables are: 1.Task implantation, figure sticking out from ground: a line or region 2.Task form: the line, whether it limits a region or not, whatever it is straight or curve. If it is curve, whether it is open or closed.

---

[1] Categorical variables herein mentioned are those which make a difference from a semiotic point of view between one equation and another, between one graph and another, in case of visual variables.

[2] Duval (1994) defines thus: "conversion is an operation which transforms a representation into a register change (p.222). It is also: the transformation of such representation into one of different register where all or only part of the former content is preserved". (1988, p. 21).

And three particular variables: 1.Direction of line's slope (goes up, goes down). 2.Angle and axes (symmetrical partition, major angle, minor angle). 3. Position on the y axis (cuts above the axis, below the axis or on origin).

We notice the following changes in its school treatment about straight line:

a) Graphism associated to the Euclidean straight line where position has no referents. Only the figure-form exists. Position is contemplated as a visual variable acquiring value when it is associated to other elements as a tangent or a bicectrice. The kind of process is called here an *Euclidean straight line.*

b) The straight line is treated almost like the Euclidean straight line. With almost complete freedom as to position, restricted only by the figure-ground, this is, by the coordinase axes. The equations are not the most important subjet here. Relationship between figure-ground and figure-form is given on a figural manner. This kind of process is called here *post-Euclidean straight line.*

c) Straight lines` graphism is associated to linear equations. Equation use and its variables determine the straight line's position. The straight line's characteristics are manifested through points such as (0,b) and/or (-b/m, 0) or by the slope m of straight line $y = mx + b$. Visual variables are not considered, since the position is now expressed algebraically. We call this treatment *algebraic straight line*[3] .

In the mathematics straight-line process, two principles can be noted: 1) The straight line is formed by an infinite number of points; 2) The straight line can be extended indefinitely to both sides.

These principles are strictly theoretical but they have a gestalt importance that is often missing. Every attempt to represent them figuraly or algebraically is but an attempt to build an infinite set with a finite number of elements through a finite process as well, however is not possible miss these gestalt characters when they build this straight line.

For the current paper we must consider some works have demonstrated that when handling definitions, are influenced by certain kinds of examples called prototype, which are more popular than the others are. There is also a tendency to increase secondary or irrelevant characteristics in the definitions (Hershkowits, 1989; Vinner, 1991).

The problem

The main problem we have used for our observation is the following; the task is about geometric description on graphic register:

---

[3] Front a point of view of the historical development of the mathematical knowledge the stages can be a) Euclidian line, b)Analytic line and c) Afin line, but here we want to consider the school treatment by programs.

6.Draw a line where all the abscissas of the points are positive, this is, where x is always x > 0.	

In this task we only used one way in the double link of the conversion between algebraic and graphic registers and the algebraic register is support with the natural language.

In the sense of Duval (1988), we describe the general visual variables in our task: 1. Task implantation: a line; 2.Task form: Straight line. About particular variables we have: 1. Angle and axes: symmetrical position; 2. Lines slope: infinity; 3. Position on the *y*-axis: no cut.. The draw conditions are: $x = k,\ k > 0$

Categorical variables	Visual variables
x	Parallel straight line to *y*-axis
k	Cut *x* axis at *k*
k>0	The line put on the right side of *y*-axis.

Methodology

In early papers, we have been doing research with highschool students between 15 and 19 years old, in relation to the graphication of points, straight lines and semiplanes, with special attention to their ordinate spatial character and observing the relationship between figure-form (graph) and figure-ground (axes) under a coordinate-based treatment, on the one hand, and to the relationships established by its graph, on the other. In the present paper we want to ask about how the student think the build a straight line no typical, where they have to show theirs conceptions about the straight line in the Cartesian plane. Our samples were:

Sample	School	Age	Number	Questionnaire
A	CCH Sur	16	87	C2
B	CC Sur	17	77	C6
C	Sciences Department	19	42	C7

The sample C students are highschool graduates coming from different schools that answered the questionnaires on the first day of class at the Sciences Department, Math division, University of Mexico.

Students of the other samples were on 5o semester but the ones who are 17 years old were repeating the course, our intention with different samples was to see if the typical descriptions are present in different school information and age stages.

Localization of points on the plane may be considered a condition for graphicating point-to-point, although it is not enough for conversion between representations because the continuity gestalt condition is lacking. The following table shows performance when locating points on the plane.

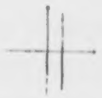
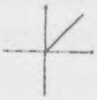
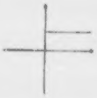
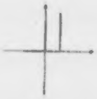
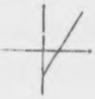
Sample	Point (4,7)	Point (-5,8)	Point (-4,-7)	Point (0,9)
A	95.4%	88.5%	94.2%	
B	89.6%	85.6%	84.45%	74%
C	95.2%	97.6%	97.6%	95.2%

We have introduced in all of the questionnaires we have worked with a legend elucidating the difference between ordinate and abscissa to prevent confusion about axes from being the most relevant error. Table results indicate that once the difference is made, the task is reduced to applying the localization algorithm.

The problem dealt with in the present paper is proposed as a pencil and paper task on the Cartesian plane where we work with whole units around the origin. Students distinguish positivity on the plane as we have noted on their localization results, and they have built straight-line graphs point-to-point, supported by their equations as part of the highschool curriculum.

We consider their answers reflect their interpretation of the question in terms of their own conceptions since they have no background on questions like this; therefore, they are not influenced by their teacher's indications.

Below we have the answers to the problem:

Answers	Graphics	Sample A	Sample B	Sample C
I Right answer		0%	5.2%	40.5%
II		62.1%	26%	19%
III		1.1%	29%	16.7%
IV		3.4%	6.5%	7.1%
V		24.1%	9.1%	2.4%
Others		3.4%	10.4%	14.3%
Absent		5.7%	13%	0%

The nest table show the relevant and no relevant aspects in the answers of the students we classify them like follow:

Relevant aspects:

$x = k$
$k > 0$

0% in A, 6.5% in B and 40.5% in C

No relevant aspects:

In II the points are: $(x,y) = \{ y = x, x > 0 \}$ 62.1% A; 26%, B; 19% C
In III $(x,y) = \{ y = k, f(x) = y, x > 0 \}$ 1.1%A; 29% B; 16.7% C
In IV $(x,y) = \{ y = mx + b, 0 < m$ from infinity; $x > 0\}$ 24.1%A; 9.1% B;2.4% C
In V $(x,y) = \{ x = k, k > 0, y > 0 \}$ 3.4% A; 6.5% B; 7.1% C

In all wrong answers the students missed a hidden aspect, the continuity of the line, this aspect own to the *treatments*[4], it is not a visual variable but it is necessary for focus the visual variable that we need for the conversion. The number of no relevant aspects that students consider in their answers are: one in IV ($y > 0$), two in II and V , and three in III, besides the continuity condition (see the next before table).

Analysis and Conclusions

Previous results evidence that almost all of the students in sample A and B and almost 60% of sample C students make no difference between a straight and a cut straight line and, once the student incorporases this idea, he might think that, complementarily, the negative abscissas may be avoided if axis y is not exceeded as in answers II, III and V.

If we consider the item referring to the semi-straight line at 45° , we may say that this prototype example is rather relevant in the teaching of Analytic Geometry. In fact, it cannot be missing in school texts or in class. This straight line has a figural characteristic in the sense that its special slope is synonymous of its angle.

We found that the wrong answers start when they no consider the gestalt continuity condition about the continuity and that the straight line can be extended indefinitely to both sides.

Among the answers given to our question, answer II has anchorage on the origin of the coordinate axes, this point is privileged because it makes the difference between positive and negative abscissas.

The cut straight line of answer III seems to be stopped by the x axis; the same happens with answer V. Anchorage is produced on the same axis in both cases.

How the requested answer is interpretad in the exercise might be rooted in the school legend that says: *positive to the right, negative to the left,* which is used when establishing positivity on the real straight line.

An anchorage would explain answer IV on the horizontal axis where the legend could be *positive above, negative below.*

In the present case, as far as figural treatment is concerned, we observe that students alienate figure-ground from figure-form in their graphic description. The straight line is treated like the post-Euclidean straight line with almost complete freedom as to position, restricted only by the figure-ground, this is, by the coordinate axes, though not in its coordinate character but as limits to positivity, the same way zero is the cutting limit between positive and negative numbers.

---

[4] Trearment of one representation is when it hava a converting inner the own semiotic register.

We may draw the following conclusions:

1. The figure-ground (the axes), in the case of no correct answers, is reduced to a general reference of the plane's positivity in one dimensional terms. Reduction is very important in the figural treatment produced by our students, because it is the reference, that the axes support, brings meaningfulness to the straight line's position, the reference's frame make the difference with a straight line in the Euclidean Space, they use the straight line like a post Euclidian striaght line.

The straight line's position, for students that proposed some types of cut straight lines as solution, is linked to the legend *positive above, negative below;* on one side and other side *positive to the right, negative to the left.* When they use both conditions separate in fact they use it like a one dimensional element, defining like a single ordinate not like a couple.

2. Most of our students -mainly the youngest- use a cut straight line related to a prototype straight line where the general visual variable regarding the problem solution is not considered by them. Their conception were building with only one part of the defintion, they miss that 1) The straight line is formed by an infinite number of points and 2) The straight line is extended indefinitely to both sides.The most important cause of mistakes in this task were the usefulness miss of this two conditions.

3. In the three samples they use the straight line and the cut straight line equally, according to convenience. This happen more in sample with youngest students but the oldest too made it. The straight line's figural properties as a geometrical object (using part of it or parting it) is used in solution without even noticing the difference.

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LEARNER PERFORMANCE AND FUZZY GENERALISATIONS: KEY ISSUES IN PROFESSIONAL DEVELOPMENT RESEARCH

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In this paper we discuss two methodological issues that emerged through a teacher education research project at the University of the Witwatersrand (Wits). (i) If improved learner performance is an important indicator of professional development impact on the teacher, how can this be established? (ii) Given the complexity of teaching, and related qualitative methods in researching professional development, what kind of knowledge claims can be made? What might be their status? From our empirical experience, we argue that it is feasible to generalise about take-up from, and impact of, a professional development programme through 'fuzzy generalisations' (Bassey, 1999). We also raise concerns about one-off learner tests and inferences about professional development.

In 1996, the University of the Witwatersrand launched a Further Diplomas in Education (FDE) programme in Mathematics, Science and English Language teaching with the following goals: to broaden and deepen teachers' subject knowledge, pedagogic subject knowledge and educational knowledge; to extend teachers' reflective capabilities; to facilitate professional growth; and to enable access to further education. The team responsible for the development of the FDE set out to develop the programme through research[1]. Undertaking the simultaneous development of, and research into, a programme was a considerable challenge, exacerbated at the time by a funding climate in South Africa in which teacher education was expected to demonstrate impact: education funders were demanding indicators of impact such as measurements of improved learner performance.

The push to learner performance as a significant component of research in contexts of educational change has recently gained further momentum. In their analysis of education research methodology, Taylor and Vinjevold (1999) observe that changing forms of educational activity do not necessarily lead to improved learning. From this they assert two implications for research methodology. First, "the final test of the effectiveness of teaching/learning strategies lies in the outcomes of learning" and this is best gauged from learner performance on "carefully designed tasks". Second, observation needs to go beyond descriptions of form (p. 66). These are not new ideas. The issue is how is this testing and observation to be appropriately done, where appropriacy includes ethical research practice and possibilities for capturing the complexity of teaching and learning. In *teacher education* research, the underlying demand for accountability raises a key question. What kinds of data and analysis enable valid claims about learner performance and classroom practice and *how do these lead to inferences about the effects of a teacher education programme*? The double inference from learner performance to INSET effects is particularly problematic. We also need to ask whether INSET programme success can be inferred from qualitative descriptions of classroom practices on the basis of observation and/or videotext? What are the possibilities for generalised claims about substance (as opposed to form) of teaching and learning practices which, we would argue, require in-depth case studies? In short, what methodologies need to be employed to infer the worthwhileness of investment in in-service professional development programmes?

This paper begins with a brief discussion of the research methodology that evolved over the three years of the research project (See Adler and Reed, 2000, for more detail). The focus then foregrounds mathematics and turns to two broad methodological issues: (i) If learner performance is an important indicator of the impact of professional development, how can this be established? (ii) Given the largely qualitative methodologies employed in professional development research, what kind of knowledge claims can be made and what might be their status?

Research methodology:

The overall aims of the FDE research project were three-fold: (i) to investigate teachers' take-up from the FDE programme in Mathematics, Science and English Language Teaching and to what extent and how this shaped the quality of their classroom practices; (ii) to contribute to knowledge about formal in-service professional development (INSET); and, (iii) to feed back into the FDE programme's curriculum development through research. The research team set out to describe and analyse continuities and changes in classroom practices within and across some participating FDE teachers over time, in relation to conditions in which teachers work and their pupils learn. Methodologically, while the research has 'project evaluation' elements to it, it is more appropriately described as a practice-based (Lampert and Ball, 1998), case study of cases (Bassey, 1999). The *FDE programme is the overall case under study*. It is a case of formal, in-service professional development designed to improve the practice of the practitioners in the programme, but also to contribute to policy and practice in the wider field of teacher professional development. *The teachers we worked with constitute a collection of particular cases*. The selection of 25 teachers across maths, science and English language, and their location in 10 urban/rural, primary/secondary schools is discussed elsewhere (Adler, Lelliott and Slonimsky, 1997). Our unit of study was the 'contextualised teacher', or the 'teacher-in-school'. If take-up from the programme and the quality of classroom practices was to be understood, these needed to be contextualised and personalised, with a description of what happened *over time* with *this* teacher in *these* kinds of circumstances i.e. a set of case studies. However, given the goals of the FDE programme, the study also needed to enable the identification of patterns or trends across teachers and contexts, or a cross-case analysis. The overall research project therefore focussed on a relatively small number of teachers in the programme. There were 10 mathematics teachers in the study, a number small enough to enable us to look in depth at each teacher, and large enough for us to be able to identify patterns and trends across teachers.

There are also two dimensions to the *practice base* of the study. The study is embedded in the practices of the FDE as a mixed-mode delivery, professional development programme. It was carried out by a team of researchers, most of whom are practitioners in the programme. The research was also carried out in classrooms, on and with classroom practitioners. It was our intention from the outset to learn about ours and the teachers' practices by investigating and theorising practice (Lampert and Ball, 1998) in local settings in all their complexity and diversity.

In relation to classroom practice, the key question that framed the overall study was: How are teaching and learning affected by resources (material; time; socio-cultural), by teachers' subject knowledge, by teachers' pedagogic knowledge and by teachers' reflective capabilities? Data was gathered from: school inventories, classroom observation schedules, supplemented by field notes; learner classwork and homework tasks and projects; tests administered to learners; videotapes of some lessons; audiotaped interviews with teachers and school principals; questionnaires and narratives completed by the teachers. The research team worked in subject teams in the selected schools in the Northern Province and Gauteng for one week in each of the three years (1996-1998) of the project, with the data collected in 1996 serving as a base-line (see Adler, Lelliott and Slonimsky, 1998). The team attempted to develop portraits of each teacher, capturing as fully as possible, the texture of the teacher's practice. Each portrait was similarly structured to facilitate cross-case analysis and the identification of patterns of 'take-up' within and across the three subject areas in the first instance, and then across the whole set of cases. Analysis of the data collected in 1997 and 1998 also began with the development of teacher portraits. By 1998 we had identified key issues that we believed were central to the FDE programme, teacher development and curriculum change in South Africa, and to in-service professional development more widely. We proceeded to analyse the data over the three years according to the following key themes: the nature, availability and use of material and cultural resources as a function of programme take-up and the context of teachers' work; the critical issue of the relationship between teachers' knowledge-bases and development of high order knowledge and skills in their learners; the challenge of language-in-education policy and practice, particularly code-switching as a teaching and learning resource across contexts where English language infrastructure varies; teachers' take -up of the forms and substance of learner centred practice; whether and how in-service professional development plays a role in teachers becoming reflective practitioners.

Discussion of each of these is summarised in Adler, Bapoo, Brodie, Davis, Dikgomo, Lelliott, Nyabanyaba, Reed, Setati, Slonimsky (1999). The research team faced considerable challenges as the research unfolded. Practical constraints continually raised concerns about the potential for mismatch between our epistemological and methodological assumptions, our research intentions and goals, and on the ground realities. Many of these have been discussed elsewhere (Adler et al, 1998; Adler and Reed, 2000). The focus here is on the challenges of an accountability context discussed in the introduction to this paper.

Learner "performance" as indicator of INSET success

What does learner performance on "carefully designed tasks" (Taylor and Vinjevold, 1999, p.66) tell us? About the learner? About the teacher? About the curriculum? About national standards? Learner performance on carefully designed tasks can tell us about any or all of these ... depending. It depends on the nature of the tasks set, when the assessment occurs, where the assessment occurs and how often. These assertions are not new. As Saljö and Wyndam's (1993) study reveals, task "performance" is a function of the task and the learner in a particular setting at a particular time. There are thus significant issues in reading learner competence from single tests or tasks without significant attention to context, let alone moving from learner performance to the teacher's competence.

Learner *performance* is typically accessed through some form of testing. While it was the original intention in the study to test learners, the complexities of doing this were completely underestimated. We kept coming back to how a particular test or set of tests would be an appropriate or adequate means of assessing learners (and which learners?) over three years, and moreover, in such a way that the impact of the FDE programme on the quality of a particular teacher's practice could be inferred. For example, tests for the same learners at two different times in the year should show learning gains, but we could not see how to legitimately establish any kind of causality between learner gains and their teacher's participation in an in-service programme. More appropriate could be either comparative testing of programme teachers' classes with other similar classes in their schools, or with similar classes in different schools where teachers were not involved in the programme. But we did not believe we could control for intervening variables, nor that such an endeavour was conceivable. In addition, any of these tests meant the construction of new items in order to assess new forms of knowledge and skills valued in the programme. The reliability and validity of any new test could not be accomplished within the time frame of the project. As Cooper and Dunne (2000) have argued, there are significant validity issues in more complex forms of math assessment, particularly context-embedded items.

We considered the use of existing standardised tests e.g. tests that had been constructed by the Human Science Research Council (HSRC) prior to the first democratic elections in 1994. Aside from inappropriate cultural contexts to contextual mathematics items, there were few that assessed mathematical processes beyond set procedures. Our frustration and difficulties with test development does not mean that it is not important to design effective means for studying learner performance as an indicator of learning gains. As Jansen (1996) and Black and Atkin (1996) argue, there are limitations in evaluations of educational innovations when student learning information is lacking.

Despite difficulties, the FDE team did not abandon assessment of learner performance as an additional indicator of teacher learning. Each teacher observed had a set of his or her classwork books examined to ascertain the kind of written work that was being covered by learners - a coverage that could not be discerned from two or three lesson observations. Learner books are not direct indicators of learner performance. They nevertheless can reflect the kind of mathematics valued by the teachers through inscription and attempts at practice/ and mastery. These added to observations of mediated content during lessons. A detailed "Pupils' Written Work Schedule" was constructed, refined in 1997 and 1998 and used to illuminate learner performance in the subject through all their written work accumulated between February and August each year. Classwork books, homework books, test books, exam papers, test papers and scripts of nine learners in a purposefully selected class (three good, three average and three poor) were examined and recorded in the schedule. In addition, some testing was conducted in Grades 7, 8 and/or 9 classes in each of the three subjects. In 1997 the tests used were constructed and conducted by members of the research team. These were exploratory, both in terms of how they were used, and what they revealed. In 1998, we accessed Grade 7 mathematics tests that had been developed as part of a project geared towards more appropriately normed tests than those available to us through the HSRC. We built on Grade 9 tests from 1997 and conducted learner tests in classrooms where teachers were teaching math at either Grade 7 or 9.

We learnt several lessons from this testing. Testing some learners revealed to us how this additional data provides for triangulation of data within case studies. The test performance of learners in different teachers' classrooms by and large confirmed and thus strengthened the accounts of teaching and learning practices analysed and built into the teacher portraits. In instances where there was a mismatch between our independent test assessments and what we observed in learners' written texts, including their in-school testing, we were able to explore these with the teacher and develop insights to enrich the overall portrait of teaching and learning. Testing learners as part of researching teacher development and INSET effects can be illuminating. This is "the good" side of such testing.

The bad side of the testing for us was that, in general, our independent test results confirmed the TIMSS messages with regard to levels of performance in mathematics and science across our schools. Our results were not at odds with test results obtained by most of the teachers themselves in their own testing. The bad is that this situation persists. Our broader data assisted us in seeing that pupil performance was not in any simplistic way, a reflection on the teachers' knowledge-base. For example, one of the secondary level mathematics teachers, a teacher who demonstrated extraordinary take-up from the FDE programme, worked in an over-crowded, impoverished context. Her learners arrived in her Grades 8, 9 and 10 classes considerably under-prepared for the levels at which she was expected to teach and assess them. No wonder then, that on her own tests, let alone the independent tests we administered, performance was extremely poor. This learner performance tells us something about the state of the nation. However, to infer teacher quality and INSET programme quality from such "results" is extremely problematic.

The "normed" tests we used were themselves problematic. We found that some of the math items were ambiguous. This raised serious questions about reading off learner competence from their performance on tests that included such items. In addition, and as is the case with reform anywhere, for some of the learners, the form of the test items was unfamiliar. A simple prompt by a researcher on one occasion enabled a correct response by the learner. Crude analysis of test performance then will misrepresent learner knowledge. Testing is not simply a matter of "carefully designed tasks" but crucially a function of the testing context, including learners' familiarity with the tasks. Test validity is a serious research endeavour. Our concern as a result of our experience is not that testing should not be done, but that research and development be undertaken to develop instruments appropriate to various processes of research. Wilson and Berne note similar in professional development research in the USA where the capacity of researchers to tie measures of teacher learning to measures of student learning is challenged by "the lack of robust and standardised measures of student learning" (1999, p. 197).

Testing is not the only indicator of learning gains. Our experience suggests that close analysis of learner written material is more illuminating than one off tests in the contexts of teacher education research. Close examination of learners' classwork and test books, alongside the tests and examinations that teachers set and their marksheets were illuminating of depth and breadth of coverage over time by the teacher and the kinds of knowledge forms that were inscribed by learners hence indicating what is valued as knowledge within the school setting. We did not need independent tests, over and above such analysis, to reveal to us key challenges for teachers and hence the FDE programme. Learner written texts revealed learners' limited exposure to knowledge, and how teachers, for a range of reasons, were not covering required areas of learning, nor enabling learners to engage with knowledge at anything beyond the most superficial levels of recall and repetition. We noted difficulties with selection, sequencing and grading of tasks across most of the teachers, and this was evidenced across learner written texts. This observation tells us not only about teaching practices, but also about assumptions in the FDE programme. Like the government of the day, the programme had not grasped the full extent of the breakdown of a culture of teaching and learning across schools in struggles fought against apartheid education in the 1970s and 1980s.

Teacher education research: the status of resulting knowledge claims Bassey (1999) defines a case study as a "Study of singularity conducted in natural settings" (p.22). Earlier, we discussed how and why it was necessary to work in depth with a few teachers: qualitative multifaceted observation was required if we were to do any justice to the complexity of teaching as a social practice. How then do we, on the basis of diverse case studies, even with cross case analysis, make claims about teacher up-take from the FDE programme that extend beyond the specific research teachers, and hence to overall programme effects? Bassey (1995) distinguishes two kinds of empirical study in educational research: the search for generalisations (requiring investigation of large populations through carefully selected samples), and the study of singularities (case studies). The implication here is that case studies cannot lead to generalisations, and thus that they are limited in their use in educational policy and planning. Bassey argues that it is possible to develop what he describes as fuzzy generalisations from carefully conducted case studies. He uses the term "fuzzy generalisation" for a statement that makes no absolute claim to knowledge but hedges its claim with uncertainties. It arises when an empirical finding from a case study such as *In this case it has been found that* is turned into a qualified general statement such as *In some cases it may be found that* or *If we do x rather than y then teachers may learn more.* Bassey suggests that if educational researchers disseminate their findings in the form of "fuzzy generalisations" they are inviting teachers and education policy makers to enter into a discourse with these generalisations. Entry into such discourse is likely to be facilitated by access to an 'audit trail' - the evidence in support of the fuzzy generalisations which the case study has produced.

Bassey's argument for "fuzzy generalisations", and even weaker claims in the form of "fuzzy propositions" arises out of his extensive educational research experience, where he has seen numerous studies of quality not impacting on teachers and policy makers precisely because findings are deemed too specific. Our findings in the FDE research project , and the status of the related claims we believe we can and should make about the FDE programme as a whole, INSET practice in South Africa, and INSET practice more widely, resonate with Bassey's notion of "fuzzy generalisations". Indeed, "fuzzy generalisations" appear to be constitutive and reflective of other teacher education research.

In their review of "highly regarded" published research on "teacher acquisition of professional knowledge in" the USA, Wilson and Berne (1999, p. 194) identify a number of common themes, one of which is particularly pertinent here. They identify a concern with the labour intensity entailed in the qualitative nature of the research (hence expensive in human and related financial terms from our point of view), and the substantial commitment it demands in terms of examining teacher talk, and classroom practices. They point out that "[E]ach research project *struggles with ways to document teacher knowledge*" (1999, p. 195, emphasis added). Because of the complexity of classroom practice and the qualitative, case study nature of much of the research, documenting and hence evidencing teacher professional development is difficult. Claims made ("... programs ... were likely to ...") are tentative, "fuzzy generalisations" in Bassey's terms. To return to the FDE research, we will draw on our analysis of teachers' take-up of language practices (Setati, Adler, Reed and Bapoo, forthcoming) to illustrate how we documented, evidenced, and then drew out recommendations at the level of the FDE programme, and fuzzy generalisations in relation to INSET policy in South Africa, and INSET research and development more widely.

We documented extent and frequency of teachers' and learners' code-switching practices over the three years of study. In the final year we also examined the production and reception of expressive and discourse-specific language. We used structured classroom observation schedules, unstructured videotape of lessons, structured observation of learners' written texts and teacher interviews. We found increased use of code-switching by teachers and learners in most classrooms, in particular increased drawing on learners' main language(s) as a resource. We learned from the teachers that their code-switching practices are intentional but dilemma-filled, particularly in the face of the dominance of English in the South African context. We also found widespread 'take-up' by most teachers of forms such as group work, and hence increased possibilities of learning from talk (i.e. of learners' using language as a social thinking tool). However, our observations in 1998 raised the question as to whether most of the teachers were complementing this shift to *learning from talk* with strategies for *learning to talk* i.e. learning to talk and write formal mathematical discourse. We also found that while the above were general patterns across all the teachers, they concealed important attenuated differences across teachers in different contexts, levels and subjects. For example, because their primary goals differ, there was more code-switching by math teachers than English language teachers. There was less code-switching and more focus on using and modelling English in primary than secondary mathematics classes as primary teachers carry out their dual functions of teaching the subject, and developing learners' proficiency in English. This dual role and emphasis on English was complicated further in rural schools i.e. schools with limited English language infrastructure.

These 'findings' from our case study of cases, led us to the following recommendation for the FDE programme, and fuzzy generalisations for INSET policy and practice. The intent here is to inform ongoing curriculum review in the FDE programme, and to invite teachers and policy makers to "enter into a discourse" (Bassey, 1999, p.52) with these generalisations.

* At the level of the FDE programme, we need to pay more explicit attention to possible journeys from exploratory and informal talk in the main language towards formal mathematical talk and writing in English

*At the level of educational policy in South Africa, findings from our research *suggest* that some of the dominant 'messages' in current curriculum documents *may* need to be reviewed. For example, one of these messages in Curriculum 2005 is that group work is 'good' as it encourages exploratory talk and co-operative learning. The issue of how teachers and learners are to navigate the journey from informal spoken language (in the learners' main and/or additional languages) to formal, written mathematics in English is not addressed.

* At the level of INSET: What we have shown from our study of FDE teachers in multilingual contexts is that firstly, take up is attenuated across contexts. This *suggests* the need for more serious engagement in teacher education with the possibilities of, and constraints on, what are typically presented as panaceas for 'good practice'. The different English language infrastructures, levels and subjects in and with which teachers work *appear to be* significant for shaping INSET possibilities and constraints. We need further research and development programmes that dis-aggregate schools and classrooms along these three different axes (Setati, Adler, Reed and Bapoo, forthcoming, emphases added).

In the presentation of this paper, I will present additional information on the overall research methodology employed in the FDE professional development project, as well as some of the "audit" trial that lead to the fuzzy generalisations about changing language practices and their effects. Hopefully, through this reflection on issues emerging from a specific research project, this paper will contribute to the development of theory, practice and research, in their inter-relationship, in mathematics teacher education more broadly.

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NOTES

1. The team that worked on the research over its three years, and to whom debt is owed for all research outcomes, including this paper, was: J Adler, A Bapoo, K Brodie, H Davis, P. Dikgomo, T Lelliott, T Nyabanyaba, Y Reed, M Setati, and L Slonimsky.

Ethical practice in mathematics education research: Getting the description right and making it count

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Abstract *Building on the work on ethics in educational research in recent publications, we present a framework for ethical practice in mathematics education research. In particular, we discuss what are the implications of claiming or denying a particular piece of research as acceptable within the community. We argue that researchers must be aware for whom they advocate, thus making it count. We present a map with which researchers should engage the ethics of their practice, and we suggest that they must consider whether they are getting the description right.*

Howe and Moses (1999) have presented a detailed history of ethical practice in educational research, and Sowder (1998) has done much the same in a collection on mathematics education research. In the latter, Sowder also discussed issues of ethics in the dissemination of research findings. Together with these authors we wish to recognise: that educational research is always advocacy research inasmuch as unavoidably it advances some moral-political (and so ideological) perspective; that educational researchers must be able to defend what their research is for (respect for truth); that the research must have points of contact with insiders' perspectives (voice, respect for persons); and that it militates against race, gender, class and other biases (respect for democracy) (Bassey, 1999, p. 37).

Summarising these overviews, we propose that researchers should engage with the following questions:

- On whose behalf is the research advocating? Is it against racism, sexism, classism etc?
- What is the research for? Can the researcher defend the research? On what grounds?
- Does the research incorporate the insider's perspective?
- Is the research reflexive?
- Does the research take care of those being researched, especially avoiding their oppression?
- Does the research draw on refined notions of consent, autonomy, and privacy?

Our concern in this paper is to extend these ethical positions and to ask: On what grounds can one claim that a piece of research is or is not *mathematics education research*?

Sowder (1998) began her chapter with some scenarios. We will do the same here and through them we will present the main themes of our argument. The first is a fictitious scenario but familiar to many of us in the issues it raises. The second scenario is a perspective on TIMSS. We conclude with presenting a map for researchers to elaborate an ethical framework for their research and applying that map to the two scenarios (for a more developed account see Adler & Lerman, in press).

Scenario 1 (fictitious) - A research dilemma

All over the world there is a tendency towards reform in mathematics teaching and learning which takes for granted the four following features:

- Rich mathematical tasks
- Relating mathematics to real life experiences and practices
- Learner-centred practice (valuing and working with learners' mathematical meanings)
- Inquiry-based classrooms.

These reform initiatives are being researched and developed and, while emerging from practices in the developed world, they are nevertheless the object of desire in the developing world, despite substantive contextual differences. The underlying assumptions in the reform, and in much of its related research, is that these four features of mathematical classroom practice will lead to appropriate, meaningful and more successful mathematical learning.

Imagine a situation where the dominant forms of schooling are over-determined by selection rather than education. In poor countries there are enormous constraints on wide provision of public services (like health care) and public goods (like education). As Mwakapenda (2000) so vividly describes of Malawi, when only 10% of primary school leavers gain access to public secondary schooling, teaching and learning practices are inevitably driven by the forces of selection. Processes of democratisation and development – increasing equitable access to improved social and economic goods – in such a context are significantly different from those in the developed and dominant world. It goes without saying that mathematics education reforms will be shaped by such divergent conditions.

Imagine a mathematics education researcher from a developing context as described above, at the level of, say, PhD. As is often the case, this person gains entry into PhD study in an institution in the developed world, and is sponsored by the State Department of Education in his country. He enters a world where what counts as a problem in mathematics education is framed by the reform movement described above. He decides, after considerable exploration of the field, to study the implementation of inquiry-based mathematics teaching. He believes, as a result of his reading, discussion, and reflection on the educational situation at home, that inquiry-based approaches offer potential for improving mathematics teaching and learning in his country. He communicates with relevant parties at home, teachers are reported to be interested. He develops a programme and a set of materials that he believes are appropriate to his home context and he returns to set up the project, including at this stage, a series of workshops with a selection of Grade 7 mathematics teachers (the final year of primary school). During this time, he obtains their agreement and support for the project. Indeed, the teachers appear to enjoy the workshops where inquiry-based mathematics learning is modelled and issues discussed. The teachers share with him how they have been challenged mathematically and pedagogically. They express positive views of the potential for such practice in their own classrooms and a willingness to implement these ideas. He then spends a short period of time with one of the teachers in her class and together they try out activities in her classroom. On the basis of this piloting, he modifies and then leaves a set of materials for all the teachers to try out and develop and reflect on in their respective Grade 7 classes and returns to his academic institution.

Armed now with what has been agreed by his institution as sufficient ground-work and piloting, he proceeds with designing the next and critical phase of the research, the collection of data related to teachers' implementation, and so interpretation, of inquiry-based mathematics learning. Three months later, as planned, he returns home, this time with a range of research tools (instruments) and a carefully formulated participant observation design process for data collection and analysis. To his dismay and frustration he finds, across all the teachers, that the materials have barely been touched – an occasional activity had been tried. Moreover, term dates have been unexpectedly changed. Instead of a process being underway where he could now work with teachers to interrogate their interpretations of inquiry-based mathematics teaching, the teachers are focussed on preparing their learners for the kinds of assessments they will face at the end of their primary schooling. Teaching is restricted to providing practise with algorithms for the operations on common and decimal fractions. What is more, the extended time he had thought would be available for participant observation has been curtailed by changed examination times. He now faces considerable practical, methodological and ethical challenges.

He could continue with a modified exploration of inquiry-based mathematics. He could, for example, organise additional time with learners and teachers from one or two schools, after school hours, where he himself teaches mathematics in an inquiry-based way. Through this research strategy he might be able to identify and describe the kinds of activities learners engaged with, how and with what effects. His overall description and explanation is, nevertheless, likely to proceed from a starting point of 'failure' in relation to mathematics education reform by the educational system in his country and include a description of how and why the teachers were unable to implement inquiry-based mathematical learning. The description would keep intact a decontextualised sense of the potential benefits of inquiry-based mathematics teaching, and lead to recommendations for how school mathematics needs to change in his country, and what is needed to support this change.

By contrast, he could abandon his orientation to inquiry-based mathematical learning and reorient the study so as to understand why and how testing has come to over-determine considerations of epistemology and pedagogy, and how and why the timetable changed, so 'disrupting the data' (Valero & Vithal, 1998). This would be a difficult decision to take. Given time constraints for the study, he would need to proceed with a rolling plan for interviews, observations, where time for developing and piloting instruments was curtailed. If he travels this road he is likely to elicit data related to the selective function of mathematical performance, and to a range of socio-cultural and political conditions that shape the forms of school mathematics practice in Grade 7 in his country. His description and explanation of what happened through his research activity is more likely to focus on wider educational issues than strictly mathematical ones. He is also likely to be able to explain resistances in the system (as opposed to resistances in the individual teachers) to the intended 'reforms'. In other words, to explore and understand what happened would require redesigning the study, and most critically, zooming out of inquiry-based mathematics and into the wider educational practices in which the teachers are positioned.

How should he proceed? Which route should he follow? Depending on where he shares his quandary, he is likely to experience quite diverse and unsettling responses, particularly if he presents a preference for the latter approach. In the wider educational arena he could be challenged as to his competence to take this more sociological and systemic approach to the research. He is likely to share this concern. At the same time, in the community of mathematics education research, he is likely to experience reactions like: "Well, this is no longer mathematics education research" (one of us was witness to precisely this negative reaction when a similar situation was raised for discussion at an international mathematics education research forum).

What we are raising here is that getting the description right and making it count across diverse interests are ethical issues that need to inform the practices of the mathematics education research community.

Let's assume that because of this ethical standpoint, and within his financial and time constraints, the researcher proceeds along the more challenging path. He makes this choice despite not being an apprenticed sociologist and aware that it might well undermine goals for his own development and entry into the community of mathematics education research. He sets out to explore and explain teachers' practices in their mathematics classrooms with tools from the interpretative turn, and so to chart a less clear methodological path. As intimated above, he finds his description of teaching practices are framed by an analysis of the educational system in his country, fiscal constraint, and its overall examination and selection processes. The knowledge produced becomes more about how the teachers interpret and explain their mathematical practice within such systemic enablements and constraints, rather than about teachers' understandings of, and approaches to, inquiry-based mathematics. He goes on to include recommendations for a serious localisation of the notion of "inquiry-based" mathematics, and a speculation that a description of its forms and functions is likely to be substantively different from that which permeates dominant mathematics education discussion.

From our concerns with ethics in this chapter, this emergent description is 'right'. But our experience is that it does not easily count in the dominant field of mathematics education research. There is always the additional question: Is this research *mathematics* education research? From a research perspective this can be re-interpreted as: Can and will it add to the knowledge base in mathematics education?

Some, including ourselves, would answer in the affirmative. Despite limitations that are inevitable given time and financial constraints, this research could and should inform the knowledge base in mathematics education. The position here is that insights into the challenges of reforming the teaching and learning of mathematics in school lie precisely in an understanding of how mathematics takes shape in teaching and learning situations across school contexts. Such insights entail more than a grasp of the mathematics of the reforms intended, and their interpretation. Critically, getting the description 'right', and making it count for its participants, entails coming to grips not only with didactical transposition (Brousseau, 1989) but with recontextualisation processes inevitable in schooling. Curriculum change involves changes in how knowledge is classified and framed, and so too in relations of power and social control (Bernstein, 1996). Curriculum change will inevitably be contested terrain. It thus requires an in-depth understanding of school mathematics, and schooling itself, across diverse contexts.

This story and the questions it provokes are about the worth of the research reported, its quality, its boundaries and its methods, its financial constraints, and ultimately about ethics and values. We will return to this scenario in our concluding remarks.

Scenario 2 - TIMSS

This second description will be brief for reasons of space. In a globalising world, international comparative assessments make sense. They provide benchmarks for both internal and external comparisons. Such arguments have been made both by the key organisational hub for TIMSS (Plomp, 1998) as well as wider afield (Nebres, 1999). As it re-entered the world in 1994, participating in TIMSS in 1995 was an attractive option for South Africa. Here was a possibility for setting up a benchmark against which progress by the post-apartheid Government could be mapped and judged (Howie, 1998). The results of TIMSS are now well known, and need no rehearsal here. The question we pose is the broader ethical one that drives this paper. Did TIMSS get the description right? Keitel and Kilpatrick (1999) pointed to four problems: that the direction of the study has been over-determined by psychometric expertise; that financial support for the study influences the goals; that control over the framing and dissemination of results necessarily affects the results; and that there is an assumption that curricula across widely diverse contexts can be compared through learner performance presented as an average.

We wish to raise some questions, following their critique, and reflecting our concern with getting the description right and making it count:

- why were countries ranked as in a league table?

  - whose interests are served by this?
  - what kind of description is this?
- on whose behalf is TIMSS advocating?
- Given that the results served conservative agendas in so many countries, and that researchers are thought not to be responsible for how their work is used, where does responsibility begin and end in mathematics education research?

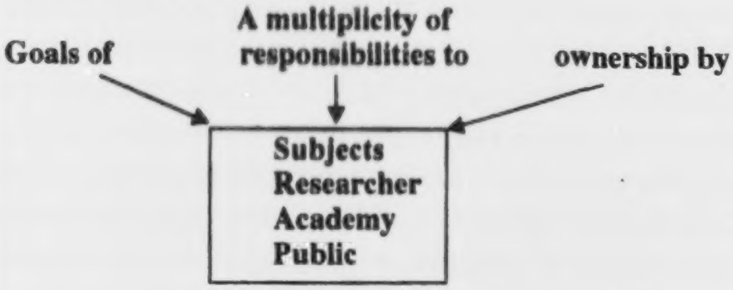
These criticisms of TIMSS are known. Why are we repeating them? Our point is that in the light of the above criticisms, TIMSS cannot get all of the description right, and in its omissions lie significant ethical issues, and thus for whom does this research count?

Concluding remarks

There has been a distinct Southern African focus in this paper. We are aware of the danger that some readers might marginalise the ethical issues we have raised because the history of the region, indeed the continent, is full of very dramatic inequalities, exploitation by the 'developed' world, and so on. We insist, though, that whilst inequalities might be more stark in Southern Africa than in many other places, inequalities and injustices are just as pervasive and ubiquitous in every part of the world and within every society, if sometimes less obvious. The cultural capital of success in school mathematics is common across the world: so too is the failure of so many students from working class and disadvantaged groups in mathematics. It is precisely the high levels of inequality that throw ethical issues into relief, issues that need to be confronted by all mathematics education researchers wherever they are.

We consider that educational research should be seen as located in a knowledge-producing community (Usher, 1996). What comes to the fore is the engagement with others and with history in an enterprise that should meet, as well as perhaps challenge, sets of socially constituted standards and values. Research communities, like all communities, are fragmented, with sub-groups, established and new paradigms, tensions, disputes, and boundary conflicts. These are indications of a normal healthy research community: the modernist image of a unified scientific group achieving universally accepted answers to universally agreed research questions is no longer expected. The complexity of the research enterprise is thus captured in the notion that it is a social practice.

Research can then be seen as a map:



Research must take account of this multiplicity. Thus, in our **first scenario**, our researcher's dilemmas concern:

Goals: to modify his study of inquiry-based mathematical learning, or to examine why testing overcame other issues. He needs to be aware of the goals: of his subjects, the teachers, to be supported in their struggles, not undermined, and not to have their trust broken; of the academy, to do what informs *mathematics education* research in ways that uphold if not develop the ethical standards of the community; and of the public, who want the best for their children, whatever that may mean, and who want their privacy respected.

Responsibilities: to his subjects, the teachers, to advocate for them, not to hold them up for criticism when they joined him in his plans in good faith, as his understanding of their situation changed; to himself, in gaining a PhD within the community to which he belongs and at the same time to be true to what matters to him in his research; to the academy, to advance knowledge of the teaching and learning of mathematics in its widest sense whilst challenging the community to recognise and value the research issue; to the public, to do research that takes care of teachers and students whilst informing for future policy.

Ownership: by the subjects, who see themselves to some degree as participants in the research, to improve the learning of their students, but pulled also by many other, perhaps stronger, constraints; by the researcher, who demands that his community also claim ownership through him; by the academy, that this *matters* to mathematics education research; by the public, that they should share in researching their schools.

In our **second scenario** of international comparisons, perhaps a question mark should be raised over the whole enterprise. If it does proceed to a FIMSS then the researchers' dilemmas concern:

Goals: of the subjects for the improvement of their life chances and therefore for the international publication of balanced, moderated results; of the researchers to endeavour to take account of all the factors that impinge on the results; to the academy to represent the subjects fairly so that the results are valid; and to the public not to misrepresent and do harm to their communities, including the educational community.

Responsibilities: to the subjects, to advocate for them; to the researchers themselves to produce democratically valid, generative results; to the academy to take responsibility for how their research might be used; to the public to represent appropriately the subjects of the research so as not to mislead and perhaps inadvertently encourage negative stereotypes.

Ownership: by the subjects, that they are included in the research at least by appropriate representation; by the researchers, again to worry about how their research will be used and by whom; by the academy to ensure that such research is for the researched as well as for the research community; and by the public so that the large sums of money spent will be of value to them in improving the life chances of their children and their community.

The community is very successful in locating and engaging with issues and challenges where the mathematics is prominent. We are convinced that the community also needs to be more open to seeking questions and answers where the mathematics recedes behind a myriad of intersecting social and political issues. Let us be clear: social and political issues are not an irritation that gets in the way of research in mathematics education. We consider it our ethical responsibility to seek out these settings for research. Otherwise we collude in denying access to power and control over their lives for the majority of students.

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BACK TO THE BASICS: RE-INTRODUCING ε - N DEFINITION OF THE THE LIMIT OF A SEQUENCE INTO CALCULUS 1 INSTRUCTION

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Abstract. The paper describes a teaching experiment whose goal has been re-introduction of ε --N definition of the limit of a sequence into standard calc 1 instruction. It identifies the limit schema as the coordination of two processes, both involving epsilon. It is suggested that this coordination is the basic mental construction involved in the construction of the limit schema. The role of certain student misconceptions on the development of the understanding of the concept is also discussed.

Introduction. Why is it important to understand the fundamental concepts of mathematics from their basic definition? In particular, why is it important to develop a good understanding of the ε - N definition of the limit of a sequence among Calculus students, if, the majority of them are not math majors? It is so, in our opinion, because the understanding of the limit represents one of the more important achievements of modern mathematics, which helped to clarify its conceptual basis, and as such belongs to the area of general education. It is significant in acquainting students with the means of treating "infinity" with the help of *only finite processes* [7], and as such has a general educational and philosophical value.

Recent research [4], [5], [6], and not so recent [9] of students' understanding of the definite integral as the limit of Riemann Sums reveals serious absence of the knowledge of the relationship between the geometrical construction of Riemann Sums and the corresponding numerical sequence of partial sums, in the thinking of students of Calculus. It became clear that a mathematically correct understanding of the definite integral at the Calculus II level necessitates a precise understanding of the concept of sequences and their limits. The teaching experiment described in this article is first in the series devoted to the appropriate reorganization of the curriculum and instruction in that direction.

Whereas students' difficulties and treatment of this theme are quite well known (section below), yet at the same time there is a relative absence of reports about the successful incorporation of this knowledge into instruction, to advance its understanding among students of calculus. Our teaching experiment has been designed to bridge that gap.

Teaching experiment described in this was performed at Instituto Tecnologico de Monterey, Mexico, D.F. during the Fall 2000, with the specific goal of investigating the effectiveness of re-introducing ε - N definition of the limit of a sequence into standard Calculus 1 instruction. The discussion below will be presented in more of a conceptual manner rather then in a quantitative manner (although some broad assessment will be provided), due to the very short time which has elapsed since the experiment has been completed - yet some results are sufficiently important, and necessitate presentation in a preliminary fashion.

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Literature review. Numerous researchers have found students to have seemingly inevitable cognitive and epistemological obstacles both with limits and the connected ideas of continuity, and differentiability [2],[7],[9], [10], [14]. In particular, Orton in [9], quotes some of the students he interviewed regarding the meaning of sequences in the context of Riemann sums, as saying that "the computation obtained in this way is approximate because the sequence never reaches the limit". This particular problem is related to the general issue of whether the sequence ever reaches the limit, and is discussed at length by [2],[7] and [10]. The authors of [7] discuss also another students' misconception, the neglect of the sequential order, which in our experiment turned out to be quite essential. The students often approach the limit in the context of ε - N definition, saying that the larger N, the smaller the distance ε between the term and the limit - a description, in some sense, reciprocal to that incorporated by the definition of the limit of a sequence.

The difficulties with all these concepts seem to be so profound that Cottrill et al [3] remark, "We have not…found any reports of success in helping students to overcome these difficulties".

In this bleak landscape of the instruction and understanding of limits, two brighter results are worth mentioning. In her work, "The effects of writing assignments on second semester calculus students' understanding of limit concept", [13], Walhberg mentions a measurable improvement of student understanding under the influence of writing assignments. Orton [8], on the other hand, mentions that when students were introduced to limits of sequences with the help of the Wallis technique [12] and apart from their interpretation in the context of the Riemann sums, they were quite adept in finding the limit as well. The Wallis technique of finding the limit of sequences was used for the first time in the Aritmetica Infinitorium [12] on the occasion of finding the area under a parabola. It consists in estimating the limit L_e of the sequence given as a list of terms followed by the decomposition of each term a_n into the sum of L_e + "additional term". The additional term, can, in many cases, be written generally as $\frac{c}{f(n)}$ where f(n) is a linear function of n, so that this term can be shown to tend to 0 as $n \rightarrow \propto$, and that, consequently, the limit of the sequence is indeed L_e. In terms of the ε - N definition of the limit, the additional term is easily to be seen as the $a_n - L$, the difference between a term and the limit, whose absolute value can be made smaller, according to the definition, than arbitrary positive ε. This suggests that there is a natural relationship between the Wallis technique and the ε - N definition, which could be exploited to students' advantage.

Design of instruction. The principles of instruction during the teaching experiment had three components, all suggested by either positive results indicated in the literature or by the critique of certain didactic practices:

1. The development of the intuition of the limit was based on the Wallis technique, which, as the time progressed, was joined by the standard technique of finding the limit from the general term by dividing it by the highest power of n and "sending" n in the new expression to infinity.
2. Using the positive and measurable results found by Walhberg [13] about the role of writing assignments in the promotion of understanding of limits, the process of written explanation and discussion of the results was introduced at the very beginning of the discussions of sequences. Every assignment and every limit related problem during the partial tests had both computational, and conceptual written components in order to encourage the dialectical relation between these two aspects of understanding.
3. **A substantial increase in the understanding of the limit concept and a coordination of all aspects involved therein, (more precisely, a substantial increase in the scope of the limit schema, explained in section below)** was effected by the introduction of
   a) the eps - N definition of convergence (sequences $a_n = \frac{f(n)}{g(n)}$, where f(n), g(n) are linear functions of n, constant sequences $a_n=c$ and the alternating versions of both: $(-1)^n c$, $(-1)^n \frac{f(n)}{g(n)}$), its negation as the basis for the discussion of non-convergence (alternating, not converging sequences i.e. $a_n = (-1)^n * 5$) and by the precise definition of divergence, and of
   b) the Heine definition of the limit of a function which states *that a function has a limit L at x= a if for every sequence x_n which has a limit at a, the sequence $f_n = f(x_n)$ has a limit L.* The strength of this definition (which is equivalent to ε-δ definition of Cauchy) consists in that it allows to unify the instruction of sequences, limits of functions, continuity and other concepts depending on the notion of the limit, around one central idea, that of the limit of a sequence. Its application (or of its negation) to the different types of discontinuities, to the limits at infinity (functions with horizontal asymptotes) and to infinite limits (functions with vertical asymptotes) allows for the simultaneous usage (thematization) of all three types of sequences discussed in a). Such a consequent use of the concept is a necessary condition, according to Piaget [17] as well as to other Piaget-based frameworks such as APOS [1], for the formation of an abstract object from a particular concept. Thus the main role of this definition was not so much to have students master its different applications, but rather to provide the ground where the definitions and usage

of convergent, non-convergent and divergent sequences in a natural mathematics context might provide a nourishing soil for the construction of the final object from the schema under investigation.

Theoretical framework. There are several definitions of a schema in the math education literature [1], [17]. For the purpose of this research the definition of the schema contained in the APOS theory [1], will be used:

a schema is a collection of processes, objects, and other schema which can be organized in a structured manner, that is used to deal with a certain category of mathematical problem situations. The structure of a schema gives it coherence in the sense that the individual has some means of understanding what kinds of situations a particular schema can be used to deal with.

The schema of the limit of a sequence described in the definition of that limit:

The sequence a_n *has a limit L, if for every* $\varepsilon>0$*, there exists N such that for all* $n > N$*,* $|a_n - L| < \varepsilon$

is the result of the composition of two processes, the process of choosing an ε which is understood as a distance, hence, is part of a topological schema, and the process of establishing the relationship between ε and N through the inequality contained in the definition, where ε is seen as an independent algebraic variable of the function which determines suitable terms a_n (with $n>N$) of the sequence. The student has to start the first process by choosing some particular ε understood as distance, then he or she has to change its meaning to that of an independent variable which can take on any positive real value, and then finally, to demonstrate the understanding of the values of n so determined, he or she has to again go back to the concept of the distance; hence again changing the meaning of ε from that of a variable to that of the measure of a distance. Such a coordination of mental processes has been described by Dubinsky [8] as one of the basic components of reflective abstraction.

We have taken the successful coordination of these two processes as the evidence that a foundation of the schema of the limit of a sequence had been constructed within the mental apparatus of the students. We have also taken the successful application of the schema to the non-standard problem situation as evidence of the coherence of the schema foundation required by the definition.

As a tool of assessment to determine whether students have the required understanding of the discussed coordination of two processes we have used the following credo that many veterans of mathematics teaching profession say to students: You don't know whether you understand a concept until you can explain it in words. Indeed, anyone who has ever been in the position of learning a new mathematical concept and has been forced to explain it "in his/her own words" can recall the effort needed to follow, or even to formulate one's own thoughts and to give them the adequate verbal form of explanation while "thinking aloud" about a solution of a problem in question. This effort is precisely the effort of finding the meaning for the concept in question while the thought and the word are mutually accommodating to each other. This process of mutual accommodation between thought and word, under the name of a verbal thought, has been considered by Vygotsky [15] as the unit of meaning of a concept, which underlined his theory of concept formation. It makes sense therefore to take this relationship as the criterion for understanding of the ε-N definition of the limit of a sequence, especially since, explaining the meaning of procedures and discussing concepts in written words was one of the basic instructional strategies during the experiment.

Thus for the purpose of this research, only a student, who, while presented with the problem:

Find the limit of the sequence $a_n = \frac{3n+2}{2n-1}$ *and prove using ε-N definition, that the number you found is indeed the limit.,*

was able to show the necessary computational competence together with being able explain precisely why his or her calculations constituted the required proof, was classified as understanding the definition of the limit.

The sequence .9, .99, .999,… sequence was taken as an instrument to assess the coherence of the schema. This type of a sequence had not been discussed in class during the course.

Methodology. The teaching experiment took place in a section of Calc 1 class with 32 students participating and taking the final exam. The sample of 10 students was chosen among them for the clinical interview which had lasted approximately an hour. The sample in this cycle of the experiment was representing better motivated students in the class. The criterion here was the judgment of the instructor; in the next cycle a well tested measure of motivation will be used.

The data for the study of the understanding of the ε -N definition of the limit of the sequence consists of transcribed clinical interviews at the end of the term, just before the final exam, and answers to two questions concerning the limit of sequences from the final exam. Whereas the full analysis of the data awaits the publication of the full report, here, the analysis of two questions from the clinical interviews will be presented and supported by a similar question from the final exam.

Interview questions

1. a) What does the statement "**the limit of the sequence** $\mathbf{a_n} = \frac{3n+1}{4n-3}$ **is** $\frac{3}{4}$"mean to you?

b) How one could prove that the number you found is indeed the limit of that sequence?

2. Find the limit of the sequence $a_n=\{.9, .99, .999,\ldots\}$. How would you prove it is, indeed, its limit?

Final Exam question

3. a) Find the limit of the sequence $a_n = \frac{3n+2}{2n-1}$

   b) Prove using eps-N definition of the sequence that the number you found is indeed the limit of that sequence (start your work by writing the definition of the limit of a sequence)

   c) Explain your work.

The questions 1 and 3 were supposed give information about students' understanding of the limit in the context of ε -N definition, while the question 2 was designed to provide the information about their ability to apply that definition in the unfamiliar context, and to check their mastery of the Wallis technique in conjunction with ε-N definition.

Data analysis. As mentioned above, the concept described by the ε-N definition is the result of the composition of two process, the process of choosing epsilon and the process of establishing the relationship between ε and N. Clear articulation of the relationships between these two meanings of the epsilon, has been taken in this research as the indication that a certain conceptual whole has been mentally constructed by the students and that it can constitute the foundation of the construction of the schema of limits of sequences. Thus the student responses of the type (accompanied by the correct computations)*[6]:

In order for the value of ε to reduce itself, is necessary that N is larger than 7/4e + 1/2 . This will give me the value closer to the limit than the value of ε I desire.

Or of the type:

Well, epsilon served as the base for..., knowing the closeness which we want with epsilon, we can obtain that so great should be N to obtain that particular closeness were considered as evidence of such a coordination.

On the other hand, the responses like:

- *With that we can see that when ε is smaller, N is greater.*
- *With this we know that this great have to be the values of N to approach the limit.*

were not considered such an evidence, even if they had elements of truth in them.

In the interviewed sample of students, 50% demonstrated such a construction while during the final exam about 25% demonstrated it.

These results seem to suggest that the **coordination of the two processes constitutes a dividing line between students who had constructed the foundation of the schema and those who had not completed such a construction.**

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[6] Translated from Spanish by authors of the presentation.

Consequently, the coordination of mental processes should receive a special attention during the instruction.

Coordination of these two processes is most probably made still more difficult by the duality between the visual and algebraic approaches with which students are presented in the attempt to synthesize the concept of a converging sequence. The graphical representation of the limit and of the sequence seems to be clearly described by the statement that as n is creasing to infinity, the distance between the term and the limit becomes smaller. This formulation, suggested by the graph of a sequence, and possibly, by the notation $\lim_{n \to \infty} a_n$, is reciprocal to that suggested by the algebraic definition, where epsilon is an independently chosen measure of the distance and N is its function. Thus students who are unable to transcend the "temporal"[7] view upon the sequence and its limit could not make the connection with the formal definition at all. Below are samples of thinking of these students who, while computationally able, could not transcend the difficulty and hence did not construct the required schema:

- *N should be sufficiently large for ε to be sufficiently small. We want to demonstrate that the distance between a_n and L is small and is decreasing when n is approaching infinity..*
- *N should be sufficiently large for ε to be sufficiently small.*

The presence of this type of confusion was already known the authors of [7]; however its significance for the construction of the limit schema was not as clear as it is now.

Summary of the results

As mentioned above, the teaching experiment described above is the first in a cycle of experiments whose goal is to successfully reintroduce the formal definition of the limit (taking into account the research conducted on this topic in the past several years) into the standard calculus instruction and the assessment of the results has to be viewed from that point of view. Whereas a 25% success rate in the class is not very high in terms of what could be achieved with respect to understanding the concept, yet we have obtained several invaluable insights, which will help to modify the instruction in the Spring Semester 2001. It is clear that a serious emphasis has to be made on the coordination of two processes participating in the definition. Moreover, it seems that among the discussed misconceptions of students, the one which needs to be dealt with most urgently is the temporal versus sequential conflict [7] upon the relation between epsilon and N. Many students who have constructed the basic schema of the limit, were not clear about the meaning and significance of the sequential order. There are certain students' excerpts, which suggest that the emphasis on the development of input-output meaning of a function can be a help in this particular process of understanding.

The other commonly occurring misconception namely, the concern for the last term of the sequence, does not seem to impact students understanding of the definition very drastically at the Calculus 1 level, and in our opinion the strategies designed to deal with it should be dealt with in Calculus 2 when that misconception starts to impact seriously the understanding of the definite integral [6].

Let us add that the Wallis technique contributed to high rate of success on question 2 of the interview. Every student who had coordinated the two processes was also successful on the application of the schema to a_n=.9, .99, .999…

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COGNITIVE LOOPS AND FAILURE IN UNIVERSITY MATHEMATICS

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This paper explores one case study, typical of a large number of students, in the transition from school to university. This category of student has developed a view of mathematics as fundamentally procedural. This means that they are able to reason about specific objects, but cannot (and do not attempt to) acquire a meaningful conceptual understanding of university mathematics. We show that this procedural approach, adapted to the university level, leads to a loop in which the student's focus keeps him removed from the central ideas of university mathematics.

Background

Tom is good at doing mathematics, just not at university mathematics!

Tom was one of eighteen first year students taking part in a study of the effects of two different styles of teaching first term Analysis at a UK university (Alcock & Simpson, 2001). One was a standard lecture course to over 100 people, supported by assignments. The other involved students in small groups working through carefully structured questions which built up the analytic theory in the course (Burn, 1992). The study gathered data in numerous ways, though the data presented here comes from biweekly semi-structured interviews which the student attended in pairs. These covered their feelings about the course, general discussions about the mathematics they were encountering and a task-based section in which they were presented with a problem related to recent material. The data were examined using a grounded theory approach (Strauss & Corbin, 1990) and classes of common features emerged under categories such as the student's informal facility with the material, their approach to reasoning about general objects and their view of their own role as a learner.

Patterns in the data across these categories led to the classification of "types" of learner, with the largest factor in a student's mathematical development being their view of their own role as a learner and their resulting approach to the course. One clear category in this sense is those whose view we might call procedural (or instrumental in the sense of Skemp, 1976) – these students see mathematics as set of fixed procedures to be learned. This paper will examine the consequences of this procedural approach using Tom as an exemplar.

School and university mathematics

The procedural approach is associated with beliefs that are well documented in students at school level (Schoenfeld, 1992). While this may not be desirable, in school it is viable to believe that mathematics consists of procedures to be learned, that problems have only one "best" solution and that teachers are the expositors and arbiters of correctness (all of the eighteen students in the study attained an A in A-level mathematics and half of them exhibited some or all of these beliefs). In contrast university mathematicians see mathematics as being about concepts to be understood, problems with many forms of solution which demonstrate different insights, and correctness warranted by formal deductions (Tall, 1995, Moore, 1994).

Tom is typical of this procedural category. We will use his case (and illustrations from similar individuals) to demonstrate that those who bring these beliefs from school may attain some facility at the level of handling specific objects, but beyond this they tend to have a focus on detail, to be inflexible in their use of procedures and to have little meaning associated with them.. We then consider how these characteristics lead such students to ignore and avoid changes in requirements at university level.

Tom's background

In the first interview Tom credits his eventual success in A-level to the fact that:

> "…we got to practise exam-style questions, and got used to it and just went through lots of examples really. I think that's what you needed, lots of examples to practise."

and says he thinks the best way to learn a new piece of mathematics is:

> "Definitely step by step examples going through it. Like the teacher gives you step by step examples, so you can go away and look at them when you do questions and say 'oh this is how you do it'. And get used to the method."

This background was quite typical of this category of learner. Zoe, another procedural learner, expresses similar ideas:

> "I just like the way we were taught at school…Where we, we have our notes, and then we practise so many times that it's just sort of drummed into us. You just don't forget it that way, it's like, I don't know…"

From this viewpoint there is an implicit reliance on the fact that teachers will provide procedures, and little onus the student to assess their suitability in novel situations. There is considerable security for the student in this approach; the responsibility definitely lies with an outside authority and a student who diligently learns what they are told can expect to do well (cf. Perry, 1970, Copes, 1982). However this ceases to be the case at university level, where these beliefs actively interfere with any learning on a more conceptual level, as we now see.

Effects of procedural beliefs

Focus on detail, not concepts

One result of procedural learning is that such students tend to focus on remembering the detail of what they have done rather than thinking about what this means in terms of the concepts studied. For example in week 3 Tom is asked what they have been working on in Analysis recently and says:

> "Erm, we're learning about, different proofs with erm, modulus. All of those, like *a* minus *b*, all in bars equals - doesn't, is less than, *a* minus *b* outside and things like that."

This appears to be a description of the triangle inequality, although it is hard to tell because he does not state the final result. Again Tom's response is typical of this category of learner. In this excerpt Wendy is describing a homework question that she found difficult:

> Wendy: You had, you had to erm, find an increasing and a decreasing sequence that erm, converged to the same limit,
> Xavier: Subsequence, wasn't it?
> Wendy: Yes, subsequence and prove that the, sequence would converge to the limit. And it was like, we had to find one that, a decreasing sequence that converged to the same limit as an increasing sequence. And all this other stuff going round that, you had to use and…oh no!

Everyone tends to become less coherent when trying to describe something that they didn't understand well. However Wendy is very focused on the instructions they "had to" follow; at no point does she state the result that this was supposed to lead to.

This lack of focus on conceptual relationships also shows up when Tom does attempt to use concept terms in sentences:

> Tom: Erm, he's just been giving us sequences, and we're supposed to be proving them.
> *Laughter.*
> Interviewer: Proving what?
> Tom: Just, proving that they're true. The sequences tends to a limit, and…

His meaning is relatively clear at the second attempt, but his first does not make sense and he clearly is not fluent in describing the work in these terms.

Informal use of the ideas from the course

This is not to say that procedural learners don't learn anything. Again Tom is typical, reasoning informally using the ideas from the course when asked to think about specific objects.

For example in week 5, considering the sequence given by $a_n = \frac{1+\cos n}{nx}$, he says:

> "Well, because it's cos n, the greatest cos n can be, is, 1, so the greatest that this can be, is, on the top is 2, so, whereas the bottom, is always going to get larger and larger."

And later:

> "Then the bottom's just going to, increase and increase and increase and even if, x is negative it's going to, be increasing by so much, that the top is going to become irrelevant, and it's just going to tend to zero."

He does not notice the problem with $x=0$, but does reach a correct conclusion about the eventual behaviour of the sequence for the remaining cases using strategies of considering the extreme values of the cosine function and making informal comparisons between variable quantities.

Procedures as immodifiable wholes

Difficulties arise for Tom when he tries to prove his assertions algebraically. Typically of the procedural learners in this study, he has procedures available, but if they are not applicable in very standard form he does not try to adapt them to fit the new situation. This is frustrating to listen to since he can identify problems with some precision. In this excerpt he is explaining why he doesn't know how to write down an answer to the above question:

> "Because we can't really, bring that nx up to the top, on the other side, because we don't know, whether x is positive or negative. And we can't really use the squeeze rule, because, what happens - we don't know what x is. It could be like 0.0001. And in that case, with it being on the bottom, it would increase the number."

His first approach is, in his own words, to "make n the subject", showing some awareness of the routine for showing that the definition of convergence is satisfied (though whether he understands that this is what he is doing is open to question). Further thought would make either this or his other suggestion viable: he could treat positive and negative values of x as separate cases, or use the fact that if (a_n) tends to zero then so does (ca_n) (for any real c) in conjunction with the "squeeze rule". What seems to happen is that when a particular algebraic manipulation routine does not give him a straightforward path to the desired result, Tom simply stops. We suggest that he does not think his role involves being creative in mathematics, so that once a procedure is invoked he simply follows this and will not attempt to adapt it.

Lack of meaning associated with a procedure

In the week 9 interview the students are asked to establish for which values of x the series

$$\sum \frac{(-x)^n}{n}$$

converges.

By this time Tom's earlier level of meaningful understanding has collapsed under his search for procedures. Whereas before we saw him instantiate a suitable procedures but lack the inclination to adapt them, here he attempts to use an inappropriate and incorrectly recalled "procedure" on the basis of surface similarities with the given question. He says:

> "Oh right, I know, I remember, that the sum of 1/*n*, tends to log *n* plus, lambda *n*. Or, it's the sum of, brackets minus 1 close brackets, to the power of *n*. Divided by *n*, tends to, log *n*, plus lambda, *n*. And I think it's the first one."

He appears to be explaining a confused memory of the process of finding the partial sums of the series $\sum\frac{1}{n}$, which form a sequence $\log n+\lambda_n$ and so demonstrate that the series is unbounded. What he says would not be correct even if it were appropriate, since it would make no sense for anything to tend to the variable quantity $\log n+\lambda_n$, but Tom apparently remains unaware of this. Also he says he "thinks" that the first expression is the one he wants, but although he is uncertain of this there is no attempt to do any reasoning to check; he seems to rely entirely on recall. He goes on:

> "Okay this is just the formulas, not what they tend to. Formula 1 is the sum of 1 over *n*, and formula 2 is the sum of brackets minus 1 close brackets *n*, over *n*. To the power of *n* or whatever. So, I think, formula 2, is very similar to the question, except for, *x* has been given a number, 1, and we've just got to use, formula 1, to create formula 2, to see what it tends to."

The strategy of trying to manipulate a new situation so that it resembles a known one is good, and Tom's outline of a possible solution path is clearly explained. However, in addition to the apparent lack of meaningful understanding of what it is that he is suggesting, by this stage he has lost track of the question. The limit of the series is not required; only the values of *x* for which it converges. Failing to monitor progress toward a solution may be simply a weakness in this particular area; perhaps he finds the material hard and cannot keep sufficient in his working memory to be able to make this judgement as well. However the evidence in this study suggests that these factors are all related for a learner of the procedural type and, we will see, form a loop.

A cognitive loop

Tom uses the phrase "we've got to" twice in the above excerpts, and we have seen that his use of procedures is inflexible and sometimes inappropriate. This suggests a continued belief that there exist externally-provided rules prescribing what should be done when, rather than natural approaches which he himself could generate based on the intrinsic meaning of the mathematics. We suggest that his lack of progress is due to being caught in a loop involving two complementary aspects – lack of meaningful checking and attending to procedural detail rather than concepts.

Tom does not feel the need to check whether his suggestions "make sense" in a meaningful way. He assumes that the algebraic procedures he is shown are generally useful; having faith that these will work with similar-looking examples, he does not attend to the concepts that the algebra expresses. This actually renders him more or less incapable of performing such checks; he has no meaning for the manipulations so he cannot assess whether the concepts involved in a new situation are really comparable. However Tom does not notice this, because that isn't what he's trying to do, and so on.

The challenge for the teaching at this level is that this does not cause him any great distress. He finds enough in the new mathematics to maintain his beliefs; university mathematics is not all proofs, quite a lot of it involves applying techniques to particular examples and at least at the beginning of the course Tom can cope with this. Combined with the fact that he is not looking for links between concepts, this means that he does not feel he is failing to understand.

A challenge for teaching

A student's progress is limited by what it is that they are trying to do, and while procedural learners have individual strengths, on the whole they fare badly in this first course in Analysis. For example, Wendy can cope with the question about series, using generic examples to reason about different cases:

Wendy: Erm, if, if you take x is between nought and 1, say x equal to a half, erm, and put it into the series, you'd get, minus a half, plus a half squared over 2, minus a half cubed over 3, and so on…
Xavier: So that's going to be smaller.
Wendy: Erm, the terms are decreasing in size, so…x n, is bigger than, x n plus 1 (*writing*)
Xavier: And tending to zero.
Pause (*writing*).
Wendy: Converges?

This makes it appear that she is keeping up with the course, but in fact she shows serious weakness in forming more general arguments and a lack of understanding of important formal categories. For instance, it is very difficult to persuade her to consider sequences which are not monotonic (see Alcock & Simpson, 1999) and she ends the course believing that a convergent series have its sequence of partial sums eventually constant. Even Tom is not the weakest. Zoe, the other student mentioned earlier, has even less concern for meaning than Tom. Here she describes the approach she and Yvonne take to doing their assignments:

> "What we usually do is when we have our notes, like sprawled out everywhere (*laughs*)…copy bits from here and here and like, put it all together. And hope it turns out right."

Unsurprisingly, they find it difficult to get started even on the questions where Tom has some success, and it often appears that the interview is the first time they have given any meaningful thought to the work.

Of course there are students who have quite different goals. Jenny, for instance, expends a great deal of effort on her assignments and says that she "feels like a fraud" if she hands in something she does not fully understand. However the prevalence of procedural learners and their failure to attend to the aspects of mathematics their university lecturers see as central – the marriage of conceptual understanding and formal deduction (Yusof and Tall, 1995) – begs the question of the role teaching can play in helping students negotiate the transition from school to university.

The obvious direction in which we must now move is to address the question of breaking the loop generated by lack of meaningful checking and attendance to procedural detail. How do we help Tom to be as good at university mathematics as he is at *his* mathematics?!

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What are we trying to achieve in teaching standard calculating procedures?

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Abstract

In societies that depend on calculators and computers for all important calculations, questions must be asked about the purpose of written calculation. Reforms in arithmetic teaching have led to a shift from the repetition and rehearsal of algorithmic approaches to a focus on developing pupils' own methods, but mastery of standard procedures remains a fundamental goal. This presents a dilemma for pupils who try to replicate a taught procedure which may not be their most effective solution strategy. This paper reports a study involving English and Dutch pupils (n=535) and highlights the way their efforts to implement taught procedures can inhibit more appropriate strategies using number sense.

Background

Classroom activities today involve pupils in observing patterns and explaining relationships so that they develop understanding of connections among different numbers and operations gaining a 'feel' for numbers often referred to as 'number sense' (Anghileri 2000). This term 'number sense' is widely used across the world in reform documents (NCTM 1989; AEC 1991) and refers to 'flexibility' and 'inventiveness' in strategies for calculating. It is a reaction against overemphasis on computational procedures and reflects 'new numeracies' (Noss 1998) that are more relevant to the skills and understanding needed in our social and working lives. At the same time, computational procedures remain an important element of the curriculum, for example, the National Numeracy Strategy for England states that 'at least one standard written method of calculation should be taught in primary schools' as these 'offer reliable and efficient procedures which, once mastered, can be used in many different contexts' (DfEE, 1998: 52).

English and Dutch teaching approaches

The 'standard' procedures to be taught will vary from one country to another and will reflect the teaching that has led up to the stage of written calculations. England and the Netherlands have different teaching priorities culminating in different standard methods. The role of place value is emphasised from an early age in England where 'understanding about place value is required as a sound basis for efficient and correct mental and written calculation' (SCAA 1997: 4). In the Netherlands, in contrast, holistic approaches to numbers include the development of counting skills as the basis for calculating (Beishuizen and Anghileri 1998). For the operation of division these contrasting approaches culminate in different written procedures. Standard written methods for division in England are based on the traditional algorithm while the Dutch approach involves repeated subtraction with appropriately chosen multiples (chunks) for developing efficiency (Anghileri 2001). Progression from informal methods to standard written procedures is more clearly evident in the Dutch approach as will be shown in the results reported in this paper.

Purposes of written calculations

Ruthven (1998) identifies two distinct purposes for using pencil and paper for calculating: 'to augment working memory by *recording* key items of information' and 'to cue sequences of actions through *schematising* such information within a standard spatial configuration'. The traditional algorithm is structured to 'direct and organise' (Anghileri 1998?), providing a highly efficient written method for solving problems but is not easy to reconcile with 'the way people naturally think about numbers' (Plunkett 1979). The formal procedure is also prone to errors in some cases due to its incompatiblity with intuitive approaches (Anghileri and Beishuizen 1998; Anghileri 2000). The Dutch Realistic approach uses contextual problems as a starting point and a standard procedure is evolved from informal approaches based on repeated subtraction (Gravemeijer, 1994) with whole numbers retained at all stages.

Comparing the effectiveness of different approaches

Effectiveness of the different teaching approaches is compared in a study in the two countries. In cities with similar cultural characteristics, whole classes of year 5 pupils in ten English schools (n=276) and in parallel grade 4 classes in ten Dutch schools (n=259) were asked to write solutions to ten division problems. Pupils completed written tests in January involving five word problems and five symbolic ('bare') problems with similar numbers (Table 1).

Table 1 - Ten problems used in the first test

context	bare	type
1. 98 flowers are bundled in bunches of 7. How many bunches can be made?	6. 96÷6	grouping: 2-digit divided by 1-digit - no remainder
2. 64 pencils have to be packed in boxes of 16. How many boxes will be needed?	7. 84÷14	grouping: 2-digit divided by 2-digit - no remainder
3. 432 children have to be transported by 15 seater buses. How many buses will be needed?	8. 538÷15	grouping: 3-digit divided by 2-digit - remainder
4. 604 blocks are laid down in rows of 10. How many rows will there be?	9. 804÷10	grouping: 3-digit divided by 10 - remainder
5.1256 apples are divided among 6 shopkeepers. How many apples will each shopkeeper get? How many apples will be left?	10. 1542÷5	sharing: 4-digit divided by 1-digit - remainder

The numbers were selected to encourage mental strategies and to invite the use of known number facts. The tests were repeated in June to establish changes in pupils' strategies. In the second test, numbers in the context and non-context problems were interchanged to reduce the influence of memory.

Results

Solution strategies were classified into 8 categories. Low level strategies involved making tally marks or repeatedly adding or subtracting the divisor **1(S)**, partitioning the divisor or dividend (or both) **2(P)**, or use of small multiples of the divisor in low level chunking **3(L)**. Efficiency gains were evident with repeated subtraction of large chunks **4(H)**, or use of the traditional algorithm **5(AL)**. Where there was a solution but no working the strategy was classified as mental **6(ME)**. Some solutions involved the wrong operation **7(WR)** or the strategy was unclear **8(UN)**.

Overall success was greater for the Dutch pupils who successfully completed 47% of the items in test 1 and 68% in test 2. English pupils successfully completed 38% in test 1 and 44% in test 2 (Table 2).

Table 2: Percentage of questions successfully completed

Dutch test1	English test1		Dutch test2	English test2
47%	38%		68%	44%

English pupils persisted longer in using low level strategies 1(S), 2(P) and 3(L) with 28% of attempts in test 1 and 22% in test 2. Dutch pupils used these strategies for 33% of items in test 1 but this reduced to 13% in test 2. With the large numbers involved pupils struggled to reach a successful solution using these strategies.

The most popular Dutch strategy in both tests involved repeated subtraction of large chunks, 4(H), which was often structured in a standard written format and was used for 41% of the items in test 1 and 69% in test 2. English pupils used the traditional algorithm, 5(AL), most extensively with 38% of items in the test 1, and 49% in test 2 attempted using this approach. The Dutch standard method, 4(H), led to a correct solution in 74% of attempts while only 47% of the English attempts to use the traditional algorithm 5(AL) were successful.

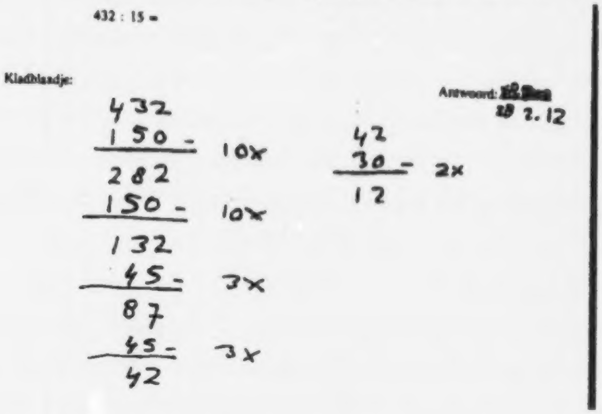
Mental methods were used equally by the Dutch and English pupils (11% of all items) with almost equal success (6% Dutch/5% English).

Progression

In the solutions of Dutch pupils there was evidence of progression from repeated subtraction of the divisor, to subtraction of small multiples/chunks which often involved long calculations, to efficient use of large multiples/chunks in a standardised written procedure (Beishuizen and

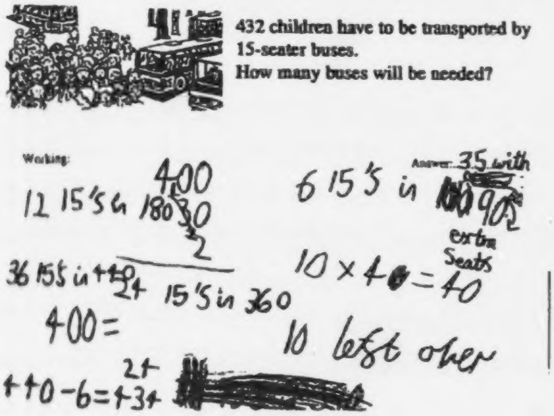
Anghileri, 1998). At all stages, whole numbers were used and the written structure developed was the same for 1-digit and 2-digit divisors. Pupils written solutions showed extensive use of the structured written procedure at different levels of efficiency (Figure 1).

Figure 1: Dutch written procedure for division


432 : 15 =
Kladblaadje:
432
150 - 10x
282
150 - 10x
132
45 - 3x
87
45 - 3x
42
42
30 - 2x
12
Antwoord:
28 r. 12

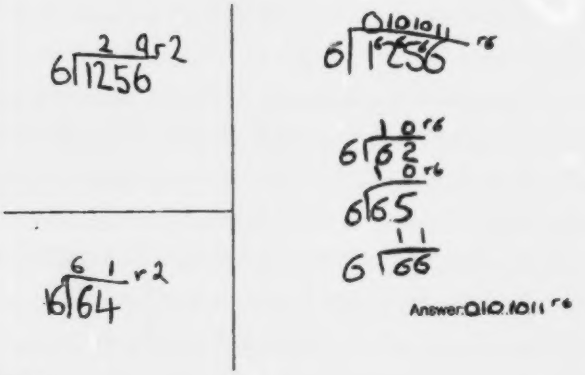
Progression was not evident in the English strategies where idiosyncratic written methods based on mental strategies did not appear to relate well to the traditional algorithm. Informal methods generally lacked any written structure and it was evident that difficulties arose for some pupils in following through their own working to give a correct solution (Figure 2).

Figure 2: Unstructured recording of an English pupil


432 children have to be transported by 15-seater buses.
How many buses will be needed?
Working:
400
12 15's in 180 30
2
36 15's in 440
24 15's in 360
400 =
24
440 - 6 = 434
Answer: 35 with
6 15's in 90
extra Seats
10 x 4 = 40
10 left over

For a 2-digit divisor, attempts were made to partition the divisor or to operate on separate digits. A typical example was the problem 64 ÷ 16 which was solved by first dividing by 10 and then by 6, or as 6 ÷ 1 = 6 and 4 ÷ 6 = 1 r 2 (wrong use of the commutative rule) (Figure 3). Some pupils appeared to get stuck trying to divide 60 by 16. Use of the formal procedure appeared to preclude any return to an informal approach and inappropriate results written in the answer space suggest that a written procedure had been followed which took no account of the approximate answer (Figure 3).

Figure 3: Errors with the traditional algorithm



Single digit divisors

In addition to having a procedure that related well to informal thinking about division, better results for the Dutch pupils may be explained by the fact that they meet division by a 2-digit divisor in grade 4 (Y 5) while most English pupils will meet only 1-digit divisors. Results were compared for those items involving only a single digit divisor. Scores in test 1 were close for the English and Dutch pupils with averages of 45.5% and 47.25%. Both were more successful in dividing the 2-digit numbers that in dividing the 4-digit numbers. In test 2 the English results improved to average 55% over the four problems while the Dutch result was 71% successful.

Improvements were similar for the items, 96÷6 and 98÷7, with Dutch/English increase in correct answers 8%/5% and 22%/21% respectively. For the 4-digit numbers, 1256÷6 and 1542÷5, the Dutch improvements were higher than those of the English children, with increases of 29% and 36% compared with 2% and 10%.

English pupils used the algorithm with low success rate for the 4-digit numbers. The Dutch pupils used repeated subtraction with large chunks and although the success rate is not as high for the 4-digit numbers, differences were less marked (Table 3).

Table 3: Percentage use and effectiveness of the most popular strategies for test 2

	Problem	96÷6	1256÷6	98÷7	1542÷5
English test 2	traditional algorithm 5(AL)	66 (51)	67 (21)	66 (52)	70 (34)
Dutch test 2	repeated subtraction of large chunks 4(H)	78 (69)	72 (50)	76 (69)	71 (52)

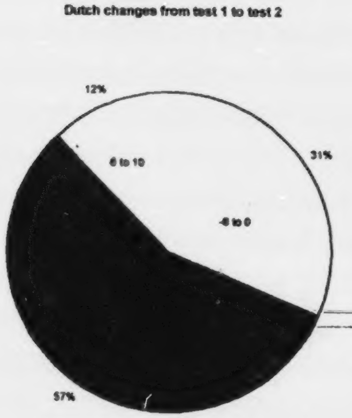
The figures in brackets give the percentage of correct attempts.

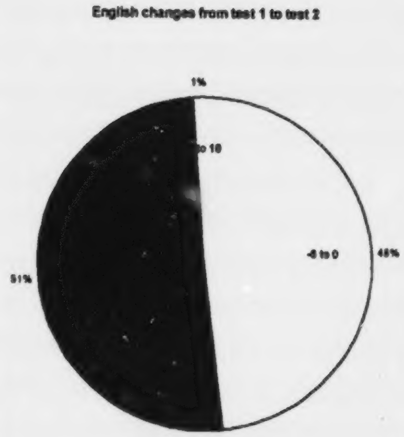
Errors by the English pupils included missing digits in the answer, but also many confused attempts often leading to impossible (and sometimes bizarre) answers (see Figure 3).

Overall improvements

When individuals' scores were compared for test 1 and test 2, changes in score varied from +9 (e.g. 1 correct in test 1 and 10 correct in test 2) to –6 (e.g. 8 correct in test 1 and 2 correct in test 2). Again, the scores of Dutch pupils showed better improvements with 69% improving their score while almost half of the English pupils (49%) showed no improvement or a deterioration (Figure 4).

Figure 4: Changes in score from test 1 to test 2


Dutch changes from test 1 to test 2


English changes from test 1 to test 2

Conclusions

The Dutch approach to written division calculations, involving repeated subtraction using increasingly large chunks, builds progressively on a mental strategy and retains whole numbers at all stages. The success of the Dutch pupils reflects their mastery of an increasingly efficient approach that has the flexibility for individuals to use the knowledge of multiplication facts that they have. On the other hand, the traditional algorithm extensively used by the English children, introduces a schematic approach that focuses on separate digits with their true value implicit, rather than explicit. Not only is the traditional algorithm more difficult to understand and prone to errors, but also progression to division by a 2-digit divisor requires substantial adaptation that is not intuitively clear to pupils. There is no flexibility in the choice of multiplication facts that can be used and links with mental methods are not clear.

When a standard procedure for calculating is taught in school it appears to take precedence over informal methods and implementing the procedure can be at the expense of making sense of a calculation. A problem such as 64÷16 caused great difficulty to the English pupils because it does not respond readily to the traditional algorithm which was used in preference to informal approaches. Instead of *recognising* the number relationships involved, pupils used a *procedure* cued by the operation.

Developing efficient procedures that relate to pupils' knowledge of numbers and to their intuitive understanding is crucial for developing the confidence that will encourage pupils to work on making sense of problems they meet. When presented with a meaningful problem the two approaches illustrated below show how the algorithm can lead back to the original calculation while a procedure that is better understood can encourage a solution that goes beyond the minimal requirements of a pure arithmetic calculation.

A farm shop sells about 72 eggs each day. How many days will 300 eggs take to sell?

$$\begin{array}{r} 0\,0 \\ 72\overline{)\,3\,{}^{3}0\,{}^{30}0} \end{array}$$

$$\begin{array}{rl} 300 & \\ -144 & (2\text{ days}) \\ \hline 156 & \\ -144 & (2\text{days}) \\ \hline 12 & \end{array}$$

So it will take 4
days and a bit so
probably about 11:00
on the 5th day.

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NEGATION IN MATHEMATICS: OBSTACLES EMERGING FROM AN EXPLORATORY STUDY

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ABSTRACT: In a previous research we have elaborated three schemes of behaviour to classify the difficulties of students in the interpretation and formulation of mathematical negation. This report provides a detailed analysis of one of these schemes, particularly its link with common practice, showing that some of the greatest obstacles in learning the meaning of mathematical negation are the difference between negation in mathematics and in natural language, and the tendency to classify into particular categories, that take into account the differences more than the analogies.

1.Introduction

In the works which involve the mathematical negation, the university students in the first year of the faculty of sciences often incur in errors. Such errors witness students' difficulties in understanding of mathematical negation and the presence of an obstacle to accept indirect proof.

There are currently few works in mathematics education directly concerned with negation. However, we would like to quote Thompson (1996), who report on the problems with proofs by contradiction due to the difficulties of negation. Thompson sees the correct formulation of negation as an important prerequisite for the use of indirect proof. Moreover, he draws up a list of some typical mistakes students make when negating a sentence. In Barnard (1995) there is a more detailed list of typical mistakes in recognising negations.

The theoretical framework which we refer to is Antonini (to appear). This framework, based on the notion of scheme of Piaget (1967) re-formulated by Vergnaud (1990), distinguishes three different schemes - scheme of the opposite, of the possibilities and of the properties - which guide the behaviour of the students in situations that involve the mathematical negation.

After a short description of the three schemes, we intend to widen the analysis of one of them (the scheme of the possibilities), by hypothesising about the existence of elements that, on the one hand, strengthen it and, on the other hand, can hinder the passage to the scheme more suitable to mathematics (scheme of the properties).

2. Method

The results we present in this report are related to a research study which, in this early stage, is essentially exploratory and based on a collection of data which can be furtherly refined. The goal of this first phase is to implement a framework of theoretical reference for a more detailed analysis. The author was support tutor in the first year of the calculus course, held at Pisa University in Italy. On this occasion, different types of students' behaviours were observed through questionnaires and discussions in the classroom. Furthermore, a number of interviews were conducted and then recorded with students of the fourth year of the degree course in physics at Pisa University.

3. Negation schemes

In this paragraph we shall try to classify the behaviour of students faced with situations concerning negation.

In this respect, we shall use Piaget's notion of scheme (Piaget, 1967), as re-formulated by Vergnaud (1990):

> "On peut distinguer:
>
> 1) des classes de situations pour lesquelles le sujet dispose dans son répertoire, à un moment donné de son développement et sous certaines circonstances, des compétences nécessaires au traitement relativement immédiat de la situation;
> 2) des classes de situations pour lesquelles le sujet ne dispose pas de toutes les compétences nécessaires, ce qui l'oblige à un temps de réflexion et d'exploration, à des hésitations, à des tentatives avortées, et le conduit éventuellement à la réussite, éventuellement à l'échec.
>
> [...]
>
> Appelons 'schème' *l'organisation invariante de la conduite pour une classe de situations donnée*".
>
> (Vergnaud, 1990)

A first classification of students' behaviour when facing with mathematics negation resulted in the definition of three mental schemes (Antonini, to appear), which can be briefly described as follows.

SCHEME A (of the opposite): The negation of "x is p(x)" is "x is q(x)" where q is "the opposite" of p. Examples of opposites are increasing-decreasing, even-odd, all-none, major-minor.

The subject is often aware that besides p and q there are other possibilities as well, but these are considered "exceptions", "extreme cases".

We observe that a concept and its opposite are strongly linked by analogies, symmetries, oppositions, and are often two aspects of the same concept (e.g. monotonicity: increasing-decreasing; order relation: major-minor).

The example which follows is designed to illustrate these features:

Interview to Vincenzo (4th year Physics student)
1.Int: [...] I say: f is an increasing function. What is its negation?
2.Vinc: f is decreasing (he answers immediately, without thinking).
(...)
3.Int: Well, you must prove by contradiction a theorem whose thesis affirms that f is an increasing function.
4.Vinc: by contradiction...
5.Int: Let us suppose by contradiction that ...
6.Vinc: That f(x) is decreasing.
7.Int: That f is decreasing; o.k. you start: let us suppose that f is decreasing.
8.Vinc: I must show ... (pause)
9.Int: Well, you start from decreasing f and then, what happens?
10.Vinc: I must show that I reach a contradiction ... of the hypotheses.

11.Int: Ok, at a certain point we reach a contradiction of the hypotheses.
12.Vinc: Yes.
13.Int: And so? Have you finished the proof?
14.Vinc: So… the fact is that it is not decreasing … therefore it is increasing …
15.Int: Are you sure?
16.Vinc: Yes.

Vincenzo uses the idea of the opposite to construct the negation of "f is increasing". In the last part of the protocol we notice how "f is decreasing" is for the subject the logical negation of "f is increasing". As a matter of fact, in 14 Vincenzo says: f "is not decreasing … and therefore it is increasing".

SCHEME B (of the possibilities): If x is not $p(x)$ then it can be $p1(x)$, $p2(x)$ or $p3(x)$, etc. In other words, the statement "x does not possess the property p" means that there are various possibilities. Whilst there were only two possibilities for scheme A, for scheme B the negation dissolves in a moltitude of different cases.
What follows are some examples in which the students formulate the negations proposing various possibilities rather than using an opposite.

1) Written questionnaire (freshman Science students). One question was: "What is the negation of a proposition? (Try to give also an example)".

A reply:
"For example: $f(x)>0$, negation: $f(x)\leq 0$.
In this case negation can include both < and =, that is all the possibilities in which the proposition does not occur. For example, if we were to deny that an f is increasing we should say instead that it is decreasing or constant or increasing and decreasing …" (underlining is ours).

2) Test (freshman Science students)
"We know that a certain function g is not strictly decreasing in the interval (-2,+3). Is there anything that you could define as being certainly true? (Give a justification for your answer)."
A reply:
"The fact that g is not strictly decreasing does not say anything about the function because it leaves the possibility of g being both increasing and constant and also that it is decreasing but not strictly or even increasing and decreasing at the same time.
Therefore nothing absolutely true can be said about function g". (underlining is ours).

SCHEME C (of the properties): If p is false we look for a property q common to all x for which $p(x)$ is false.
This scheme is commonly used in mathematical reasoning. On many occasions it leads to a really efficient behaviour from the operative point of view , for example if we want to deny that "f is increasing", we could say that "there exist x, y such that $x<y$ and $f(x)\geq f(y)$": this property is common to every non-increasing function, whether they are constant, increasing, discontinuous, etc.
On the contrary this scheme is a very rarely used by students, who seem to prefer scheme B.

4. Dominance of scheme B

The proposed schemes represent a first attempt to classify the behaviours. We observe that, in keeping with Vergnaud (1990), the behaviour of a subject can be guided either by different schemes in various situations or by different schemes in the same situation.

The obtained results show that scheme B is very common. This scheme corresponds to some aspect of the everyday language. It is also easily extended to mathematical context, even though it leads to poor and wrong behaviours.

In this paragraph we intend to analyse more in detail the nature of scheme B. We think that this analysis allows to interpret and better understand the students' behaviours guided by this scheme.

We retain that two are the fundamental elements that strengthen scheme B and hinder the acquisition of scheme C:

1) the natural tendency to a certain type of classification;

2) the impossibility with the natural language, unlike that mathematical, to always express the negation in the affirmative form - for instance, in Italian, without the use of "non".

4.1. Classification

As far as classification process is concerned, previous studies have highlighted two different aspects which are fundamental. By referring to two different studies, we show two different aspects of the process of classification:

a) the necessity to differentiate (Mariotti-Fischbein, 1997);

b) the classification in *cognitive categories* (Lakoff, 1987).

a) In a study on definitions, Mariotti and Fischbein (1997) point out that the students tend to classify some solid figures on the basis of differences more than analogies:

> "The figural differences between a parallelepiped and a hexagonal prism lead to a classification which aims to separate the two classes of objects whilst, in the standard mathematical classification, the class of the parallelepipeds is [illegible] in the more general class of the prisms. In order to get such structural conceptualisation, differences between the particular objects should be overcome in favour of analogies between them." (Mariotti-Fischbein, 97) (underlining is ours).

The need to express the negation as a unique property therefore conflits with the natural tendency to classify different objects in different classes.

As a matter of fact, the objects which do not have a particular feature may represent even enormous differences.

Let us give an example: the set of non-continuous functions is made up of functions which are very different one from the other. There are functions discontinuous at a particular point but with right and left finite limits, functions unboundied at a point, functions discontinuous at every point, functions continuous only at one point, etc. Mathematicians have classified the various types of discontinuity in three species. In each species there are functions which have some features in common.

The subject using scheme B, even though he realizes the possibility of many different cases, does not manage to assemble all the elements for which a proposition is false in a unique whole. He gathers them in cases or possibilities; each possibility contains elements with some common characteristics. Describing "non-p" with a list of possibilities therefore helps to overcome the difficulties encountered by treating with the differences. The more the objects are different, the more the necessity to find a common property (to generalise) conflits with the necessity to differentiate:

> "The process of generalisation requested by a theoretical definition conflicts with the need of differentiating. Difficulties arise when theoretical constraints state the equivalence between 'different' things, requiring to cancel the variety once for all." (Mariotti-Fischbein, 97).

b) There are different studies of the nature of human categorization and of the nature of the categories (see Lakoff , 1987):

> "From the time of Aristotle to the later work of Wittgestein, categories were thought be well understood and unproblematic. They were assumed to be abstract containers, with things either inside or outside the category. Things were assumed to be in the same category if and only if they had certain properties in common. And the properties they had in common were taken as defining the category." (Lakoff, 1987, p. 6)

Nevertheless, the formulation of Aristotle finds a series of great limits in the explanation of some experimental data. More suitable to explain the collected experimental data is the *prototype theory* of Eleanor Rosch (see Lakoff, 1987, chapter 2).

From the cognitive point of view, one category is not so much determined by the common characteristics of all of its elements but by the similarity with a particular element, said *prototype*; if an object is too different from the prototype it cannot belong to that category. All this is particularly valid for the so-called *basic-level* categories, defined in Lakoff (1987, p. 46). The categories at the basic-level are the ones better differentiated and it is at this level that a great part of our knowledge is organised.

> "The complements of basic-level categories are not basic level. They do not have the kinds of properties that basic-level categories have. For example, consider non-chairs, that is, those things that are not chairs. What do they look like? Do you have a mental image of a general or an abstract nonchair? People seem not to. How do you interact with a nonchair? Is there some general motor action one performs with nonchairs? Apparently not. What is nonchair used for? Do nonchairs have general functions? Apparently not.
>
> In the classical theory, the complement of a set that is defined by necessary and sufficient conditions is another set that is defined by necessary and sufficient conditions. But the complement of a basic-level category is not itself a basic-level category." (Lakoff, 1987, p. 52)

In this theoretical framework we can explain the tendency of the subjects to behave by using scheme B in situations of negation: the objects that do not belong to a given category (ex. the non-increasing functions) do not set up a category, being composed of too much different elements and often deprived of a prototype, but are an union of categories, each of which has a good prototype (ex. decreasing, constant and periodic functions).

4.2 Language

The second element that strengthen scheme B and hinders C is the lack of adequacy of the natural language to express a negation in affirmative form. In the natural language, to express the negation, often we can only affirm "non-p" (for example, sentences like "I did not travel by train" or "it does not rain" can be unlikely formulated in affermative forms). Instead, in mathematics it is often possible to re-phrase "non-p" in affirmative way, removing any trace of negation (for example, "f is a non increasing function" is the same as saying that "there exist x, y such that $x<y$ and $f(x)\geq f(y)$"). This is therefore a newness, that requires a specific educational approach.

5. Analysis of a protocol

The following protocol represents a very good example.

Interview to Carlo (freshman engineering student)
1.Int: What is negation?
2.Carlo: It is the opposite.
3.Int: What do you mean?
4.Carlo: If I say "switched on", the opposite is "switched off". But if I say "it is raining", there is no opposite, I can only say that "it isn't raining".
5.Int: Do you know what a proof by contradiction is?
6.Carlo: It means proving that the opposite of a thesis cannot be true. I don't know whether this is always possible, I think it is possible only in those cases in which I only have two possibilities, like "switched on" and "switched off".
7.Int: If I have a theorem whose hypothesis is that f is an increasing function. How would you begin a proof by contradiction?
8.Carlo: Let us suppose that it is decreasing … (pause) … no, because there are other cases which are not included …
9.Int: And so?
10.Carlo:…(pause)… Well, I should identify something in common … I mean a property which is common to all the non increasing f, then prove that the f of the theorem cannot have that property, and therefore is increasing. But the proofs by contradiction turn out better when I have only two cases, like "switched on" and "switched off".

1-4: Carlo distinguishes two types of negation: the first one is a typical opposite ("switched on" - "switched off"), the second an asymmetrical case. Only in the first case the language makes it possible to describe the negation in affirmative terms (not switched on = switched off); in the second case "I can only say that it isn't raining."

5-6: Carlo retains that the proof by contradiction can be done only when there are two "possibilities", but it is important to underline that, while "switched on" and "switched off" are possibilities, "increasing" and "non increasing" are not considered as such, in accordance with that said above about the cognitive categories.

7-8: After having applied the opposite of increasing to build the negation, Carlo realizes that there are other "cases" (possibilities).

9-10: The subject builds by himself the idea to formulate the negation in terms of property. Nevertheless Carlo is not sure that this is really possible ("I should identify") and, actually, he does not try to do it. Our hypothesis is that the familiarity with the mathematical language could help to overcome this obstacle.
Finally, Carlo returns to the idea that the proof by contradiction "turns out better" in cases of opposites.
We can observe that the difference that Carlo underlines between the two types of negation is not considered from a logical point of view. In fact, given a proposition p, there are always only two cases: p and non-p (as "switched on" and "switched off"). The difference underlined by Carlo can be only explained with the fact that, while, in any case, we are in the presence of only two properties, we are not necessarily in the presence of only two possibilities (as “increasing” and “non increasing”).

6. Conclusions

We have described three schemes that guide students’ behaviour in situations which involve negation. These behaviours are sometimes guided by a single scheme, other times by the consecutive combination of various schemes.
These schemes are not correct or mistaken in themselves, since any of them can lead to results which can be both correct and incorrect; however they can be more or less adequate to the solution of the mathematical problem that we intend to deal with.
We can also observe that the difficulties that may lead to the use of schemes A or B are of a very different nature: while scheme A may lead to errors (see Vincenzo protocol), scheme B may lead to a block of mental processes (see Carlo protocol). From the didactic point of view, identifying the involved scheme can give important indications to differentiate the kinds of didactical approaches.
Moreover, scheme B is the most diffused scheme among the students, and for this, it requires an in-deepth study; in this article, we have suggested a first analysis of this scheme, and we have described two elements that strengthen it and hinder the passage to a scheme mathematically more refined.
Further investigation are required in order to fully highlight the complexity of the mental processes involved in mathematical negation.
The knowledge of elements like those we have described, of their origin and of their link with the common practice, is deemed very important as it provides the teacher with good indications for the construction of didactical approaches for the introduction to mathematical negation and to proof by contradiction. We retain extremely meaningful statement 6 of Carlo protocol, that individualises a narrow bond between the schemes of negation proposed and the problem of the learning of proof by contradiction.

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The Students' Processes of Transforming the Use of Technology in Mathematics Problem Solving Tools

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The students' use of technology plays an important role in their learning of mathematics. Here we report the work shown by high school students who participated in problem solving activities that involves the use of dynamic software (Cabri-Geometry). A task proposed by the students themselves is used as a means to illustrate three different approaches that appeared during the students' work. Each approach shows diverse mathematical processes and resources that helped them explore and solve the task.

Recent curriculum proposals identify the use of technology as a powerful tool in the learning of mathematics (NCTM, 2000). There are different ways in which technology can be used by students, in particular, the idea that with the help of some software or calculators they can achieve easy representations, explore different cases, and find loci or trajectories of points (segments or figures) seems to be attractive in designing students' learning activities. What type of mathematical resources do students need in order to show an efficient or significant use of technology during their learning experiences? When does the use of technology become a powerful tool for students? These are questions that provide information to explain what students achieve in classroom that promote the use of technology. In this paper, we document features of students learning that show a process of students' adaptation in the use of technology. At this latest stage, students not only search for different approaches to represent and solve problems, but also they explicit redesign or formulate their own questions or problems.

Conceptual Framework

In a changing and demanding society the study of mathematics becomes an important need for all students; however, as Romberg and Kaput mentioned:

> the changes make it imperative that any answer to the question "What mathematics is worth teaching"? Be periodically considered". … regardless

> of the specific content, the aims of mathematics teaching can be described in terms, as teaching students to use mathematics to build and communicate ideas, to use it as a powerful analytic and problem-solving tool, and to be fascinated by the patterns it embodies and exposes (pp.15-16).

The use of technology can play an important role in helping students represent, identify, and explore behaviors of diverse mathematical relationships. An important goal during the process of learning mathematics is that students develop an appreciation and disposition to practice genuine mathematical inquiry during their school learning experiences. The idea that students should pose questions, search for diverse types of representations, and present different arguments during their interaction with mathematical tasks has become an important component in current curriculum proposals (NCTM, 2000). Here, the role of students goes further than viewing mathematics as a fixed, static body of knowledge; it includes that they need to conceptualize the study of mathematics as an activity in which they have to participate in order to identify, explore, and communicate ideas attached to mathematical situations.

> ...Students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge (Carpenter & Lehrer, 1999, p. 23).

It is also recognized that instructors should provide a class environment that promotes students' experiences in reflecting, conjecturing, and persisting. In this context, the design and implementation of tasks, which favor the use of these experiences, continue to be a great challenge in problem solving instruction (Santos, 1998). We document that the use of technology eventually becomes a powerful tool for students to make sense of information, to propose conjectures, and to examine different approaches to the problems. Students were encouraged to work as a community in which they valued not only personal contributions, but also the participation as a group. Students' engagement in processes of inquiring and explaining became the key ingredient while working with the tasks.

Methods and Procedures

Sixteen grade 12 students participated in a four weeks seminar that included two sessions per week (2.5 hours each session). The general idea of the seminar was to employ dynamic software to solve mathematical tasks initially provided by the instructor. Later, the same students were encouraged to propose their own tasks or problems. During the first two sessions the instructor gave a general introduction to the use of the software and illustrated the use of some commands to the whole class. In general, a student worked individually first, later in small groups of four members, and at the end of each session there was a general discussion with the whole group. Students could also exchange files and receive feedback from other participants. For the analysis of the students' work, we have chosen a task that was proposed by a small group. This task was solved during the last two sessions of the seminar. Throughout the analysis, we attached some comments or observations to describe particular students' behavior that appeared during this implementation; however, there is no attempt to show a detailed analysis of transcripts of their work. Instead, we identify a set of observations that illustrate mathematical relationships that emerged from students' interaction with the task. In some cases, the teacher' participation played an important role in orienting the students' discussion which eventually led them to propose and examine those relationships.

Origin of the task. An important activity that appeared during the sessions was to ask students to formulate their own questions or problems. So, during the students' interaction with tasks or situations, they were free to explore connections or change original statements to examine and document the behavior of other relationships. A member of a small group mentioned that in order to formulate questions, it was important to identify basic properties attached to different figures. For example, what do we know about rectangles? They have four sides (two pairs of parallel sides, perpendicular sides, four right angles, two diagonals, one center (diagonal intersection), and attributes such as areas perimeters and include pair of congruent right triangles (Pythagorean Theorem). Indeed, students agreed that in order to represent a task via the software, it was important to think of all figures in terms of properties and then select proper commands to achieve particular representation.

Can we construct a rectangle if we know only its perimeter and one of its diagonal? This was one question proposed by one student to the whole class. Three different students' approaches emerged from the students' work in this task. Although in all of them the use of technology appeared to be relevant, we focused on identifying two approaches in which the software functioned as powerful tool not only in achieving the solution but also in exploring other geometric properties of present figures.

Solution Process shown by three small groups

How can I represent the perimeter geometrically? What information does the perimeter provide about the sides of the rectangle? ¿How the perimeter information is related to the diagonal? These were some of the initial questions discussed within a small group that eventually led students to represent basic information and use the software dynamic to connect such information. The important stages are described next:

(i) Students represented the semi-perimeter as segment AB and chose point Q on it. That is, a + b is segment AB where a & b are sides of a rectangle. With this information they constructed the corresponding rectangle EHGF (figure 1). Here a = AQ = EH and b = QB = HG.

(ii) Students realized that by moving point Q along segment AB, a family of rectangles with a fixed perimeter was generated. Indeed, they decided to find the locus of point G when point Q is moved along AB (figure 2).

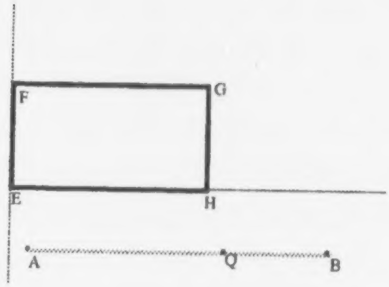


Figure 1

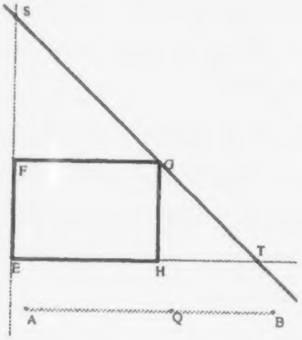


Figure 2

(iii) They found that the locus was the segment ST and explained that when point Q becomes point B, them ET will become segment AB. Similarly, when point Q coincides with Point A, and then segment ES becomes AB. That is, they noticed that the rectangle they wanted to find was one of those that can be inscribed in the right triangle EST. Indeed,. they realized that the rectangle could be drawn in two different positions except when the rectangle became a square (figure 3).

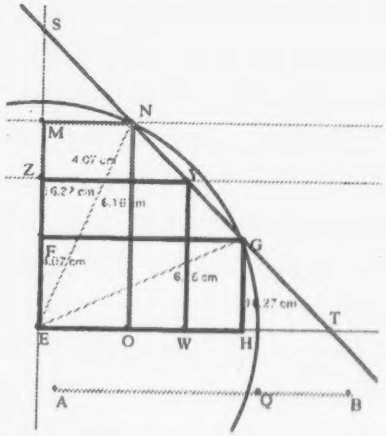
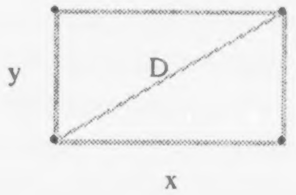


Figure 3

Another approach shown by two small groups was to focus on the algebraic representation of the situation. That is, they decided to use x and y for sides of the possible rectangle and wrote the following equations:



Here, a student suggested to graph both equations, he mentioned that since P and D were given numbers, then the first equation represented a line and the second a circle. They showed the following procedures and representation:

$$y = -x + \frac{P}{2}$$
$$x^2 + y^2 = D^2$$

Here, it is important to mention that students spent significant time analyzing the cases in which it was not possible to construct the rectangle. Eventually the graph became a referent to explain the existence of such rectangle (the circle might intersect the line in one point, the case of square, two point, the above figure, and no intersection points). The other small group which followed this approached gave an algebraic explanation regarding the solution of the system of equation.

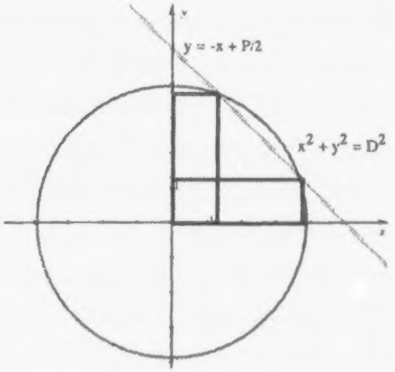


Figure 4

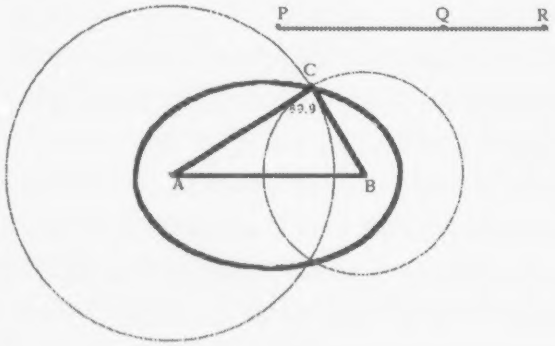


Figure 5

Yet another students' approach was to construct a family of triangles with perimeter equal to the sum of two sides of the rectangle plus the length of the diagonal. Here, they chose the given diagonal as a fixed side of the triangle and the other two sides of the triangle as the semi-perimeter of the rectangle. The software became a powerful tool to find the family of triangles with fixed perimeter (figure 5).

Segment AB represents the given diagonal and segment PR is the semi-perimeter. Students drew two circles, one with center on A and radius PQ and other with center on B and radius QR. These two circles get intercepted in C. The locus of point C when Q is moved along PR is an ellipse (foci A & B and constant PR). Here, students focused on finding the triangle with angle ACB a right angle. To find it, they drew a circle with center, middle point of the diagonal AB, and radius half of the diagonal. The intersection of the ellipse and the circle will determine the vertex of the right triangle. Here, they also noted that there were cases in which angle ACB never became a right angle. In this case, it is observed that there is no intersection between the circle and the ellipse.

When these solutions were discussed within the whole class, it was evident that students realized that the use of the software provided a means to explore the task from diverse angles and perspectives. In particular, they were surprised that a variety of mathematical resources and ideas were present in each approach. At the end of the session, a student asked: Can we construct a rectangle if we have its diagonal and its area (instead of its perimeter? Here, students again were ready to explore this question through the software.

Remarks

The work shown by students during their interaction with the task illustrated different mathematical qualities that allowed them explore strengths and limitations of their approaches to the task. For example, the dynamic approach in which students focused on finding the locus of the fourth vertex provided enough information for them to identify all rectangles with fixed perimeter. Here, they introduced the diagonal information to find the rectangle. The students' algebraic approach relied on a static representation in which they basically represented a particular case and discussed other possibilities of behavior of the two graphs in terms of the graphs intersection. The third approach in which students decided to construct a family of triangles with a fixed perimeter combines both a partial representation of the rectangle, that is a right triangle and the power of technology to find all of them with a fixed base (the diagonal). When students moved points, found diverse loci, assigned measurements, and formulated and supported conjectures, it was clear that the software became a powerful mathematical tool for the students.

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THE USE OF REAL WORLD KNOWLEDGE IN SOLVING MATHEMATICAL PROBLEMS

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The focus of this paper is the issue of reality in relation to mathematical non-routine problems. Findings pointed to three gaps: the gap between the world of in-school mathematics and the out-of school world; the gap among "realities", different realities between teachers and students but also different realities among the teachers and among the students; the gap between teachers' theoretical knowledge of the kind of problems they should teach and between what they actually teach in class.

Introduction

Mathematics has been a large part of the school curriculum. Students study mathematics from kindergarten through elementary and secondary school. Mathematics is also an integral part of our everyday life and much of the mathematics knowledge is acquired outside of school. Thus, it seems reasonable for students to use everyday life considerations when solving mathematical problems in school and use mathematics learned in school when dealing with everyday life situations. Moreover, it seems reasonable that the two mathematical worlds - the in-school world and the out-of -school world, will complement one another. However, Resnick (1987) describes the gap between learning mathematics in school and the out-of school world, saying that children frequently do not bring to school knowledge that was not studied formally, while knowledge acquired in school is not used adequately outside school. She concludes that the gap between "school intelligence" and "practical intelligence" should be minimized. Schoenfeld (1992) lists students' beliefs about the nature of mathematics. Students believe that mathematics learned in school has little or nothing to do with the real world. Nunes, Schliemann and Caharrer (1993) explored the use of "school mathematics" and "street mathematics". They found that unschooled street-vendors, fishermen and carpenters performed their street mathematics calculations competently. On the other hand, students did not know how to use mathematics studied in school and sometimes arrived at absurd solutions. Young street vendors who had some schooling performed their street mathematics calculations much better than when trying to solve the same problems using their school mathematics. These young street vendors solved problems they encountered in their jobs more easily than when the same problems were given to them in school. The detachment of school mathematics from the reality outside of school is documented in several research studies in which it was found that school children had difficulty in applying real world knowledge in problem solving (for example: Säljö, 1991; Greer, 1994). This difficulty is not unique to students. Verschaffel et al. (1997) found that pre-service teachers also had the same difficulties.

One of the reasons for this gap may be embedded in the mathematics itself. Resnick (1987) suggests that at school children learn rules and symbols but tend to lose the relationship to what these symbols represent. The detachment of the symbols from what they symbolize may cause difficulties in applying real world knowledge in mathematics lessons and vice versa. Another reason for this gap may derive from the repertoire of problems that appear in the textbooks and are studied in school. It is well known that teachers rely "heavily" on textbook problems. Researchers criticize the "stereotyped" problems studied at schools (for example, Nesher, 1980; Reusser, 1988; Gravenmeijer, 1997). Russell (1996) criticizes textbooks' problems for not being "real life applications". She gives an example of a problem found in a textbook in which the student is asked to find the average length of the world's seven longest rivers. Russell claims that the problem is "silly". "Why would we want to know the average length of these seven rivers?" Children should be given the opportunity to solve practical and realistic problems, otherwise they become bored, often causing them to dislike problem solving. On the other hand, when Nesher and Hershkovitz (1997) gave students non-routine problems taken from their lives, children did use real world considerations when solving those problems. (The problems dealt with dividing pizzas among children in a summer camp).

Our research adds one more piece to the puzzling relationship between school mathematics and everyday life. It is part of a larger research which investigated knowledge, beliefs and attitudes of elementary school teachers with different professional backgrounds toward non routine problems, as well as attitudes and beliefs of sixth graders to these problems (Asman, 2000). In four out of the 11 problems used in the research real world considerations had to be taken into account. In this paper we report some of the findings regarding these four problems.

Methodology

30 elementary school teachers were interviewed. Each interview lasted for about one hour and a half. After some personal information concerning the teacher's background, each teacher was asked several questions regarding her beliefs and perceptions about problems in general. For example: why do we teach problem solving? Or, what is a "good" problem? Then each teacher was presented with 11 non-routine problems one at a time and asked to solve the problem. If the teacher was not able to solve it, or solved it incorrectly, she was given some hints by the interviewer who helped her to arrive at a reasonable solution. At this stage the teacher was asked about her beliefs regarding the non-routine problem she just solved: Would she give the problem to her class? In an examination? Has she encountered such problems in textbooks? In workshops for teacher development? After going through all 11 problems the teacher was asked for some feedback on the interview. Each interview was recorded and then transcribed.

In order to find out how do different professional backgrounds affect teacher knowledge and beliefs, we chose the 30 teachers as follows:

- Ten pre-service teachers specializing in mathematics, at the end of their third year of college. (PST)

- Ten in-service teachers, teaching in the higher grades of elementary school. (T)
- Ten in-service teachers teaching in the higher grades of elementary school, who had participated in long-term mathematics intervention programs. (TT)

265 sixth grade students participated in this research. They studied the 11 non-routine problems in class. Their teachers (T and TT teachers) who participated in this research study taught them. The students answered a short questionnaire regarding their beliefs and attitudes toward the problems.

In addition we observed two six grade classes studying these problems.

The Problems

The following are the four problems which in their solutions one needs to take into account everyday life considerations:

1. A bus can hold 40 students at the most. How many buses will be needed to transport 175 students? (Transportation Problem)

2. When looking at a 35° angle through a magnifying glass that magnifies four times, what size angle will you see? (Angle Problem)

3. Four families live in Dan's building, altogether they have 10 children. What is the average number of children in each family? (Average Problem)

4. John's parents bought furniture for their dining room. The table cost $1.1 and each chair $0.4 . What is the total amount they should pay if they bought one table and four chairs? (Furniture Problem)

Results

The Transportation Problem

This problem is well known in research literature and was first used (with different data) by Carpenter et al. (1983). The exercise 175:40 is only the first step in solving this problem. The second step is adjusting the result to real life.

All 30 teachers solved the problem using everyday life considerations. For example, a (T) teacher said that five buses are needed. However, she would try to seat more students in each bus since she works in a neighborhood where the parents can not afford to pay for five buses. Another teacher (PST) commented that although each bus can hold 40 students there is no need for them to crowd the children and sit 40 in four buses and 15 in the fifth and they can divide the number of students more comfortably among the buses. Another teacher (T) said they could order four buses and one minibus in order to decrease the expenses. It seems that teachers used their experiences in ordering buses for field trips, which is something teachers frequently do. As for the students, many solved the problem just by performing the mathematical exercise, without using everyday life considerations, saying that 4.375 is the number of buses to be ordered. Some realized that 4.375 was an unrealistic number of buses, thus they wrote "4.375 – illogical", not knowing how to adjust the answer to everyday life or not knowing that the answer could be different from the result of the exercise they had just performed. This situation seems not to be a part of students' reality. One TT teacher commented that her students would probably know how to solve such a problem, since before going on a field trip they used to calculate the expenses per student. Thus, she said, for them this problem would be a "real" problem and a significant one.

The Angle Problem

Two numbers are involved in this problem (35 and 4), but in order to solve it one is not supposed to do any calculations with these two numbers. The use of real life considerations should be the clue to this problem. One needs to understand that when looking through a magnifying glass, the *shape* of the object being viewed does not change, thus the angle should remain the same. But even if the solver did not use everyday life considerations and obtained 140° as the answer her/his common sense should start working since they started with an acute angle, which turned into an obtuse angle. One should wonder what would happen to a right angle? Would it turn an angle of 360° when viewed through a magnifying glass? And what about a 150°? What would it look like?

23 teachers (4 T, 9 TT and 10 PST (all 10 PST were exposed to this problem in one of their courses)) said that the angle would remain the same. Some of them explicitly indicated that they relied on considerations from everyday life, as in the following explanation given by a pre-service student:

> *"I used my real world knowledge, I wear eyeglasses. When I put on my eyeglasses do I see a different angle? But I am not sure that children would see it the same way as I do"*

7 teachers (6T and 1 TT) gave 140° as their answer. During the interviews all of them were led to the correct solution. One of the teachers (T teacher) explained:

> *"It is a difficult problem, a magnifying glass is something that I have never worked with nor have the students."*

Both teachers expressed their concerns about the way students will deal with this problem, since this is probably not from students' real life.

Many students did not look for reality when solving this problem. It seems that the numbers and the "key words" that appear in the problem confused them. Many of them explained that they were distracted by the words "magnifies four times". They were surprised how simple this problem actually was.

One student who did take into account that the problem reflects out of school reality, explained her solution:

> *"The angle would not change in degrees, when we look at a bug through a magnifying glass, does it change its shape?"*

The Average Problem

Relying on everyday life considerations (without knowing the meaning of the term average) might be an obstacle in correctly solving this problem. This was the case with 13 teachers (3 T, 2 TT and 8 PST). They used their "real world knowledge" but since *"We deal with children and not with tomatoes"* and since *"there is no half a child"* 2 ½ cannot be a reasonable answer. Some teachers rounded off the number either to 2 or to 3 so it would make sense. These teachers did not know that the average is a statistical datum and is not representative of reality. They were surprised to find out that the correct solution is that each family has an average of 2½ children. They complained that in their textbooks, when dealing with an average concerning people, the average is always a whole number, therefore they had never encountered such a problem. A (TT) teacher who solved the problem correctly said that since the material in textbooks is not authentic she brings her class authentic materials from the newspapers, like information about surveys. From this resource her students can learn about mathematics in real life.

In the classes we observed most students tended to round off the quotient. However there were students who left 2 ½ as the answer not paying attention at all to the possibility that it might be problematic. Only one student said that he saw in the newspapers that when dealing with averages regarding human beings it is o.k. to write fractions. This might take us back to the 13 teachers. We may wonder how could it be that they did not pay attention to the numbers in the newspapers, on the radio or television. Or is it that they did pay attention, but "their reality" in which we can not have 2 ½ children as an answer to a problem was the deciding factor in their reasoning?

The Furniture Problem

In this problem the prices are extremely unrealistic. The first purpose for including this problem in the research was to find out if teachers and students would notice the absurd prices. The second purpose was to find out more about teachers' beliefs regarding the question: Should the data that appears in math problems, stand the tests of reality? Only 5 teachers (4TT and 1PST) noted that the prices were unrealistic. The fact that most teachers solved the problem, expressed their beliefs toward it but were not bothered at all by the extremely unrealistic prices, illustrates the gap between in-school mathematics and out-of-school mathematics. When the interviewer raised the issue of the extremely unrealistic data, 25 teachers agreed that word problems should reflect reality and therefore if they had noticed that the data were unrealistic in the first place they would have changed the numbers. Five teachers (2T, 1TT and 2PST) thought that it does not matter if data reflect reality:

> *"I do not care about the numbers in the problem. It is important for me that they would be simple. I care about the strategy of solving, not the numbers since students calculate with calculators" (TT)*
> *"I would not change the prices since the problem is for an exercise, not a test of reasoning". (PST)*

Teachers who thought that prices should be changed to real ones expressed their opinions saying;

> *"I would change prices since we must bring reality into class". (TT)*
>
> *"We should adapt prices to reality since children constantly ask why should I study this or that? They are motivated to study only if they are convinced that what they learn will help them in everyday life or in the future. As a child I did not want to study stuff that I was not sure that it would be applicable in my everyday life. If the mathematical problem reflects a real problem from everyday life, the child will be interested". (PST)*
>
> *"A child should be prepared for real life. He should understand the value of things. If he vandalizes a desk in class, he should know its value, and this is surely not $1.1." (T)*

The last sentence is an authentic example from the real life experiences of a teacher who is teaching in a low socio-economical neighborhood. She explained why a child in her class should know the real price of furniture.

As for the students, most of them did not notice anything peculiar in this problem. In one of the classes we observed a student made a remark about the "funny" prices. This opened a discussion in class, whether prices should be real. Most students thought that the numbers in the problems are not important since teachers give problems in order to find out if the student knows how to solve and calculate. Only a few students thought that it would be better if the data would match reality.

One (TT) teacher reported that when she gave this problem to her students, a student (weak in mathematics) laughed and said that the prices are senseless. In the discussion held in class, as to weather the prices should be real, he was quite assertive saying that prices should be changed to real ones. The teacher was surprised since this student never showed any interest in mathematics lessons. The teacher explained that this student is helping his father in his shop after school and prices are part of his reality.

Discussion

It seems that the findings in this research may point to three gaps regarding the issue of reality and problems in mathematics. The first gap has to do with "theory and practice", between what teachers believe in theory and what they do in class. When asked about their beliefs of problem solving in general and "good problems" in particular, almost all teachers said that problems should be from students' lives, indicate authenticity, and promote reasoning. However, when asked to give examples of problems that they teach in class, most examples did not reflect these characteristics. Moreover, only very few paid attention to the unrealistic prices in the Furniture Problem. It might be that others did pay attention, but did not say anything since the numbers in the problem do not have to be realistic for them.

The second gap is the gap between in-school mathematics and the out-of-school world. This gap was prominent for example when teachers and students did not use their out-of-school knowledge when solving problems in school. Furthermore, unrealistic data did not bother them as long as the problem was solvable.

The third gap that emerged is the gap between different "realities". We could see that teachers' reality was different from students' reality, but we could also see different realities among teachers and different realities among students. It seems that when problems are from the solvers' reality it might reinforce their willingness to be involved and might increase their chances to solve the problem correctly. A (T) teacher aptly described this gap mentioning the student's world, her world and her husband's world:

> *"......"good problems" are surely not those in textbooks, which are irrelevant to a child's life or culture. There are some problems that do not belong to the student's cultural world. This is a disguise, it is not his world, therefore he does not understand it, and it frustrates him. ...sometimes I do not know myself what they mean, and I have to ask my husband for an explanation. If it is not even from my world, how could the child possibly understand it? For example, problems which involve filling a swimming pool with two pipes. Neither my students nor I have witnessed such a situation. The solution should be logical too, not with difficult fractions, and not with four or more digits after the decimal point. Such numbers may be relevant for my husband who is an engineer and works with small particles, but surely not for our students".*

There are probably many sources and many reasons for these gaps. The gap between what teachers say and what teachers do (a well-known gap among teachers in general) may be partly explained by the materials they use in class. During the interviews teachers complained about the textbooks, saying that textbooks are their main source for problems but criticized the textbook problems for being "stereotyped", not authentic, and boring. However, the TT teachers who participated in long-term mathematics intervention programs, reported that they also have other sources for word problems and that they do not rely on the textbooks only. The textbooks might also be the reason for the gap between in school mathematics and everyday life out of school. Another reason for this gap might be students' learning to play according to different rules. They have the "in school rules" to solve mathematical problems in class and "out of school rules" to deal with out of school situations which involve mathematics. They probably are familiar with considerations from everyday life but they do not think that they are supposed to use them in school (Bishop and de Abreu, 1991).

The existence of the gap of different "realities" is very clear. But does it mean that the teacher has to generate problems to fit the reality of all students in class,? Is it possible? Is it the right thing to do? Must all problems be realistic? Wouldn't academic mathematics be lost?

It seems that more research is needed in order to answer these questions and to more deeply explore the puzzling relationship between mathematics and everyday life.

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THE EFFECTS OF INSTRUCTION ON LIKELIHOOD MISCONCEPTIONS

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This paper reports an investigation into the effects of instruction in probability concepts on the decision making strategies of twenty-four 11-12 year olds. The instruction, based on small-group practical activities, had an overall positive influence on performance in specific probability tasks. It was also found that the particular experiences within the small groups of students had a strong influence on decision-making strategies in the final 'test' tasks. Groups that experienced sets of random outcomes in their activities that were not representative of the structure of the sample space tended to use inappropriate reasoning in later tasks.

Research into probabilistic reasoning has identified various strategies used by people in situations including sequences of randomly generated outcomes, and in particular involving the expectation (or prediction) of the most likely 'next' outcome. One such strategy that has received considerable attention is *representativeness* (for example: Fischbein & Schnarch, 1997; Shaughnessy, 1981), which is the expectation that a random set of outcomes should be representative of the composition of the sample space. Amir, Linchevski & Shefet (1999), working with 11 and 12 year olds explained that;

> The 'representativeness' heuristic includes two distinct and independent dimensions: the tendency to expect a sample space to reflect the numerical proportion of the parent population; the tendency to expect the sample not to be too orderly, to look 'random'. (p. 2-32)

Closely related to *representativeness* is the type of thinking known as *negative recency* (Fischbein & Schnarch, 1997) where there exists the expectation that as the frequency of a particular outcome increases the probability of that outcome occurring again decreases. For example, when repeatedly flipping a coin, a run of heads would lead to the expectation of the next flip being a tail.

Another, little studied, influence on decision-making is the confirmation or refutation of the 'prediction' by the actual next outcome. Truran (1996), working with a *known sample space*, analysed the changes in prediction of primary and secondary students in regards to the next outcome. One finding was that when the more-likely outcome was predicted, it didn't really matter whether the next outcome confirmed or refuted that prediction. However, if a less-likely outcome was predicted, the subject was highly likely to change the prediction, particularly if the following outcome refuted the less-likely prediction. Similarly, Ayres & Way (1998, 1999), working with *unknown sample spaces*, found evidence that upper primary-aged students would change their prediction patterns according to how successful they were in their predictions. Although students would choose the most frequently occurring outcome under specific conditions, they would change strategy if their predictions were not rewarded. Consequently, Ayres & Way (1999, 2000) argued that children may be influenced in their probability judgments by confirmation or refutation of their predictions rather than by their knowledge of the overall situation.

In the context of mathematics education, the apparent instability of students' understanding of basic probability concepts such as randomness and sample space, makes the influence of instruction an important area for investigation. Watson & Moritz (1998) reported that after five years inclusion of probability and statistics in an Australian state's curriculum (primary and lower secondary), no improvement in students' performance in chance measurement tasks was found. However, Jones, Thornton, Langrall & Mogill (1996, 1999) found clear indications of improvement due to instruction in chance concepts in Grade 3 children. The Jones et al. (1999) study suggested several key aspects of instruction; overcoming misconceptions about sample space and its relationship to likelihood, the application of part-part and part-whole reasoning, and the development of language to describe probabilities. Further research into significant experiences that may help or hinder the development of understanding of probability concepts is required. For example, little is known about the impact of commonly used classroom-teaching techniques, such as group practical work and whole class discussions, on the development of probability ideas.

The study reported here is the latest in a series of investigations into probabilistic reasoning and utilises the established 'controlled randomness' video (see Ayres and Way, 1999) as an assessment instrument . The aim of this particular study was to carry out an initial investigation of the effect of instruction, based around small-group practical activities, on the basic decision-making strategies of upper-primary students.

INSTRUCTION PERIOD

Participants. Twenty-four grade-six students (11-12 year-olds) from an independent Sydney (Australia) primary school for girls participated in this study. No student had been formally taught probability theory in any mathematics classes. The students were drawn from the top mathematics class of the grade and a state-wide numeracy test indicated that each participant ranked above the State average on this test. The students were randomly assigned to six groups

Procedure and materials. The instruction period consisted of two 1-hour sessions over consecutive days. Because of space restrictions, this paper will focus on the first session which consisted mainly of a small group practical activity. Initially (approximately ten minutes), a whole-class discussion with one of the researchers (Investigator 1) was completed on the basic ideas associated with likelihood and chance. Although the class had received no previous school-based instruction on likelihood, it was clear from the discussion that many students had a reasonable understanding of likely and unlikely events in the real world. Furthermore, some students demonstrated knowledge of theoretical probabilities associated with obtaining a head in coin-flipping ("1 in 2", "fifty-fifty") or a six in dice-rolling ("1 in 6").

Each group received a paper bag containing ten coloured tiles. Two groups received a bag containing 5 Green, 3 Yellow and 2 Blue tiles; two groups received a bag with 6G, 3Y and 1B tiles; and the remaining two groups received a bag with 7G, 2Y and 1B tiles. Ratios varied to investigate the impact that these differences might have on the overall results. Prior to the start of this phase, a demonstration of a game was given by the researcher to the whole class using a bag with colours in the ratio 1: 3: 6. A total of four games were completed. The only information given to the students was that the bags contained green, yellow and blue tiles. Students were required to predict the colour of a tile before it was withdrawn from the bag. Students took turns within groups to select a tile, which was returned to the bag before the next selection. Students were told that it was a game and the winner would be the one with the most correct predictions overall. Students were required to record each prediction and the actual colour that occurred in a booklet. A game consisted of five predictions. After each game, students were required to tally their correct predictions and discover the winner(s) for that game in their group. Furthermore, after each game, students were asked to discuss the game within their groups and to ascertain why the *winner* won, and to record this reason/discussion in their booklets.

On finishing the last game, students were asked to count the number of correct predictions they made over the four games and to discover who the overall winner was in their group. They were then asked to reflect on winning strategies and how they could have improved their own predictions. After all tasks were completed, students were allowed to examine the contents of the bag. This session was closed with a whole class discussion around the main idea that there were more greens in the bags, therefore the best prediction strategy was to choose green.

The second session (which will be reported in more detail at a later date), focused on two central themes related to the structure of the sample space and its relationship to likelihood. Firstly, selecting the most frequently occurring colour in the outcomes was a good strategy. Secondly, this strategy does not always work as less likely events can occur. A practical activity based on the strategy of choosing the most frequently occurring colour over a number of trials was completed.

Results. The mean number of greens (there were more greens in each bag than any other colour) predicted by each group is reported in Table 1. There was a significant difference between the groups on this statistic, measured by a 1-way ANOVA; $F(5.18) = 5.43$, $p < 0.01$. In terms of the number of correct predictions made (see Table 1), a 1-way ANOVA also revealed a significant difference between groups; $F(5,18) = 3.67$, $p < 0.05$. Whereas these results are not necessarily surprising considering the differing bag contents; it is clear that Groups 2 and 3 observed a reduced frequency for green to what might be expected from the theoretical probabilities: 35% compared with 50% for Group 2, and 40% compared with 60% for Group 3. The small number of occurrences of green influenced both groups' selection of green and ultimately their prediction successes, as there was a correlation of 0.89 (Pearson product-moment coefficient, $p < 0.001$) between prediction success and choice of green. Students who regularly chose green were more successful in their predictions. The written responses from individuals indicated that students from Groups 1,4, 5 and 6 believed that green was the dominant colour in the sample space and students would enhance their prediction rates if they chose more green. In contrast, Group 2 concluded that luck was involved and success depended upon being able to "spot the patterns". Group 3 generally believed that there were more yellows in the bag and success depended upon "knowledge of previous games".

Table 1: *Quantitative Group data for instructional phase*

Group	Ratio of G:Y:B in Bag	Actual colour (G:Y:B) outcomes over 20 trials	Mean number of greens predicted over 20 trials	Mean number of correct predictions over 20 trials
1	5: 3: 2	12: 5: 3	9.8	8.3
2	5: 3: 2	7: 7: 6	7.3	6.8
3	6: 3: 1	8: 8: 4	6.5	5.3
4	6: 3: 1	12: 7:1	11.8	9.5
5	7: 2: 1	15: 3: 2	13.5	11.5
6	7: 2: 1	13: 4: 3	11.8	9.8

Whereas, many students (particularly in groups 1, 4, 5 and 6) were able to give reasons consistent with an understanding of likelihood, the winner of Group 2 (9 out of 20) gave a very unexpected answer during an interview.

> I worked out a theory. The teacher (researcher) is English and he pulled out a yellow tile. My dad's English and I also pulled out a yellow tile. Alison's dad is Australian and Australia is on the opposite side of the world to England, therefore she would pull out a blue tile and she did. Maria's dad is Greek, therefore she should pull out a green tile and she did.

In Group 2, students were taking turns selecting from the bag. The winner changed her choice according to who was making a selection. By coincidence, she was the most successful of her group, and this misconception became her "successful" strategy. Overall, quantitative and qualitative data revealed that most students demonstrated a good understanding of likelihood in this domain. However, it became evident that selection strategies and the reasons given by individuals tended to converge within the groups.

TEST PERIOD

Participants. The 24 participants were the same students who completed the instruction period. However, two students (one each from Groups 1 and 3) were absent.

Procedure and materials. To test the effectiveness of the instructional phase, the students were given prediction tasks based around a video recording. The video, previously constructed by Ayres and Way (1998, 1999) with pre-ordained outcomes, featured a presenter making thirty selections of coloured balls from a box with replacement. Students were required to predict the next colour after observing the five previous selections. In all, six predictions were required. In the video, 19 whites (63%), 7 blues (23%) and 4 yellows (13%) were drawn from the box. However, the emerging colour sequence was manipulated so that the less likely outcomes (blue and yellow) appeared consistently (four out of five) at the prediction locations, whereas the most likely colour (white) appeared only once. As a consequence of this design, students found that predicting a number of whites was not a successful strategy and switched from using "the most likely strategy" to strategies based on misconceptions such as colour patterns and negative recency (see Ayres and Way 1999, 2000 for more detail). In this present study, it was anticipated that this particular task would prove a considerable test of student beliefs in employing the "most likely" strategy consistently without reverting to common misconceptions.

Previous research by Ayres and Way (1999, 2000) also found that knowing or not-knowing the sample space made no overall difference to the prediction strategies employed on this video task. This aspect was also investigated in this study by randomly assigning students to two groups. One group (*sample space known*) was informed that the box contained 10 balls (6 white, 3 blue and 1 yellow), which was approximately equivalent to the experimental probability of the sequences. In contrast, the second group (*sample space unknown*) were only given information about the colours of the balls (some white, blue and yellow balls in the box). Both groups observed the same video recording. After each prediction, students were required to give reasons for each decision. All students were told that the prediction tasks were a game and they should try to predict as many correct colours as possible.

Results. A record of each student prediction was made. Previous studies by Ayres and Way (1999, 2000) found that students often changed their strategy over the last three predictions compared with the first three predictions. Consequently, the number of whites chosen in the first and last three predictions were also reported (see Table 2). A 1-way ANOVA (known v unknown ratios) with repeated measures (first 3 and second 3 predictions) were completed on this data. For the main effect, knowing the sample space made no difference; $F(1, 20) = 1.88$, $p > 0.05$. However, students chose significantly more white balls during first 3 predictions than during the second 3; $F(1, 20)$ 7.12, $p < 0.05$. Consistent with previous research with Ayres and Way (1999, 2000), this particular outcome sequence caused many students to switch from using the "most likely" strategy to strategies based on common misconceptions. Qualitative data (due to space restrictions, this data will be reported more extensively at a later date) also confirmed an increased use of misconceptions over the last three predictions, such as "its yellows turn" and "the colours are forming a pattern". However, nearly half the students (45%) continued with a strategy of choosing the most frequently occurring colour for the last 3 predictions.

Table 2: Mean number of white balls selected in Video Test

	Ratio Known (n = 11)	Ratio Unknown (n = 11)	Combined Group Totals
First 3 Predictions	2.6 (0.5)*	2.3 (0.8)	2.5 (0.7)
Second 3 Predictions	1.9 (0.9)	1.6 (0.9)	1.8 (0.9)
Overall Predictions	4.5 (1.2)	3.9 (0.9)	4.2 (1.1)

*Note: Standard deviations are given in brackets.

To explore the influence of the initial group instructional activities further, prediction means were calculated for each of the instruction groups (see Table 3). Although there were no significant between-group effects (group numbers were very small) on 1-way ANOVAs for these measures, group means did vary considerably. On the crucial measure of the second set of predictions, Groups 4, 5 and 6 chose almost twice as many whites as Groups 1, 2 and 3.

Table 3: Mean number of white balls selected by each instructional group

Instruction Groups	First 3 Predictions	Second 3 Predictions	Total
1	2.7	1.3	4.0
2	2.0	1.3	3.3
3	2.7	1.0	3.7
4	2.5	2.3	4.8
5	2.8	2.0	4.8
6	2.3	2.5	4.8

It was noticeable that there appeared to be a match between the group data for this measure and some of the statistics shown in Table 1. Correlation calculations revealed that the number of whites predicted on this trial was significantly correlated to both the actual number of greens (dominant colour) that occurred during the group activities ($r= 0.45$, $p < 0.05$) and the number of correct predictions made during these activities ($r = 0.55$, $p < 0.01$). Both fairly strong positive values suggest that the differing experiences within groups had an effect which transferred into the video trial.

CONCLUSIONS

The main aim of this study was to test the effectiveness of an instructional period, featuring small group practical activities, on primary aged students' development of probabilistic reasoning. Whereas, students generally showed a good understanding of likelihood, when faced with an unrepresentative set of outcomes (the video trial) many students reverted to strategies based on misconceptions. In a similar fashion to the previous studies of Ayres and Way (1999, 2000), students chose less white balls (the most commonly occurring colour) over the final three predictions compared to the first three, indicating that their use of the "most likely" strategy was reduced. However, it is worth noting that the prediction rates of white in this study were higher than previously found. For example, in the Ayres and Way (2000) study with grade 8 students, overall mean values for this video outcome sequence was 2.0, 1.4 and 3.4, for the first 3, second 3 and total predictions respectively. In the Ayres and Way (1999) study with grade 6 students, the mean values were 1.5, 1.2 and 2.7. Both sets of data were lower than those found in this study (2.5, 1.8 and 4.2). These comparisons suggest that the instructional period in this study, as short as it was, may have helped students developed a better understanding of chance.

Of considerable interest in this study was the group effects which appeared. During the small group activity, students were exposed to different sample spaces, which produced varying sets of outcomes. Some groups were more successful in predicting than other groups, with success rates apparently dependent upon the observed outcomes. Groups which observed a set of outcomes representative of the sample space tended to demonstrate a better understanding of likelihood than groups who observed less likely outcomes. Furthermore, these differences during instruction appeared to influence decision making during the final video trial.

The implications of this study are as follows. Firstly, instruction in this domain seems to have, at least in the short term, a positive effect. Secondly, the types of outcomes observed in these random-based activities, seem to have an influence on future decision making strategies. Consequently, we believe that any instruction of this nature must ensure exposure to different types of outcomes to provide students with the opportunity to develop a clearer understanding of the relationship between sample, randomness and likelihood. Furthermore, because the groups had different experiences, teachers need to be aware of the possibility of collective misconceptions forming within groups.

Finally, it must be acknowledged that this was a small study based on a particular sample of students. It is our aim to investigate these findings further, using a broader sample of students and modified versions of the instructional period.

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SYMBOLIZING DATA INTO A 'BUMP'

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Abstract

In this paper we analyze how concepts and symbolizations co-develop in the case of statistical data analysis. The focus is on the development of distribution, which ranges from a very concrete, intuitive understanding to a formal, mathematical concept. Examples from teaching experiments with 11 to 12-year-old students illustrate how their concept of distribution develops in relation to what the graphs they use and make mean for them. In particular we discuss an episode in which a student symbolizes data into a so-called 'bump' and we give examples of how other students reason with this 'bump' in connection to distribution.

Introduction on Symbolizing

Symbolizing as a field of research has been receiving more and more interest within the community of mathematics educators (Cobb et al. 2000). Our point of departure is that the students' way of symbolizing and what these symbolizations come to signify develop in a dialectical way. The learning-teaching process is organized in such a way that the conceptual development benefits from the development and use of symbols, and vice versa. In teaching experiments on statistical data analysis we aimed for the gradual emergence of the multifaceted concept of distribution rather than a collection of loosely related concepts and graphs. In this paper we analyze how symbolizations and their meaning co-evolve in the case of distribution and graphs of data sets (cf. Meira 1995).

The Concept of Distribution

Basically, the notion of distribution refers to how data are distributed in a space of possible values. Mathematically seen, distribution could be defined as a frequency or density function. In this paper we use the term 'distribution' for the whole range from a very concrete, intuitive level to the statistical concept. The concept of distribution is tightly connected to many other statistical concepts such as frequency, skewedness, spread, and even mean and median. It is also highly interwoven with certain visual images that 'show' distributions, the most famous one being the bell-shaped curve of the normal distribution.

In statistical data analysis we are not very interested in individual cases; instead we focus on group characteristics such as center and spread, or skewedness of data. Even if we want to use the mean or median we have to take the whole distribution into account (Zawojewski & Shaughnessy 2000). For example, if there are many outliers or if the distribution is very skewed, we probably will not use the mean. From the numerical data it is hard to see how they are distributed, so we need to look at the shape of the data.

There are two other reasons we focus on distribution. First, students tend to see data as individual cases instead of values of a variable (Hancock et al. 1992). That is why they find it difficult to see group characteristics. Focusing on the shape of the data is one way of dealing with this problem. Second, we wanted students to reason proportionally instead of absolutely with parts of the graph. By proportional reasoning we mean reasoning with proportions as opposed to absolute numbers. If students, in comparing parts of two samples with different size, reason with absolute numbers then we call it absolute reasoning, whereas if they reason with proportions we call it proportional reasoning (Cobb (1999) calls this multiplicative reasoning). If students focus on the shape of distributions they are supported in reasoning more globally with groups and are maybe led away from absolute numbers. A helpful tool for this purpose is the box plot since it shows proportions, for instance where the middle 50% of the data is, without showing individual data points.

For statisticians distribution has a clear experiential meaning. For 11-year-old students, however, distribution initially is not on the horizon. Still, young learners can deal, for example, with questions concerning the way data are distributed. They can solve problems that involve looking at how the data are spread out or bunched up. To foster this kind of reasoning demands that the tasks in statistical data analysis must have certain features. First, they have to conduct the students' reasoning to characteristics of the distribution, even if the students do not talk in terms of distributions. Second, these characteristics must be expressible both in terms of the context and with statistical concepts. Anticipating the next section we mention that just one problem concerning the life span of batteries initialized discussions around center, spread, outliers, majority in terms of the context. A sample of a good battery brand has a high mean and a sample of a reliable brand has small spread and few outliers (figure 1). Of course none of these notions were very precise yet, but still they formed a basis for the development of more formal concepts.

In solving such problems the students used so-called statistical minitools. These software tools have been designed for the teaching experiments of Cobb, Gravemeijer, and others (Cobb 1999; McClain et al. 2000). The minitools do not contain any ready-made conventional statistical graphs. Instead the students can structure the data in various ways they might use when they just have pencil and paper. Still, some grouping options are precursors to conventional graphs such as using equal interval width underpins the histogram and using four equal groups underpins the box plot (figure 2).

Value Bars Come to Signify Data

The first step of symbolizing was inscribing data as case value bars, which offers a visual way of dealing with the data (figure 1). The horizontal bars are motivated by a sense of linearity that many variables have. The first data set analyzed by the students concerned the life span in hours of two battery brands. The task was to decide which brand was best and to report an analysis to the Consumer Reports.

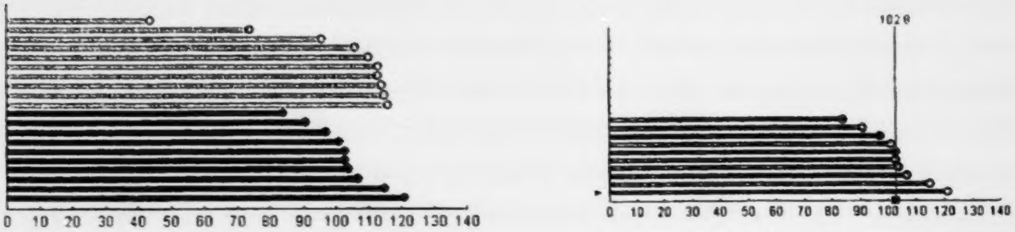

Figure 1. *Value bar graphs in minitool 1. 1a: Life spans of two brands of batteries in hours. The upper ten and lower ten are of different brands. 1b: Visually finding the mean with the value bar.*

Already in this type of graph it is visible, for statisticians, that the distribution of the first brand is skewed and has outliers; the second brand has smaller spread and has a symmetrical distribution. Here distribution is symbolized on a very concrete level. The students did not talk of distributions but they discussed the outliers of the first brand and the high maximum of the second brand. Some argued that the first brand was better since 'it has more higher values' and others opposed that the second brand 'has less bad outliers'. The second brand was considered more reliable or predictable (compare this with the notion of consistency on which Cobb (1999) and Sfard (2000) report). Students were very well able to invent data sets that could be of a very good but unreliable brand or a bad brand with the same spread as one they had encountered before.

After a few lessons the students developed a visual way of estimating means by mentally cutting off 'what was too much on the right side' and 'giving it to the left side', as they expressed it (see figure 1b). Such an activity was made possible by the inscription of the case value bars. We conjecture that students would not have done this with numerical data or with dots in a dot plot. This supports our claim that every symbolization influences the way students see the data or the context problem. On the other hand, a suitable problem may give rise to developing a new symbolization that helps in answering a question. This exemplifies the dialectical co-development of meaning of the context and of the symbolization (cf. Meira 1995).

Dots Come to Signify Value Bars

As the preceding section shows, the students developed a statistical language that was situated in the context of battery life spans and other problems. In thinking about such problems the students focused on the end points of the bars. These end points of the bars in minitool 1 were to collapse down onto a horizontal axis in minitool 2. The dot plot appears here as an image of a variable; the dots get a place in a space of possible values. This dot plot is also one step closer to the conventional graphs in which a unimodal distribution appears as a hill-shaped curve. As mentioned above some grouping options were close to conventional graphs such as histogram and box plot (figure 2), both being helpful tools in describing distribution.

In analyzing data with this second minitool, students further developed their statistical language. Since we wanted students to view data sets as a whole with certain characteristics we hoped that they would start looking at the shape of the data in minitool 2. Unfortunately they did not talk of hills as students in other experiments did (Cobb 1999); they only talked of majorities and still reasoned absolutely in some cases.

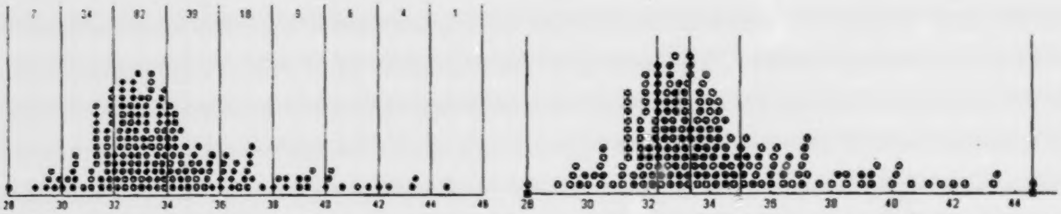

Figure 2. *Dot plots in minitool 2. 2a: The option of equal interval width is used to organize the data, underpinning the histogram. 2b: The option of four equal groups underpins the box plot.*

Symbolizing Data into a 'Bump'

We tried another route that turned out to be more promising. The basic ideas of symbolizing and guided reinvention (e.g. Gravemeijer 1994) suggest that students also should create their own graphs. For the eleventh lesson we therefore asked them to represent their weight and height data in a graph that would be clear for a particular purpose: a balloon rider had to decide how many seventh-grade students could join a balloon ride and she did not just want to know the mean.

The students came up with many different graphs resembling minitool 1 and minitool 2, but also a scatter plot of weight and height, and one graph we will discuss in more detail. For the designed learning process we focused on graphs that could help students in seeing a data set as a whole, or in other words, could help them in constructing distribution as an object-like entity which they could reason with. This was why

Michiel's graph (figure 3a) was discussed extensively. He explained his graph as follows.

Michiel: Look, you have roughly, averagely speaking, how many students had that weight and there I have put a dot. And then I have left [y-axis] the number of students. There is one student who weighs about 35 [kg], and there is one who weighs 36, and two who weigh 38 roughly. And then I have put, yeah/
Teacher: Have just put dots.
He then explained in more detail what the dots stood for. The dot above 48, for example, signified that four students had weights around 48 kg. After discussing other graphs the teacher asked the following question.

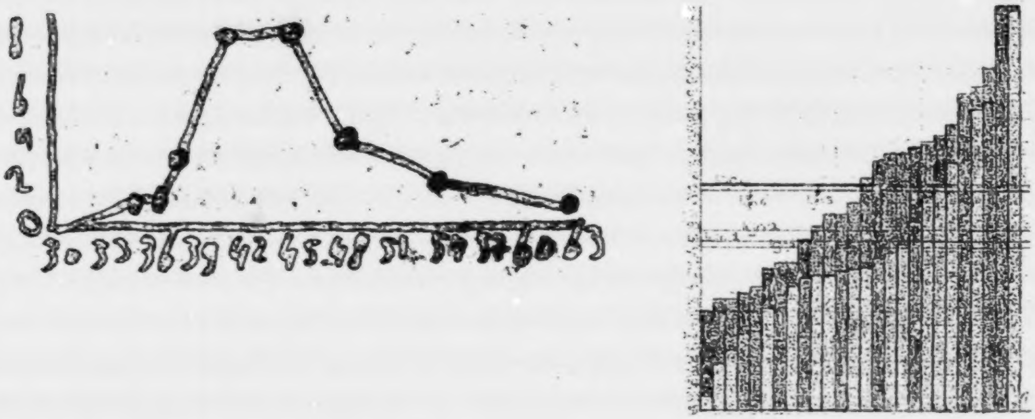

Figure 3a and 3b. *Michiel and Elleke's graphs of weight data. Elleke's includes height represented by the darker and higher bars. Her graph is called a value bar graph.*

Teacher: *What can you easily see in this graph [of Michiel]?*
Laila: *Well, that the average, that most students in the class, um, well, are between 39 and, well, 48.*
Teacher: *Yes, here you can see at once which weight most students in this class roughly have, what here is about the biggest group. Just because you see this bump here. We lost the bump in Elleke's graph.*

It is the teacher who uses the term 'bump' for the first time. Later in the discussion one student explained where the bump in Elleke's graph was.

Nadia: *The difference between ... they stand from small to tall, so the bump, that is where the things, where the bars are the closest to one another.*
Teacher: *What do you mean, where the bars are closest?*
Nadia: *The difference, the ends [of the bars], do not differ so much with the next one.*
Another student commented on this.

Eva: If you look well, then you see that almost in the middle, there it is straight almost and uh, yeah that/ [teacher points at the horizontal part in Elleke's graph].

Teacher: And that is what you [Nadia] also said, uh, they are close together and here they are bunched up, as far as height or weight is concerned.

Eva: And that is also that bump.

From these excerpts and further analysis of the episode (Bakker 2001) it became clear that these students did not just rely on visual aspects of the bump. They were able to relate the different graphs to one another by thinking about what the bars and dots signified. From the analysis of this lesson the question remained whether the bump just signified the majority for the students or that it signified a characteristic of the whole distribution.

Reasoning with the 'Bump'

From consequent lessons we inferred that the bump not just signified the majority. We give a few examples of how students reasoned with the bump. What is interesting in this respect is that many students used the term 'bump' even for the straight part of value bar graphs for where most data were. It became clear that they related this visual characteristic of the symbol also to statistical concepts such as outliers and sample size.

Laila: But then you see the bump here, let's say.

Ilona: This is the bump [pointing at the straight vertical part of the lower ten bars, like in figure 1b].

Res.: Where is that bump? Is it where you put that red line [the value bar]?

Laila: Yes, we used that value bar for it (...) to indicate it, indicate the bump. If you look at green [the upper ten], then you see that it lies further, the bump. So we think that green is better, because the bump is further.

Here the bump seems to have become a reasoning tool.

One question in a class discussion was what a graph of eighth-graders' weight would look like. Some of the answers follow.

Luuk: I think about the same, but another size, other numbers.

Guyonne: The bump would be more to the right.

Teacher: What would it mean for the box plots?

Michiel: Also moves to the right. That bump in the middle is in fact just the box plot, that moves more to the right.

Turning to a different question, the researcher (the author) asked the class how the graph would change if not just their own class but all seventh–graders in the province were measured. He was curious if the bump only signified the majority or that it was also linked to outliers and sample size. Earlier in the class discussion Elleke had mentioned that with more students one has more chance for outliers.

Elleke: Then there would come a little more to the left and a little more to the right. Then the bump would become a little wider, I think.

Res.: *Is there anybody who does not agree?*

Michiel: *Yes, if there are more children, then the average, so the most, that also becomes more. So the bump stays just the same.*

Albertine: *I think that the number of children becomes more and that the bump stays the same.*

Nadia: *I think that if there are less children, you have more chance for outliers. Maybe some are very thin and some very heavy or so. But I think that it stays roughly the same.*

A few students were able to see in figure 1a which distribution was 'normal'—defined in an informal sense—and which was skewed. They were exceptions though.

Albertine: *Oh, that is normal (...).*

Nadia: *That hill.*

Albertine: *And skewed if like here the hill is here [the upper ten bars].*

Conclusions

In the process of symbolizing data were first inscribed as value bars in the first minitool. The end points of the bars collapsed down onto an axis and formed a dot plot. It was shown that every symbolization has its advantages. The bars helped the students in seeing data as values of a variable and made it reasonable to the students to find means in a visual way and the dots made it easier to structure the data in other helpful ways. Finally, one of the students' graphs led to interesting discussions about bumps. At this stage, the learners were able to reason in a more global way without focusing on individual data points and they also argued in a more multiplicative way.

As we demonstrated, focusing on the concept of distribution, being a multifaceted notion, has the advantage of being strongly related to almost all other statistical concepts. In this way we could help students to gradually build up their understanding of these concepts in close relation to one another. We also showed how students came to construct an intuitive understanding of distribution in close relation to how they come to signify meaning to a series of graphs. As the examples illustrate the students related statistical concepts to characteristics of the graphs. It is clear that the students' way of symbolizing and what these symbolizations come to signify develop in a dialectical way, but also that the development of a concept, in this case distribution, cannot be separated from the development of the symbols.

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Comparative Study of Cognitive Units in Mathematical Thinking

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In this paper we consider the theory of 'cognitive units' introduced in 'Cognitive Units, Connections and Mathematical Proof' (Barnard and Tall, 1997) and compare it with other theories of cognitive construction. We find that it includes a theory of 'cognitive compression' to reduce the cognitive strain involved in thinking about more complex concepts that is related to the 'varifocal theory' of Skemp. It generalises the notion of process-object and schema-object encapsulation to give a broader theory of rich cognitive units, with connecting links that minimise cognitive strain and maximise thinking power.

Cognitive units, connections and compression

The notion of 'cognitive unit' that we consider here is 'a piece of cognitive structure that can be held in the focus of attention all at one time' (Barnard and Tall, 1997). We are particularly interested in cognitive units with rich internal links that allow them to be thought of as a single entity. For instance, the equations

$$P = QR, \quad \frac{P}{Q} = R, \quad \frac{P}{R} = Q$$

may initially be conceived as three separate items, together with links between them carried out by performing the same operation to both sides. Later this perception may become compressed into a single relationship between the quantities involved, seen as different ways of saying the same thing (Barnard, 1999).

Such pieces of cognitive structure do not exist in a vacuum. Intimately connected with them are other ideas in the mind that may readily be called to the focus of attention in turn. Our theory is that rich, compact cognitive units allow the thinker to manipulate these ideas in efficient, insightful ways, whereas students with diffuse structures will not find it so easy to make connections between concepts that are themselves diffuse and vague. There is already considerable evidence to support this thesis (Barnard & Tall, 1997; Chin & Tall, 2000; Crowley & Tall, 1999; McGowen, 1998). Not only do such units operate as a form of shorthand, linking many aspects of a complex structure, they also carry along with them connections that are able to guide their manipulation. Such activities may set up new links which may in turn become increasingly strong so that new cognitive units may be formed, building a network of nested mental structures that may span several layers of thought. This offers a manageable level of complexity in which the thought processes can concentrate on a small number of cognitive units at a time, yet link them or unpack them in supportive ways when necessary.

These ideas are closely related to Skemp's (1979) "varifocal theory" of cognitive concepts, where a concept may be conceived either as a global whole, or viewed under closer scrutiny to reveal various levels of detail. He referred to this conceptual detail as the interiority of the concept. Powerful cognitive units have a rich interiority appropriate for the task in hand.

The building of cognitive links in such a way that one item in the focus of attention can refer at will to a variety of closely connected pieces of knowledge will be termed ***compression***. In some cases a physical compression is known to occur in brain activity:

> As a task to be learned is practiced, its performance becomes more and more automatic; as this occurs, it fades from consciousness, the number of brain regions involved in the task becomes smaller. (Edelman & Tononi, 2000, p.51)

More generally, however, a process of *long-term potentiation* occurs in which connections between neurons become more easily activated and therefore resonate together as a single structure (Carter, 1998, p. 159). As mathematicians we see an analogy in which the brain has not only 'substructures' that work as a unit, but also 'quotient-structures' caused by the identification of separate units. A possible example is the cortical system for vision which has twenty or so separate regions each of which perform a specific task (eg recognising colours, changes in shade, edges, orientation of edges, movement of edges, identifying objects, and so on). We suggest the manner in which these disjoint areas are connected together to give a unified visual perception may be considered as such a quotient structure. As mathematics educators involved in thinking we have no physical evidence for this, so we pass the metaphor on to neurophysiologists to evaluate its usefulness. Meanwhile it is our role to collect evidence that intimates how mathematical structures may be held as manipulable cognitive units with an interiority that is able to both guide manipulation of the unit and also be subsequently expanded without loss of detail.

Process-object duality

Process-object duality is at the heart of several theories of mathematical development, for instance the encapsulation of process into object (Dubinsky, 1991) or the reification of process into object (Sfard, 1991). Unlike Skemp's theory which sees the schematic structure consisting of object-like concepts which are linked by properties and processes, these encapsulation theories describe how sequences of activities can become routinized into thinkable processes which are then in turn conceived as mental objects. This is described by Asiala et al (1997) as follows:

> According to APOS theory, an action is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. When the individual reflects on the action and constructs an internal operation that performs the same transformation then we say that the action has been interiorized to a process. When it becomes necessary to perform actions on a process, the subject must encapsulate it to become a total entity, or an object. In many mathematical operations, it is necessary to de-encapsulate an object and work with the process from which it came. A schema is a coherent collection of processes, objects and previously constructed schemas, that is invoked to deal with a mathematical problem situation. (Asiala et al, 1997, p. 400.)

Sfard (1991) proposed a corresponding sequence, affirming that operational mathematics (use of processes) almost invariably precedes structural mathematics (use of objects). Dubinsky and his colleagues followed the process-object sequence for several years before the strict sequence was loosened:

> … although something like a procession can be discerned, it often appears more like a dialectic in which not only is there a partial development at one level, passage to the next level, returning to the previous and going back and forth, but also the development of each level influences both developments at higher and lower levels. (Czarnocha et al, 1999, p. 98.)

Gray & Tall (1994) took a different view of the relationship between process and object. They saw the role of the *symbol* as being pivotal in the thinking process in a very special way. A symbol such as "3+4" could act as a pivot between a process (of addition) and the concept (of sum). This immense power—which is characteristic of symbolism in arithmetic, algebra and calculus— allows the thinker to switch between using the symbol as a concept to think about or as a process to calculate or manipulate to solve a problem. They formulated the notion of *procept* as a combination of process and concept evoked by a single symbol. This theory saw the notion of procept becoming richer (in interiority, to use Skemp's terminology) as different symbols and processes represented the same object, for instance, 6 as $5+1$ or $2+4$. From this range of associations it is possible to compute, say $8+6$, because 6 is $2+4$ and $8+2$ gives 10, and 10 and 4 gives 14. The notion of 'procept' was extended to include all the triples of process-object-symbol that have the same object in a given cognitive context. For instance, 6 is a procept which embraces $3+3$, $5+1$, $2+4$, and so on. Later in the development of the individual, it might also come to embrace $12/2, \sqrt{36}$, $3 \cdot 5 + 2 \cdot 5$. In our terminology, a procept is therefore a special case of a cognitive unit that grows with interiority as the cognitive structure of the individual gets more sophisticated. This compression of mental schemas or schemes into a cognitive unit features in a range of theories of cognitive development.

Schema-concept duality

Mathematical thinking involves two different kinds of mental activity which are both referred to as schemas or schemes. One is a *sequential* action scheme that occurs in time and is stabilized by long-term potentiation, strengthening and coordinating cognitive links such as the "see-grasp-suck" scheme in the young child. Another refers more specifically to the physical structure of the brain, which offers a *multi-connected* schema in which many possible links are available at any given time. The process-object theories seem to focus more on the first of these, theorizing that sequential schemes are encapsulated or reified as mental objects. The second type of schema offers a more subtle way of building up mental concepts that can operate flexibly as cognitive units.

Crowley and Tall (1999) consider how the "linear equation schema", for formulating and solving linear equations may represent the same idea in different forms:

- the equation $y = 3x + 5$,
- the equation $3x - y = -5$,
- the equation $y - 8 = 3(x - 1)$,
- the graph of $y = 3x + 5$ as a line,
- the line through (0,5) with slope 3,
- the line through the points (1,8), (0,5).

For some college algebra students these may all be compressed into a single cognitive unit, with the various representations just alternate ways of expressing the same thing. But it is also clear that there are students who see the structure as consisting of distinct ideas with procedures (that they may not be able to carry out) required to get from one thing to another. A student with such a diffuse view of linear equations may therefore have a partial schema for relating the various representations but not a global schema that easily sees them all as essentially the same cognitive unit.

Dubinsky and his colleagues extended their APOS theory so that, in addition to a process-object construction, there was also a schema-object construction:

> As with encapsulated process, an object is created when a schema is thematized to become another kind of object which can also be de-thematized to obtain the original contents of the schema.
> (Asiala et al, 1997, p.400)

In this way, highly connected mental structures are built up at different levels of detail, connected in various ways.

The distinction between procedural thinking that allows limited success in familiar contexts and conceptual thinking that is more adaptive in new problems has long been a subject of study. Hiebert and Carpenter (1992) proposed two alternative metaphors for cognitive structures—as vertical hierarchies or webs:

> When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships. (Hiebert & Carpenter, 1992, p. 67.)

Skemp's formulation would allow the nodes in a web to be seen as varifocal hierarchies, thus allowing the two structures to be used together. Likewise a concept could be considered as a web of connected ideas, allowing both hierarchical and web-like structures to coexist in a single structure.

However, the notion of webs and nets are still simplified metaphors for a far more sophisticated mental system. Greater subtlety is essential to be able to reflect on the way we think in mathematics. Consider for example, the statement: $\sin 60^{\circ} = {}^{\sqrt{3}}\!/_{2}$. This may be conceived by an individual as a cognitive unit and linked to a picture such as that in figure 1.

This in turn is related to many other ideas such as "the angles in an equilateral triangle are all equal", "the angles in a triangle add up to 180°", "an angle in an equilateral triangle is 60°", "the line joining the vertex to the midpoint of the base (of an isosceles triangle) meets it at right angles", "if the side is two units, half a side is 1 unit", "Pythagoras' Theorem", "$a^2 + b^2 = c^2$", "$b^2 = c^2 - a^2$", "the square of $\sqrt{3}$ is 3", "$1^2 + (\sqrt{3})^2 = 2^2$", "the sine of an angle is opposite over hypotenuse", "the opposite is $\sqrt{3}$, the hypotenuse is 2", etc. In this way we see a single cognitive unit linking

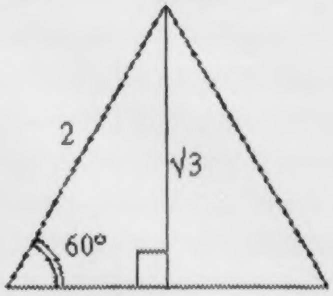

Figure 1: Relationships within an equilateral triangle

theorems about triangles, definitions of the trigonometric functions, algebraic representation of a sum of squares, numerical facts about a specific triangle, and so on. It is possible to formulate these partly in terms of hierarchies, for instance, Pythagoras' Theorem has "$1^2 + (\sqrt{3})^2 = 2^2$" as a special case, the definition of sine also includes this special case in terms of $\sqrt{3}/2$ as "opposite over hypotenuse". The many links involved also relate to other ideas; the definitions of trigonometric formulae relate to notions of similar triangles having sides in the same ratio; the trigonometric functions have relationships between them such as $\sin^2 + \cos^2 = 1$. Processes of the brain allow these ideas to become intimately connected in such a way that they are easily linked and manipulated.

Professional mathematicians build up highly subtle cognitive units packed with meaning. By focusing on commonly occurring properties which prove useful in making deductions, they build up a range of different theories based on generative concepts translated into chosen systems of axioms. For example

> ... the concept of a group captures the essence of the notion of symmetry and is connected in a precise way to the concept of an equivalence relation, which itself is a precise abstract formulation of the notion of 'sameness' with respect to a given property. Not only are the properties defining a group sufficiently general to be satisfied by a large variety of relatively concrete mathematical objects, but they are also sufficiently special to have lots of powerful consequences at the abstract level. Thus a group is a precisely defined concept which sits at a major junction in the mathematical network of relations. Indeed, one of the most beautiful features of mathematics is the way it allows such precision at even the deepest levels of abstraction. (Barnard, 1996)

Performing cognitive compression

A common method of compression is to use words and symbols as tokens for complex ideas. In particular, words can be used in a way that allows a hierarchical structure to be conceived. For instance, a square is a special case of a rectangle, which is itself a parallelogram, which is a quadrilateral. In the early stages a square and a rectangle may be seen as quite different entities, both having four right angles, but a square has all sides equal, whereas a rectangle has only opposite sides equal. The cognitive unit "rectangle" develops greater sophistication and interiority, growing from meaning just a perceived figure, to a whole collection of rectangular figures (including squares) in any orientation. The realization that squares are special cases of the class of rectangles is an example of cognitive compression where one class of objects is subsumed (for certain purposes) within another.

Language supports the communication and refinement of ideas, both between individuals and within the mind of the individual. It allows verbalized properties of perceptions to be used as a foundation for the development of cognitive units that are sophisticated mental idealizations. These include the notion of a point having position, but no size, or a line having arbitrary length and no thickness. These go on to play their role in proving relationships in elementary Euclidean geometry and perhaps later in more abstract forms of geometry.

A cognitive unit may, or may not, be associated with a natural visual label. For example, in contemplating a mental image of angles written around a circle, in degrees and in radians, there is no need to convert, say, $\pi/4$ radians to 45 degrees (by multiplying by the factor $\frac{\pi}{180}$). The symbols $\pi/4$ and 45° are simply different labels for the same angle. On the other hand, the steps of algebraic manipulation involved in seeing that 'a linear combination of a linear combination of quantities x and y is a linear combination of x and y' is an idea that may be compressed into a cognitive unit without there being a clearly defined label associated with the manipulation process.

Compression can also occur in other ways. For instance, when a collection of ideas or symbols is 'too big' to fit into the focus of attention, it can sometimes be 'chunked' to group into a single unit using some kind of alternative knowledge structure. The four digit number 1914 may be seen not just as a number, but as the year at the beginning of the first world war. This kind of associative link may be used to chunk numbers together into sub-units that can now be held in the limited short-term memory. Most individuals would find the 12-digit number 138234098743, impossible to remember on a single hearing. However, the twelve-digit number 246819141918 can be 'chunked' and remembered as the sequence '2, 4, 6, 8' followed by the dates 1914–1918 of the First World War.

Although it is possible to formulate a range of possible compression strategies, individuals do not use them all to the same extent. This leads to different individuals processing mathematical ideas in ways which may have very different outcomes that may lead to success in some and failure for others. Gray and Tall (1994) noted that children who use the longest counting process—count-all (count one set, count the second, put them together and count them all)—could also remember certain "known facts" such as 1+1 is 2 or 2+2 is 4. But none of these ever put together "known facts" to obtain "derived facts", such as "4+3 is 7" because "4+4 is 8", or "12+2 is 14" because "2+2 is 4". Instead, they always computed arithmetic problems (whose answers were not immediately known facts) by counting. We hypothesize that the sums performed by these children are seen as counting processes and not as meaningful cognitive units with any interiority. The "known facts" for them were isolated and not in a sufficiently rich compressed form which could be mentally manipulated as cognitive units. On the other hand, children who were able to derive new number facts from known related ones were able to perform arithmetic in a far more flexible way which used numbers as compressed cognitive units with powerful interiority.

Another bifurcation occurs in elementary algebra. Here an expression such as "$2+3x$" stands for a potential arithmetic operation such as "add 2 to the product of three times whatever x is". This can cause discomfort for students who feel that a problem "must have an answer" as it does in arithmetic. They are therefore faced with manipulating expressions as mental objects that have only a potential, rather than an actual, internal process of evaluation. This can lead simply to procedural compression in which students learn to carry out a solution process by rote ("collect together like terms", "get the numbers on one side and the variable on the other", "simplify to get the solution", etc). Others are able to conceive the algebraic expressions as entities that can be manipulated. They may go on to conceive of the equation itself as a cognitive unit expressing a given relationship, with a "solution process" as a cognitive unit that can be unpacked to give an efficient route to the solution.

Krutetskii (1976) studied this curtailment of mathematical reasoning, in which capable students would compress their solutions in a succinct and insightful manner.

> … mathematical abilities are abilities to use mathematical material to form generalized, curtailed, flexible and reversible associations and systems of them. These abilities are expressed in varying degrees in capable, average and incapable pupils. In some conditions these associations are performed "on the spot" by capable pupils, with a minimal number of exercises. In incapable pupils, however, they are formed with extreme difficulty. For average pupils, a necessary condition for the gradual formation of these associations is a system of specially organized exercises and training.
> (Krutetskii, 1976, p. 352.)

Great success in calculation may be developed with a huge range of connected ideas, some meaningful, some rote-learnt, as Nobel Prizewinner, Richard Feynman reports:

> I memorized a few logs and began to notice things. For instance, if somebody says, "What is 28 squared?", you notice that the square root of 2 is 1.4 and 28 is 20 times 1.4, so the square of 28 must be around 400 times 2, or 800. If somebody comes along and wants to divide 1 by 1.73, you can tell them immediately that it's .577 because you notice that 1.73 is nearly the square root of 3, so 1/1.73 must be one-third of the square root of 3. And if it's 1/1.75, that's equal to the inverse of 7/4 and you've memorized the repeating decimals for sevenths: .571428… . (R. Feynman, 1985, p. 194.)

Mathematical proof

Mathematical proof involves cognitive units and connections of a more general type than those encountered in elementary mathematics. (Barnard & Tall, 1997). In addition to sequential procedures of calculation or symbol manipulation found in arithmetic and algebra, mathematical proof often requires the synthesis of several distinct cognitive links to derive a new *synthetic* connection. For instance, in the standard proof that $\sqrt{2}$ is irrational, having written $\sqrt{2}=a/b$ as a fraction in its lowest terms, the step from "$\sqrt{2}=a/b$" to "$a^2=2b^2$" is an elementary sequence of algebraic operations, but the step from "a^2 is even" to "a is even" requires a subtle synthesis of other cognitive units, such as "a^2 is either even or odd" and "if a were odd, then a^2 would be odd." These synthetic links constitute an essential difference between the elementary procedures of arithmetic or algebra and the more sophisticated linkages involved in mathematical proof.

Students often say that they can follow proofs when the lecturer goes through them in class, but they are unable to construct proofs for themselves when required to do so for homework. One explanation of this phenomenon (Barnard, 2000) has to do with the shifting of focus through the different layers of detail in the cognitive units to be manipulated: statements, statements within statements, expressions within statements, symbols within expressions, etc. In a lecture, the lecturer may implicitly specify the level of items that are to be the primary objects of thought at any stage. For example, in a proof by induction on n of a statement $P(n)$, the distinction needs to be made as to when $P(n)$ is to be thought of as a compressed item within the statement, "$P(n)$ implies $P(n+1)$", or when it is to be unpacked for a finer grained manipulation.

> It is this focus shift of compression and expansion that often lies at the heart of the difficulty when students try to construct proofs for themselves. It is a bit like knowing when and how to change gear while driving. When students ask the seemingly bizarre question, "How do you do proofs?", they may simply be reacting to a predicament similar to that of trying to drive without awareness of the existence of gears.
> (Barnard, 2000.)

The wider challenge in mathematics education is how we can help students to construct appropriately linked cognitive units that are flexible and precise to help them build mathematics as a coherent and meaningful structure. These cognitive units arise naturally in human thinking and take on a wide range of roles – general strategies, specific information, routinized sequences of steps, linked together to produce mathematical thinking. Without cognitive units of appropriate manipulable size, thinking becomes diffuse and imprecise and is far less likely to be successful. Even with the development of manipulable cognitive units in individuals that give current success, there will still be challenges requiring intelligent reconstruction to cope with novel situations.

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INVESTIGATING MATHEMATICAL INTERACTION IN A MULTILINGUAL PRIMARY SCHOOL: FINDING A WAY OF WORKING

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How is it possible to investigate student interaction in a multilingual mathematics classroom? As researcher, I have little access to students' language, culture or experience, making it difficult to make claims about their meanings or interpretations as they work together on mathematics tasks. This paper discusses the development of a methodology based on the discursive psychology of Edwards (1997), which makes possible an investigation of the participation of English Additional Language (EAL) learners in mathematical interaction. The aim is to find a way of working which avoids making assumptions about what students mean by what they say. Analysis proceeds from looking at what participants attend to, to examine how they think together.

Introduction.

Although there has been considerable interest in the role of language in mathematics education, there is much less work which considers the teaching and learning of students who are still in the process of acquiring the classroom language. This paper concerns students in a UK primary mathematics classroom for whom English is an Additional Language (EAL) [1]. How do EAL students participate in the complex environment of the mathematics classroom? And how can this be investigated?

The literature contains much that relates to these questions, though little which addresses them directly. Several quantitative studies have used written instruments as a basis for comparisons between different language groups (e.g. Dawe, 1983) or to compare language proficiency with mathematical attainment (e.g. Cocking and Mestre, 1988; Clarkson, 1992). These outcome-based studies give little insight into learning *processes*. There has been a modest amount of work which examines interaction in multilingual mathematics or science classrooms (e.g. Warren and Roseberry, 1995; Moschkovich, 1996; Gibbons, 1998; Setati, 1999) although this work does not generally problematise the difficulties of making interpretations as an analyst in such culturally complex environments. More generally, there is a wider body of work which examines mathematics classroom interaction from different perspectives, examining such notions as mathematical communication (e.g. Steinbring, 2000), argumentation and the role of narrative (e.g. Krummheuer, 2000). As this work is typically based in monolingual classrooms, issues of cultural heterogeneity tend to be overlooked.

Complexity.

One of the aims of the research reported in this paper is to develop a methodology which does take account of students' cultural, linguistic and social histories in analysing data. On a substantive level, I am seeking to explore the nature of EAL students' participation in mathematical interaction as they engage with tasks from their mathematics lessons. I am seeking evidence of what EAL learners *can* do in mathematical interaction, rather than conceiving of their language status as a barrier to be overcome. In this paper, however, the focus is primarily methodological.

In considering students' classroom talk, I start from the position that I cannot have direct access to students' personal, subjective meanings, since these meanings are related to students' "individual histories" (Bruner, 1996: 14), their unique social and cultural experiences. The individual experience of each student is a crucial part of how they make sense of the world around them, including the world of the mathematics classroom, since this experience entails the participation in and enculturation into patterns of language and behaviour, which in turn allow the interpretation and production of situated meaning (Bruner, 1990: 19). Given that EAL students in the UK come from a wide range of cultural, linguistic and social backgrounds and have often experienced life in several parts of the world, sometimes in difficult or traumatic circumstances, it is clear that it is unrealistic to assume that a researcher is in a position to understand students' interpretations.

The discursive psychology of Derek Edwards (1997) (also Edwards and Potter, 1992), which draws on conversation analysis and ethnomethodology, offers both a theorisation of language in use and a methodological approach to analysis with the potential to address the problem set out above. Language is conceptualised as primarily "a medium of *social action* rather than a code for representing thoughts and ideas" (Edwards, 1997: 84, original emphasis). Social action is foregrounded as the primary function of language, which is seen as having evolved through social interaction, and therefore as being structured both by and for social interaction. From the plurality of possible forms and modes of expression at any given moment of interaction, only one utterance can emerge. The path taken through this plurality of expression is determined by the social action and interaction of the participants. So for example, an utterance designed to persuade will take a different form from an utterance designed to attack, even if the 'content' is the same. The patterns of language through which these different actions take their form derive from each individual's experience of social interaction, their cultural and linguistic history. Thus, rather than attempting to analyse what students mean, discursive psychology seeks to examine how meaning is constructed and situated in discourse. Analysis of classroom discourse asks "not *what* do children think but *how* do children think" (Edwards, 1993: 216, original emphasis).

Discursive psychology also offers an approach to discourse analysis which emerges from the theorisation of interaction presented above. The process of analysis is based on the principle that language-in-use makes explicit that which participants are concerned with, and as a result, makes their interpretations available for analysts (Sacks, Schegloff and Jefferson, 1974: 728-729). Developing these ideas, Edwards and Potter (1992) outline five distinctive aspects of the discourse analysis of discursive psychology:

1. Analysis is of naturally occurring talk and prepared texts.
2. Analysis is concerned with the content of talk and its social organisation. This includes seeing talk as sequential and analysing utterances within the sequential context in which they occur.
3. Analysis is concerned with action, construction and variability. Different ways of talking are used in different circumstances and for different rhetorical purposes.
4. The rhetorical organisation of talk and thought is designed to counter potential alternative versions which may arise. The form of an utterance is determined by the action it is designed to perform, including the prefiguring of potential future courses of interaction.
5. It is the consideration of 'cognitive' issues such as intention or meaning in terms of how they are dealt with in discourse that leads to this approach being characterised as 'psychological'. The focus is on looking at how participants construct and rhetorically deploy psychological concepts in interaction. This is not to deny that people have intentions or meanings, but to argue (Edwards 1999: 272) that we can only examine how such notions are interactionally employed in different ways to suit different occasions and thereby accomplish different social actions.

In this paper I offer an example of this approach to analysis of interaction. First however, it is necessary to provide some context.

Research context.

I have been visiting the Year 5 (aged 9-10) mathematics lessons in a multilcultural urban UK primary school of approximately 150 students from a variety of cultural and linguistic backgrounds. In Year 5 (1999-2000) there were six students recognised as EAL. Initially I had hoped to record EAL students as they worked in order to obtain records of naturalistic interaction. I particularly wanted recordings of student-student interaction, rather than of the more heavily cued exchanges between students and the teacher. As classroom-based recording proved impractical the approach was modified: small groups of students were withdrawn from the classroom and recorded while they worked on a task together. Although not identical to classroom situations, the teacher frequently asks students to work together in this way. Furthermore, the task and the combinations of students selected were based on classroom observations. Thus although the interaction was not completely natural, neither is it particularly artificial.

The research design at this pilot stage was based around a topic about calculators. A task, that of writing addition word problems about money, was selected from the teacher's plan for the week. Six pairs or threes of students were recorded both before and after the calculator topic as they worked on the task. The primary data consist of fully transcribed audio recordings of the interaction. The analysis offered in this paper is of the first pair of students recorded before the lesson sequence. 'Cynthia' comes from a Cantonese speaking background and arrived in the UK about 18 months ago from Hong Kong, since when she has learnt virtually all her English. Her

English in Year 5 was assessed by the school as 'stage 2' on a 4 stage scale, where 'stage 1' indicates almost no proficiency in English. 'Helena' is an English speaking African-Caribbean student.

Interaction: extract 1.

Cynthia and Helena are working together on their first word problem. As with many of the recordings in the pilot study, Cynthia and Helena start by choosing a name for a character in the problem and then proceed with much negotiation to construct their problem. The result (see [2]) consists of twenty words and reveals little of their deliberations, as recorded in more than 70 lines of transcript. As noted above, it is possible to examine 'participants' concerns' (Sacks *et al*, 1974; Edwards, 1997) in interaction, as these concerns are made available for other participants, and therefore are also available for analysts. In this paper, I will show how an examination of *what* the two students attend to as they work together leads to insights into *how* they think together. Lack of space makes it impossible to reproduce a full analysis, including the fully transcribed sequence. I shall therefore use sections of the transcript to illustrate the approach (see [3] for transcription conventions; line numbers are indicated in the text in parentheses).

In the transcript discussed in this paper, Cynthia and Helena display two broad areas of attention. One concern is with the genre (or typical form; see Gerofsky, 1996) of word problems; the way they are constructed and the nature of the language they contain. They also attend to the mathematics of the problem which in this case is closely bound up with the requirements of the task. These foci are evident in the short extract below, which comes from near the beginning of the sequence:

Extract 1a.

56	H	Daniel um	*writes*
57	C	Daniel um	
58	H	went to the shop	
59	C	n-no can/ umm/ um write that/ Daniel **work**/ n-no/ Daniel/ w=um/	
60		Daniel/ well if he work/ (...) he have/ he have/ hundred pound/ and how	
61		many/ in/ the month/ (for example) like easy one	

As the students begin their discussion, they first attend to the general form, or genre (Gerofsky, 1996) of word problems, selecting a character (56-57) and a scenario for the problem (58-61). Cynthia offers a partial version of a problem (59-61). She then offers an evaluation of her problem, switching attention to the 'easiness' of her problem "like easy one" (61), so focusing on the mathematics and the task they have been set as she interprets it. It could be argued at this point that Cynthia's problem is deficient; she has not included enough information. Does this mean she does not 'understand' what a word problem should include? Or is it that she has a suitable problem 'in her head' but is unable to articulate it? These are not questions that can be answered from the perspective of discursive psychology. Instead, we should continue to follow the sequence of the transcript, examining what it is that the participants themselves attend to. In this way it is possible to examine *how* thinking is *jointly* accomplished by Cynthia and Helena.

Extract 1b.

62 H but you've got to use it in addin'/ addin'/ addition/
63 C oh yeah
64 H so you say Daniel/ yeah it's kind of like a addition thing isn't it/
65 because/ Daniel went to work/ he had hundred pound/ a month?
66 C um/ a week
67 H oh that's ^(okay then)^ a hundred pounds a week/ how many/ how
68 many um/ how ma=how much money do he have in a month
69 C yep
70 H (okay then)

Helena's initial response to Cynthia's offer (62) attends to the mathematical nature of what she has heard Cynthia suggest (which is not necessarily the same as what Cynthia thinks she has suggested), which is related to the task as she understands it. She states what is ostensibly part of the instructions "you've got to use it in addin'", so evaluating Cynthia's suggestion as potentially not 'addin' and therefore not compliant with the task. By using the form of words "you've got to" she carefully manages her implied criticism as being based on an external criterion and therefore as being less personal. Although apparently citing a 'rule', Helena's evaluation is situated and constructed in the context of the ongoing interaction, designed as a part of the students' social (inter)action.

Helena continues to focus on both genre and mathematics, but goes on to develop her interpretation of Cynthia's problem by reconstructing it as a potential addition problem (64-65). In doing so she identifies something that requires clarification: "a month?" (65); Cynthia has not stated how often Daniel gets his hundred pounds. Here Helena attends to the mathematics of the problem, as well as to the generic form of word problems, the two intersecting in her question, "a month?". In responding to Helena, Cynthia supplies an extra piece of information "a week" (66), thus taking account of the recent attention to the problem genre and the scenario-under-construction, as well as the mathematics of the emerging problem and the task in hand. Cynthia's response is not seen as an act of recall, providing a missing piece of information that was omitted in her original version but which was somehow 'in her head' all along. Instead, it is viewed as a situated response specifically designed for the moment of interaction in which it was used. This shows a highly sophisticated awareness of what is going on in the discussion with Helena.

This extract concludes with Helena reformulating the question that completes Cynthia's original problem, taking account of the preceding exchange (67-68). It is not possible to say here if this is what Cynthia originally intended, but that is to miss the point. The interaction shows Helena and Cynthia thinking *together* [4] as they generate their problem. It is not possible to separate the contributions of the two students. Although words and ideas can be superficially attributed to either Cynthia or Helena, this is merely the result of the constraints of organised talk. They must take turns and speak one-at-a-time if they are to work (and think) together. Each contribution, however, must be seen as contingent on what has gone before, guided by the students' joint orientation to completing the task. Thus when Cynthia decides on "a week" (66), it is simplistic to claim that this is all Cynthia's idea, since it comes as a response to the previous discussion between both students, as well as an awareness of the potential future course of interaction.

Interaction: extract 2.

The above analysis shows how an initial focus on 'participants' concerns' (Sacks*et al*, 1974; Edwards, 1997) makes it possible to examine how thinking together is accomplished. Analysis of the rest of the transcript reveals similar insights. Extract 2 is taken from the last few lines of the transcript of the students' work on their first problem, which has evolved a little from the version apparently agreed on in extract 1. Prior to this extract, Cynthia and Helena have agreed on and Helena has written down: "Daniel has a job he gets pay £415 in a month". They are now negotiating the question which will conclude the problem.

Extract 2.

100 C how many in a week/ no oh yeah/ how many in a week
101 H (...) okay then/ how many/ how many/ how much money does he get/ in
102 a **year**/
103 C in a week
104 H a week?
105 C no that's (...)
106 H no cause/ you said in a **month**/
107 C yeah/ no/ I said/ [no/ I said/ Daniel has a job he **gets paid** four &
108 H [how many
109 C & hundred and fifteen pound in a **month**/ how many in a **week**
110 H how much he gets
111 C yeah/ how-how much he get/ on one week
112 H that's **dividing** innit
113 C oh yes that's divide/
114 H that's sort of like dividing cause there's four/ four weeks in a month so
115 that's **four** divided by (three) I mean four hundred and fifteen
116 C I'll just do/ how many in a year// [(*inaud.*)

Cynthia suggests "how many in a week" (100). Helena responds with a reformulation, emphasising the key point of difference, " a year" (102), and thus revealing the two students' current concern. The detail seems important, since the choice between 'week' and 'year' affects the kind of problem they produce – either partitive or additive. Cynthia continues to attend to this choice, saying "in a week" (103). There is a sense at this point of both students being focused on this one detail, and so on the mathematics of the problem. The discussion moves forward through Helena and Cynthia trading "contrasting versions" (Edwards and Potter, 1992: 3) of what Cynthia 'said' (106-109), with the focus remaining with the choice between ‘week’ and ‘year’. In Helena’s version of what Cynthia ‘said’, she draws attention to a detail from the agreed on problem-so-far (now written down). By setting her version up as something Cynthia ‘said’, Helena explicitly links the choice between ‘week’ and ‘year’ with what has gone before. She is now attending to the mathematical structure of the emerging problem by looking at the relationship between ‘month’ and ‘year’.

Cynthia responds by also constructing a version of what she ‘said’ - another reformulation of the problem. In fact elements of her restatement were originally said by Helena (“gets paid” (107)), but again, the point is not to check Cynthia’s claim about what she said with what she actually said, but rather it is to look at what is accomplished by her claim, which has been constructed to suit the particular circumstances in which it was made. In this latest version, Cynthia makes a clear choice for ‘week’ in the context of Daniel’s “four hundred and fifteen pound in a month” (107, 109). Helena is then able to identify Cynthia’s version as “dividing” (112) and therefore, as in extract 1, implicitly not compliant with the task. It is noticeable that Cynthia now accepts this point quite easily and along with Helena’s earlier suggestion of “how many in a year” (116). It is *not* possible to say, however, that this is due to her mathematical understanding of the argument, or because of the persuasive nature of Helena’s rhetoric, i.e. because Helena is convincing, or because she just wishes to get the problem finished, or for any other reason.

Conclusions.

Analysis of what the students attend to as their discussion unfolds reveals two foci. Firstly, they attend to the form of the problem, both in terms of the generic ‘contents’ - such as a character and a situation - and in terms of the kind of language used (see 67-68), a feature more apparent in later sections of the transcript. Secondly, there is a focus on the mathematical nature of the problem as it emerges, a focus which is closely related to the task the students were set. These patterns, which are evident throughout the transcript, are an important feature of the interaction, as they enable Cynthia to think together with Helena. It is important to note that Cynthia *does* participate successfully in this discussion, a remarkable performance considering she has been learning English for less than 18 months. One way in which Cynthia and Helena are able to accomplish this thinking together is to use language to establish joint foci of attention. Cynthia is able to do this even at her relatively early stage of English language development.

Methodologically, the above illustration of the nature of analysis demonstrates the efficacy of the discursive psychology approach to discourse analysis. The broad interest in what the participants do rather than what they mean, makes it possible to examine interaction without needing to make assumptions about meanings or intentions. Thus in the first short extract discussed above, there was no speculation regarding Cynthia’s initial attempt to formulate a problem (59-61), which would clearly be difficult to make sense of. Instead the focus was on how her problem was constructed within the flow of interaction and how it was used in subsequent discussion. The arguments and exemplification of analysis set out in this paper therefore demonstrate the possibility of analysing interaction in multilingual, multicultural settings without needing access to the languages or cultures of the participants.

NOTES.

1. English additional language (EAL) refers to any learner in an English medium environment for whom English is not the first language and for whom English is not developed to native speaker level. Native English speakers are described simply as monolingual.
2. The final written problem, typed but unedited except for the name, was: "Daniel has a job he gets pay £415 in a month. How much money does he get in a year?"
3. Bold indicates emphasis. / is a pause < 2 secs. // is a pause > 2 secs. (...) indicates untranscribable. ? is for question intonation. ( ) for where transcription is uncertain. [for concurrent speech. & for utterances which continue on a later line. ^ ^ encloses whispered speech.
4. Although the expression 'think together' has been used by Mercer (2000), my analysis does not draw directly on Mercer's work.

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MANIPULATING VIRTUAL MATERIALS AND USING "OFFICE" SOFTWARE TO DEVELOP PRIMARY MATHEMATICS CONCEPTS

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This paper describes a mathematics teaching approach which uses PowerPoint to replicate traditional non-computer teaching by manipulating virtual copies of real materials (Baturo, 2000). It reports on three case studies of primary teachers who admitted to having computer technophobia as they attempted to integrate learning technology in their classrooms even though they had undergone an extensive school-based computer skilling program. The results show that the generic software approach is a powerful way to encourage teachers to use computers in mathematics teaching, manipulating virtual mathematics materials facilitates learning, and teacher mathematics pedagogy knowledge is the determining factor in enhancing learning through computers. It is much easier to provide computer expertise than mathematics pedagogy knowledge.

Technological change is expected to transform teaching and learning. For example, the Department of Education Queensland (1995) has argued that computer technology will change the nature of student learning, the roles of both teachers and students, and "support and enhance the achievement of educational goals across the P-12 curriculum" (p.3). However, many teachers who have not grown up with computer technology have developed high levels of stress (technophobia) when faced with a teaching future that appears to be inexorably leading to the integration of learning technologies (e.g., Morton, 1996). As Eraut (1994) conceded, "using an idea in one context does not enable it to be used in another context without considerable further learning taking place" (p. 33).

According to Reilly (1997), successful teaching with computers tended to focus on knowledge-construction activities that actively engaged students in solving problems both as individuals and as members of a team. These types of activities tended to change, quite significantly, students' conceptions about the nature and discourse of the subject-matter being studied (Clements, 1994) with accompanying qualitative changes to students' mental models of the phenomena being studied (Woodruff & Meyer, 1997). As McRobbie, Nason, Jamieson-Proctor, Norton and Cooper (2000) argued, understanding of mathematics in computer related activities is dependent on the following: (1) *degree of difficulty* – computer activities have to be carefully chosen so that the mathematics being taught is within the students' zone of proximal development (Vygotsky, 1978) or the students will be unable to make the leap to the new knowledge; (2) *links to non-computer activity* – activities should integrate on and off computer activities (Kaput & Rochelle, 1997); (3) *scaffolding* – the mathematics behind an activity needs to be fully understood by teachers so that they can provide the necessary scaffolding to assist students' construction of knowledge from the computer activities (Bagley & Hunter, 1992); and (4) *reflection* - opportunities should be provided after computer activities for students to discuss and reflect on what they have done and learnt (Davis & Rimm, 1998).

In Australia, current pedagogy believes that mathematics understanding is best constructed by each child through a combination of: (1) work with materials (concrete then pictorial); and (2) discussion and reflection with peers and teacher (e.g., Booker et al., 1999). Most activity with real or concrete materials in number and space involves *sliding, joining, separating, grouping, ungrouping, partitioning, turning* and *flipping* actions. All of these actions are available on computer through mouse movements and images of the materials ("virtual materials") using the commonly available generic "office" software (e.g., *MicroSoft* Office, *ClarisWorks*) (Baturo, 2000). Real materials are multisensory (i.e., they can be seen, smelt, moved, picked up, touched, weighed) whereas virtual materials are bisensory (seen and moved) so virtual materials are more abstract than real materials. Therefore, real materials may develop a more detailed memory structure (schema) than virtual materials. However, on the other hand, mathematising is about refinement and abstraction so that the multisensory nature of real materials may actually hinder the abstraction process as the child may not know which are the salient features to focus on.

Some actions are neither as overt as they are with concrete representations nor as covert as they are with pictorial representations. For example, with respect to *numeration processes*, grouping virtual base-10 materials will require the child to activate a "selection" tool, hold down the left mouse key as s/he "draws" a box around the objects to be grouped, go to the Draw menu on the Drawing toolbar, and then select "Group" from the menu. Thus, there is indirect physical manipulation through the mouse but the regrouping process will require much more dexterity than the direct physical manipulation. Furthermore, the grouping process has to be known but held in memory as the child performs the sequence of operations that will make the transformation from ones to tens. Similarly, but slightly less difficult, actions are required for the ungrouping process, namely, select the object to be ungrouped by clicking n it, going to the Draw menu, and selecting "Ungroup" from the menu. Figure 1 shows that, from this analysis, virtual materials should provide a conceptual bridge from concrete to pictorial representations.

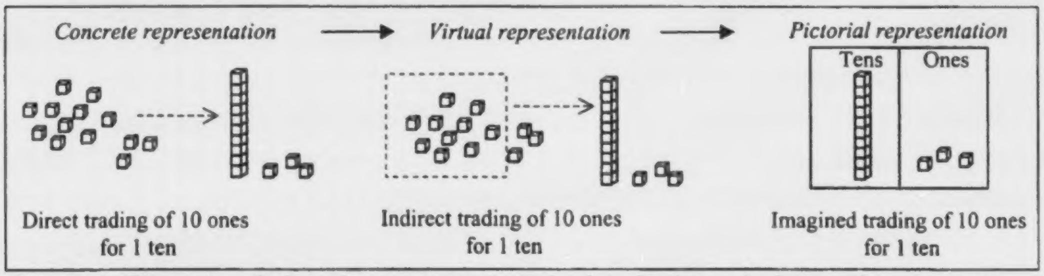
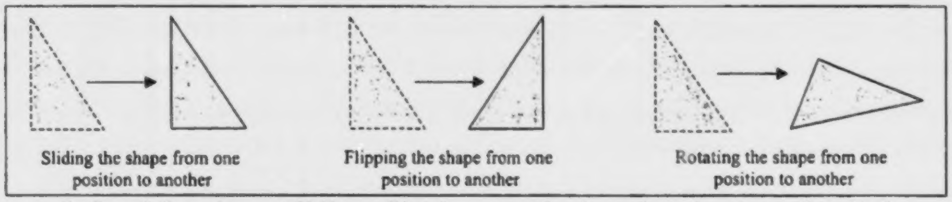

Concrete representation
Virtual representation
Pictorial representation
Tens
Ones
Direct trading of 10 ones for 1 ten
Indirect trading of 10 ones for 1 ten
Imagined trading of 10 ones for 1 ten

Figure 1. The role of virtual representations in developing whole-number concepts and processes.

Some actions are indirect. For example, for *spatial processes* (see Figure 2), the sliding actions requires the child to select the shape by placing the mouse on the shape and clicking, then to s/he simply slide the shape to a new position. For flipping actions, the child selects the shape by clicking, activates the Draw menu, selects Rotate or Flip, and then selects Flip Horizontal or Flip Vertical. For rotating actions, the child selects the shape, activates the Draw menu, selects Rotate or Flip and then selects any of the Rotate options (Free Rotate, Rotate Left, Rotate Right). Tessellations and tangrams (which require sliding, flipping, rotating actions) are spatial activities that are enjoyed by all age groups. However, assembling a class set of real materials is time-consuming. Virtual materials require only one template which can be downloaded for individual student's use. The students themselves can then quickly copy the shapes required and, with respect to tessellations, have access to a variety of colours to enhance the final product.


Sliding the shape from one position to another

Flipping the shape from one position to another

Rotating the shape from one position to another

Figure 2. Spatial actions (sliding, flipping, turning) undertaken on virtual shapes.

The project

The study used a combination of action research (Kemmis & McTaggart, 1988) and teaching experiment (Romberg, 1992) approaches in which the mentors worked with three volunteer teachers (Monica, Andrea, and Janice) in the development, implementation, and evaluation of a sequence of mathematics lessons taught with computers. Data gathered was predominantly qualitative.

Subjects. *Monica* – a Year 5 teacher with 20 years teaching experience; *Andrea* – a Year 4 teacher with 5 years experience; *Janice* – a Year 2 teacher with 12 years experience. The three teachers had been trained to use computers to support their teaching (e.g., preparing worksheets, publishing a newsletter for parents) but not as part of their teaching. All three teachers said they were severely technophobic before this training. At the start of the project, they rated their confidence in computer skills at about 4 on a 5-point scale but their confidence in teaching with computers between 0 and 1.

Monica wanted to use computers in mathematics, have the children and herself develop computer skills and have children use computer skills to learn mathematics; *Andrea* wanted to further her knowledge and use/application of technology in the classroom; and become more confident when teaching use of technology in the classroom; while *Janice* wanted to access information/ideas on how to transfer my "new" computer skills into learning situations for my class, isolate and define particular computer skills that can be taught and assessed in whole class to small group activities, and "have a go" and continue to learn about mathematics and computing.

Mentors. The three mentors consisted of the research team and *Greg*. Greg was a Year 4 teacher with expertise in the use of computers in the classroom who acted as liaison between the research team and the 3 teachers, providing "just in time" technical support when needed. The school's administration gave Greg half-time release from teaching duties for the duration of the project (6 weeks).

Procedure. There were four main stages built into the project, namely, *skilling, planning, implementing*, and *evaluating*. For the *skilling* stage, two inservice sessions (each 1-day) were undertaken with the three teachers to introduce the MS Office program, PowerPoint. The teachers' originally associated PowerPoint with high-quality presentations and high levels of computer expertise so were skeptical about their ability to acquire the skills that would be needed. For the *planning* stage, the teachers prepared a mathematics unit of four sequential lessons. The project team (mentors and teachers) then met for half a day to refine the plan, to discuss classroom management techniques with respect to computers, and to begin the construction of the computer activity in PowerPoint. Activity construction was to be done predominantly by the teacher but mentoring by the researchers was available when needed. For the *implementing* stage, the teachers conducted the computer activities in their own classroom or in the school's Year 7 mini laboratory of 8 computers. Each lesson was video-taped and all mentors were available for help if needed during implementation. For the *evaluation* stage, the research team met with the teachers and Greg to evaluate the activities in terms of effectiveness in promoting learning, and in terms of personal professional development. The three teachers were then asked to complete a questionnaire whilst Greg was asked to write a report on his role in the project in terms of the type and amount of "just in time" support, and the quality of the mentoring component. The results of the meeting and questionnaire are reported in this paper.

Cases

Monica. For her Year 5 class, *Monica* planned a sequence of lessons using tangram activities to introduce flips, slides, and turns (transformations). To this end, she developed a series of PowerPoint tasks in which tangram pieces were combined to form shapes and prepared a program that integrated on and off computer activities. The students were introduced to the virtual materials in the first week as a whole class with the use of a data projector. The students then worked on paper tangram activities and, when these were completed, went on to virtual activities (replicates of the paper activities). They were rotated through the computer activities, 6 students at a time. The classroom had three computers that were kept in a small room at the back of the class.

The students enjoyed the challenge of the paper tangram activities and were motivated to continue by the promise of computer time. With respect to the computer tangram activities, the students were highly motivated by the colourful pieces and the clear, succinct directions were easy to follow. They worked collaboratively on the computers and were soon personalising the pieces by using their own colours and exploring "what would happen if …". During the computer session, where three pairs of students were working collaboratively on the three computers, the following conversation was overheard: *These are easier than the paper ones to see where they go* (John). *I reckon the paper ones are easier 'cos you can pick them up* (Allison). Both points of view were supported by other students nearby.

Monica herself was also motivated and encouraged by her ability to develop the virtual materials that the students used. She successfully introduced the necessary understandings of PowerPoint and the school's network structure to enable the students to retrieve, manipulate (flip, slide and turn) and save the tangram activities. However, some students experienced difficulty solving the tangram puzzles and the tangram activities were not explicitly connected to transformations.

The tangram activities were not sequenced; they did not move from simple to complex, increasing the number of tangram pieces and providing increasingly less detailed templates. They did not differentiate between puzzles that involve flips (more difficult) and those that did not (less difficult). The on and off computer activities were also at different levels of difficulty. The paper tangram activities had solutions provided on their templates, the computer activities did not. Finally, there was no explicit teaching of the role of flips, slides and turns in the formation of the puzzles. The result of this was that a section of the class needed support to solve any problems and the achievement of the students was less than expected.

Andrea. For her Year 4 class, *Andrea* developed a unit of work on polygons. She had taught the mathematical properties of polygons earlier and the computer activities were viewed as a means of assessing the extent to which the students understood the concepts. The first activity provided a range of 1-D, 2-D, and 3-D shapes (see Figure 3) and the students were required to sort them into Polygons/Not Polygons. Later activities involved students constructing their own polygons. The students worked in pairs; half the class at a time, on 8 computers in a small laboratory while the teacher with whom Andrea shared a double teaching space supervised the other half of the class. .

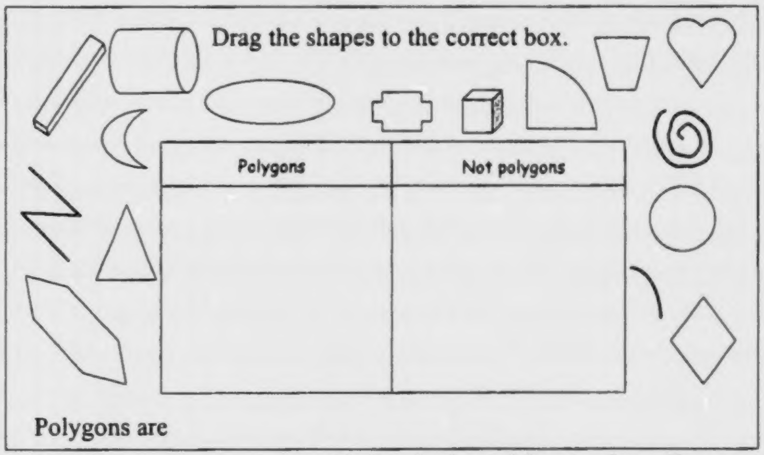


Figure 3. A polygon assessment activity designed by Andrea (Lesson 1).

Andrea tried to direct the activities so that all students listened to her and then did part of the activity. She was very structured (scaffolded) in the way she implemented the lessons because, unlike Monica, she was teaching the mathematics concepts and requisite technology skills simultaneously. These skills consisted of knowing how to retrieve and save files, and how to use "click and drag", text boxes, and the Draw, AutoShape and colour features of PowerPoint. She also was at pains to ensure both students in each partnership had time on the computer and that there was a period at the end of each activity where students reflected on what they had learnt. They were required to type these reflections into text boxes, an activity that revealed that the students had acquired the appropriate knowledge and language.

Although Andrea's initial lessons did not allow time for students to explore the enhancing features of PowerPoint (e.g., colouring lines and shapes; playing with different fonts), students nevertheless did so. One student showed his group how to change the colours of the lines and the enclosed space and before long all students were exploring the Format feature. This activity highlighted the motivational power of the computer, the collaborative nature of student learning when engaged on a task, and the need to let students explore first and learn later.

Andrea realised that this aspect of her lessons needed attention in her review. She stated that there was not a lot she would change except have smaller groups and more investigation in the mini-laboratory situation ("I used a similar teaching style as in the classroom - very structured and controlled - perhaps give children more experimentation time at the end of lessons").

Janice. In her Year 2 class, *Janice* developed materials to introduce and reinforce two-digit numeration. She used the data projector to introduce simple "click and drag" PowerPoint skills and then rotated the children through activities on three computers where they moved virtual base-10 blocks to form two-digit numbers or numbers and words to label pictures of base-10 blocks. She also taught the children to open folders and save the results of the manipulations. Although there were some difficulties with the computer hardware, the students were so highly motivated that they kept coming back to the activities during their free time (lunch and before school).

Janice had been fearful that her Year 2 students would not be able to save, retrieve and "click and drag". This proved unfounded; the children quickly acquired computer skills. Where Janice needed support from the researchers was in developing activities that appropriately sequenced numeration development and which ensured all connections between materials, language and symbols were made.

Reviewing the study

In the review of the study, three findings became evident. First, for technophobic teachers, the replication of traditional mathematics activities via PowerPoint provided a bridge from the acquisition of computer skills to the implementation of classroom activities.

Reflecting on her achievements, *Monica* stated, "Well, I'm now a PowerPoint junkie". She said she was confident in using her computer skill with the class and that she hoped that her children realised the value of computers in mathematics. She proudly said, "They liked my PowerPoint creations!" *Andrea* also indicated that she had become a lot more confident overall and had tackled projects and basic teaching activities that she would not have previously. She stated, "I feel more able to tackle using technology to integrate other subject areas". *Janice* simply cried, "MORE!!!" She described how she used the multimedia projector and prepared tasks involving specific computer skills ("click and drag, copy and paste") in PowerPoint. As she said, I developed an understanding of "where I've come from" to "where I want to go". At the end of the study, *Monica* was very positive, "I am confident to use PowerPoint in my preparation and implementations of my program"; *Andrea* described how she was very stressed at first but gained confidence as the lessons went better than expected; while *Janice* described how she was initially concerned, learnt to have confidence in her own planning, and was "most impressed with the eagerness of the children to access computers and work folders in their own time and to complete mathematics and computer tasks".

Second, where there were no mathematics-education difficulties associated with sequencing or activity type, the virtual materials provided a powerful medium for mathematics learning. Both the weaknesses and strengths of the cases reinforced the importance of ensuring activities: (1) were within students' *zone of proximal development* (Vygotsky, 1978) – the jump to complete set tangram puzzles was too difficult for some students in Monica's class; (2) integrated *on and off computer* tasks (Kaput & Rochelle, 1997) – the links between real and virtual base 10 blocks and between the paper and virtual tangrams were a source of understanding for students in Monica's and Janice's students; (3) were appropriately *scaffolded* and *reflected upon* (Bagley & Hunter, 1992; Davis & Rimm, 1998) – this strength of Andrea's activities appeared to be the reason for her success in the concept of polygon.

Third, teachers' mathematics pedagogic knowledge remained a major determining factor in enhancing learning when computers are integrated into the mathematics classroom. The study showed that learning is maximised when instruction takes account of: (1) *sequencing* and *connections* – this was a particular problem for Monica and the tangram activities; (2) *interpretation* and *construction* – Janice needed support to ensure her virtual base-10 blocks activities exhibited both these; (3) *sharing* and *recording* findings – this was needed by Monica to go beyond the puzzles to flips, slides and turns; and (4) *creative extension* – Andrea found she had to add this to her activity sequences. In the study, the teachers believed they lacked expertise in using computers to teach mathematics but not in teaching mathematics itself. Therefore, they were more receptive to advice regarding technology than mathematics instruction. This was particularly evident in the response of Monica when asked what she would do differently if they were to attempt the same project again. She stated that she would learn the features of PowerPoint well beforehand and practise it more ("I would spend more time with giving skill lessons to children, then I'd move slowly through the activities over a longer time").

Becker (1994) claimed that difficulty in accessing suitable software (time spent searching, getting it funded through the school) has contributed to many teachers' reluctance to incorporate computer learning in their mathematics programs. However, Sarama, Clements and Jacobs-Henry (1998) argued that teachers' beliefs about computer learning were of more concern. Their research showed that if teachers believe that mathematics cannot be taught effectively with computers, then they will resist attempts to incorporate them in their classrooms. Thus, there is a need to provide mathematics computer activities that teachers feel are easy to develop, do not require specialist software, and will promote positive learning outcomes. The manipulation of virtual materials described in this paper meets this need. There were difficulties particularly with respect to sequencing of mathematics content but no problems with confidence in using the computers.

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CONFLICT BETWEEN PERCEPTION, COGNITION AND VALIDATION AS YEAR 12 AND UNIVERSITY STUDENTS ANALYSE THE PROBABILITY OF AN EVENT

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Eighteen Year 12 students and 2 cohorts of final-year BEd students (74 students) were shown a "fair" (equiprobable outcomes) spinner with three noncontiguous colours and asked whether each of the three colours had the same chance of "being spun". Half of the Year 12 students either gave unequivocal incorrect responses derived from inappropriate considerations of sector size or number of sectors per colour, or vacillated between correct and incorrect responses and were unable to make a decision (equivocal). These findings were echoed with the university students although their incorrect responses tended to be more unequivocal than equivocal. Validation through trialing (with the university students) did not help as the results did not show exactly $^1/_3$ for each colour and, in fact, were interpreted as supporting an incorrect response.

Hawkins and Kapadia (1984) identified four types of probability, namely: (1) *theoretical* – derived from making assumptions of equal likelihood; (2) *frequentist* – calculated from observed frequencies; (3) *intuitive* – generated from personal belief and perceptions; and (4) *formal* – calculated precisely from the mathematical laws of probability. Of interest to this study are the latter two which relate to intuitive and analytic cognitions (Fischbein & Schnarch, 1997). They defined the intuition cognition as "self evident, directly acceptable, holistic, coercive and extrapolative" (p. 96) which was distinguished from the analytic cognition by "the feeling of obviousness, of intrinsic certainty" (p. 96).

Probability ranges from 0 (impossible event) to 1 (certain event) so possible events are represented numerically by fractions (part of a whole). It is well-documented (Behr, Harel, Post, & Lesh, 1992; Nik Pa, 1989; Payne, Towsley, & Huinker, 1990) that continuous area models are more conducive to facilitating construction of the part/whole notion than discrete set models. Therefore, when developing the part/whole notion of probability, it seems reasonable to begin with spinners (continuous area model in which all possible outcomes are visible) than with coins, dice, marbles, tickets, or playing cards (discrete set models where all possible outcomes often need to be held in memory).

The literature is replete with misconceptions in students' probabilistic thinking (e.g., Fischbein, 1975; Fischbein & Schnarch, 1997; Hawkins & Kapadia, 1984; Jones, Langrall, Thornton, & Mogill, 1999; Kahneman & Tversky, 1972; Piaget & Inhelder, 1975). Fischbein and Schnarch's (1997) study set out to determine whether 7 main known misconceptions (e.g., representativeness, negative and positive recency effects, compound and simple events) diminished, increased or remained stable across the years (Grades 5, 7, 9 & 11). They found that the only stable (and frequent) misconception across the ages was related to compound and simple events. However, the example given refers to the comparison of events in two similar sample spaces. Jones et al. (1999) referred to this type of comparison of events as Level 3 whilst the probability of an event in one sample space was classified as Level 2. They found that even after persistent instruction, misconceptions at Level 2 remained stable for some Year 3 students.

Apart from the subjective and experiential beliefs that students invoke when analysing probability tasks, one of the major problems related to the teaching/learning of probabilistic notions is the difficulty of validating responses because of the large number of trials required. This can be extremely time-consuming and the results are not always persuasive for students who have a deterministic view of mathematics. (See *Discussion* for elaboration of this point.)

This study explores a very elementary probabilistic notion, namely, the probability of an event in a single sample space (Level 2 – Jones et al., 1999) using a spinner (continuous part/whole area model) with Year 12 students who were individually interviewed on the task to determine the extent of their conceptions (including misconceptions). The study was replicated, to some extent, with university students but was extended to include validation of their conceptions.

Study 1 – Year 12 students

Background. Sixteen Year 12 students comprising 8 students from an algebra-based university entrance mathematics (designated as UM) and 8 students from a "social" mathematics course that involved no algebra (designated as SM) were involved in this study. Within each of the mathematics categories, there were 4 males and 4 females with 2 high- and 2 low-achievers in each gender group.

This paper reports on one (see Figure 1) of several elementary probability tasks that were undertaken with the students in semistructured individual interviews conducted out of school in the student's home. The tasks incorporated both continuous area models (spinners) and discrete set (marbles) models.

- *Is this a fair spinner? How can you tell?* [Contingent: *Does each colour have the same chance of being spun?*]
- *If you were playing a game with this spinner, is there a colour that would give you a greater chance of winning? Why?*

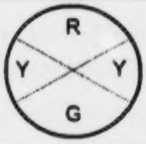

Figure 1. Spinner used to determine the robustness of student's probability notions. ("Fair" is used to denote equiprobable outcomes.)

The spinner used in this study was designed to be provocative, that is, to provoke conflict between intuitive and analytic cognitions. To help students invoke the part-whole fraction notion of probability, a simple area model (familiar to Queensland students) was used but was made more difficult because students were required to *reunitise* (Behr et al., 1992; Baturo & Cooper, 1997, 1998, 2000) either: (1) the red and green sectors as two parts, each of which was equal to the yellow part, thus realising the spinner was actually partitioned into sixths; or (2) reunitise the two yellow sectors as one sector, thus realising the spinner was actually partitioned into thirds. Therefore, although the spinner had only three colours to consider, the *noncontiguous* nature of the colours increased the difficulty level of the task (Jones, 1974). Furthermore, to provoke conflict between intuitive and analytic reasoning, the first question was designed to promote analytic reasoning (a consideration of "fairness"– equal chances) whilst the second question was designed to promote the intuitive reasoning that is often invoked by games and winning.

Thus, the task had three main purposes: (1) To determine the robustness of the students' analytic reasoning in determining the probability of an event; (2) To ascertain whether the students displayed any conflict between visual perception/intuitive cognition (the amounts of colour do not look equal) and analytic cognition (knowing that if the two yellow parts were adjacent, they would cover the same amount of area as each of the other two colours); and (3) To determine whether the student's dominant form of processing was intuitive or analytic.

Each interview was videotaped then transcribed into protocols for analysis in terms of intuitive or analytic cognitions.

Results. The students either responded with a firm conviction regarding the correctness of their response or they vacillated with their answers. To indicate the conviction or the vacillation, the responses for this task were categorised as *unequivocal* (immediately stated and incontrovertible) or *equivocal* (ambivalent, indeterminate) with correct and incorrect subgroups within each category. (See Table 1 for the results.)

Table 1

Task Results in terms of Equivocation and Correctness of Responses

	Form of response	
	Unequivocal	**Equivocal**
Correct	Andrea & Michelle (UM/High) Ben & Camille (UM/Low) Brendan (SM/Low) Sarah (SM/High)	Matthew (UM/High) John (SM/High)
	Cognitive processing dominant	*Visual perception strong but not dominant*
Incorrect	Karoline (UM/Low) Malcolm (UM/Low) Nicholas (SM/High) Marney & Joe (SM/Low)	Eddy (UM/High) Jane (SM/High) Kerri (SM/Low)
	Visual perception dominant	*Conflict between types of processing*

Only half of the students gave the correct response (including those 2 in the equivocal correct category). Of the two students in the *equivocal correct* category, Matthew's (UM/H) initial response was negative, but his change to a positive response was almost instantaneous, perhaps indicating that, to him, intuitive reasoning is still a powerful factor in processing information but not so powerful that it dominates his analytic cognitive processing. The other student, John (SM/H), took about 10 seconds before responding but his explanation was rather interesting.

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. . . because there's an even amount of each colour; like these two [red and green] *have the same odds, right, but . . . these two* [2 yellow] *have got to vary between each of their coordinates* [indicating the width of each yellow section with the fingers of each hand]*; add these* [the 2 yellow] *both add up and I think they would equal the green and the red.*

Of these 8 students who gave correct responses, there was an equal number of the higher-level UM and lower-level SM students, an equal number of high and low-performing students, as well as an equal number of females and males. Therefore, this study did not find that course, achievement, or gender impacted on the ability to process the probabilistic notion of equally likely outcomes analytically.

The incorrect responses were based on strategies related either to number of like colour sectors or to sector area. For example, of the five students in the *unequivocal incorrect* category, Karoline (UM/L), Marney and Joe (both SM/L) maintained that yellow had more chance of being spun than either red or green because there were two yellow portions and only one red and one green portion. Nicholas (SM/H) and Malcolm (UM/L), however, had no doubts that red or green would have a greater chance of being spun than yellow because each of them had a larger area for the needle to land on than yellow.

The remaining students, those in the *equivocal incorrect* category (Eddy, Jane, and Kerri), fluctuated between intuitive and analytic processing. Ultimately, though, the intuitive cognition was more dominant than the analytic cognition. Their protocols reveal the conflict invoked by the task.

Eddy: *No, maybe because these two colours* [yellow] *are right opposite so these two* [yellow] *would have more chances. But if these two* [yellow] *joined together are the same as these two colours* [red and green] *then it would be a fair spinner but these two* [red and green] *would have more chances.*

Jane: *I think if you put those two* [yellow] *together, they'd probably be the same as the others but I think that the red and the green are probably more dominant. Like your object* [indicating the needle on the spinner] *is more likely to land on the red or the green.*

At this stage, Jane's (SM/H) understanding of fair and unfair as they applied to spinners was investigated. She was first shown a spinner which was half blue and half orange and asked if this was a fair spinner to use in a game. Jane said that it was because you had the same chance of landing on either colour. She was then shown another spinner which had all equal parts (4 blue, 3 red, 1 green) and all colours were contiguous. Jane said that this was a fair spinner, too. When asked if she were playing a game and could only win if she spun green, she said: *Oh, no, it wouldn't be fair then. Oh, do you mean to look at the colours? No, not fair* [referring to the original spinner in Figure 1] *because there's two of them* [yellow].

Kerri's responses were indeterminate on all the tasks and were therefore difficult to probe as the following protocol reveals.

Kerri: Um . . . that is not a 50% fair spinner. It's probably 2 thirds.

I: What colour would you prefer to have in a game?
Kerri: Prefer or most likely?
I: Well, if you say that's not a fair spinner, then one colour must have more or less chance of occurring.
Kerri: Well, it'd be the two yellows because they're smaller and there'd be either the red or the green.
I: Are you saying that red or green would have more chance than yellow or – ?
Kerri: Definitely (interrupting). *Yes.*

Study 2 – BEd students

Background. The task in Study 1 was tendered for discussion in a tutorial/workshop with 2 cohorts of final-year BEd students (39 and 35 students). As for the Year 12 students, responses were either correct, unequivocally incorrect or equivocally incorrect. (The conflict provoked vociferous and robust arguments as each group of students tried to convince the others that their thinking was appropriate.) With respect to the incorrect responses, the university students had the same misconceptions as the Year 12 students. No new misconceptions were proffered.

Validation results. The spinner (with colours) was shown on an overhead transparency and then a transparency copy of the spinner partitioned into sixths but without colour was placed on top (see Figure 2). The students were then asked what the probability was of getting red, green or yellow. For these students, the fact that they could see that each colour had 2 sixths (or 1 third) of the area did not offset their initial intuitive cognitions regarding the fact that one of the colours was split and therefore red or green had a better chance (because of sector size) or yellow had a better chance because there were two parts, albeit smaller parts.

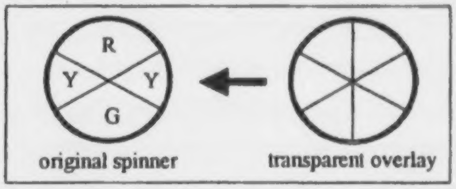

Figure 2. Attempt 1 to "prove" that all outcomes are equiprobable on the given spinner.

The students were then shown the spinners in Figure 3 and asked if any of them were "fair". All agreed that Spinners A and B were fair (A because the colours were contiguous; B because the noncontiguous allocation of the colours was "even") but continued to maintain (or be indecisive) that Spinner C was not.

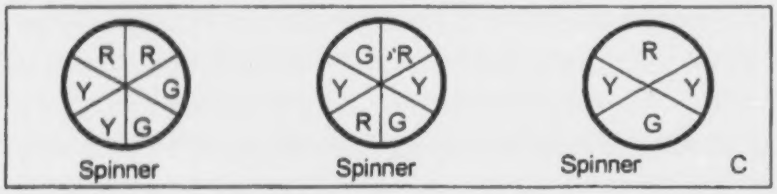

Figure 3. Attempt 2 to "prove" that all outcomes on the original spinner are equiprobable.

Each class decided that, if this occurred in their teaching career, they would ask their students to undertake an experiment with Spinner C. The BEd students were allocated to 10 groups and each group was provided with a model of the spinner. Each student in the group was asked to spin the needle 10 times and to record his or her result for each spin.. Table 2 shows the outcomes of this trial as well as those undertaken by a second cohort.

Table 2
Results of An Experiment Undertaken by Two Cohorts of BEd Students to Validate Predicted Outcomes for Spinner C

	Outcome		
Results	Red	Yellow	Green
1st cohort ($n = 390$)	136	117	137
2nd cohort ($n = 350$)	116	119	115

Note. In this table, n refers to the number of trials in the experiment.

With respect to the first cohort, the students who had thought that the spinner was not fair because red or green would be more likely to be spun than yellow felt vindicated by this result. They believed that the results supported their prediction that the spinner was not fair. With respect to the second cohort, the students who thought that the spinner was not fair because yellow, with its two parts, had more chance than either red or green also felt that the results vindicated their reasoning. The first cohort was unimpressed by the suggestion that there should have been more trials whilst the second cohort was unconvinced that the results were very close to a third of all trials (indicating a deterministic view of mathematics).

Discussion and conclusions

Cognition. As both studies showed, misconceptions with respect to the spinner in Figure 1, which had only three possible outcomes to consider, were evident in a large percentage of students. Half of the Year 12 students and many of the BEd students gave either an incorrect response or vacillated between correct and incorrect responses when shown the spinner. The incorrect responses revealed that students had two main misconceptions which were an artefact of the task, namely: (1) the larger sectors (red and green) had more chance because they were "dominant" (Jane); and (2) the two smaller yellow sectors had more chance because they gave 2 chances whereas the red and green sectors gave 1 chance only. The students who gave unequivocal incorrect responses appeared to be operating from intuitive cognitions based on comparing either the size of the parts or the number of like colours. That is, they were estimating chances using a part-part ratio schema rather than measuring probability with a part-whole fraction schema (Fischbein, 1975).

This result supported Fischbein and Schnarch's (1997) study in which they found that the probability of an event produced stable and frequent misconceptions across age levels. It also extended their findings by showing that these misconceptions continue into adulthood. Furthermore, the task used in this paper was more simple than the one used by

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Fischbein and Schnarch because only one sample space had to be considered, indicating that the problem is deep-seated.

Validation. Validation of reasoning was not successful as the results of the 2nd study showed. Neither encouraging reunitising by overlaying a transparent replica of the spinner showing sixths (see Figure 2) nor a consideration of structurally isomorphic spinners (see Figure 3) was sufficient to persuade students to focus on the fraction schema embodied in the task (i.e., analytic reasoning). The university students all stated unequivocally that Spinner A and Spinner B (see Figure 3) had equally likely outcomes (red green, yellow) but maintained that Spinner C (task spinner) did not. Spinner A had all contiguous parts whilst Spinner B had all noncontiguous parts. However, Spinner C had some contiguous and some noncontiguous parts thereby producing the conflict between equality (2 red, 2 green, 2 yellow), inequality through sector size (red and green both larger than either yellow) and inequality through number of sectors (1 red, 1 green, 2 yellow). These inequalities appear to be linked to the part-part notion of ratio rather than to the part-whole notion of fraction.

Validation through experiment was equally unsuccessful for two reasons: (1) the insufficient number of trials produced skewed results (see Table 2), thus inadvertently supporting a misconception; and (2) the students' deterministic view of mathematics was so entrenched that they were dissatisfied with any result that did not exactly show 1 third of the trials for each colour (as for the 2nd cohort of university students).

Teaching and learning. The major implication for teaching and learning is that probability schemata must be connected explicitly to fraction schemata through language, exemplars, and symbols. In this study, the students who vacillated between correct and incorrect responses clearly were unsure as to whether to trust their perceptual/intuitive processing (comparison/ratio) or their cognitive processing (fraction). Eddy, the top-performing mathematics student in his school was obviously perplexed by his inability to decide. He had a well-developed fraction schema but it seems as though he did not realise the validity of this cognition, possibly because his probability learning experiences did not focus on the connection between probability and fractions. As Study 2 showed, this situation is exacerbated by problems with validation. Neither logical argument nor experimentation may convince students of the errors in their answers.

Probability, possibly more than any other mathematical domain, is plagued by a plethora of informal and formal *language* to denote possible events. A diagram such as that in Figure 4 was found to be useful for the BEd students in this study because it provided an organisational framework for plotting the language "mathematically".

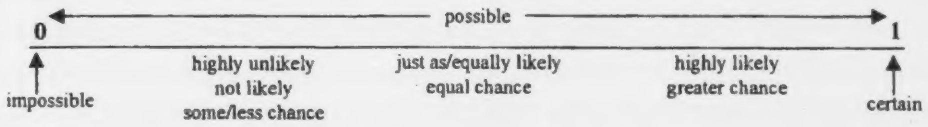

Figure 4. A continuum of formal and informal language ranging in meaning from impossible to certain

Teachers need to be aware of the strengths and weaknesses of common probability *exemplars* and should be guided in their use by sound pedagogical principles rather than by their real-world appeal. For example, the sequence of probability exemplars should follow the sequence of exemplars used to develop the part/whole notion of fractions, that is, continuous area models such as spinners before discrete set models such as marbles. However, spinners can be partitioned in different ways and the outcomes (colours, shapes or numbers) can be arranged either contiguously or noncontiguously. If contiguous parts only are used, students may inadvertently come to rely on the intuitive and inappropriate comparison/ratio schema. Therefore, provocative tasks such as the one in this paper should be incorporated to provoke conflict between intuitive and analytic cognitions to provide insights into the appropriateness of student's thinking.

In the early stages of learning, the common fraction recording facilitates connection to the fraction schemata required for processing probability tasks. Unlike decimals or percents, common fraction *symbols* indicate the total number of outcomes (denominator) and the number of outcomes under consideration (numerator).

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MOVING SYMBOLS AROUND OR DEVELOPING UNDERSTANDING: THE CASE OF ALGEBRAIC EXPRESSIONS

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Abstract

This paper deals with the complex relationship between the meanings and symbols of algebraic expressions. The study reported here uses a theoretical model to interpret some basic dynamics of algebraic thinking in the analysis of students' protocols. The strong difference noticed in protocols by students who have been given very different instructional approaches, suggests further investigation on the sources, mainly in view of educational implications.

Introduction

The complex relationship between the meaning of algebraic expressions and their symbolic representation is a major issue in research. Many authors have pointed out the incapability of relating symbolic expressions to their meaning (see, for example, Linchevsky and Sfard, 1991, Mac Gregor and Stacey, 1996). In the inadequacy of such a relationship are the roots of many misunderstandings, wrong performances and blind manipulations with algebraic symbolism. A consequence of this is that many secondary school students do not master the sense of those symbols which they have learned to handle formally. On the other hand, some students, even if clever "algebraic calculators", seem to be unable to see and use algebra as a means suitable for understanding generalisations, to grasp structural connections, and to argue in mathematics (Boero, 1994).

Our study is originated by noticing strong differences in protocols provided by students: on the one hand "moving symbols around", on the other "good mastery and understanding". This "surface element" has been recognised in need of further investigation, by means of a theoretical model apt to interpret some basic dynamics of algebraic thinking. This application has revealed to be very useful to guess hypotheses about the origin of such great differences in the students' behaviour: as a result we assume that such differences are strongly related to differences in the teaching styles. This report aims at describing the work we have been carrying out, in view of further development as far as school practice is concerned.

The theoretical framework

The relationship between thought and language is a key point in didactic research. Algebraic thinking is generally recognized to be inseparable from the formalized language with which it expresses itself. However, it is "reductionist" to believe that algebraic thinking exists only at this level. In such a case everything would be reduced to a manipulative mechanism, which often does not work in students' hands.

Vygotsky's theory, with reference to verbal language, suggests a functional interaction between thought and language. They are considered as intertwined and mutually dependent aspects of the same process.

In the specific case of algebraic language, a vygostkyan point of view is taken by Radford (1999). According to Radford, instead of seeing signs as the reflecting mirror of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required by the contextual demands in which individuals find themselves located.

Furthermore, signs bear a kind of "embodied intelligence", in that they were historically built for some purpose and, as such, carry patterns of previous reasoning. Signs, tools and other cultural artefacts, do not speak for themselves, but acquire life in the life of individuals through acts of communication and interaction where they become endowed with meaning.

So, *becoming endowed with meaning* is at the very core of our investigation.

In the following, we will adopt the theoretical model by Arzarello, Bazzini and Chiappini (1993, 1994, 1995), aiming at interpreting the very nature of algebraic thinking.

This model has taken its inspiration from the epistemological triangle by Frege, which specifies the distinction between sense and denotation of a given expression.

In particular, an algebraic expression (E) incorporates in its writing the mathematical object involved (the denotation) and the way in which such an object is expressed (the sense). For example the expression $y = x^2-2x-3$ denotes a set of couples which satisfies the given relation. This expression activates, for example, the sense (S) of finding y by starting from x, squaring x, subtracting 2x and finally subtracting 3. A different sense (S') could be that of thinking of x and y as coordinates in a Cartesian plane (hence the graphic of a parabola))

Furthermore, if we transform $y = x^2-2x-3$ into $y = (x-1)^2-4$, this new expression (E') activates a new sense (S''), which points out the coordinates of the vertex of the parabola.

Thus, algebraic transformations are closely related to the activation of senses: doing algebra means interpreting expressions and relating them with senses, coherently with the given denotation.

The theoretical model outlined here provides us the means to approach our investigation of the students' protocols, when they face algebraic formulas.

Aims and methodology of the study

In this report we are mainly interested in the differences that students show when required to face algebraic expressions and to relate them to meaning.

Additionally, we are also concerned about the level of awareness students have when dealing with algebraic language and its relationship to instruction.

For this purpose, we addressed our attention to students coming from different instructional treatments and we tried to relate their typical behaviours to the didactic interventions they had received previously.

Our assumption is that different instructional approaches highly influence the mastering of algebraic expressions, especially as far as the triangle sign-sense-denotation is concerned.

In the following, we report the description of four paradigmatic protocols: the first two show a limited and incorrect reading of algebraic expressions, while the others witness to understanding.

Legend: I for interviewer, S for student.

Student A (**Mario**, 16 y.o., medium scores, High School, humanistically oriented course, Liceo classico, traditional teaching).

I - $-8x-3x^2+11=0$*, what is this?*

S -It is a second grade equation; it could be the equation of a parabola.

I- Which kind of equations do parabolas have?

S- $ax+bx^2+c$.

I - What?

S- Equals 0.

Comment: there are conflicting ideas about the word *equation*. The writing $-8x-3x^2+11=0$ is not connected to its denotation. The equation of a parabola is evoked, due to similarity in writing. When requested to make it explicit, the student is not able to link the denotation (i.e. the couples of numbers) of the parabola to its algebraic representation.

Student B (**Stefania**, 16 y.o., high scores, High School, scientifically oriented course, Liceo Scientifico, traditional teaching).

I writes

$y= x^3 + 6x$

$y= x^3+ 3x^2$

and says;" *Compare these two functions, try to say when a function is "greater than the other" (questo è un modo per dire: quando i punti del grafico di una funzione stanno sopra i punti del grafico dell'altra).*

S.: I would make a system (and she puts {)

{ $y=x(x^2+6)$

$y=x^2(x+3)$

I: Solving a system means finding the common solution of the two equations, I have asked you just to compare, that is saying when, for example, $x^3+6x>x^3+3x^2$.

S: and y, where does it go?, Ah, it is the solution.

I: It doesn't matter

S: So, why is there a y and then it disappears?

I: If you consider the system, you have $y=y$*, thus also* $x^3+6x= x^3 +3x^2$*, do you agree?*

S: Yes

I: So, go on.

S: I should solve the inequality $x^3+6x> x^3 +3x^2$*, so*

$-3x^2+6>0$ *and* $-3x(x-2)>0$ $-3x>0 \rightarrow x<0$ $x-2 >0 \rightarrow x>2$

and this is what she wrote

0 2

------------------------

+ - -

- - +

- + -

$0<x<2$

I: What does that mean?

S; Perhaps, if I substitute a value which is between 0 and 2, the equality is true. But I don't know if it is true, or whatever. That is, maybe the inequality is true.

Comment: Stefania feels lost in front of symbols. This task is not a standard task for her, however, she has all the knowledge needed to solve the problem.

As for student A, this student evokes senses starting from a very superficial reading of the formula. The writing of the functions evokes the sense of solving a system, which is totally out of place here. Also, in following the procedure, no connection between signs, senses and denotations seems to exist.

Student C (**Davide**, 14 y.o., medium scores, Junior Secondary School, teaching strongly oriented to understanding and verbalizing)

I: Take the function $y = x^2+3$*: what does it remind you of? Do you first think of numbers which change or how the graphic might be?*

S: $y = x^2+3$ *I first think of the graphic, which is a raised parabola with the x axis of 3 cm. This helps me to understand how it works. In fact, with positive numbers,* x^2 *increases y of 0, 1, 4, 16... and with the negatives it increases on the opposite side. Then +3 does not allow the values to reach zero* $0^2+3=3$*...*

At the beginning, I think of the formula without doing any calculation, because I know how it works in each part...

I: Namely?

S: $y = x^2+3$ *is like taking two equal numbers, multiplying them and adding 3 to the value: 3 increases the value. Within the negatives, the two numbers which have been multiplied become a positive in any case, because we know that + x+=+ and -x-=+.*

This is the reasoning when facing a formula, after having seen the formula I see the graphic which represents the formula itself.

Finally Davide concludes:

The function is a unique thing. It is important to consider it as a whole, however it is important to understand each part, in order to grasp it globally.

Comment: Davide clearly shows a holistic view of the function $y = x^2+3$. The numbers which change and the related graphic are closely related and Davide is able to pass from one sense to another easily. This student also shows a global approach to the notion of function in general (*the function is a unique thing...*).

Student D (**Federico**, 14 y.o., high scores, Junior Secondary School. Same class and same teacher as Davide)

Federico has the same task as Stefania (student B): He is required to compare these two formulas from an algebraic and graphical point of view, then to guess how the graphic is and finally he has to draw a sketch.

$y= x^3 + 6x$

$y= x^3 + 3x^2$

Here is Federico's solution

Algebraic comparison

- *the first part is the same in both formulas*
- *they are two curves, because the increment from one value to another is greater every time*
- *if x is positive, then y is positive too, because there are no "minus signs" in either formula*
- *if x is negative, then in the first formula (the right side) is negative too, because* $x^3=x\cdot x\cdot x$ *and* $((-x)\cdot(-x)\cdot(-x))$ *and also* $6(-x)=$ *neg. In the second formula y is positive from* -3 *to 0, because* $(-3)^3+3\,(-3)^2= -27+ 3\cdot 9= -27+27 = 0$. *From* -3 *down, i-e.* -4, -5, *etc, y becomes negative because the first piece becomes greater and greater, after 3, than the second one.*
- *the first formula is greater than the second one from 0 to 2 and smaller from 0 down and from 2 up.*
- *If x=0, then y=0 in both formulas, because* $0^3+6\cdot 0=0$ $0^3+3\cdot 0^2=0$

Graph comparison

- *they are two curves, because the increment from one value to another is greater every time*

and he does the following calculation

$y= 2^3+ 6\cdot 2=20$ $y= 3^3+6\cdot 3=27+18=45$

$y= 3^3+ 3\cdot 2^2=20$ $y= 3^3+3\cdot 3^2=27+27=54$

- *they pass through the origin, because there are not + with fixed numbers which are not multiplied by x*
- *the first curve passes through the first and third quadrant, because when x is positive, then y is positive too; when x is negative, y is negative too. The second curve passes through the first, second and third quadrant, because if x is positive, then y is positive. If x is negative from 0 to* -3*, then y is positive because the first term is negative and the second positive, but the second term is greater because it is multiplied by a number greater than x, then it goes to the negatives, because the first term becomes greater than the second, in terms of numbers, not of signs.*

Predicting the curves

- *they are two curves, because the increment from one value to another is greater every time*
- *they pass through the origin*
- *the first curve passes through the first and third quadrant*
- *the second passes through the first, second and third quadrant.*

Finally, he draws the graphs correctly

Comment:

This student has a great mastery of algebraic symbols and he relates them to the graphical representation. Different senses are activated and they work together.

As in the case of Davide, Federico is able to manage the triangle sign-sense-denotation. This allows him to approach and solve the problem easily, notwithstanding the lack of advanced mathematical techniques (this task was given at the end of junior secondary school).

Provisional results and implications for teaching

The protocols show quite different behaviours: students A and B seem to move symbols around, while students C and D are much more concerned with understanding. The theoretical model allows us to read such behaviours in terms of the sign-sense-denotation relationship. Starting from an algebraic formula, different senses can be activated: some of them are coherent with denotation, some others not.

Students A and B evoke senses starting from a very superficial reading of the formula. Mario evokes the equation of a parabola incorrectly, due to similarity in writing. Similarly, Stefania evokes the sense of solving a system, when facing the writing of two functions. For both students, no connection between signs, senses and denotations seems to exist.

Things go differently for students C and D, who approach and solve the task from a very global point of view.

Davide approaches the function $y = x^2+3$ by considering the changing numbers and the graphical representation. He is also able to pass from one sense to another easily. A similar behaviour is noticeable in Federico's protocol: both students master algebraic symbols and are able to relate them to the graphical representation of the functions. In short, the triangle sign-sense-denotation is handled fruitfully.

Let us remind that the NCTM Principles and Standards 2000 suggest "…Being able to operate with algebraic symbols is also important because the ability to rewrite algebraic expressions enables students to re-express functions in ways that reveal different types of information about them" (p.301).

As already pointed out, the analysis of the protocols has suggested a closer investigation on the origin of such great differences

Our main assumption now is that previous instruction plays a major role. In fact students A and B come from a traditional approach: they are able to solve standard tasks (for example that of solving equations) but feel lost when required to handle symbols and meaning. On the other side, students C and D (who are two years younger) have been provided with innovative teaching, strongly oriented towards understanding and verbalising. From early algebra, these students have been asked to compare different expressions and different senses, with special attention to link geometrical investigation to the representation by numbers and letters. Furthermore, the use of metaphors have been highly encouraged: teaching and learning algebra has been framed in an *embodied cognition* perspective (Bazzini, Boero and Garuti, 2001). Further investigation on long term research (Malara, 1999) is needed to confirm our assumption: specific teaching experiments have to be designed to suit this purpose.

Finally, focussing on implications for teaching, we remind that, according to the NCTM Principles and Standards 2000, algebra is much more than moving symbols around. Doing algebra is not just formal manipulation, but rather a competence which deeply involves understanding. In this perspective, new emphasis should be given to research studies on the mutual relationship between algebraic expressions and their meaning and, consequently, on related educational choices.

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PROFESSIONAL DEVELOPMENT THROUGH CLASSROOM COACHING*

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Abstract

This paper presents preliminary results from an ongoing study of classroom coaching in elementary mathematics classes. Seven teachers who have been involved in a professional development program for several years are released from the classroom to work as coaches. I have been observing the coaches in their coaching work, and also observing the teachers whom they are coaching. The purpose of the research is to ascertain whether coaching is effective in improving instruction in mathematics. In this paper I identify three styles of coaching I have observed, and discuss their promise for promoting classroom change.

Purposes

The main purpose of this ongoing project is to investigate the efficacy of classroom coaching in improving instruction in elementary mathematics classrooms. The coaches involved in this study have been participants in a state-funded professional development program for a number of years. That program includes three major aspects:

- an intensive 3-week summer institute focusing on mathematics content, pedagogical content knowledge, and leadership skills;
- summer lab schools for children organized and run by participants, who themselves, with staff support, provide professional development for team teachers who teach the classes;
- comprehensive follow-up activities including workshops with leading national and international mathematics educators.

Part of the leadership development strand has included training in classroom coaching, using a peer coaching model. With private foundation funding, the coaches in this study have been released from classroom duties to be full-time coaches in mathematics in their districts. This ongoing study has been designed to ascertain the impact these coaches are having in the classrooms in which they work, and indirectly, the impact of the professional development in which they have participated. In particular, the study was designed to document how coaches worked, how they interpreted their roles, and how they affected the teachers with whom they worked.

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Background

In the last edition of the *Handbook of Research on Teaching*, the chapter on mathematics education (Romberg & Carpenter, 1986) hardly mentions research on in-service teacher education. As Grouws pointed out (1988), and as is still the case, there is little information available about the overall design features of in-service education programs which maximize changes in teacher beliefs and ultimately classroom practices. Grouws called for studies that focus on the impact of various features of in-service education on classroom practice. More recently, the *Handbook of Research on Teacher Education* (Sikula, 1996) does not even include coaching in the index of the volume. The meager research that has been reported in mathematics about classroom coaching as a means of professional development predicts considerable promise for this technique. For example, Becker and Pence (1999a,b) identified classroom coaching as the most important component of a professional development program for secondary teachers. In these studies, the coaching was done by the authors, who also designed and implemented the whole professional development program. Coaching that was intended as a non-evaluative mechanism for identifying the impact of the professional development itself became the most important aspect of the in-service for participant teachers. Those studies concluded that coaching might itself be a worthy, though time-consuming and expensive, planned component of professional development.

There are a number of models of coaching extant within the educational community. For example, Evered and Selman (1989) define coaching as conveying a person from where he or she is to where he or she wants to be. The metaphor of an old stagecoach communicates this perspective. In this model the teacher is considered a thoughtful decision-maker who, through support and collaboration, can further develop her/his ability to reflect on and improve instruction. A second model is content-focused coaching (Institute for Learning, 1999), which focuses on the content of the lesson in relationship to issues at the core of the teaching-learning process. From my reading and viewing of videotapes in which content-focused coaching is used, it appears to be a bit more directive, in that the coach may use the pre-conference to "teach" content to a teacher who seems to lack content knowledge related to the lesson, may interrupt the lesson and even take it over, and may provide her/his own solutions during pre- or post-conferences. However, both models have the following characteristics in the ideal situation: a pre-conference to discuss the lesson and its goals and the teacher's focus for the observation; an observation of the lesson in which the coach records as much data as possible; and a post-conference to debrief. Coaching might also include demonstration lessons, co-teaching, or joint lesson planning. In this study I applied aspects of both models during observations of coaching sessions as seemed appropriate. That is, I focused on interactional moves of the coach, such as listening skills, strategic questions, and use of feedback, as well as content specific moves, such as clarifying the goals of the lesson, anticipating and diagnosing difficulties, or reflecting on students' attainment of lesson goals.

Methodology

This was a qualitative study using participant observation techniques (Glaser & Strauss, 1967). Observation sessions varied depending upon what the individual coach had planned and how s/he worked with teachers. For example, one coach, Lewis, is working with two fourth grade teachers at the same school. They plan lessons together in a meeting a day or two before the lesson. Then the coach views half of the lesson with one teacher and half with the other, and holds a joint post-conference with both teachers during lunch. Because of scheduling and prohibitive distances involved, in this case I meet with the coach before the lessons to determine what was discussed in the pre-conference. Then we jointly observe the classes, interacting with the children as they work on activities. I observe the post-conference, providing my input when asked or when it adds to discussion of, for example, student work. In this case I am more on the observer end of the participant-observer continuum.

In another case, I spend the whole morning at a school with the coach, Nellie, and the two fifth grade teachers with whom she is working. We have a brief pre-conference with each separately, one before school, the other during a break, identifying areas of focus for the observation. We observe the whole mathematics lesson of each teacher, with the coach making notes that she hands to them during the post-conference. The post-conference usually takes place with both teachers during lunch. There are other variations but space limitations preclude discussing these.

All notes from observations and interviews with teachers and coaches are typed and expanded, with patterns and questions to investigate further identified as work progressed (Glaser & Strauss, 1967). The aim is to identify patterns of coaching work and its impact on teachers, and subsequently, to ascertain how participation in the professional development program has affected the coaches and their work. Data include field notes, interview transcripts, and artifacts from the classrooms such as assessments.

Data Sources

The study is ongoing during the 2000-2001 academic year. Seven coaches are being observed, each with at least one teacher, for a total of 13 teachers and 7 coaches. Due to space limitations this paper will discuss the case of three coaches, here called Lewis, Nellie, and April to preserve anonymity.

Lewis is a former middle school teacher who has been with the professional development project for three years. This is his third year as a coach. Lewis is a European American male who has been teaching over 20 years. He works in a small district of eight K-8 schools in northern California. This district uses *Mathland* as its curriculum at the elementary level, and *Connected Mathematics* for middle school. *College Prep Math* is also used for two eighth grade algebra classes. Lewis is working with two fourth grade teachers, Sally and Susan, both young and relatively new teachers.

Nellie is a former intermediate school teacher who has been with the professional development project for two years and is in her second year as coach. Nellie is a European American female with over 30 years of teaching experience. She is working with two female fifth grade teachers, one in her first year of teaching. Nellie works in a small K-8 district in an urban area with a very diverse student population. Many students are emerging English learners, especially in the primary grades.

April is a former primary teacher who works in a small-city unified (K-12) district. April is a European American female with 15 years of experience. She has been in the project for five years and a coach for three. April is working with two first and two second grade teachers and I am also observing two other teachers with whom she has worked in the past (third and fourth grades). Here I will be focusing on her work with the primary teachers.

Results

At this point in the study three different modes of coaching have been identified based upon how the coaches interact with the teachers with whom they are working and on how they seem to define their role. I am calling these: coach as collaborator, coach as model, and coach as director.

Coach as collaborator. Lewis is an example of what I am calling a "coach as collaborator." He endeavors to be one of the group of three who are working on this lesson together. Thus the post-conferences tend to be about the structure of the lesson rather than specific as to how each teacher implemented the planned lesson. In fact, by viewing half of each lesson for Sally and Susan, Lewis cannot really ascertain how the second teacher developed the core of the lesson [he does switch order each visit]. Lewis does not keep written notes from the lessons, and does not give the teachers written feedback. However, he works closely with children, frequently asking questions, and seems to have a good sense of what they are understanding. For example, in one lesson the teachers were developing multiplication facts greater than 10; they wanted children to work them out without use of the standard algorithm. In Sally's class, as students shared their methods orally, it was clear that this was difficult for those who knew the algorithm. One girl even verbalized the whole standard algorithm by visualizing it in her head (the problem was 12x6). Both Sally and Susan noted in the post-conference that students seemed wedded to an algorithm. Lewis had noticed in his questioning of children in both classes that many did have other strategies for figuring out 12x6. Lewis suggested to the teachers that they ask children to find more than one way to do the problem to get them beyond an algorithm. Sally and Susan liked this suggestion, and in later observations, both were observed asking for more than one way in other contexts.

Although much of his work is collaborative, it is clear that Lewis has a slightly different role from that of the teachers. He provides performance assessment practice items for teachers' use, scores them for the teachers, and does the class presentations of the problems and the rubric scoring to help children get familiar with that type of testing. Although Lewis does not provide feedback specific to how a teacher organized the lesson, he does concentrate on what students seemed to understand. By being active in the classroom, watching and questioning students, he gleans considerable information about student understanding to share with teachers. From Lewis' perspective, perhaps the most important part of his role is encouraging and facilitating the team planning and reflection that are occurring. Without his presence as coach, this level of collaboration would not be taking place. The planning time forces each teacher to think through the lesson, its goals, and how they plan to implement them beforehand. Because they are working as a team in this way, they have a mutual responsibility for the lesson and its pros and cons. The teaming that Lewis has encouraged has extended to consistent planning throughout the week, even when he is not visiting. Thus Lewis' model encourages the elimination of the isolation many teachers in the USA feel by working alone in their own classrooms.

On the other hand, lack of specific feedback to each teacher precludes Lewis from the possibility of influencing the teachers' teaching strategies. A lesson may be the same but may be implemented in quite different ways. Thus Sally has a need for full control at all times in her classroom, so that she shows students exactly how she wants them to do problems. This discourages multiple methods of solution, such as sought for 12x6. Susan's more open style generates more ways of solving problems. Peer visits or feedback on pedagogy might provide both with more ideas on instructional strategies that would lead to further mutual professional growth.

Coach as director. Nellie is an example of what I term "coach as director." Nellie's model of working with Harriot and Debra is much more directive. Although in the pre-conference she asks them what they would like her observation to focus on, Nellie feels free to interject her opinion on something that occurred in the lesson even if that was not the specific focus of the observation. For example, in one observation of Harriot, the teacher had the children start on a two-day lesson in which they had to measure 100m outside the room. After some measurement of the room and estimation of 100m, she asked how they would actually measure 100m. By very persistent questioning and major hints, she finally got someone to say what she had intended: to make a longer measuring instrument than the meter stick using register tape. Nellie felt that Harriot should have given students more opportunity to develop their own method rather than leading them to "her" way. She **told** Harriot that after the observation, and wrote it in her notes to her. While Lewis might have raised a question and collaboratively worked with the teachers to come up with alternative strategies, Nellie was very explicit as to what Harriot should have done.

Although Harriot and Debra teach at the same school, there has been no attempt at team planning and in fact on each visit they have been teaching totally different units of content. Nellie seems to view her role not so much as "fixing" teachers but very directly providing guidance and alternative strategies that she believes will work. As Nellie is much older than either Harriot or Debra, the relationship seems to be a motherly one, in which direct guidance is accepted rather than resented. Thus Nellie seems to have quite good rapport with both teachers. However, I have yet to see either teacher subsume Nellie's suggestions into her own repertoire of teaching strategies. Perhaps Nellie's style is not supporting instructional change in the way she might like because she is not promoting thoughtful decision-making and self-reflection with these two teachers.

Coach as model. April exemplifies "coach as model." April has developed a unique way of working with teachers new to her. First she presents several model lessons, leaving the teachers materials and ideas on how to continue that work until her next visit. Then she moves into modeling a peer coaching model, in which she is the teacher and the classroom teachers act as coach for her. Then she plans to facilitate, by covering their mathematics classes, their serving as peer coaches for each other.

For example, I observed two second grade lessons that April did several weeks apart. In the first lesson, April was investigating growing patterns. She first modeled finding the first five steps in a geometric pattern on the overhead projector, engaging the children in finding the pattern and describing how it was growing. Next children were given pre-made patterns to copy with cubes, then extend to the fourth and fifth steps. Patterns ranged in difficulty and were exchanged as children completed them. April did this lesson in both classes for each teacher, then left the materials behind asking them to give children more practice in finding growing patterns and extending them. At the next lesson, a two-day one, children had to complete five steps of a pattern with cubes, then color in the first five steps on inch grid paper, then make a poster of their pattern and a description of how it grew. This lesson ramped up the concept as children had to also fill in a table showing how many cubes were used at each step; this was also modeled with the whole group. Interestingly, April adjusted her instruction of the second lesson in the second class as one aspect, looking for patterns in a 100s chart, confused the children. This difficulty and her adjustment provided interesting topics for discussion after the lesson.

Thus April is acting as a model on several dimensions. She presents exemplary lessons and is always prepared with materials, manipulatives, and everything needed for the lesson. Her lessons always begin with a whole-group activity in which she models what she would like the children to do. She clearly does long-range planning as teachers can infer from the work she leaves them to do. She wants them to do peer visits, so she first models that to help them understand and feel comfortable with it. Perhaps it is April's background in primary school that makes her affinity for modeling so ingrained.

Summary

This study identified three different ways of coaching. These could be considered to range on a continuum from less to more directive. April is perhaps the least directive as she tries to stimulate professional growth in teachers through modeling. Lewis is still quite non-directive, but he does raise questions regarding instruction that he tries to work through with the teachers through collaborative dialogue. Nellie is the most directive of this group, explicitly giving her opinions and suggestions even if not requested. Since I have yet to see the teachers with whom April is working teach on their own (this is a slow development April has planned), I cannot judge directly how her approach will impact their teaching. But teachers seem somewhat awed by what their children can do mathematically and seem eager to emulate April's approach to classroom discussion. Lewis seems to have had a positive affect on the teachers with whom he works by encouraging their own decision-making and by encouraging the collaboration which they have transferred to all lessons. Nellie's approach seems to have the least potential for stimulating teacher growth; no substantial changes in teachers' instruction is evident at this point in the study.

Of course these are three individuals who have a personal style that must match their own personality. I would not want to generalize that everyone should work in one style. However, these three cases are thought-provoking and stimulate questions that will be investigated with the rest of the sample of coaches:

Is there a style of coaching that is most efficacious in promoting growth in teachers?

Is there a range of skills and dispositions that are needed by a coach?

What is effective coaching?

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ADAPTING PIRIE AND KIEREN'S MODEL OF MATHEMATICAL UNDERSTANDING TO TEACHER PREPARATION

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Abstract

This theoretical paper examines a conceptual framework for studying prospective teachers' understanding of what and how to teach high school mathematics. The framework draws from Pirie and Kieren's model of mathematical understanding and incorporates ideas from Schoenfeld, Ma, and Lave. Several conjectures are made for future study in relation to folding back and primitive knowledge.

1. 0 Focus. The focus of this theoretical paper is to examine the model for mathematical understanding proposed by Pirie and Kieren (1994) with respect to the preparation of high school mathematics teachers. We adapted the model for teacher preparation to accommodate for prospective teachers' understanding of school mathematics and their understanding of strategies to teach school mathematics.

2. 0 Conceptual Frameworks. Among three distinct types of research frameworks, Eisenhart (1991) proposed that conceptual frameworks bring more flexibility to the research process than those based on theory or practice. She argued that the adoption of frameworks built exclusively from a single theory excluded the reporting of a broad range of results whereas conceptual frameworks provided a more comprehensive approach to the intended inquiry. The conceptual framework described in this paper scaffolds several broad ideas related to teaching and learning. They include what prospective teachers plan to teach [school mathematics], and how they plan to teach [teaching strategies]. The framework is intended to capture the dynamic process of learning over time and task.

Schoenfeld (1999) expressed the need to bridge the two dominant theoretical frameworks in education - cognitive and social. He proposed that theories or models of content competencies and acting-in-context have potential to span the divide between the two current research traditions. Further, Schoenfeld proposed that researchers consider how to maximize their contributions to fundamental understanding and, at the same time, contribute to practice. Studies of teachers in context, conducted by Schoenfeld's Berkeley team, are cited as examples of research directed toward the development of a theory of thought and action. This study draws from this proposition with respect to studying prospective teachers' understanding of teaching high school mathematics in the context of teaching tasks. Related to Schoenfeld's work is the Ma (1999) investigation of elementary teachers' content knowledge [thought] and strategies [action]. Ma chose to examine what Chinese and American elementary teachers' knew about topics in school mathematics and the strategies they selected to teach those topics. This approach gave more depth to her investigation of teachers' understanding of teaching mathematics.

The model of mathematical understanding proposed by Pirie and Kieren (1994) is a keystone of this conceptual framework. Modifications were made to accommodate for the cognitive and social dimensions of what the prospective teachers understand about school mathematics and teaching strategies. From the model of mathematical understanding we drew heavily upon the first five activities in the model: 1) primitive knowledge, 2) making an image, 3) having an image, 4) property noticing, and 5) formalizing. The meanings of these activities, as defined by Pirie and Kieren, were maintained for this framework. Their model was used to examine processes related to mathematical understanding, but we are proposing to adapt that model to study prospective teachers' understanding of what and how to teach. We expanded the Pirie-Kieren model to accommodate for the tasks of teacher preparation. Tasks such as lesson planning, teaching, and assessment all require an understanding of what and how to teach. A prospective teacher may have formalized the school mathematics related to rate of change, but is making an image of what teaching strategies to use to teach rate of change. The conceptual framework considers the understanding of school mathematics to be a function of mathematical understanding in which prospective teachers access their primitive knowledge of mathematics to make connections with what is taught in high school. Depending on the amount of mathematical study, the prospective teachers' primitive mathematical knowledge can be extensive. The primitive knowledge of teaching strategies, on the other hand, may be quite limited for the prospective teachers. This dichotomy will be discussed later in this paper. The process of folding back, taken from the Pirie-Kieren model, is a critical feature in changing the prospective teachers' understanding of teaching.

3.0 Related Literature. In this section we will briefly review two areas of the literature that Schoenfeld (1999) suggested as having potential to bridge the cognitive versus social divide. We consider those who have used frameworks built around content competencies and those who have approached their inquiry from an acting in context perspective.

3.1 Content competencies. Skemp (1976) argued that for mathematics teachers to make a reasoned choice between instrumental and relational approaches to instruction (how to teach) they require a relational understanding of the mathematics (what to teach) itself. His ideas laid the groundwork for many mathematics education research efforts related to content competencies. Ball's (1990) research illustrated the importance of relational or conceptual understanding when prospective elementary teachers attempted to develop application problems of division of fractions. In a study of Chinese and US elementary teachers, Ma (1999) noted that "a teacher's subject matter knowledge may not automatically produce promising teaching methods or new teaching conceptions" (p. 38). She suggested that new ideas about teaching (how to teach) cannot be realized without strong support from subject matter knowledge (what to teach). We assumed that prospective teachers' knowledge of what and how to teach is enriched with both instrumental and relational understanding of mathematics.

3.2 Acting in context. Lave (1991) claimed that the decontextualized practices of schooling contradicted her theories of learning in context but suggested that a view of peripheral participation may prove valuable in analyzing learning in school settings. She defined peripheral participation as actions within a community of practice over time and discussed changes in the apprentices' thoughts and actions. A number of researchers in mathematics education have situated their studies within the classroom to analyze students' or teachers' thoughts and actions within specific contexts over time. Maher and Martino (1996) reported on a five-year study of the thoughts and actions of one student's developing understanding of mathematical proof. An international study of prospective teachers' understanding of area within the context of a lesson planning activity reported a variety of thoughts that apprentices as peripheral participants bring to the community of practice (Berenson, et al. 1997). In a much larger study, Eisenhart et al. (1993) examined the many contexts of the educational communities affecting practice to better understand the field experiences of one prospective teacher. Decontextualized practice, the key term in Lave's description, is of great importance to the conceptual framework proposed here. We contend that to maximize prospective teachers' experiences as apprentices in the community of teaching high school mathematics, their preparation must engage them in the tasks of teaching. This does not imply that all tasks occur in the classroom with students; teachers can engage in a variety of tasks outside the classroom, such as developing lesson plans and creating assessments. These tasks are repeated in one form or another and occur over time as the prospective teachers participate peripherally within the community of mathematics teaching.

4.0 Activities of Understanding. In this section, descriptions of the first five Pirie - Kieren levels of understanding are given with some data from previous research. Together they are intended to clarify the application of these levels to our conceptual framework. In the previous study, prospective teachers were asked to plan a lesson to introduce the concept of rate of change to an Algebra 1 class and to relate the concept to ratio and proportion. The lesson plan was viewed as acting in context while content competencies were needed to complete the task. Individual interviews were conducted before and after the planning task. Subjects were 19-21 year old undergraduates with Grade Point Averages between 3.2 – 3.75 [out of 4] who had completed 6-8 university mathematics courses.

4.1 Primitive knowledge. Primitive knowledge was defined by Pirie and Kieren as all that is known mathematically before coming to the new learning task. There were several aspects of primitive knowledge accessed during the interviews including college mathematics, college physics, college chemistry, school mathematics [what], and teaching strategies [how]. We defined school mathematics of this task to refer to the specific mathematics (rate of change) and the related mathematics, such as, slope, fraction, ratio, proportion, and rate.

4.2 Making an image. At Pirie and Kieren's second level, *image making*, one uses primitive knowledge in new ways. In their model, one's image-making activity is an important step toward the understanding of a mathematical concept.

Amy's Lesson: An Example of Image Making

What to teach: Interpreting changing rates

How to teach: Collecting and representing distance/time data

The following excerpt is an example of Amy's image-making process in terms of what to teach [school mathematics]. She began tentatively to explain the relationship between the circumference of a circle to pi as an example of a ratio. The interviewer's prompt caused Amy to revise her formalized definition of a circle's circumference, folding back to make an image of the relationship between the circumference and the radius.

A: … what I was thinking about is the way that pi is related to … kind of makes a ratio out of the circumference and the radius, maybe.

I: What happens if you solve algebraically for *pi*?

A: [Considering the formula $c=2\pi r$,she divides both sides of the equation by $2r$ to solve for *pi*] Yeah, the circumference over $2r$. I think that is what I was thinking about.

One teaching strategy developed by Amy involved students in collecting distance/time data using toy cars and stopwatches. Making an image of how to teach, she tried to explain her ideas to the interviewer. In the process, she remade her image of how to teach, deciding that a table of data was better than recording the data directly onto the graph.

A: … they want to go another 50 and maybe do it slower or faster. And then plot the next point, how long it took them, for the second time, and then do it a third time for another 50 centimeters. … And you know, now that I think about it, it might be even easier to have a little chart of first time, second time, third time and the distance for each time already given.

4.3 Having an image. According to Pirie and Kieren, *image having* occurs when a person uses a mental construct to assist his/her understanding without having to repeat related activities. One has an image when he or she can mentally manipulate ideas to consider different aspects of the learning tasks.

Chris's Lesson: An Examples of Image Having

What to teach: Comparing distance/time ratios and calculating rates of speed

How to teach: Collecting and calculating distance/time data

Throughout the pre-planning interview Chris had an incorrect image of ratio as fraction.

C: I remember from the early years doing shading in squares and parts of the circle. And how much of the squares this covered. That's the majority of what I remember.

Then she drew a representation to explain her meaning of ratio. She drew a 3x3 square, shaded 4 out of the 9 squares, and explained that the ratio of shaded parts was 4/9. The lesson planning process helped Chris to fold back, changing her image of ratio as fraction as is shown below.

Chris had an image of how to teach rate of change that involved her students going outside to collect walking and running data using meter sticks and stopwatches. She was able to mentally manipulate her image of data collection to describe how those data would be used for teaching back in the classroom.

C: And then when you come back inside, [we will] talk about what we did and put up an example on the board of how we can use these numbers to find a rate of how fast we were traveling. … if Jane ran 12 meters in 3 seconds what is her rate in meters per second? So that is like two ratios.

4.4 Noticing properties. Pirie and Kieren explained that *property noticing* occurs when one manipulates or combines aspects of an image to identify related properties or contexts. This level of understanding is closely related to the images that the prospective teachers made and had about their lesson plans.

Sam's Lesson: An Example of Property Noticing
What to teach: Solving missing value problems
How to teach: Students working problems and teacher demonstration

Sam planned for her students to use hands-on materials while working in small groups to solve a missing value problem. Explaining her solution, she noted that one way to solve the problem was to use the simplest whole number ratio and that another option was to find the unit rate. This demonstrated her noticing properties about the ways proportions can be solved and ratios can be simplified.

S: I would give them the trophy problem that was in the book and then each group would have blocks… and fake dollars…If they struggled with this, I would … help all of them see the ratio of 3 blocks to 2 dollars and then go through my solution of the tall trophy…Our rate of cost here is $2 per 3 blocks. We could also break this down into … I can show them how we could call it $2 for 3 blocks or 67 cents per block.

Sam had noticed properties of teaching with concrete materials and their usefulness in teaching mathematical ideas to high school students. She based these properties from her lab experience in a high school classroom.

S: [The lab teacher] didn't use hands-on manipulatives, and I could see a lot of students struggle with that. I would use hands-on manipulatives.

4.5 Formalizing. The activity of *formalizing* occurs when a person identifies common features of the image he or she has made. Examples include finding patterns, applying an algorithm, or creating a formula.

George's Lesson: An Example of Formalizing

What to teach: Unit conversions of rates

How to teach: Calculations, data collection, graphing

George formalized his meaning of rate of change to be the slope of a line, a derivative, and speed, but when describing what he would teach, he focused narrowly on only one aspect of rate of change, unit conversions.

G: I looked at [the lesson from] more of a physics standpoint than a trig-math standpoint. I figured the most important thing about proportions and rate of change is the units because if you have the units wrong the answer is wrong. I would start with some basic unit measurement and conversion type deals.

George planned to ask students to collect time/distance data using toy cars, stopwatches, and tape measures and then work through a series of unit conversions. The interviewer's comment below encouraged George to fold back to make a new image of how to teach.

I: Would you have them do any graphs along the way to look at representations like the one you drew earlier?

G: I think that after we did all the units and stuff, we'd graph the speed of the car. I think I would do one on the board or overhead and say OK, so this is time and draw a straight line like I did before. Depending on whether or not they had slope yet, I would introduce that to the class.

4.6 Folding Back. As defined by Pirie and Kieran, *folding back* is a revision process that occurs when a new issue is raised with respect to one's current understanding. It is a recursive process in which a person folds back to an inner level of understanding in such a way as to inform and change that inner level. For example, Amy folded back to her primitive knowledge to remake her image of how to teach rate of change to include a table-making activity for her students. She also folded back from her formalized knowledge of school mathematics to remake her image of the relationship between circumference and radius. We consider folding back to be critical to the development of prospective teachers' understanding of what and how to teach high school mathematics. In George's example, the interviewers were unable to precipitate his folding back from the formalized level of unit conversions to remake his image of what to teach about rate of change.

5.0 Conjectures and Implications. In this paper we were prepared to discuss and apply Pirie and Kieren's first five levels of understanding activity. As our research expands to involve tasks of teaching in the classroom, results may inform the last three levels of prospective teachers' understanding: observing, structuring, and inventising. From another perspective, perhaps only expert teachers attain these outer levels and that they do not pertain to apprentices. We have made several conjectures for future discussion among ourselves and other researchers. The first conjecture involves the critical nature of the process of folding back to extend and expand prospective teachers' understanding of what and how to teach. The second and third conjectures consider the retrieval and storage functions of primitive knowledge of school mathematics and teaching strategies.

Conjecture 1. Folding back is critical to the development of prospective teachers' understanding of what and how to teach and can be prompted by conversations *with others.* This function of understanding appears to be critical to the preparation of teachers and one that we will explore more fully in future research. The interviewing processes of the lesson plan task opened a dialogue between the mathematics educators and the prospective teachers in such a way as to encourage the apprentices to rethink their knowledge of school mathematics and approaches to teaching. An important feature of the dialogues was that they were focused on the lesson planning task, thereby situating the conversations in context. The implications of this conjecture to teacher preparation is that extended dialogues between educators and apprentices can increase understanding if situated within apprentice tasks involving what and how to teach.

Conjecture 2: Deep stacks of primitive mathematical knowledge may hinder the process of folding back to revise images of what to teach in school mathematics. The difficulty may stem from one's memory search functions or the ability to unpack one's knowledge base of college mathematics to find the connections to school mathematics. We noted that the prospective teachers in our study were readily able to access what they had learned in college mathematics, college physics, and college chemistry, but were unable to recall what they learned in Algebra 1. Several described rate of change to be the derivative but were unable to see the relationship to Algebra 1 topics. Some were surprised to learn that the slope of a linear equation was a ratio. Perhaps the depth and/or the lack of connections of their accumulated mathematical knowledge made it difficult for them to unpack their stacks of university learning to retrieve their understanding of school mathematics. A number of studies, primarily at the elementary level, have noted the lack of mathematical understanding of prospective and practicing teachers. Ma (1999) redefined mathematical understanding as profound understanding of fundamental mathematics [PUFM]. This definition may have important implications for the practices and programs of high school teacher preparation.

Conjecture 3. Shallow stacks of teaching strategies are to be expected among prospective teachers. The lack of primitive knowledge allows the prospective teachers to search through their strategy stacks quickly. Additionally, they appear to be more receptive to interviewer's probes so as to revise their images of how to teach and add to or pack these stacks with new ideas. The interviewers' prompts promoted folding back by the prospective teachers to their primitive knowledge of teaching strategies. We concluded that their stacks of primitive knowledge were thin because of the two basic strategies used by the prospective teachers. They either planned to give information directly or to have students collect data, reserving the generalizations from the data as the teachers' roles. This lack of strategy knowledge on the part of the prospective teachers, provided opportunities for folding back to pack their understanding of how to teach. For example, one interviewer asked if the prospective teacher would consider using a graph to represent the data. Another asked what would happen if the prospective teacher placed the student activity at the beginning of the lesson rather than the end. In all instances, the prospective teachers incorporated these strategies into their lessons. A goal of teacher preparation is to pack more understanding of how to teach into the prospective teachers' primitive knowledge. We suggest that instruction in teaching strategies be associated with the tasks of teaching and the content of high school mathematics to increase prospective teachers' understanding of how to teach.

We plan to ask Susan Pirie and Tom Kieren to provide us with critical comments on our proposed conceptual framework. We will apply the revised framework to analyze data from another study on prospective teachers. Finally, we plan to revise our own practice: what to teach prospective high school mathematics teacher about teaching and how to teach them about teaching.

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INTEREST IN MATH BETWEEN SUBJECT AND SITUATION

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Interest is an important factor in learning and achievement. In my research, I reconstruct phenomena of interest change in everyday mathematics classroom education, and I identify conditions influencing the development of interest in mathematical objects. This leads to an approach to an empirically-grounded theory on interest based learning in math classes.

1. Introduction

An interested pupil pays attention in class, works in a concentrated and an engaged way, learns more, is involved in the activities, enjoys his/her involvement, wants to go on with the activity (Hidi/Berndorf 1998), and sometimes does not notice how time goes by. In this state of affairs close to that of flow (Csikszentmihalyi 1987) capacity and challenge fit perfectly (Bikner-Ahsbahs 1999 p. 52 ff). Research has shown interest to be a strong predictor of quality of experience in mathematics classes, i. e. potency, intrinsic motivation, self-esteem, importance and the perception of skill (Csikszentmihalyi/Schiefele 1993, Schiefele/Csikszentmihalyi 1995). However, interest also is a predictor of mathematical achievement. Thus it is an important factor in learning and development (Bauer 1988, Krapp 1996, Deci 1992, 1998). But how does interest arise? Undoubtedly, interest is a phenomenon of social negotiations (Renninger 2000, Krapp 1996, Deci 1992, 1998). However, there is little research analysing social processes (Deci 1992, Krapp 1996, Schiefele 1998, Wild/Krapp 1996).

In 1988, Bauer was the first to introduce interest research in German math didactics. Taking up Bauer´s reflections, Bikner-Ahsbahs has worked out an empirically grounded concept of fostering mathematical interest in everyday lessons (1999). The study was based on questionnaires about a project encouraging interested pupils in extra curricular mathematics. Referring to the empirical research on interest, the results were applied to everyday lessons. However, the social processes of the development from interest in daily classroom situations haven´t yet been investigated. Because of this lack of investigation in social processes I´ve designed a study to develop theoretical components through the reconstruction of phenomena concerning interest development, in order to explain the emergence of interest in everyday lessons, its development, encouragement and hindrance.

2. Theoretical Framework

Krapp et al (Krapp 1992) have proposed interest to be a construct describing a person-object-relation which can be seen in activities with three characteristic aspects: expansion of competence (cognitive aspect), association with predominantly positive feelings (emotional aspect) and assigning advanced value to the interest matter (valuable aspect) (1992, 1996). An interested person identifies himself/herself with the matter of interest. The self-determination theory describes this process of identification (Deci 1992, 1998). Deci regards motivation as a continuum between extrinsic and intrinsic motivation whereas interest is taken as an object-specific type of intrinsic motivation. An extrinsicly motivated activity may be increasingly self-determined leading to individual interest by taking the steps of introjection, identification, and integration (Krapp 1992, Deci 1992). Using the basic needs for competence, autonomy and relatedness, Deci is able to explain the emergence of priorities (Deci 1992, Krapp 1996). Several studies show that competence- and autonomy-supportive teaching enhance intrinsic motivation and interest (Deci 1992), and that the experience of competence and autonomy is related to the development of interest (Wild/Krapp 1996, Prenzel/Drechsel 1996). Investigations focus on two different kinds of interest: individual interest on the intrinsic side and situational interest on the extrinsic but self-determined side of motivation (Hidi/Berndorff 1998, Mitchell 1993).

2.1. Individual interest

Individual interest develops slowly but in a relatively stable and situationally independent way. (Hidi/Andersen 1992). The observation of developing processes of individual interest can only be achieved in a long-term project. Renninger (1992, 2000) compares extended interest to non-interest and attraction. Her concept of interest is associated with increased levels of knowledge and value bringing together positive and negative emotions. The connection to Krapp´s concept of interest can easily be explained in the context of mathematics education: My experiences in fostering mathematically interested pupils have shown that pupils with high individual interest like to find mathematical proofs on their own, although these processes are associated with frustration. However, pupils with low individual interest are likely to avoid such frustrations. They derive more pleasure from finding proofs in a group with the support of a teacher. As individual interest in math increases, value gets more important than the experience of positive emotions.

2.2. Situational interest

Situational interest is influenced by the environment and is therefore fickle (Hidi/Andersen 1992). Little is known about how situational interest develops and how situational interest might lead to individual interest (Hidi/Berndorf 1998).

Mitchell´s research (1993) indicates a multifaceted structure of situational interest in school mathematics with five subfacets. First of all, situational interest consists of a catch-facet and a hold-facet. Interest is caught by a situation of cognitive, sensory, or social stimulation. Mitchell shows that pupils at the age of 14 to 16 get interested in mathematical activities by using computers and puzzles and by participating in group work. For smaller children I would expect a sensory stimulating facet to catch interest, too, and my investigations indicate, that not all forms of group work are interest-catching. Even Mitchell asks the question under what conditions group work is perceived as interesting by students (Mitchell 1993, p. 433).

Environments which catch interest need not necessarily hold interest. Situational interest can be held if students perceive lessons as meaningful (*meaningfulness*) and if they feel involved in the activities (*involvement*). Thus, suitable environments and social contexts influence situational interest (Mitchell 1993). But what does suitable mean? In my study, I look for situational conditions of environmental and social interactions in natural settings which influence situational interest. The pursuit of conditions for long-term individual development then leads to conclusions about a possible development of individual interest.

2.3. Interest and learning in mathematics classroom education

Social processes of interest development cannot be observed directly. They are inferred from indicators by means of an "interaction analysis". Therefore I use the theoretical approach of the "interpretative Unterrichtsforschung" of the German math didactics (Krummheuer/Naujok 1999). Concerning this theoretical approach, the individual and social learning processes are based on the construction of meaning (Krummheuer/Voigt 1991). In everyday math classes, mathematical meanings are produced through negotiation. As they are results of interactions they are taken-as-shared. The participants of an interaction do not clarify symbols and meanings in an extensive way. They accept ambiguity as far as the interaction process can go on (Krummheuer 1992, Voigt 1995); progressive questions indicate the interest of a pupil. In the social-constructivist approach of his interactional learning theory Krummheuer conceives learning both as a creative, individual, innerpsychic process and at the same time as a socializing social process directed at convergence (Krummheuer 1992, p. 167). In this process the mathematical subject matter, the individual learning process, and the social interactions are mutually dependant. So it seems reasonable to assume that the development of interest is triggered by social interactions (cf. Deci 1998).

Learning in lessons is regarded here from a social-constructivist view as a process based on the individual constructions of meaning which are at the same time results from collective interactions. Therefore, students are active knowledge designers and also participants in a collective learning process. The teacher arranges the learning environment as an initiator of the learning process, which he/she accompanies and observes more or less. Otherwise he/she is a participant of the interactions. Mutual dependence of all the aspects suggest that the development of interest is influenced by the individuals, the learning environment, and the interactive situation.

During an interaction, individuals structure situations consciously or unconsciously according to the acquired cognitive relation schemes and regard the situation from a certain viewpoint. This *definition of the situation* changes in the course of the interaction processes. Standardized and routinized definitions of the situation are called *frames* (Krummheuer 1992, p. 24).

In everyday lessons, a common goal links the teacher and the students, namely studying of math. Therefore students are usually willing to get involved. It can occur that this willingness changes to lower activity when an activity will be refused or the attention focusses on something else. On the other hand it can happen that the activity becomes more intensive or despite disturbances the activity remains intensive. In both cases we observe an *interest-change phenomenon*. However, some of these phenomena cannot be observed in teaching situations because they take place inside a person. For that I use an instrument reflecting those inner processes.

Reconstructing these phenomena, I look for conditions fostering or hindering the unfolding of situational interest.

3. Methodological framework

According to the theoretical framework, I use a triangular design to structure my investigations. An individual and an interactional perspective are considered as additional methodological components. Concerning the object matter of interest, the process of analyses will include a subject-related view as a third perspective.

In order to gain empirically based theoretical components, the process of discovery proceeds "subsumtively" or "abductively" (Kelle 1997 p. 143 ff, Beck/Jungwirth 1999). In the case of "subsumtion" a phenomenon is assigned to a known category (e. g. meaningfulness and involvement). In the case of "Abduktion" (the German term) a phenomenon which is inexplicable by known theories, is picked up. Theoretical knowledge then serves as a basis for the construction of new explanation patterns describing the observed phenomenon.

In accordance to the difference of individual and situational interest, the core data consist of individual and situational data: For nearly half a year, the fraction lessons of a group of students at the age of 11 to 13 were observed and videotaped. Relevant situational data are taken from the video recordings. Transcripts of selected episodes are taken as documents for an interaction analysis. The individual data stem from a one-to-one correspondence of students, who want to become math teachers, with the pupils of this study.

Having identified interest-change phenomena, I reconstruct these phenomena as case studies with the methods of an interaction analysis. This approach involves the interpretative paradigm, i. e. methodological limitati[illegible] the research process using the following steps (Krummheuer/Naujok 1999 p. 66 ff, Beck/Maier 1994): transcription and interpretation of the selected episodes, common sense description, extensive interpretation, developing and proving of the interpretation hypotheses, combining the hypotheses of situational and individual data and working out a theoretical structure based on the comparison of similar and different cases.

4. Some data and first results

Up to now, only some interest-change phenomena are reconstructed. Nevertheless, first hypotheses have emerged which indicate that:

- the stimulation of interest is lowered by opposing definitions of the situation and is intensified by corresponding matters of interest.

- the construction of meaningfulness does not come about automatically but must be stimulated.
- exercises with formats that activate algorithmic-mechanical frames (Krummheuer 1992) prevent students from getting deeply involved with constructions of mathematical meaning. The construction of meaningfulness remains superficial.
- particular competition situations generally prevent the students from holding situational interest.

The data I selected confirm the last statement.

4.1. Competition situation

Competition is a rather usual form of learning mathematics in classes. These situations are meant to be especially motivating. But do they foster interest in mathematical objects? (cf. Deci 1992).

Competition must be fair for all participants. Therefore, rules have to be fixed beforehand. Usually the candidates know about the types of competition tasks they have to expect. Often a referee makes sure that the rules are followed during the contest. The primary goal in a competition situation is to win and not to get involved or to increase the knowledge of the task matter except when involvement is part of the task itself.

Rules for a learning situation are rather flexible and they can change during a learning process. Normally the teacher determines the kind of learning tasks the students have to work on.

The following translated transcript shows a competition situation with short-question-routines demonstrating typical conflicts the teacher and the students have.

The students are divided into two teams standing on either side of the teacher´s desk. The teacher allocates the task to the first two candidates. The one who solves it first may sit down, the other one has to go to the end of the line. Tobi and Anti are the next candidates.

1 T (*rubs his hands*) the ggT (*German term for largest common denominator*) (.) the ggT of 19 and 41 (...) (*looks at Anti, looks up, puts the forefinger at his mouth*) a bit more quiet
2 Tobi one
3 T (*looks at Tobi*) right why is it one´ Tobi.
4 Anti oha (*turns round and goes out of the line*)
5 Tobi um because 19 is only divisible through itself and one
6 T exactly and 41 too. (*Tobi goes out of the line.*) what ,what do we CALL these numbers 19 and 41´ (*bends down to the left as if he wants to pick up something, comes up again and points at the candidates Kia and Eric with both forefingers at the same time.*)
7 (*Kia, at the left from the T, an some students from the other team are raising their hands, Kia und Eric are the candidates, Lea und Ina are standing behind Eric. Kia raises her hand but Eric does not. The T looks at Kia first, then at Eric and then at*

Kia again.)
8 Kia prime numbers
9 T right (*does a circling movement with his right hand above Kia's head and looks at her, but Kia does not go out of the line.*)
10 Kia (*doubtfully*) oh should I sit down´
11 T yeah that was it.
12 /Eric sch (*turns*)
13 Lea oh well oh well he didn´t know that
14 /Ss he didn´t know that
15 /Eric I didn´t know that
16 Lea he didn´t know that
17 T (...) that was a question (..)scht
18 (*Eric turns round and goes to the end of the line.*)
19 /Ss that´s mean
20 /Lea that´s mean Mr K
21 (*The protest of the pupils gets louder.*)
22 T no ,a question (....)
(*The pupils all shout out at once.*)

Transcription key

S(s), T	student(s), teacher
exact	emphasized
EXECT	with a loud voice
e x a c t	prolonged
exact.	dropping the voice
exact´	raising the voice
,exact	with a new onset
(.),(..)...	1, 2 ... sec pause
(....)	more than 3sec pause
(*gets up*)	nonverbal activity
/S	interrupts the previous speaker

The teacher defines the situation as a learning situation using the opportunity to repeat the knowledge about prime numbers (3-6). Tobi is the first to give the right answer, so the next question for him cannot simply be another competition task (3-5). From the view of the students, too, the learning process is defining the situation (3-6). As usual in the contest, the teacher looks at the next candidates alternately and points at them. But by raising their hands the students do not regard the next question as a competitive one (6; 7). There are three reasons for that: The task still belongs to the previous learning context, the teacher´s movement (6) gives the task a casual status and the question does not ask for a number as it usually does in this competition, but for a term.

As Kia is allowed to sit down, the students of the other team probably feel at a disadvantage (13-24) and protest vehemently: Obviously the pupils are more concerned with winning the contest than with learning. The teacher however seems to know about the problem with the situation (17, 22), but he gets the upper hand.

From the students´ viewpoint, learning and competition situations are incompatible. Changing the situation must be clearly pointed out to avoid confusion. The teacher regards the whole situation as a learning one; competition just serves as motivation. So the teacher thinks he can change rules during a game but that is not compatible with the role of a referee.

Further transcripts show: If these conflict situations appear frequently, the possibly motivating effect of a competition can decrease, as it is already the case here with students not in the queue any more. Thus the possibly motivating effect of competition:

- can only be kept, if the teacher separates explicitly and clearly the learning from the competition situation.
- concerns only the participants of the contest and
- is primarily concerned with winning the competition.

Competition games must be fair. That means that rules must be precisely fixed. Thereby, it is impossible for students to become immersed in the task, to think deeply about it, to analyse mathematical peculiarities, ... , unless that is part of the task itself.

- Competitions with short question-answers-routines foster at best the interest in competitions or the routine in dealing with mathematical exercises but do not hold the situational interest in mathematical objects, for competition games do not allow the development of meaningfulness and involvement.

Learning situations at school require that there are individual differences among the students. It is even desirable that teachers encourage their students according to their individual differences in various ways. Further transcripts show that special encouragement in a competition situation immediately gives the impression of a lack of fairness, for students demand sticking to the rules. Competitions expecting short-term answers do not permit any personal freedom for students. Therefore, interest in mathematical objects usually cannot be developed in these situations.

4.2. Conclusions

The central criterion to win in the competition situations I've investigated so far is the promptness to solve mathematical exercises. It is still uncertain if every kind of competition in math lessons hinders the development of mathematical interest. If, above all, the quality of resolution is the criterion to win a contest, these competition situations may initiate deep, intensive, and meaningful explorations of mathematical issues, so that students might be more likely to develop interest in mathematical objects. The contest character may not always be in the centre of a competition situation in math classes. More research has to show conditions, under which competition situations are likely to foster mathematical interest.

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THE 'PARTICULAR', 'GENERIC' AND 'GENERAL' IN YOUNG CHILDREN'S MENTAL CALCULATIONS

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When asked to perform a mental calculation and then to say what was in their head when doing it, young pupils sometimes describe what they did with the particular numbers given. In some descriptions, however, pupils use the numbers as generic examples to explain the procedure that is used. In other descriptions they may simply state a general rule. Responses given in six interviews over a two year period by UK pupils aged 7 to 9 years have been analysed in terms of these categories of generality. Results suggest that the use of non-particular expressions of generality is a characteristic of pupils who are successful in mental calculation. There was evidence also that the use of these modes of expression did not necessarily bring success in difficult questions where flexibility was needed.

INTRODUCTION

Studies of the influence of classroom activities on young children's mental representations have been described in Bills & Gray (1999, 2000). In a two-year longitudinal study pupils have been asked to perform a total of 45 mental calculations in the six interviews. Their subsequent descriptions of what was 'in their head' when they performed a calculation fall broadly into three categories. After calculating 48 add 23, for instance, some descriptions have been categorised as 'particular' because the pupil simply said what they had done with those numbers, e.g. "I just added 20 then I just added 8". Other descriptions are 'generic' because they use the numbers as a vehicle to describe a procedure, e.g. "if it's 40 with 20 is 60, 48 add 20 comes 68 and then you add 3 on". The 'general' category of response makes little mention of the numbers, e.g. "I just added the tens and added the units and then added them both together". When questions answered correctly were compared with incorrect answers it was found that 'particular' expressions most frequently accompanied wrong answers. Correct answers were more often accompanied by 'generic' or 'general' expressions. Moreover, high-achieving pupils were more likely to use non-'particular' expressions than low-achieving pupils.

Responses to questions about non-numeric procedures and non-mathematical concepts were also categorised into three similarly designated categories. The data shows that pupils of all achievement levels, in mental calculation, can use 'general' expressions in non-calculation contexts. This suggests that the use of non-'particular' expressions in descriptions of mental calculations is not simply a sign of a pupil's linguistic sophistication. The analysis of responses to difficult questions suggests, however, that flexibility in mode of response is required. The implication for the teacher is that the style in which a pupil describes a calculation procedure may be a useful indicator of understanding.

ASPECTS OF GENERALITY

Procedural and Proceptual Thinking

The distinction has been made between 'instrumental' and 'relational' understanding (Skemp 1976) where relational understanding requires knowing both what to do and why, whilst instrumental understanding involves simply knowing how to do something. Gray and Tall (1993, 1994) argue that those who fail at mathematics have failed to progress satisfactorily from the procedures of counting to the processes of arithmetic and similarly fail to generalise from other learned procedures in other areas of mathematics. They distinguish between flexible, 'proceptual' thinkers, for whom a symbol is a mathematical object that can be manipulated in the mind, and instrumental, 'procedural' thinkers for whom the symbol simply signifies a procedure to be carried out.

Pupils may thus use a procedure without understanding. This view was endorsed by Harries (1997) in a study with low attaining children. He described one pupil's use of the written algorithm for two-digit addition where she failed to carry (her answer for 49 + 22 was 611). Her consideration of the correctness of the answer was based on her perceived accuracy of the process not on the basis of the objects she was working with.

Particular, Generic and General

Mason and Pimm (1984) in their paper "Generic Examples: Seeing the General in the Particular" use the term 'generic example' when a particular number is used to stand in for others and doesn't rely on any specific property of that number. More generally, a generic example has been described by Balacheff (1988,p 219) as "an object that is not there in its own right, but as a characteristic representative of the class". The use of the word 'generic' to imply a representation of something more general is common, Johnson (1987) for instance, gave a definition of 'schema' as a cluster of knowledge representing a particular 'generic' procedure, object, percept, event, sequence of events, or social situation. He suggests that this cluster provides a 'skeleton structure' for a concept that can be instantiated with the detailed properties of the particular instance being represented.

The notion that a particular instance is recognised as typical of a class and is used to represent that class is also the key to Rosch's theory of categorisation. In her view (Rosch, 1977) categories are not coded in the mind as lists of individual members of the category nor as lists of category inclusion criteria but as 'prototypes' of the most characteristic members of the categories. This theory suggests that learners construct concepts by comparing new experiences with prototypical or generic examples which represent their current knowledge. Thus our representation of the general is in terms of the particular. For the purposes of this paper 'generic' will be used as a label for both generic examples used in procedures and proto-typical exemplifications of concepts. The terms 'particular' 'generic' and 'general' will be used as categories both of modes of expression and of the mental representations that they might reveal.

Pavio (1971, p18) insisted that it is "simply asserting a truism" to say that modes of representation evolve within the individual from the more concrete to the more abstract. Luria (1982) for instance found that young children, asked for a definition such as "What is a dog?", gave proto-typical associations ("a dog barks") but older children responded with a more abstract verbal-logical category definitions ("is an animal"). In this paper 'particular' is used to signify a 'concrete' representation both in pupils responses to procedure questions, were descriptions involve what was done with particular numbers, and to concept questions where a particular object is given. 'General' is the most abstract representation for procedures, where a rule is expressed without reference to numbers, and for concepts, where a definition is given. Between these two levels of abstraction 'generic' uses a particular instance as an exemplar of the general. As previously noted, however, (Bills and Gray, 2000) literature on mental representation generally supports a view of variability within and between individuals rather than the existence of developmental levels.

METHOD

Lesson observations and pupil interviews were first conducted with two classes from Year 3 (pupils aged 7 and 8 years) in a school for children aged 5 to 11 years in a large middle-income village near Birmingham U.K., from September 1998 to July 1999. The same pupils were observed and interviewed in the following year. The 80 children in the year group had been placed in one of three groups for Mathematics based on their previous attainments. Lessons with the high attainment and the middle attainment groups were observed and a sample of 14 pupils from the first and 12 from the second was interviewed in December, March and July in each year. The samples were chosen to represent the spread of achievement levels in each group.

Examples have been given in the introduction for each of the categories of response for calculation questions. In the numerical procedure questions, such as "Tell me how to multiply by ten", pupils described what to do with a particular number (e.g. "ten times ten you add ten ten times") or chose a generic example (e.g. "like if it was 8, just add a nought on to it, so it'll equal 80") or gave a general rule ("just add a zero"). The non-numeric procedure questions evoked similar types of response. For instance "Tell me how to tell the time":

'particular'	if the big one was at the top and the small one was at the bottom it would be 6 o'clock
'generic'	I'd look at the big hand first so if it was like at 8 past I'd round it to the nearest 5, which would be 10 so I'd say like 10 past 6
'general'	The big hand points to the minutes, the little hand points to the hours.

Pupils were also asked first "What is the first thing that comes into your head when I say ...?" for mathematical concepts (centimetre, three, millions, fraction, polygon) and then to say more about each. Similarly for non-mathematical concepts (shadow, ball, adjective, Christmas, animal). The responses fell into three categories: pupils mentioned a particular object, gave a proto-typical property (also termed 'generic') or a general property that amounted to a definition. For example "three" and "ball" elicited the following:

category	“three”	“ball”
particular	A picture of the number 3	a football
generic	It’s in the 3 times table	it’s bouncy
general	It’s a number.	It’s a sphere shape and you can play with it

Over the six interviews 78 questions were used. They were classified into 10 calculation types and 6 non-calculation type. Each was presented verbally and followed by “What was in your head when you were thinking of that?”

Type	Description	Examples of questions
1	1-digit addend	17 + 8, 17 + 9 (repeated in each interview)
2	Missing addend	13 + * = 18, 30 + * = 80, 27+*=65
3	2-digit addition	48 + 23 (repeated in each interview)
4	Addition of multiple of 10	97 + 10, 597 + 10, 1097 + 10, 1197 + 10
5	Counting	What comes before 380, 2380, 12100; after 12386
6	Rounding	Round 2462 to the nearest ten, 239 to nearest hundred
7	Recent topic	What is difference between 27 and 65, 0.6+0.7
8	Recent topic	65 subtract 29, Read time (11:40), 0.1 times by 10
9	Division and fractions	quarter of 40, third of 48, 140 divided by 3
10	Multiplication	48 multiplied by 3, 47 multiplied by 5
11	Numerical procedure	Tell me how to add 23, find a third, times by ten,
12	Non-numerical procedure	Tell me how to cross road, tell the time, do subtraction
13	Maths concept, first	First thing in head when I say centimetre, three, million
14	Maths concept, more	What else can you tell me about centimetre, three, million
15	Non-Maths concept, first	First thing in head when I say shadow, ball, adjective
16	Non-Maths concept more	What else can you tell me about shadow, ball, adjective

RESULTS

Context differences

In the calculation questions there was a marked difference in the distribution of the categories of generality between questions answered correctly and those answered incorrectly (chi-square test significant, $p<0.005$). Those who gave a ‘generic’ or ‘general’ response were more likely to be correct and those who expressed themselves in ‘particular’ terms were more likely to be wrong:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
Right	184	**30**	324	**53**	107	**17**	615	**100**
Wrong	147	**51**	110	**38**	30	**10**	287	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

Descriptions of the procedure in some way other than what was done with the particular numbers are thus associated with accuracy.

Categories of generality were also compared across all aspects of the interviews and there are distinct differences between distributions in the different contexts:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
Maths Calculation	331	**37**	434	**48**	137	**15**	902	**100**
Maths Non-calc	103	**27**	206	**53**	77	**20**	386	**100**
Non-Mathematics	87	**31**	101	**36**	93	**33**	281	**100**
Totals	521	**33**	741	**47**	307	**20**	1569	**100**

Pupils demonstrate in non-mathematical contexts that they can give responses at all levels of generality but in calculation questions, mathematics-procedure and mathematics-image questions they are more likely to express themselves in 'particular' or 'generic' terms. This is emphasised when mathematics questions are grouped:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
Mathematics	434	**34**	640	**50**	214	**17**	1288	**100**
Non-mathematics	87	**31**	101	**36**	93	**33**	281	**100**
Totals	521	**33**	741	**47**	307	**20**	1569	**100**

Pupil differences

When the pupils were grouped by their level of achievement in the three written mathematics tests (SAT) conducted at the end of each year it became clear that the higher achieving pupils use more non-'particular' expressions of generality than the lower scoring pupils in calculation questions ($p<0.005$):

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
Higher SAT scores	136	**28**	253	**52**	100	**20**	489	**100**
Lower SAT scores	195	**47**	181	**44**	37	**9**	413	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

The difference between the groups is not significant in non-mathematical questions:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
Higher SAT scores	48	**32**	53	**36**	48	**32**	149	**100**
Lower SAT scores	39	**30**	48	**36**	45	**34**	132	**100**
Totals	87	**31**	101	**36**	93	**33**	281	**100**

The differences in pupils is more pronounced when grouped by their performances in the interview calculation questions. The three groups: High- (scores greater than 1 sd above mean), Middle- (scores within 1 sd of mean) and Low- (scores less than 1 sd below mean) accuracy pupils, express themselves quite differently ($p<0.005$):

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
High-accuracy	59	**32**	89	**48**	36	**20**	184	**100**
Middle-accuracy	191	**33**	305	**52**	87	**15**	583	**100**
Low-accuracy	81	**60**	40	**30**	14	**10**	135	**100**
Totals	331	**37**	434	**48**	137	**15**	902	**100**

Once again there is no statistically significant difference between the groups in non-mathematics contexts though here high-accuracy pupils use a higher proportion of 'particular' expressions than the other pupils:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
High-accuracy	22	**39**	16	**28**	19	**33**	57	**100**
Middle-accuracy	50	**29**	63	**36**	61	**35**	174	**100**
Low-accuracy	15	**30**	22	**44**	13	**26**	50	**100**
Totals	87	**31**	101	**36**	93	**33**	281	**100**

Difficult and easy questions

The facility level of questions varied from 0% to 100%. Eleven questions were answered correctly by ten or fewer pupils. When these 'difficult' questions are compared with the others the change in styles of response are similar for groups of children with different levels of success. 'High-success' pupils were correct in more than 4 of these. 'Low-success' pupils did not answer any correctly. Each group is less likely to use non-'particular' expressions in difficult questions and the swing toward 'particular' is most marked in the response of the most successful:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
High-success easy	47	**28**	91	**54**	32	**19**	170	**100**
High-success difficult	25	**42**	26	**43**	9	**15**	60	**100**
Mid-success easy	109	**31**	183	**52**	59	**17**	351	**100**
Mid-success difficult	48	**41**	58	**50**	10	**9**	116	**100**
Low-success easy	78	**48**	61	**37**	24	**15**	163	**100**
Low-success difficult	24	**57**	15	**36**	3	**7**	42	**100**
Overall easy	234	**34**	335	**49**	115	**17**	684	**100**
Overall difficult	97	**44**	99	**45**	22	**10**	218	**100**

The most successful pupils use a more even spread of 'particular' and 'generic' in difficult questions and their proportion of 'general' expressions is not as reduced as the other pupils. The sense that pupils carry on describing procedures even in questions they get wrong by failing to use the procedure correctly, is re-enforced when two similar questions are compared:

Year	Term		Correct	No response	Particular	Generic	General
3	1	597+10	17	2	6	13	5
4	1	1097+10	8	3	6	14	3

Many pupils described the procedure in similar terms in each instance but failed to deal with the thousand appropriately, 2007 being a common wrong answer.

Differences over time

There appears to be no evidence of progression from concrete to abstract when individuals' categories of generality are analysed. Some pupils were consistent in their mode of explanation but most showed variability over the six interviews. A few did move from 'particular' to 'general' in comparable questions with an accompanying improvement in accuracy. There was little change in the distribution of categories for groups of children over the two years. When the total number of responses in each category in Y3 is compared with totals for Y4 the main difference between the groups is in the number of questions that they could respond to:

	Number of responses (percentages of row totals in bold)									
	No response		Particular		Generic		General		Totals	
High-accuracy Y3	16	**15**	31	**30**	43	**41**	15	**14**	105	**100**
High-accuracy Y4	25	**21**	28	**23**	46	**38**	21	**18**	120	**100**
Middle-accuracy Y3	60	**18**	91	**27**	141	**42**	44	**13**	336	**100**
Middle-accuracy Y4	77	**20**	100	**26**	164	**43**	43	**11**	384	**100**
Low-accuracy Y3	36	**34**	40	**38**	23	**22**	6	**6**	105	**100**
Low-accuracy Y4	54	**45**	41	**34**	17	**14**	8	**7**	120	**100**
Total Y3	112	**21**	162	**30**	207	**38**	65	**12**	546	**100**
Total Y4	156	**25**	169	**27**	227	**36**	72	**12**	624	**100**

Against this background of global lack of change it is instructive to consider performance in one question, "48 add 23", used in the first five interviews:

	Number of responses (percentages of row totals in bold)							
	Particular		Generic		General		Totals	
48 + 23 Correct	15	**19**	44	**56**	20	**25**	79	**100**
48 + 23 Incorrect	15	**50**	10	**10**	5	**17**	30	**100**
Totals	30	**28**	54	**50**	25	**23**	109	**100**

The proportion of non-'particular' is higher than the average for correct answers as might be expected for a relatively easy question but the breakdown for the number of responses in separate interviews in each term in each year is more revealing:

Year	Term	Correct	No response	Particular	Generic	General
3	1	13	6	3	14	3
3	2	16	5	4	14	3
3	3	14	3	3	15	5
4	1	17	3	7	7	9
4	2	23	2	8	11	5

This shows a high level of 'recipe-following' in Y3 when the 'generic' responses reflect the use of frequently practised written algorithms. In Y4 pupils were encouraged to use more of their own mental strategies and this meant initially both more 'particular' and more 'general' responses. In the final interview the pupils were most accurate yet used a high proportion of 'particular' expressions

DISCUSSION

The initial picture presented by the data suggests that expressions at a 'generic' or 'general' level are strongly associated with accuracy. Accurate answers are most often explained in one of these modes and accurate pupils use more of these expressions than the least accurate. They are good at 'instantiating' their 'skeleton' mental representations of procedures. This does not imply, however, that achievement in mental arithmetic is due simply to an ability to express oneself in this way. The analysis of non-mathematical items shows that all pupils are capable of non-'particular' expressions of generality and, if anything, the most accurate reserve this mode for calculation questions more than in non-mathematical contexts.

When the responses to difficult questions are considered the pupils use fewer non-'particular' expressions than in easy questions. This is predictable because these questions are associated with low accuracy. Those pupils who are most successful in the difficult questions, however, show a greater flexibility by switching more to 'particular' modes of expressions. The others continued to use similar proportions of 'generic' expressions to those they had used with easy questions. This same picture of advantage in flexibility of expression emerged when the responses to one question which occurred in five interviews were examined. Initially pupils spoke predominantly in procedural terms, typically "you add the ...", but when the standard procedure was practised less often in the classroom they showed greater variation in expressions of generality and there was a higher level of success.

At one level this paper seems to provide one more measure which discriminates the successful from the unsuccessful. The least successful are less likely to describe their calculations in 'generic' and 'general' terms than the more successful. The data also demonstrates, however, that pupils can describe procedures without being aware of the correctness of their answers. They use a similar proportion of 'generic' expressions in easy questions, that they get right, to the proportion in harder questions, that they get wrong, using the same procedures. There is evidence here of 'instrumental' understanding. Teachers may have an indication that pupils know a procedure by their use of expressions of generality but need to be aware that this does not imply that the pupils understand what they are doing.

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Shifts in the Meanings of Literal Symbols

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The difference in roles played by literal symbols has been used as an organising principle for various aspects of the learning of algebra. In this paper I draw attention to the shifts in the meanings attached to literal symbols which may take place during certain problem solving procedures. I argue that students' appreciation of these shifts can contribute to their meaningful use of these procedures.

Introduction

Much of the literature on children's understanding of algebra uses as an organising principle the notion of different uses of literal symbols.

Küchemann's (1981) early classification of children's use of letters consisted of six types of response to test items: *letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalised number* and *letter used as a variable*. The last three of these six were thought of as demonstrating some kind of algebraic competence.

Usiskin (1988) describes four meanings of variable linked to different purposes of algebra. These are *generalisations*, if algebra is seen as generalised arithmetic; *unknowns* if algebra is seen as a procedure for solving certain kinds of problems; *parameters* or *arguments* if algebra is seen as the study of relationships between or among quantities; finally, *arbitrary objects which are members of an abstract system.*

In contrast Sfard and Linchevski (1994) trace the historical development of algebra through the following stages: algebra as generalized arithmetic (operational and then structural phases), algebra of a fixed value (of an unknown), functional algebra (of a variable), and finally abstract algebra (algebra of formal operations and abstract structures).

Ursini and Trigueros (1997) consider the algebraic skills necessary for undertaking undergraduate studies under three headings: *variable as unknown*, *variable as general number* and *variables in a functional relationship*. They assert that "College students should be able to cope with all of them, moreover in order to handle the variable as a mathematical object they should be able to integrate its different uses in one concept and shift between them depending on the requirement of the task" (p 256). However in this paper and in Trigueros and Ursini (1999) they focus on tasks which demonstrate understandings of one of these roles rather than of shifts between roles.

Each of these studies has a slightly different purpose for the notion of different uses of literal symbols. Küchemann uses it to classify children's responses; Usiskin links it to different understandings of the purpose of algebra; Sfard and Linchevski use it to identify stages in the historical and epistemological development of algebra and Ursini and Trigueros use it to identify and classify algebraic skills. This paper will consider, not the categorisation of algebraic activity by different uses of variables, but the kind of thinking required to move flexibly between different uses.

The study which gave rise to this paper was undertaken with sixteen and seventeen year old pupils. A key feature of the mathematics curriculum which they were studying was the introduction of problems which involved differing roles of literal symbols within the same problem, for example:

- using "standard" forms such as $y = mx + c$, $ax^2 + bx + c = 0$, where the roles of the x and y variables are familiar, but a, b, c and m are replacements for what have, up till now, been numbers
- considering functions such as $k(k-1)x^2 + 2(k+3)x + 2$ in which the roles of the two variables could be seen as equivalent but are more likely to be seen as very different because of the familiarity of one (x) and the relative unfamiliarity of the other
- using the notion of a variable point, (x, y) or (a, b) rather than a single variable, or of a variable line or curve.

Problems of this kind are often described simply as those involving parameters but this description seems to miss some important distinctions between roles of literal symbols which will arise in the later discussion. Moreover, as my second example above illustrates, the role of a literal symbol within an equation or problem situation need not be inherent to that situation but may be determined by the perspective and actions of the individual. It would be mathematically correct to describe the expression $k(k-1)x^2 + 2(k+3)x + 2$ as a quadratic in k, but much more likely that it would be seen as a quadratic expression in x with k as a parameter.

Amongst recent studies of students' work on problems involving 'parameters', Furinghetti and Paolo (1994) undertook a larger scale study which involved setting questionnaires to 199 sixteen and seventeen year olds. Amongst the points which arise from the researchers' analysis of the replies are the following:

- letters in apparently symmetrical roles, e.g. $kx > 0$, cause difficulties for students
- some letters elicit a stereotyped expectation of role

Their main conclusion, however, is that the majority of the students they surveyed had difficulty in expressing the difference between unknown, variable and parameter.

My study led me to the conclusion that very many of the problems that my students were tackling involved a subtle shift in the role played by the literal symbols; moreover it was this shift which in each case provided the power that made the solutions to these problems 'standard methods'. I will argue that it is these shifts of meaning that allow students to perform these standard methods flexibly and not by rote.

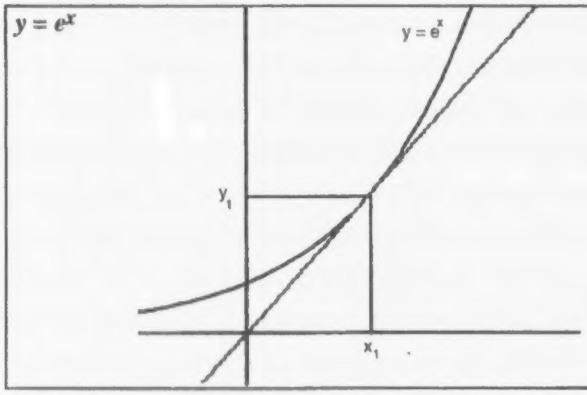
My use of the word 'shift' follows, but is not the same as that made by Mason and Davies (1988). The kind of 'shift' I am pointing to is a 'shift of attention' which occurs within the mind of the individual. By connecting them to certain classes of problems I am claiming that such shifts are associated with success with these problems and flexibility in the application of the methods of solution.

Shifts

'Variable' to 'unknown-to-be-found'

By considering the range of types of problem that the students were working on I identified four kinds of shift. The first shift is in the role of x or y from 'variable' to 'unknown-to-be-found'. By 'variable' I mean a quantity whose importance is entirely in its relationship with another quantity, rather than in its value or values. The term 'unknown-to-be-found' means a literal symbol which has a particular value which is to be discovered. For example, in a solution to the question 'Find the co-ordinates of the point where the line $x + 2y - 4 = 0$ meets the line $y = 2x - 2a + b$ ' x and y are first seen as variables whose importance is in their relationship to each other. Each can take any real value. However as soon as the learner begins to solve these as a pair of simultaneous equations, x and y take on the roles of unknowns, whose numerical values are to be found. Other examples of problems involving this shift are questions which require the co-ordinates of points of intersection with the axes or of turning points. Although it is a significant issue in earlier years at school, I found no evidence that simple problems involving this shift were still causing difficulties for students at age sixteen.

However, when this shift occurred within a more complex problem one group of students produced a very interesting response. Below is a reproduction of the work of one group on the problem "Find the point on the curve $y = e^x$ at which the tangent goes through the origin"



gradient of
tangent $= e^{x_1}$

equation of tangent $\Rightarrow y = e^x x$
$\Rightarrow y = e^x.$

$\therefore \quad e^x x = e^x.$
$\therefore \quad x = 1 \quad \Rightarrow y = e.$

On a first reading I was struck by the fact of the various meanings of x in their workings. They drew a sketch of the curve and in their diagram they recognised the *particular* nature of the point of tangency by labelling it (x_1, y_1) and the gradient of the tangent e^{x_1}. However, in their working to find the x coordinate of the point of tangency, they dropped the subscripts and wrote $y = e^x x$ as 'equation of tangent'. According to one analysis, the first 'x' in this equation refers to a particular value of x which occurs at the point where the tangent touches the curve. The second 'x' is a variable in a relationship between x and y which can be represented by a straight line.

At this stage, in their notation they have lost the distinction between x as a variable in an expression of a relationship represented by a straight line and x as an unknown coordinate of the point of tangency (as expressed by x_1). Both these meanings of x exist within the same equation.

In the next equation, $e^x x = e^x$, they equated the y-coordinate of a point on the tangent with the y-coordinate of a point on the curve in order to find the value of x at the point of contact. Now each x in the equation refers to the unknown value previously called x_1 and x is unambiguously an unknown-to-be-found. The students did not see the need to use two different letters to make the earlier distinction. However the elision between the two meanings of x did not prevent them from going on to find the particular value of x required.

This ambiguous use of x occurs because of a shift from variable in the equation of a curve (in the equation $\frac{dy}{dx} = e^x$, x stands for the first coordinate of any point on the curve $y = e^x$) to unknown particular value of x (in the statement 'gradient of tangent = e^x' x stands for the first coordinate of the point where the tangent touches the curve).

'Placeholder-in-a-form 'to 'Unknown-to-be-found'

The second shift I have identified involves the role of placeholder-within-a-form. By this I mean literal symbols which carry a special meaning by virtue of being frequently encountered in a specific context e.g. m and c in $y = mx + c$. The special status established by these literal symbols is discussed in Bills (1997). The shift in question takes place from the role of placeholder-in-a-form to that of unknown-to-be-found. For example in answering the question 'What is the equation of a straight line with gradient 3 which passes through the point (2, 8)?' a student might substitute $x = 2$ and $y = 8$ into the form $y = 3x + c$ to find a value for c. As a result of this substitution, c changes from being a placeholder for 'the y intercept' within a standard form, to being an unknown-to-be-found. This shift was also identified by Bloedy-Vinner (1994) as follows:

> 'Moreover, the meaning of a letter as a parameter or as an unknown or variable, might change throughout the process of solving a problem ..... Solving this problem ("Find an equation for the line through (2, 5) with slope 3") starts with writing an equation $y = ax + b$, where common knowledge determines that x and y are variables whereas a and b are parameters. The process continues by substituting the constant 3 for a, and solving an equation with unknown b, where constants are substituted for x and y. The process terminates by substituting the constants found for a and b, and by letting x and y be variables in $y = 3x + 1$' (p89-90)

Again, students at this level find it relatively straightforward to learn and use procedures which involve this shift. However a slightly different picture emerges when the problem offered to students is non-standard in some way. For example, in the extract below there is insufficient information given to find the equation of the line.

Paul and Trevor had asked me to give them a revision session on aspects of coordinate geometry because both had missed some of the lessons on this topic. I began by speaking to them about the general form for the equation of a straight line, $y = mx + c$, and how they would use it to find the equations of particular lines. I asked them to explain what they could tell about the equation of a straight line if they knew the line went through the point (1, 2), and, after their answer, I continued by asking them what they could say about the equation of a line which passes through (0, 4).

Trevor: c has to equal 4.

Paul: c equals 4, because 4 equals $0 + c$

Trevor: So the gradient is 0.

Liz: It tells us that c is 4, which is, you could have done that by a slightly different sort of reasoning because, c, you said to me was the point on the y-axis where it cuts.

Trevor: Yeah.

Liz: And this point (0, 4) is on the y-axis. It goes through (0, 4) then c is 4. What does it tell us about m?

Paul: That it's 0 because ....

Trevor: I don't know if it would be 0, cause you are just saying that x is 0. It still could be at an angle

Paul: We know, we know that $y = 4$, in this particular case and we know that 4 is c, so we know that mx has got to equal 0.

Liz: Yeah.
Paul: And the only way mx could equal 0 is if m is 0.
Trevor: No but
Paul: No, No, No - because x is 0.
Trevor: x is zero, so m could be anything
Paul: Yeah that's it so m could be anything

Trevor and Paul's attention throughout this extract and most of the rest of the conversation was on substituting values for x and y into an equation for a straight line in order to find the values of the placeholders, m and c, now treated as unknowns. They began by treating the point (0, 4) in the same way. My agenda was different. I wanted them to see that (0, 4) can be treated differently because it is on the y-axis, but my intervention failed to shift their attention away from the substitution they had made. In response to my question about the value of m they returned to their equation $y = 0 + 4$ and Paul deduced that m must be equal to zero. One interpretation of his line of argument is that , knowing that he was seeking information about m, he chose to treat x as indeterminate, a varying quantity which must be given freedom to vary, rather than treating it as a known value, 0. The situation was compounded by the fact that there was insufficient information to calculate a value for m. The students expected to be able to find the value of m and taking x as indeterminate rather than given enabled them to do so. However their focus on the role of m as unknown eventually allowed them to see a solution.

This incident is also an example of what Furinghetti and Paolo (1994) speak of as students' difficulties with letters in apparently symmetrical roles i.e. $mx = 0$. The confusion might be understood as one between the two different roles of the letters in an equation which gives no clues as to which letter is playing the role of unknown and which the role of given.

'Unknown-to-be-taken-as-given' to 'Unknown-to-be-found'

The third shift can occur when an *analytic* solution method is used. I use the term analytic in the sense of Klein (1968). He describes "analysis of the first kind" as a method of solving problems algebraically which assumes the unknown as known and then transforms the equation to identify the unknown. For example, in solving analytically the problem 'Find the point of contact of the tangent to the curve $y = x^2 + 1$ which passes through the origin', the first step is to name the unknown by choosing a letter to stand for the x coordinate of the point of contact. The chosen letter, a say, is then treated as given and used to form equations which express relationships between a and other quantities. Finally those equations are solved for a . The shift that takes place then is from unknown-to-be-taken-as-given to unknown-to-be-found.

A number of students worked on a similar task at my request. The adapted task was "Find the equation of the tangent to the curve $y = x^2 + 1$ which passes through the origin".

When Paul was given this question to work on he recognised that it could not be solved by the synthetic approaches he had used so far:

Paul: Because you've, you've got no place to start after you've done that. You know that it passes through the origin, it could do that, it could do that (*he indicates lines passing through the origin with different gradients*) whatever, be a tangent to the curve and pass through it. I

> don't know, you can only do it one. Well you know it passes through the origin but you've got no, no idea where it touches and you need to know where it touches to be able to get the gradient and you need to have the gradient to know where it touches. So you've got a loop which you can't, you can't solve that easily.

From my standpoint I can understand Paul's description as saying that this question requires an analytic rather than synthetic approach. An 'analytic' approach works from the unknown, treating it as known in order to discover its value. A synthetic approach starts from the known and works towards the unknown. Paul wanted to work from the known to the unknown but found that he could not do so.

I tried to encourage him to use an analytic approach. However he understood my suggestion as advising him to guess a particular value rather than to express ignorance by the use of a letter. He eventually solved the problem by a guess and check procedure, beginning by guessing a value for the gradient. Next he found the x coordinate of the point on the curve where the gradient was equal to his chosen value. Then he checked that this point on the curve also lay on the line through the origin with chosen gradient.

Paul went through this guess and check procedure twice, first starting with a gradient of 1, which he found did not fit all the conditions, and secondly with a gradient of 2, which did. His first step in each case was to find the x coordinate of the point on the curve which had the given gradient. In other words, having decided on a trial value for m he used it to find a value of x which fitted certain conditions. A little later in the conversation I encouraged him to develop this approach into an analytic one, where the unknown gradient is named as m and the testing procedure is adapted to set up equations from which the value of m can be calculated.

Liz: .. think back to what you did to start with. You said 'suppose the gradient's 1,'

Paul: oh, yes.

Liz: Now go back to that stage and think 'suppose the gradient's m.'

Paul: If you put .... $y = mx + c$ *(inaudible)*. You know it's zero, you know it's $y = mx$

Liz: yes

Paul: equals, ... ah $m = 2x$ doesn't it, because the gradient's - m is going to equal $2x$, so it's $y = 2x^2$

His earlier reasoning was along the following lines (although I have no evidence that his mental image was in terms of equations and implications)

$$\text{gradient} = 2 \quad \Rightarrow \quad 2x = 2 \quad \text{at point of contact}$$
$$x = 1$$

With m as gradient however he proceeded in this way

$$\text{gradient} = m \quad \Rightarrow \quad 2x = m$$
$$m = 2x$$
$$\text{tangent is } y = mx \quad \Rightarrow \quad y = 2x^2$$

If Paul had followed his guess and check procedure, his first stage would have been to say that the point on the curve at which the gradient is m is given by $2x = m$, so that $x = \frac{m}{2}$ at this point.

However, rather than finding x in terms of m, that is treating m as the known and x as the unknown, he took an expression for m in terms of x as his next stage. Even though it would have led him along a route parallel to that which he had already travelled when using his 'guess and check' approach, he was not able to treat m as known. This approach led him into difficulties because the x he was working with here was the x coordinate of the point of contact rather than the x coordinate of any point on the tangent. Seeing that he had derived $y = 2x^2$ as the equation of the tangent alerted him to the need to rethink.

'Unknown-to-be-taken-as-given' to 'Variable'

The fourth shift takes place when a quantity which was originally conceived of as constant, though unspecified, is allowed to vary, that is it is a shift from unknown-to-be-taken-as-given to variable. This shift frequently occurs in solutions to locus problems, for example 'A point P, co-ordinates (a, b) is equidistant from the x-axis and the point (3, 2). Find a relationship connecting a and b.' In the solution to this problem, a and b are first taken to be fixed but unspecified, so that expressions for the distances from (a, b) to the x-axis and (3, 2) can be formulated in terms of these unknown-to-be-taken-as-givens. Once these expressions have been equated the equation formed can be seen as a relationship between variables and a and b can be allowed to vary in order to map out a parabola. In this question this last stage, which represents the locus aspect of the problem, is not emphasised, because the question asks merely for a relationship between a and b. An emphasis on the locus aspect of the problem is usually accompanied by a change in notation which allows the final relationship to be expressed in terms of x and y. This notational change allows the shift to seeing the letters as variables to take place more easily because the conventional roles of x and y are as variables.

My students were set a test and one of the questions was as follows:

"A circle has centre (2, 4) and passes through the point (-1, 5). The point (p, q) lies on the tangent which touches the circle at (-1, 5). Find an equation linking p and q. Hence write down the equation of the tangent."

Of the students who made any substantial attempt at the question, all but one worked from the outset with x and y rather than p and q. Some obtained an equation in terms of x and y and then substituted p and q into it. Some did not include p and q in their answer at all.

The method used by the students was to find the gradient of the radius and hence of the tangent and then to obtain the equation of a straight line with this gradient and passing through (–1, 5). The focus was always on the tangent as a line and not on the point (p, q) on the tangent. A method which focused on the point (p, q) might have been based on expressing this fact: that the line joining (p, q) and (-1, 5) and the line joining (2, 4) and (-1, 5) are perpendicular and therefore that the product of their gradients is 1.

The students' choice of method and of letters indicates that they were following a standard procedure for finding the equation of a tangent to a circle, rather than responding meaningfully to the problem which had been set. Their understanding of the standard procedure does not include an awareness that at the outset (x, y) represents a point on the tangent, and hence they ignore the cue in the question to use (p, q) as that point. An interview with one of the students suggested that his choice of (x, y) was unconscious.

Conclusion

I have shown how considering shifts in the meaning of variables can give a new perspective on standard problems and routine procedures in this area of algebra, and illustrated how students' appreciation of these shifts can be related to their meaningful performance of these routines. In my first example students showed by their use of notation that the shift they were making was unconscious. This unconsciousness put them at risk of making an error. In my second example Paul and Trevor were able to solve a non-routine problem because they were confident in the role of *m* as unknown-to-be-found, even though it had been introduced as a placeholder. In the third, Paul was, at least initially, unable to pursue the analytic approach because he wanted to shift too soon from *m* as unknown-to-be-taken-as-given to unknown-to-be-found. Finally, students faced with a locus question use an established routine rather than approaching the question as set because their understanding of the routine does not include an appreciation of the shift in meaning of the letters.

If, as the papers I have cited earlier suggest, understanding of the different meanings of variables is a problem for students, it may be that drawing their attention to the shifts in meanings involved in some standard problems and their routine solutions would be an effective way of helping students to improve such understanding.

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Do Teachers Implement Their Intended Values in Mathematics Classrooms?

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There is little knowledge about what values teachers are teaching in mathematics classes, about how aware teachers are of their own value positions, about how these affect their teaching, and about how their teaching thereby develops certain values in their students. This paper from the VAMP project presents parts of the case studies of two Australian mathematics teachers which concern the relationship between their intended and their implemented values. As well as discussing data about these teachers' values, two possible approaches to the analysis of the interview and observational data are also presented.

At PME 24 FitzSimons, Seah, Bishop, and Clarkson (2000) outlined the Australian Research Council funded three-year project which included the goals of: (a) investigating and documenting mathematics teachers' understanding of their own intended and implemented values, and (b) investigating the extent to which mathematics teachers can gain control over their own values teaching.

Values in mathematics education are the deep affective qualities which education aims to foster through the school subject of mathematics (Bishop, FitzSimons, Seah & Clarkson, 1999; Bishop, 1996) and are a crucial component of the mathematics classroom affective environment. While accepting that values, beliefs, and attitudes are dialectically related (see Krathwohl, Bloom, & Masia, 1964; McLeod, 1992; Raths, Harmin, & Simon, 1987), our concern is with the values of mathematics, mathematics education, and education in general (see Bishop, 1996), rather than more global values such as social, ecological, moral and so forth - although these are by no means incompatible, and indeed may influence teachers' personal value systems.

As Bishop, FitzSimons, Seah, and Clarkson (1999) note, there is little knowledge about what values teachers are teaching in mathematics classes, about how aware teachers are of their own value positions, about how these affect their teaching, and about how their teaching thereby develops certain values in their students. Values are rarely considered in any discussions about mathematics teaching, and a casual question to teachers about the values they are teaching in mathematics lessons often produces an answer to the effect that they don't believe they are teaching any values at all. It is a widespread misunderstanding that mathematics is the most value-free of all school subjects, not just among teachers but also among parents, university mathematicians and employers. Mathematics is just as much human and cultural knowledge as is any other field of knowledge; teachers inevitably teach values, and adults certainly express feelings, beliefs and values about mathematics which clearly relate to the mathematics teaching they experienced at school (FitzSimons, 1994; Karsenty & Vinner, 2000). More fundamentally we believe that the quality of mathematics teaching would be improved if there were more understanding about values and their influences.

It has long been recognised that teachers are continually making decisions in the classroom (Bishop, 1976) and that they are often in the position of having to judge between two or more competing values (Bishop, 1972). It is also recognised that there are differences between the values that are officially planned and those espoused by teachers (e.g., Lim & Ernest, 1997), as well as between teachers' espoused beliefs and their actual classroom practices (Lerman, 1998; Sosniak, Ethington, & Varelas, 1991) – due in part to differential positionings as interview subjects and as teachers.

This paper, then, concentrates on parts of the case studies of two Australian mathematics teachers which concern the relationship between their intended and implemented values. One teacher is an early-learning years teacher in a suburban Catholic primary school and the other teaches at the lower secondary level in a rural government school.

Methodology and Justifications

This project relied on working with, rather than on, teachers. Initially we talked about values with groups of teachers, using video clips and written classroom incidents as prompts, in professional development settings. From among the teachers who attended these sessions, and from others who completed a circulated questionnaire, we established a small group of teachers willing to work with us in their classrooms.

The basic approach adopted with each teacher was a cycle of preliminary interview, classroom observation, and post-observation debriefing interview. This cycle was repeated on two or three days. The classroom observations were video- taped, and the interviews audio-taped. This process not only asked teachers to reflect on their teaching practices and to say what values they were intending to teach; it also asked for authentication of the teacher's analysis by seeking to observe those values being implemented in the classroom situation, devised by the teacher.

Using this strategy we studied whether the teachers could articulate their own intended values, and whether they then implemented these in their classrooms. Before each observation lesson, the teacher presented the observer with a brief lesson plan including the flow of content and the teaching strategies, and also nominated the values they were intending to teach in the lesson. During the observation lessons we looked specifically for those values being implemented, but also we looked for other values being portrayed by the teacher.

We are transcribing and analysing the audio-tapes, but the video-tapes were used to stimulate discussion with the teacher. In some post-observation interviews the video-tapes became the key memory prompting device for the teacher, who then was able to elaborate on values-related episodes for the researcher. The researcher had also noted points at which both explicit and implicit values teaching seemed to be occurring, and the use of the video-tape helped both teacher and researcher to recall the detail of these episodes. The aim of the post observation interview was for the teacher and researcher to come to a shared agreement on some particular examples of when and how values teaching occurred in a particular lesson.

In the following two case studies, we present two possible approaches to the analysis of the interview and observational data. (This project is still ongoing and more data will be available at the conference.) The first takes an holistic approach to intended and implemented values, and the second focuses more on the particular values nominated by the teacher and/or observed by the researcher.

Case Study 1 (Kay): Grade1/2 (6-8 year-old students)

Table 1

Values Teaching Portrayal Matrix

Implemented Values			
		Implicit	Explicit
Intended Values	Nominated	e.g., students to internalise rewards	e.g., "today we are going to focus on purposefulness"
	Not nominated	unconscious routine, e.g., motivating praising students	e.g., group norms, and routine procedures developed over career lifetime

This middle-class, suburban Catholic primary school, of about 400 students in total had, independently of our project, adopted a general "Value of the Week" programme; on one particular week the value was "purposefulness." The female teacher, Kay, with 16 years experience, interpreted this value as: "If you have a dream you work to it." One observed class began with two girls modelling purposefulness by attempting to build and rebuild a house of cards, while Kay led a class discussion about what they might have been thinking and feeling — a good example of what we are calling explicit values teaching.

The activity Kay called "Ticket maths" arose as a means of overcoming the lack of coverage for the topic of number skills, and to keep students gainfully employed, interested, engaged for about 20 minutes – while she worked more closely with a smaller group. It was an open-ended activity, with short questions, such as: "write all you know about the number 0" written on small slips of paper. The students could choose any ticket they liked, and as many as they liked. The teacher had tried to keep the questions within the ability range of all groups - wanting the children to experience success as well as to extend the grade 2 students. The intention was to involve literacy skills in the correct writing of numbers as well as accuracy of calculations.

In the interview prior to the lesson, Kay had indicated that her intended values were: (a) to give freedom of choice within structured activities, (b) for students to internalise rewards, and (c) for students to challenge themselves (purposefulness). These may be considered as mathematics education values, and have the potential to generate creativity and independence. The intended mathematical values were to develop number skills and a variety of means of expression for these, both written and oral. These values were to be made explicit to the children through whole-class discussion and/or the nature of the activity itself. Extracts from the transcript of the pre-lesson interview reveal these ideas.

Researcher: Why do the children like ticket maths?

Kay: Maybe it's a bit of a win-win situation … It is also good to make use of existing resources … [Last time] I didn't say 'don't get counters' and I didn't say 'do get counters'. It was interesting [to see] the children who worked without counters. … [About values] 'I'm wanting you to give your best.' … I'll be really pushing purposefulness tomorrow. It's a really good value for working. … I hate rewarding what's normal behaviour in children.

Researcher: So you want them to internalise the rewards?

Kay: Oh, yeah. …I hate behaviour being manipulated by a bit of silver paper [a silver star reward]. I want behaviour to be manipulated by your own sense of place and space. 'I'm wanting you to challenge yourself. I'm wanting you to give your best.'

The lesson observation indicated that values (a) and (c) were implemented explicitly, but that value (b) was only implemented implicitly. In addition there were other values implemented which were not nominated by Kay. For example, group norms had been established so that the children began as a whole group sitting on the floor, in order to focus on the lesson content in its fullest sense of mathematical activities and ways of working mathematically, and in accordance with this week's Value (i.e. purposefulness). In addition, other norms had been well established, such as the idea that the small focus-group was not to be interrupted if possible, and that there would be a breakup of activities within the allocated timeslot for mathematics.

The researcher's interpretation of the values implemented but not nominated by Kay include the need for motivation of the students, and the need to avoid causing the students shame or humiliation. For example, during the activity "Ticket maths" she left the small group and circulated among the other students, checking their work, praising them verbally, and even drawing 'smiley faces' on their pages. This behaviour appears to contradict her earlier espoused value of students seeking intrinsic rewards, but it is not uncommon for teachers to have to make decisions between competing values. It also happened that one student, whom Kay recognised as attempting to please her, had made an error (writing 9 x 9 instead of 9 + 9). Following her non-judgemental suggestion that he lay out the problem again with counters, he discovered his mistake.

In addition, Kay was aware of the need to justify her actions to parents and other teachers. Over a career lifetime teachers develop and formulate certain values in order to articulate them when called upon to do so, but mostly they remain tacit. In the terms of this project, we consider them as 'not nominated'.

One possible way of demonstrating the relationship between Kay's intended and implemented values is presented in Table 1 (above).

Case Study 2 (Josh) : Grade 7 (12-13 year-old students)

Table 2

Values intended and/or implemented

Value	**Intended**		**Implemented**	
	Nominated	Not nominated	Explicit	Implicit
Relevance	X		X	
Appropriate use of technology	X		X	
Mental computation	X		X	
Scientific practice [a]	X			X (unaware)
Listening		X	X (unaware)	
Accessibility of the teacher		X	X	
Efficiency		X	X	
Peer teaching		X		X
Confidence		X		X

[a] This refers to Josh's valuing of scientific practice, such as in starting a lesson with a definition, encouraging student collection of data/information to conduct the appropriate analysis, and the way in which his activity worksheet layout resembles a student laboratory work report.

Josh teaches mathematics to a Grade 7 class in a middle-class, rural secondary school, which has a student population of about 900. He was a chemist in the local dairy centre before entering the teaching profession 14 years ago. This industry experience has obviously influenced his outlook towards the purpose of education in general, and towards the inculcation of values through (mathematics) teaching in particular. In his questionnaire response to a student hypothetical question of the purpose of studying mathematics in school, Josh wrote that the school subject "will help you prepare for dealing with a range of situations throughout life – especially solving problems". In his response to another hypothetical situation of several students protesting against working in a group, his view was that while one may work better individually, "this will not always be possible, especially later in life when you are working. Everyone needs to be able to learn to get on with other people in a cooperative fashion".

Thus Josh is a teacher who perceives both the subject of school mathematics and the ways through which this subject is taught as means of preparing students for meaningful daily living. Josh's espousal of this value of 'relevance' also drove the group investigative activity entitled 'Handspan and foot pace calibration' in one of his 'measurement' lessons. In this activity, students calibrated their individual handspans and foot paces, used these to measure (in groups of two or three students) several real-life objects, compared their measurements with those obtained with a tape measure, and answered several questions given at the end of the worksheet.

Josh introduced the activity to the class in the following way:

> "Now, what we are going to do today — going to the next exercise stage, is to find out how many centimetres this hand (showing and referring to his hand), your handspan (pointing at students, referring them to their own hands) is that you can use that to measure — things when you — might not have a ruler handy ... so when you need a quick estimate of something that's — long and you want to, a bit more accurate than to just say that, oh, (pointing to a student desk) the table is somewhere between two or three metres, or one and two metres, is that right. You want to — you get it more accurately. And — particularly handy for ... people who work on farms, and that means you might be out, and you got to get back to the — to the shed or workshop to get some something, and you are going to — interrupt in the end quite a bit of your work or something. So, that's the first thing we have to do. We are also going to measure your pace, so I'm going to see how many centimetres (demonstrates striding across the front of the classroom) — you can comfortably step. Say, you can use your pace or your step to also also estimate distances."

The value of relevance Josh espoused is both mathematical and mathematics educational in nature. Mathematical knowledge is portrayed as relevant and useful knowledge; at the same time the pedagogy of this knowledge employed by Josh (e.g. the 'Handspan and foot pace calibration' activity) demonstrates that the internalisation of this value is useful for students' own present and future experiences and challenges. This value, in turn, influences Josh's portrayal of other values in his lessons, such as cooperative work, exemplified in his questionnaire response given earlier in this section, and in the following statement made in one of the post-lesson interviews:

> I want you to — to learn how to, how to work with different people. I always — go — go on to say, look, you know — the school here, seventy teachers, you know. Seventy teachers! And we got to work together. I can't just, just [say] 'Oh, I'm not going to teach — such and such people and teach someone else I happen to like'.

Here is another value which is not only nominated by Josh, but is also explicitly implemented and espoused in his lessons. Some other values, however, might have been intended but were not implemented, at least not observable by the researcher. Yet others were implemented (or portrayed) but not intended, or at least not nominated by the teacher. For example, there were several occasions on which Josh explicitly emphasised that students listen to him, or to one another. When a student asked him to read out a question again, he replied that "you got to listen here.... We do this [mental mathematics exercise through verbal dictation of questions] so that people practise listening!" Yet this is a value which Josh was unaware of implementing. Table 2 (above) lists some of Josh's values that were observed in his lessons.

Issues and Further Questions

This project has one more year to run, so this report is of work in progress. As indicated above, we are exploring various means of analysis in this new mathematics education research territory. The tables shown above can only ever present a partial picture, and we recognise the need to avoid simplistic dichotomies. For example the use of the word 'implemented' implies that a certain value is already intended to be taught by the teacher. But as we have noted with Josh, teachers may well be portraying certain values in their teaching that were not intended, and of which they were unaware. These could well include values with negative connotations. Are teachers aware of this possibility? And are there any strategies to help them overcome this problem?

We can also raise questions about the validity of the researchers' interpretations; each of us is working from multiple positionings and making subjective decisions about salient features of the data. The interviews and discussions with the teachers do help to clarify ideas, but one danger remaining is the likelihood of world-views being already shared between researchers and teachers. Is this a problem?

We believe it is not a problem because we are less concerned with discovering what values teachers are teaching, and more concerned with aim (b) above - discovering how much control teachers can gain over their values teaching. So the next phase of the project involves teachers nominating values that they are not currently implementing, or implementing explicitly, and then monitoring their attempts to do so.

We hope that by clarifying the relationship between the teachers' intended values and their portrayal of values in the classroom we will be able to offer teachers and teacher educators appropriate strategies for developing this neglected but crucial area of values education in mathematics.

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USING VALSINER'S ZONE THEORY TO INTERPRET A PRE-SERVICE MATHEMATICS TEACHER'S ZONE OF PROXIMAL DEVELOPMENT

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Abstract

The purpose of this study was to use Valsiner's (1987) constructs of the zone of free movement (ZFM) and zone of promoted action (ZPA) to understand the zone of proximal development (ZPD) of a pre-service secondary mathematics teacher during her student teaching semester. Using classroom discourse, we characterized the zones established by the teacher and identified an additional 'phantom' zone of promoted action (PZ) whereby the teacher appeared to promote certain events but in reality did not permit them to occur. We found the PZ to be a precursor to the reconfiguration of the ZFM established by the teacher and thus, potentially, a developmental necessity for the teacher. The existence of a PZ seemed to be an external manifestation of the ZPD and thus a predictor of one's readiness for professional development. We also found the teacher's capacity to construct a communal ZPD (Wells, 1999) in the classroom significant in her ability to reorganize the PZ into a true ZPA.

Theoretical Perspective and Purpose

In a Vygotskian perspective on learning, the zone of proximal development (ZPD) is postulated as the space characterizing one's potential capacity for development with the assistance of a more knowing other (Vygotsky, 1962). Moreover, the ZPD is predicated upon how that more knowing other organizes, or "scaffolds", the task at hand. As such, it is important to understand one's ZPD in order to design appropriate instruction. However, in spite of the significance of the ZPD in characterizing one's growth (Oerter, 1992), it is largely inaccessible as a diagnostic for instruction. As such, we need a more accessible route into the ZPD so that we can more effectively scaffold an individual's development.

The purpose of this study was to explore Valsiner's (1987) zone theory as a way to understand the novice teacher's ZPD. Within the context of child development, Valsiner (1987) extended the notion of a ZPD to include two additional zones of interaction: the zone of free movement (ZFM) and the zone of promoted action (ZPA). The ZFM is an "inhibitory psychological mechanism" (p. 99) set up by the adult to constrain the freedom of the child's choices of thinking and acting. The ZFM limits access to different areas of the environment, determines availability of objects in the environment, and constrains ways of acting with objects in the accessible area. It ultimately canalizes the direction of development for the child, providing a framework for cognitive activity and emotions. In essence, the ZFM addresses the question of what is allowed by the adult (or teacher). On the other hand, the ZPA describes a "set of activities, objects, or areas in the environment" (Valsiner, 1987, pp. 99-100) with which the adult attempts to persuade the child to act in certain ways. It is defined by what is being promoted by the adult without obligating the child to comply. The ZPA is contained within the ZFM, since theoretically the adult cannot promote what he or she does not allow. The ZFM and ZPA are seen as equivalent when the child's only choice becomes what the adult requires.

The ZPD is connected to the ZFM and ZPA in that learning is optimized only when the child's ZPD lies completely within the options the adult makes possible. However, a portion of the ZPD always lies outside the ZFM because only part of a person's potential can be realized in a given environment at a given time (Oerter, 1992). Since the ZFM and ZPA are socioculturally determined, the child can only develop those aspects of the ZPD that are advertised by the adult (ZPA) and not prohibited (ZFM) by the culture in which he or she is acting. This anomaly exists because in every social interaction where development is promoted, a "canalization occurs that excludes other options of the ZPD" (Oerter, 1992, p. 193). In other words, when a particular experience is promoted (and hence allowed), some other event is necessarily excluded. We include here our rendering of Oerter's (1992) depiction of the relationship between these three zones (See Figure 1).

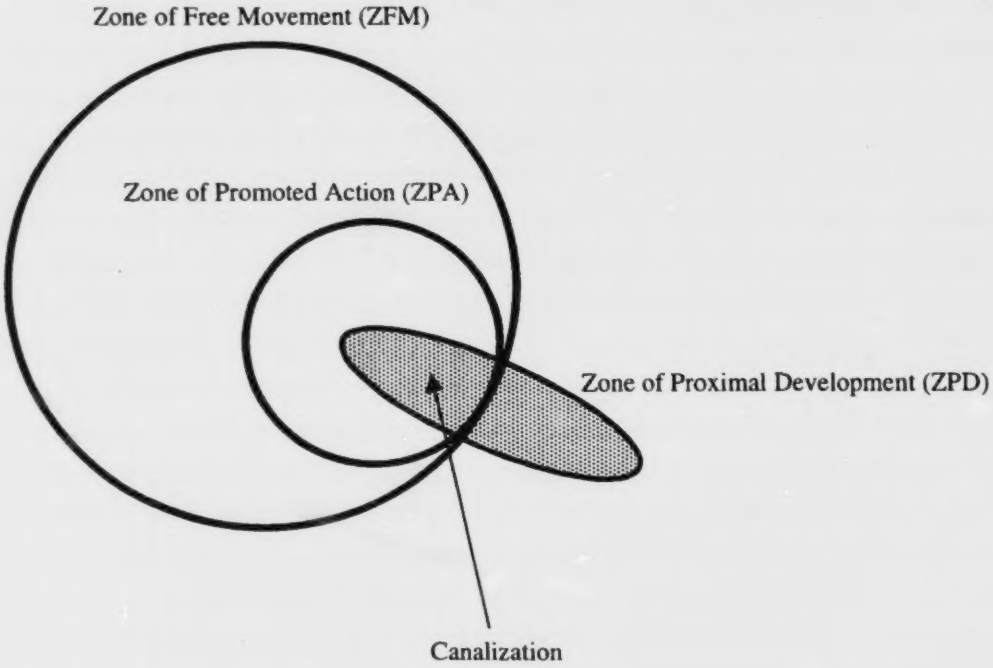
Zone of Free Movement (ZFM)
Zone of Promoted Action (ZPA)
Zone of Proximal Development (ZPD)
Canalization

Figure 1. Relationship between the ZFM, ZPA, and ZPD.

The classroom provides a further illustration of how these zones interact. For example, if a teacher chooses to promote only individual seatwork comprised of repetitive mathematical exercises, then he or she is necessarily excluding the possibility of an open, inquiry-based classroom from the ZFM/ZPA complex. As a result, the student's full potential of development within the ZPD cannot be realized because he is not exposed to collective mathematical inquiry. In this case, the learning environment (ZFM) necessary for establishing a ZPD in which students realize their potential for mathematical thinking based on activities of public conjecturing and argumentation fails to be established. The implication is that the individual's ZPD would not be fully contained within the ZFM, thereby creating a canalization in his or her development.

If we consider that the ZPD is ontogenetic (i.e., within the learner) and the ZFM/ZPA complex is microgenetic (i.e., between the learner and the environment), (see Lightfoot, 1988), then it seems reasonable to explore an individual's ZPD by identifying the ZFM and ZPA in which development occurs. We have previously used this approach to describe the complex interactions occurring in the professional development of science teacher interns (e.g., Carter, Westbrook, & Wheatley, 1998). The data from those studies led us to conclude that professional development was most likely to occur in situations where the cooperating teacher structures a large ZFM, thus increasing the likelihood of overlap between the intern's ZPD and ZFM. Applying the zone concepts in that context provided a glimpse into the conflicts that arise when the beliefs and practices of a novice teacher are not congruent with those of a more experienced teacher acting as the more knowing other.

Elsewhere, we have used Valsiner's (1987) zone theory to examine patterns of discourse established by pre-service mathematics teachers (Blanton & Westbrook, 1998). Our inquiry showed that these teachers funneled students' thinking through leading questions that restricted the ZFM and established a ZPA organized around the teacher's conceptions. Such patterns of discourse appeared to promote verbalizations in the classroom that provided the illusion of sense making, yet instead established cognitive boundaries in the classroom. In the study reported here, we again use an analysis of discourse in a novice teacher's classroom to extend this previous inquiry into the practical and developmental applications of Valsiner's zone theory. Our premise here was that the way the novice practitioner organized the ZFM/ZPA complex *in the classroom* would additionally inform us about his or her development within the ZPD. That is, by identifying what a teacher allowed (ZFM) or promoted (ZPA) in an instructional context, we could better understand that teacher's trajectory of development and thereby more effectively scaffold his or her development. In this sense, we claim that the ZFM and ZPA provide a more accessible route into the teacher's ZPD.

Methodology

At the time of this study, the student teacher participant, Mary Ann (pseudonym), was in her final academic year of teacher preparation and was student teaching in an urban school. She was assigned to a 7th-grade classroom in which she taught general mathematics and pre-algebra. She had a supportive cooperating teacher whose intent seemed more collaborative than authoritative. We observed Mary Ann approximately once per week during the one-semester student teaching practicum. Each visit was documented by field notes and audio and video recordings and consisted of two classroom observations and a 45-minute interview.

We used (verbal) classroom discourse to identify the ZFM/ZPA complex that Mary Ann established in her classroom. In particular, the analysis focused on identifying what Mary Ann allowed or promoted during the course of instruction and how this might have shifted throughout the practicum. Our focus on discourse draws on a genre of research in teacher education in which classroom discourse analyses have been successfully used to understand teachers' developing practices and to identify the social and cognitive aspects of the learning environment (see, e.g., Blanton, Berenson, & Norwood, in press; Peressini & Knuth, 1998; Wood, 1995). The tapes, transcripts, and the researchers' notes served as the primary data sources for this study. The research team independently and collectively analyzed the transcripts, viewed the videotapes, and developed assertions from classroom discourse about the zones established by Mary Ann.

Analyses and Interpretation

In the process of identifying the ZFM and ZPA established in the classroom, we found that part of the ZPA could be illusionary. That is, Mary Ann at times established what we describe as a '*phantom' zone of promoted action* (PZ), or, a zone in which she seemed to promote actions that in actuality were not permitted. We characterize this zone as a phantom zone because it reflects an apparent contradiction in how the ZPA and ZFM interrelate. That is, in theory the ZPA should be contained within the ZFM; one can only promote what is at least allowed. However, the PZ represents precisely that which appears to be promoted (hence allowed) but in fact is not. In this sense, we maintain that part of the ZPA can be illusionary and thus exist outside the ZFM/ZPA complex.

For example, we found discourse patterns in Mary Ann's early practice that appeared to promote the illusion of sense making, yet established cognitive boundaries in the classroom. We include the following protocol as representative of the discourse characterizing Mary Ann's early classroom practice during her first month of student teaching (see also Blanton, Berenson, & Norwood, in press). In it, Mary Ann led a whole-class discussion with her students about the following problem:

> Alex had $5 left in his wallet after he spent $12 on snacks and souvenirs at the Jubilee. How much money did he take to the Jubilee?

1 Teacher: How much money did he spend?
2 Jim: Twelve dollars.
3 Teacher: Twelve dollars. OK, if he spent \$12, would he be minus or plus?
4 Students: Minus.
5 Teacher: Minus. He's going to be minus. So how much money he took to the Jubilee is an unknown. It's something that we don't know…. So, how do we represent unknowns?
6 Students: Variable.
7 Teacher: A variable. OK, what variable do [you] want to use?
8 Mark: M.
9 Teacher: OK, so m is the amount of money he has, and how much did he spend?
10 Carol: Twelve.
11 Teacher: Twelve dollars. And how much did he have left over?
12 Students: Five.
13 Teacher: (Mary Ann writes the equation '$m - 12 = 5$' on the OP to be solved.) OK, what was the very first step [in solving this equation]? What was the very first step that I gave you yesterday Chad?
14 Chad: Isolate the variable.
15 Teacher: Isolate the variable. OK, how did we isolate the variable?

We infer from this episode that, while Mary Ann promoted the illusion of sense making, her forms of questioning necessarily restricted students' ZFM. That is, by asking short-answer, leading questions (e.g., 1[1], 3, 7, 13) Mary Ann seemed to be restricting students from essential problem-solving activities such as purposeful conjecture and argumentation. In essence, she promoted students' (verbal) participation, yet her questions placed cognitive limits on their thinking. In this sense, we would argue that Mary Ann established a PZ in that students were invited to participate in class discussions, but only in a manner that funneled their responses toward Mary Ann's approach for solving the problem. Thus, while it was clear to us that Mary Ann valued having students participate in her class, we maintain that she promoted a type of interaction with students in her early practice that necessarily canalized the development of their mathematical thinking in a direction which treated mathematics as a procedural body of knowledge.

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[1] Numbers refer to lines in the protocol.

In Figure 2, we identify the ZFM, ZPA and PZ established by Mary Ann in her classroom during her early practice. We note that the nature of the relationship between the ZPD and the ZFM/ZPA complex suggests that those areas contained within the PZ established by Mary Ann represent areas not developed within the student's ZPD.

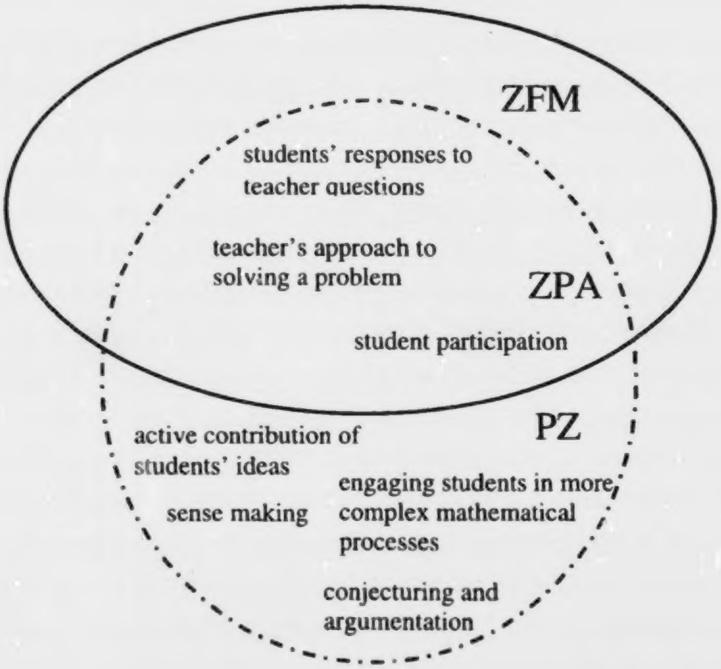
ZFM
students' responses to teacher questions
teacher's approach to solving a problem
ZPA
student participation
PZ
active contribution of students' ideas
sense making
engaging students in more complex mathematical processes
conjecturing and argumentation

Figure 2. The ZFM, ZPA, and PZ established in Mary Ann's early classroom practice.

The PZ became significant in our understanding of the ZFM/ZPA complex and thus Mary Ann's ZPD. In particular, we found the PZ to be a precursor to the reconfiguration of the ZFM established by Mary Ann and as such, we posit that it constituted a developmental necessity for her. For example, Mary Ann ultimately expanded her ZFM by promoting (and permitting) students' activities of conjecture and argumentation. Thus, what had been a PZ for Mary Ann transitioned into a set of events (e.g., argumentation and conjecture) that ultimately she promoted and allowed. Thus, we argue that the PZ potentially signals a necessary, although not sufficient, condition for development. As such, we suggest that the existence of a PZ may be an external manifestation of the ZPD and thus a predictor of one's readiness for professional development.

We find further support for this claim in Vygotsky's theory on concept formation. In particular, we suggest that the construct of the PZ parallels Vygotsky's (1962/1934) notion of *pseudo-concept*, which he used in describing the development of higher mental functioning in children. Vygotsky argued that the pseudo-concept is an essential bridge in children's thinking to the final stage of concept formation. While the pseudo-concept a child possesses is phenotypically equivalent to that of an adult, it is psychologically different. As a result, the child is able to "operate with [the concept], to practice conceptual thinking, before he is clearly aware of the nature of these operations" (p. 69). Applying this to our study, we concluded that the PZ signals that desirable teaching practices might be within the teacher's ZPD although the teacher might not yet have a fully articulated understanding of those practices. As such, the PZ is therefore a zone of possibility, or a ZPD where with mediation there is potential for growth. Thus, identifying the existence of a PZ in a teacher's practice helps us to understand his or her potential for growth. As such, we maintain that when a teacher is operating within a PZ, she is in a place of transition, or a pseudo-conceptual stage, and has the potential for professional development.

Finally, we conjecture that Mary Ann was able to reorganize her ZFM because she allowed herself to make sense of *students'* ideas, and as she did so, she became engaged by students *as a learner*. Through this process of co-participation with students, she seemed to shift towards more student-centered inquiry. We thus argue that a teacher can potentially effect the transition of the PZ into a true ZPA if she is able to construct a communal ZPD (Wells, 1999) as Mary Ann did. Wells' notion of a communal ZPD entails a community of inquiry whereby "jointly undertaken activity creates a context in which *all* participants – teachers … as well as students – can assist each other in their zones of proximal development" (p. 312). He further argues that "for learning to occur in the ZPD, it is not so much a more capable other that is required as a willingness on the part of all participants to learn from and with each other" (p. 324). From this perspective, we found that Mary Ann's participation in a classroom communal ZPD facilitated her reorganization of the ZFM in a way that diminished the PZ.

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METAPHORS IN TEACHING AND LEARNING MATHEMATICS: A CASE STUDY CONCERNING INEQUALITIES

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ABSTRACT: In this paper an embodied cognition perspective is considered in order to frame teaching and learning problems concerning inequalities. The nature and functions of some "grounding metaphors" are discussed, as well as the possibility of enhancing their use by students.

1. Introduction

Since the beginning of the eighties metaphors have been reconsidered as crucial components of thinking (see Lakoff and Johnson, 1980). The relevance of body-related metaphors in mathematical thinking has been clearly stated by Lakoff and Nunez (1997) and Nunez et al (1999) with some examples concerning, in particular, natural numbers and continuity.

Nunez (2000) describes *conceptual metaphors* as follows: "*It is important to keep in mind that conceptual metaphors are not mere figures of speech, and that they are not just pedagogical tools used to illustrate some educational material. Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference*". Considering *conceptual metaphors*, Lakoff and Nunez (2000) (see also Nunez, 2000) make a distinction between *grounding metaphors* (i. e. conceptual metaphors which "*ground our understanding of mathematical ideas in terms of everyday experience*") and other kinds of conceptual metaphors (*Redefinitional metaphors, Linking metaphors*).

This paper has three aims:

- to show how different kinds of *grounding metaphors* can intervene (as crucial tools of thinking) in novices' approach to inequalities:

- to discuss possible refinements of the idea of a *grounding metaphor* deriving from the analysis of students' behaviour and related to the cultural variety of possible everyday life source domains;

- to investigate how *grounding metaphors* can become a legitimate tool of thinking for students.

In order to fulfil these aims, we will consider a teaching experiment performed in two VIII-grade classes (13-14 years old students) with the main purpose of detecting the young students' potential in dealing with inequalities approached in a functional way (i.e. by considering them as comparisons of functions) . We will consider also four Ph. D. students' behaviour in similar tasks.

2. Grounding Metaphors and Communication Metaphors

In mathematics as well as in other domains, many metaphors are frequently used with pure communication purposes: we will call them *communication metaphors*. For instance, in a sentence like "*The proof of this theorem was like an obstacle race: once some progress was made, immediately another difficulty appeared*" the *obstacle race* metaphor fulfils a pure communication purpose which has no mathematical content. *Communication metaphors* can also be employed in order to substitute for technical espressions which are not shared by the interlocutor: this happens frequently in a popularisation situation.

Can metaphors fulfil other functions, in particular thinking tool functions?

The examples produced in recent papers by Lakoff and Nunez (1997; 2000), Nunez, Edwards and Matos (1999) and Nunez (2000) show how some metaphors (*conceptual metaphors*) can function as thinking tools, in particular as ways of thinking about peculiar mathematical objects; these authors go further in hypothesizing that conceptual metaphors (in particular, *grounding metaphors* - i.e. those referring to everyday body actions and relations) are not exceptions, but usual ways of thinking in mathematics.

The distinction between *communication metaphors* and *grounding metaphors*, and the very existence of grounding metaphors in a given mathematics domain, are relevant for educational purposes: indeed communication metaphors could not solve deep learning problems and so they would demand weak engagement by teachers. On the contrary, teachers should legitimate and enhance the use of grounding metaphors as tools of thinking. Special attention should be paid to the existence and functions of grounding metaphors in those mathematics domains where difficulties of learning are greater: indeed ignoring specific grounding metaphors could be one of the reasons for students' difficulties (see the analysis of the difficulties inherent in the learning of continuity, in Nunez, Edwards and Matos, 1999).

In this paper we deal with grounding metaphors in the domain of inequalities; as we will see in the next section, for different reasons (including students' difficulties) this subject is poorly covered in current teaching. The challenge coming from an *embodied cognition* perspective (Lakoff and Nunez, 1997; Nunez, 2000) is to ascertain at what extent grounding metaphors can support the teaching and learning of this subject and overcome some of the students' difficulties.

Another problem concerns the necessity of establishing whether novices' *grounding metaphors* do substitute more advanced thinking tools; in this case their importance should be only temporary in the students' career. For this reason in our experimental study we considered both novices (VIII-graders) and university students (in particular, four Ph.D. students in mathematics) engaged in structurally similar tasks (even if the level of difficulty was different: see Annexes).

3. Rationale of the Experimental Study

In spite of their importance in pure and applied mathematics, inequalities constitute a neglected and ill-treated subject in secondary school curricula. In most countries, inequalities are taught as a subordinate subject (in relationship with equations), dealt with in purely algorithmic manner which avoids, in particular, the difficulties inherent in the concept of function (see Assude, 2000; Sackur and Maurel, 2000). As a consequence, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (cfr. Boero, 2000; Malara, 2000) at the entrance of the university mathematics courses in Italy most students fail to solve easy non standard inequalities like $x^2 - 1/x > 0$.

Concerning the mathematical relevance of the current teaching of inequalities in school, we may observe that it does not take into account the importance of inequalities in pure and applied mathematics (for instance in the case of the concept of limit or to deal with asymptotic stability) and the fact that in many cases equations are solved with approximation methods which are based on inequalities (thus reversing the usual approach to inequalities in school - as a subject subordinate to equations).

This brief presentation brings to the following conclusion: the prevailing manner of teaching inequalities in school neither is efficient (as concerns the results, in terms of the capacity to deal with a large set of rather simple but non standard inequalities) nor fits relevant aspects of the professional (mathematicians') practice about inequalities. In order to try to find the reasons of this situation and elaborate tools for overcoming it, we can observe that the functional aspect plays a crucial role in mathematicians' work, both for equations and inequalities. This fact is often neglected in traditional teaching. From a functional point of view, inequalities fully involve difficult concepts like *variable* and *function* in situations which need a complex treatment. As suggested at the beginning of this section, we can recognize that the traditional teaching of inequalities avoids the "function" concept and reduces the difficulties ineherent in the *variable* concept and the complexity of the solution process by treating inequalities as a special case of equations.

Keeping in mind this analysis, when planning the teaching experiment with two VIII-grade classes our basic cultural choice consisted in treating equations and inequalities from a functional point of view, i.e. approaching them as special cases of comparison of functions.

Here some details about the classes, educational choices and classroom activities are reported:

- 36 VIII-grade students (divided into two classes) were involved; as usual in Italy, they had started to work with the same mathematics and science teacher in grade VI;

- the didactic contract established in grades VI, VII and at the beginning of grade VIII was coherent with the methodological choice of a cooperative, participated, guided enrichment of tools and skills in the planned activity. A rather common routine of classroom work consisted in individual production of written solutions for a given task (if necessary, supported by the teacher with 1-1 interventions), followed by classroom comparison and discussion of students' products, guided by the teacher and, possibly, by the adoption of other students' solutions in similar tasks. Another aspect of the didactic contract included the exhaustive written wording of doubts, discoveries, heuristics, etc.;

- the approach to the concept of function was built up through activities involving tables, graphs and formulas and mainly concerning geometric entities (lengths, perimeters, areas, etc.). After some initial activities on functions as machines (*operational view*) and then as x to y correspondences (*correspondence view:* see Slavit, 1997 for a survey about these different *views*), point-by-point drawing of graphs was discouraged, while making hypotheses about their shape (starting from their formula) was greatly encouraged;

- the role of the teacher was to help students through 1-1 interactions and manage students' classroom discussions. In particular, students were encouraged to communicate their ideas with words, gestures, graphs drawn at the blackboard, etc..

Collected materials from the two classes consisted in individual protocols, audio-recordings of classroom discussions and detailed teachers' notes. The same materials were collected in the case of the four Ph. D. students in Mathematics involved in this study for comparison purposes (see end of Section 2).

4. Grounding Metaphors and Inequalities

In the Annexe 1 some excerpts of a student's solution are reported; they are representative of a large set of protocols deriving from the written solutions of the 36 VIII-grade students considered in this study. Also a solution from a Ph. D. is reported (Annexe 2), to show impressive similarities between the novices' strategies in dealing with open problems concerning inequalities and the efforts of an expert young mathematician in dealing with a similar, more difficult task (in fact, a task not covered by learned procedures).

Different metaphors surface in students' protocols. For space restrictions we will consider only one crucial step of the solution: the search for *pivot points* around which the direction of the inequality changes. In many protocols we find one (or more) of the following metaphors:

- dynamical reference to increasing and decreasing values, and the necessity of a meeting point supported by consecutive dynamical gestures of one hand (firstly indicating increase, then decrease, or vice-versa); words are coherent with this body dynamic representation: students speak about *going up* and *going down* of the two functions (*formulas, graphs*, etc.), and thus they must "*meet in one point*" (*meeting metaphor*): "*one graph goes up steeper and steeper from below and must meet the other which increases and then goes down*" (see Annexe 2).

- reference to the imagined (or drawn) shapes of the two graphs, and the necessity of a meeting point supported by static crossing of the two arms; again words are coherent with this static body representation: the two graphs "*must have one point in common*" (*intersection metaphor*);

- *balance metaphor*: in this case the idea of a possible equilibrium between the values of the two functions drives the student's attention towards values of x which are near to satisfy the equation. A reference to physical trials performed in order to reach the equilibrium point is evident. We may note that the balance metaphor was used by students to guide the search for the equilibrium point in different ways: in particular with numerical trials on the two sides in order to approach the equilibrium point; or through a regular movement from left to right (see Annexe 1 for an example).

5. Discussion

In our opinion, the reported excerpts and the examples of the metaphors surfacing in VIII-grade students' attempts to find the 'pivot points' (as well as in the Ph. D. students protocols: see Annexes) raise three relevant questions:

Can we speak of grounding metaphors?

The communication function does not seem to be the most important function in the students' protocols: metaphors "*project the inferential structure of a source domain onto a target domain*" , according to Nunez's description of conceptual metaphors (see Section 1), and the relationships established in the different source domains (for instance: balance equilibrium) serve as crucial references to infer conclusions in the target domain (functions and inequalities). In particular, the necessity of a point belonging to both graphs derives from the necessity that can be experienced in the source domain (see later for a detailed analysis).

Can we consider the grounding metaphors used by students as spontaneous, or may we identify their origin in classroom activities?

The knowledge of students' background brings to the hypothesis that words and gestures (strongly encouraged by the teacher during the previous classroom activities on functions: see Section 3) allowed different kinds of grounding metaphors concerning functions and variables to become legitimate and spread in the classroom. We would like to make some comments about *legitimacy* and *spreading*. *Legitimacy* means that students were allowed to overtly reason through those kinds of grounding metaphors. This is not frequent in mathematics teaching (even in lower grades): abstract reasonings are privileged. *Spreading* means that overt, legitimate gestures and words were freely adopted by the schoolmates, according to their personal needs; by this way different grounding metaphors became accessible as thinking tools for dealing with variables and functions. For instance, the balance metaphor was produced by the author of the first protocol (Annexe 1) in a previous situation; the teacher promoted a discussion about it; then it spread in the classroom (9 students used it in the task reported in Annexe 1).

If this perspective is appropriate to describe what happened in the two classrooms, the teacher's role seems to be crucial in order to provide students with the opportunity of accessing powerful grounding metaphors. We can imagine that potentially every student can use them; but this potential does not translate into an effective, appropriate use if this use is not legitimate and supported by appropriate signs (particularly words and gestures). From the research point of view it would be interesting to better understand the specific role of signs (words and gestures) in the appropriation (or activation) and functioning of the grounding metaphors considered.

In the case of the search for pivot points how can we distinguish between the three kinds of metaphors described above?

The very nature of the source domains show important differences between the different metaphors. All of them enter the definition of *grounding metaphor* quoted in Section 1, but the nature of the evoked *everyday experience* is not the same in the different cases: again considering the three grounding metaphors surfacing in the research for the pivot points, in the first case we can recognize a reference to an everyday experience concerning crossing of movements. It is interesting to observe that the gestures for the description of this situation are the same that we would use in describing the crossing of two people climbing and descending a staircase. A physical experience concerning ordinary life supports the necessity of a *meeting point*. In the second case, a familiar situation of a necessary crossing of two continuous lines is evoked: the activity of drawing lines on plane surfaces provides the support for the visual necessity of a *common point*. In the third case, an everyday life technological situation is evoked: a technological tool (the balance) provides the physical necessity of an equilibrium point (in fact, the *pivot point*). These remarks suggest the consideration of different kinds of *everyday experience*, with different relationships between *culture* and *body*: an immediate relationship in the first case, a visual culture-mediated relationship in the second case, a technology-mediated relationship in the third case. In other words, we could speak of different culture-mediated necessities for a *pivot point* in the three cases.

Even these remarks suggest some educational implications: a variety of everyday experiences should be recognized and encouraged by the teacher in the classroom as legitimate sources of grounding metaphors for crucial mathematics concepts and situations.

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Annexe 1: some excerpts from the first part of an VIII-grade student's solution to the following problem:

Compare the following formulas from an algebraic and a graphical pont of view. Make hypotheses about their graphs and motivate them carefully, finally draw a sketch of their graphs. A) $y= x^2- 4x +4$ B) $y= -x^2+4$

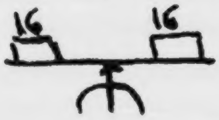
"Due to the fact that x^2, *that is x multiplied by itself, is there, we should get two parabolas. The presence of +4 makes them start from +4 when x=0*

The first curve will meet twice the second: at +4 (on the y axis) and in another point to be found. The first curve will go down, in the first quadrant, below 4 when the weight of 4x will be greater than x^2 *and will go up again when this situation will be over.*

When does the first curve go below +4? Moving from 0 on the right, the first curve will go below +4 till when -4x will balance x^2 *, that is they will become equal*

$$\begin{tikzcd}
x^2 \arrow[r] \arrow[d] & 4\cdot x \arrow[d] \\
x\cdot x \arrow[r] \arrow[d] & 4\cdot x \\
4\cdot 4 \arrow[r] \arrow[d] & 4\cdot 4 \arrow[d] \\
16 \arrow[r] & 16
\end{tikzcd}$$

è tutto equilibrato



The second parabola is a very usual parabola, but it is translated "upwards" by +4 and made negative as an effect of -x^2 which makes everything negative. It will remain over the level 0 for a while, until +4 will remain greater than -x^2. But when will x^2 equilibrate +4? (drawing: again a schematic representation of a balance). *When -x^2 will give a number that will annihilate +4, that is -4. When? -2·2+4=-4+4=0 [...]*"

Annexe 2: a Ph. D. student's solution for the following problem: "*To find where xsinx>x^2-1*". The student is invited to tell aloud what he thinks.

(from audiorecording and notes taken by the interviewer)

"*It is evident that the parabole overcomes the other function when x grows in absolute value, because xsinx cannot be bigger than the absolute value of x. It is a parabole compared with two straight lines outcoming from the origin and moving upwards* (he makes gestures showing the two curves, then he makes a sketch on paper). *The problem is what happens near to zero. Indeed if x=0 I see that the parabole is below the other function. But... xsinx... it is a pair function, a symmetric function... Well, I can consider only the positive side of the x axis. Here I imagine... x^2-1 is like x^2 lowered by one* (gestures in the air: a parabole then a lower parabole). *OK, the other function goes up and down, but definitively it will remain below the parabole... I have already said it* (he points to the drawing). *Now I must coordinate what happens near to zero and what happens at large* (he carefully draws the parabole $y=x^2-1$, the y=-x and the y=x straight lines). *I must be more precise, and see where xsinx meets the x axis* (he makes a sign for 1, 2,3,4, 5, 6, then he makes a sketch of the graph of xsinx for x>0, saying: "*it goes up and down between these two straight lines*"). *It looks fine. Oh, oh, this sketch is not precise enough - I must find where xsinx meets x... OK, sinx=1, it is here (he makes a sign on the straight line by going up from the value of approx. 1.5 on the x axis, then he draws a more precise graph of y=xsinx between 0 and 3). xsinx=x^2-1 ... no precise solution, but a solution do exist, I see here, one graph goes up steeper and steeper from below and must meet the other which increases and then goes down. But I should get no more than one solution... Let us see: if x=1.5, x^2-1 makes 2.25-1, that is 1.25... bigger than one, but not so bigger... It means that the meeting point is near to 1.5 on the left... OK, my drawing was OK! I can see that x^2-1 overcomes xsinx out of this interval* (he rapidly completes the drawing by symmetry on the left, and makes symmetric gestures with the two hands to indicate the two symmetric parts of the x axis, out of the central interval)".

FROM THE DECIMAL NUMBER AS A MEASURE TO THE DECIMAL NUMBER AS A MENTAL OBJECT

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Abstract. *We have carried out some explorative studies about introduction of the concept of decimal number, in the upper elementary school. In order to achieve this objective, we have designed a classroom practice that engaged students in a sustained mathematical activity which requires an extensive use of the ruler to accomplish different functions (measuring, drawing segments, ordering and approximating decimal numbers). In this paper we presented an explorative study with fifth-grade children about the role of the ruler as concrete model as well as conceptual model in passing from the decimal number as a measure to the decimal number as a mental object. The emergence, in terms of prospective learning, of some of the properties of the decimal number, and in particular of the property of density on the real number line, is also presented.*

Framework

In recent years connections are advocated between mathematical content and the home cultures of learners, as well as between different branches of mathematics, various disciplines in which mathematics is used, historical roots of mathematical content, and connections with the real world and the world of work, see e.g. Civil, 1995. Some considerations are anyway important. Mathematics is a part of students' social and cultural lives, and the mathematics classroom has its own social and cultural life, see Boaler, 1993. Indeed, "academic" mathematics can also be viewed as a form of ethnomathematics, involving particular cultural practices, see Presmeg, 1998.

The task of connecting students' everyday contexts to the classroom is not easy. Many educators argue that transferring ideas from one context to another is hard because the two context differ in some significant ways. Just as mathematics practice in and out of school differs, so does mathematics learning. Schliemann, 1995, pointed out that "*school learning focuses on individual cognition, pure thought activities, symbol manipulation, and general principles, while out-of-school learning is characterized by shared cognition, tool manipulation, contextualized reasoning, and situation-specific competencies*". In particular in out-of-school mathematics practice, persons may generalize procedures within one given context but be unable to carry these procedures forth to another context since problems tend to be context-specific. Generalization, which is an important goal in school mathematics, is not usually a goal in out-of-school mathematics practice.

The role of cultural artifacts

The critical problem of how to manage at school the relationship between everyday knowledge and school mathematics has been the subject of our studies for some years now. How can we at the same time benefit from what children already know and avoid the limitations that are typical of the everyday mathematical experience? How can we design better opportunities for children to develop mathematical knowledge that is wider than what they would develop outside of school, but that preserves the focus on meaning found in everyday situations (see Schliemann, quot.)? How should we proceed to lead children to develop new understandings about underlying mathematical concepts and structures and their potential generalizability?

Although mathematics learning and practice in and out of school differ in significant ways, we deem that those conditions that often make extra-school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed while some differences between the two contexts may be inherent, many can be narrowed if in the classroom we create and promote learning processes that are closer to the ones that occur in the out-of-school mathematics practices. That can be implemented in a classroom, for example, by encouraging the children to analyze some 'mathematical facts' that are embedded in opportune 'cultural artifacts' (Saxe, 1991). We are talking here of objects that have relevance for the children and that are meaningful because they are part of the children's real life experience and refer to concrete situations. This enables children to keep their reasoning processes meaningful, to monitor their inferences. As consequence, they can off-load their cognitive space and free cognitive resources to develop more knowledge (Arcavi, 1994). We can thus make use of the children's familiarity with the artifacts we have chosen as being objects that belong to the students' daily experience, and allow the children to express their intuitions and produce their own anticipations, in the sense of "*prospective learning*" as described by Freudenthal, 1991. These anticipations precede and can be functional to any systematic learning process. Further, if we use the artifacts in a certain way (cf. Basso & Bonotto, 2001, and Bonotto, 2001), we can develop classroom activities of "*realistic mathematical modeling, i.e., both real-world based and quantitatively constrained sense-making*" (Reusser & Stebler, 1997), and we can overcome the clear limitations of classical word-problems (cf. Verschaffel & De Corte, 1997).

Like in the Realistic Mathematics Education perspective of the Dutch school of taught (cf. Gravemeijer, 1994), we deem that progressive mathematization has to lead a student to algorithms, concepts, and notions that are rooted in the individual's learning history, a history that starts with informal, experientially real knowledge.

The formal approach to the introduction of decimal numbers in elementary-school classrooms

In previous studies (Bonotto, 1993; 1996), we analyzed the conceptual obstacles 10- and 11-year-old Italian children encounter in ordering decimal numbers. Our findings are consistent with those of classical research studies (Nesher, & Peled, 1986; Resnick, et al., 1989). It was hypothesized that such findings may depend not only on the inherent difficulties of the subject matter but also on the teachers' conceptions and educational strategies. Many teachers introduce decimal numbers by extending the place-value convention. They tend to spend little time to let the children understand the meaning of the decimal number symbols and reflect on the decimal number properties and relationships. Efforts to connect decimal numbers and decimal measures are insufficient. As a consequence, children do learn to carry out the required computations, but they have difficulties in mastering the meaning of decimal notation and between fractional and decimal representations, and finally in ordering sequences of decimals.

Measuring activities as an alternative introduction to decimal numbers

According to innovative instructional approaches, we mantain that children's decimal number understanding can be fostered in rich classroom environments, where learners can transfer their out-of-school knowledge and utilize familiar tools (such as the ruler) to accomplish a recurrent set of mathematical activities, and where they can share some minimal presuppositions about the problem definitions and the goals. "*The roots in the student's reality are expected to foster the meaningfulness and usefulness of the so-developed mathematical knowledge*" (Gravemeijer, 1997).

We propose that a set of measuring activities that require an extensive use of the ruler can offer the children good opportunities to move toward the construction of an encompassing numerical structure, which integrates in a consistent whole both the natural and the decimal number systems.

The ruler is a cultural artifact which can offer the children a first approach to the decimal number as the result of a given measurement. On the ruler, "mathematical facts" are represented through its signs: the natural number sequence is visible, and some fractional parts are marked. Therefore, the ruler can offer a "situation-specific imagery" of the additive structure of the written decimal number notation, which supports the child's progressive understanding. For example in order to draw a 3.15dm segment, the child first draws a 3dm line and marks the final extreme, then she/he adds a 1cm line to it, and finally a 5mm line, and expresses each affixion as 'plus', or 'and'. The child can understand that if there are two decimal digits after the decimal point, then there are units, plus tenths plus hundredths, and that each digit specifies how many parts of a given magnitude are included in the addition. The learner is expected to form images out of her/his actions through the use of the ruler, and to visualize relevant properties. The child can map this visualization onto the decimal number representation to attribute a meaning to the decimal digits after the decimal point; her/his ability to solve ordering problems is enhanced.

As to regard the basic characteristics of the teaching/learning environment that have been designed and implemented in the classroom, a set of activities based on suitable cultural artifacts, in particular the ruler, on interactive teaching methods and on introduction of new sociomathematical, in the sense of Yackel & Cobb, 1996, were combined in an attempt to create a substantially modified teaching/learning environment. This environment focused on fostering a mindful approach toward realistic mathematical modeling, i.e. both real-world based and quantitatively constrained sense-making, see Reusser & Stebler, 1997.

The research

A previous exploratory study (see Basso, Bonotto, Sorzio, op. cit.) concerned the introduction of the concept of decimal numbers, in the normal classroom curriculum, with third-grade children. In order to achieve this objective, we have designed a classroom practice that engaged students in a sustained mathematical activity which requires an extensive use of the ruler to accomplish different functions (measuring, drawing segments, ordering and approximating decimal numbers). The ruler has been a mediational role in their understanding of the additive structure underlying the standard written decimal notation. The results obtained showed how they correctly measured and expressed lengths with numbers containing only one digit after the decimal point; where numbers with a second digit following the decimal point were concerned, the children had difficulties in distinguishing between the decimal digit value – which represents how many parts of a given magnitude there are – and the meaning to be attributed to each decimal digit position – which represents its magnitude. For example the whole group of digits after the decimal point referred to the decimal unit directly following the main unit that was being measured. The fact of measuring with a ruler, however, did offer the children a concrete anchor and clearly illustrated their errors, allowing them to autocorrect themselves.

Later we decided to carry the study further, to evaluate the influence that the preceding activities with the ruler had had on the children. The same children were thus given poblems, in both the fourth and fifth grades, which involved comparing, ordering and approximating decimal numbers. The idea was also to start the children thinking about the structural properties of a line of numbers.

Research objectives: data are gathered and analyzed concerning:

- the children's understanding of the signs and intervals on the ruler;
- the children's understanding of the additive structure underlying the standard written decimal notation;
- the children's process of detachment from the representation on the ruler and from the presence of a given unit of measure;
- the children's understanding of the relation between different units of measurement;
- the possibility for the children to grasp that the approximation is a limit of the physical instrument;
- the children's understanding of the density property of the enriched decimal number line ;
- the passage from the number as a measure to the number as a mental object.

Subjects: 21 fifth-grade children (aged 10-11 years) in a small school in a village (northeast Italy) participated in this explorative study; they had first started measurement activities in the second grade. Data were gathered from participant observations and children's written works.

Procedure: Each student was given a sheet that listed three tasks:

1. *Write down at least two measurements between 1dm and 2dm.*
2. *Write down at least two measurements between 1.2dm and 1.3dm.*
3. *Write down at least two measurements between 1.9dm and 2dm.*

Explain how you found these numbers.

Discussion

We briefly present some significant extracts from the written work that show

i) the role of concrete model (characterized by a particular symbolic code-system), as well as the conceptual model of the ruler, in passing from the number as a measure to the number as a mental object;

ii) the emergence, in terms of prospective learning, of some of the properties of the decimal number and, in particular, of the property of density on the real number line.

N.1 (Moreno): 1st answer*: 11cm – 12cm – 13cm – 14cm – 15cm – 16cm – 17cm – 18cm – 19cm*

The measurements I got without looking at the ruler were all between the 10cm and the 20cm marks.

2nd answer*: 1,21dm – 1,23dm*

I got them by adding anotherdigit after the decimal point: the millimeters.

N.2 (Daniele): 1st answer*: 1,1 – 1,2 – 1,3 – 1,4 – 1,5dm*

To get these numbers I thought that between 1dm and 2dm there are the little pieces that are smaller that are centimeters.

2nd answer: *1,21 – 1,22 – 1,23 – 1,24 – 1,25dm*

I have to go from 1.2 to 1.3 and all I have to do is add the smaller pieces that is the millimeters.

In these first two protocols (like in about 30% of the class's total), one sees that the students still reason very much along manipulative and procedural lines linked to the image of the ruler.

In fact, numbers/measurements that fit within the required interval, arrived at via mental operations, were clearly thought out placed in the physical spaces of the artifact/instrument. This kind of reasoning highlights some of the characteristics of the ruler as model. Mentally the children refer to the physical act and manipulation of the ruler involved in measuring a segment of, say, 1.21dm: first they draw a 1dm-long segment and mark off the end point; there they add a 2cm segment and then a 1mm one. In this case the physical manipulation they had previously carried out helped them to understand the meaning of a number written with a decimal point and expressing units of measure, and it helped them to understand the additive structure underlying the standard written decimal notation.

In other protocols (about 50% of the total), the model offered by the ruler proved to be more flexible. Here the ruler no longer served just as a visual and tactile model (as it had in the activities of the preceding years), but as a model that induces thinking in terms of relations between numbers and quantities involved, in a certain sense more "conceptual". A process had been set off from thinking on the basis of concrete, material objects to thinking on the basis of mental, mathematical objects.
Let us look at the examples given by the following two protocols.

N.3 (Simone): 1st answer: *1,3dm – 1,4dm – 1,5dm – 1,6dm – 1,7dm*
I went like this: in 1dm there are 10cm. Therefore 14cm is more than 1dm and less than 2dm. Since we're doing marks in decimeters, I did 1.4dm.
2nd answer*: 1,21dm – 1,23dm – 1,24dm – 1,25dm*
I went like this: what's smaller than centimeters is millimeters.
In this case we have to use millimeters because if you add or take away centimeters you don't get a number between 1.2dm and 1.3dm.

N. 4 (Pamela): 1st answer: *1,3dm – 1,5dm – 1,7dm – 1,9dm – 1,06dm*
1,32dm – 1,54dm – 1,73dm – 1,99dm – 1,05dm
To get these measurements I didn't make my brain work very hard, I thought of them, first, thinking of all the possible ones, and then out of these, taking the ones that seemed better to me. For example, 1.3dm equals thirteen centimeters while 1.32dm equals thirteen centimeters plus two-tenths of a centimeter. Therefore one decimeter plus three centimeters makes thirteen centimeters.
2nd answer: *1,23dm – 1,24dm – 1,25dm – 1,26dm but also 1,2321dm*
For example 1.23dm consists of one decimeter, two centimeters and 3 tenths of a centimeter. And the same goes for 1.2321dm which means 1dm, 2 centimeters, 3 tenths of a centimeter, 2 hundredths of a centimeter and 1 thousandth of a centimeter, that is 12 centimeters and 321 thousandths of a centimeter.

Pamela carries out transformations going from one unit of measure to another in a meaningful way using fractional expressions. On can see the distinction starting between decimal digit value – which represents how many parts of a given magnitude there are – and the meaning to be attributed to each decimal digit position – which represents its magnitude.

In the following answers, the fact that the subdivisions on the ruler are decimals becomes evident, as does the notion of subdivisions that can go on infinitely, even if one cannot actually see them on the ruler. Thus, the density property of decimal numbers in the number line intuitively emerges. Here are some examples.

N.5 (Selenia): *One could go on writing numbers infinitely, because there are always littler spaces that you don't see on the ruler because it would be impossible to* *see them all because they're infinite. Because you only see centimeters and millimeters on the ruler.*

N.6 (Veronica): *You can have 1,22dm – 1,23dm – 1,24dm – 1,25dm – 1,26dm – 1,27dm – 1,28dm – 1,29dm ...*
I found them like this. First of all between 1.2dm and 1.3dm there are ten spaces and therefore I can go like this. There aren't only these measurements, but there are infinite ones because every space continues to be divided in 10 parts.

N.7 (Sara): 1st answer: *1,2dm – 1,4dm – 1,6dm – 1,9dm – 1,5dm ...*
It was very simple to find them I thought what numbers there are between 1dm and 2dm and among so many numbers I chose these, because the interval between one decimeter and two decimeters is decimal and you always divide by ten.
2nd answer:_*1,21dm – 1,22dm – 1,263dm – 1,25dm – 1,299dm ...*
I used the same method as before and if the teachers give me another task like this to do, I'll use the same method again. It's like a cycle that repeats itself ... because the spaces inside a centimeter are infinite.

Conclusion and open problems

In this paper we presented an explorative study in the upper elementary school about the role of ruler in passing from the number as a measure to the number as a mental object; the emergence, in terms of prospective learning, of some of the properties of the decimal number, and in particular of the property of density on the real number line is also investigated.

Concerning the first point of our analysis, the results show the passage from mathematics understood as an instrument and incorporated in certain cultural artifacts, like the ruler, and mathematics as an object of study. The use of this cultural artifact first gave meaning to the operations of measuring, to numbers-measures and to decimal number notation in general (cf. Basso, Bonotto, Sorzio, op. cit.); then it favoured integration between acting and thinking in mathematics.

As to regard the second point, and that is the emergence of prospective learning, we hold, in agreement with Freudenthal, that anticipatory intuitions must be encouraged and not stymied: "*Prospective learning should not only be allowed but also stimulated, just as retrospective learning should not only be organized by teaching but also activated as a learning habit*" (Freudenthal, op. cit.). Mathematics teaching should try to exploit both forms of learning: "*prospective*" learning (which makes the fullest possible use of the student's intuitions, and to the greatest extent possible encourages anticipation of results), and "*retrospective*" learning (which makes use of "old" or previously acquired knowledge, revisits it, and re-composes it in new contexts). "*Just as prospective and retrospective learning aims at an integration of past and future learning processes, so does intertwining learning strands locally, yet with a view on the involved learning processes as a whole*".

There remain some problems that Freudenthal highlighted which still need to be confronted and solved, e.g.,

- how do mental objects develop into concepts?
- what criteria are there by which to judge if the process has taken place?

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"I'D BE MORE LIKELY TO TALK IN CLASS IF ...": SOME STUDENTS' IDEAS ABOUT STRATEGIES TO INCREASE MATHEMATICAL PARTICIPATION IN WHOLE CLASS INTERACTIONS

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Abstract *This paper reports on some exploratory research with a single class of students. They were given the opportunity to express their views on a variety of teacher strategies used in whole class interactions. The students' responses highlight that developments in the curriculum which support a more problem-based, exploratory approach would be welcomed, particularly if accompanied by opportunities for sharing ideas. This would reduce the shame, public and private, generated by 'getting it wrong'. Gender and class issues are mentioned but not discussed.*

There is considerable interest currently in the UK about the use of whole class teaching in mathematics. This interest has been provoked both by international comparisons (see Jaworski and Phillips 1999) and by government policy (see Brown *et al* 2000). Previously, there appears to have been an identification of this practice with a traditional, expository approach to learning, with discussion only occurring in the context of small group work (cf Groves and Doig 1998). However, some teachers are re-examining their use of whole class interaction, trying to include in the ensuing talk opportunities both for a more personally dialogic response from students and also for a more equitable one.

As a contribution to this debate and concentrating on the affective dimension, we offer the reflections of some school students on their experience of whole class questioning and what they feel would make their mathematics education a more participatory experience for them. Our concern with increased participation stems not simply from a concern with equity but also because of a belief that participation itself is a defining aspect of learning (Lave and Wenger 1991). We note, however, that the value of increasing participation is dependent on just what it is that is being participated in. Research on the different 'classroom traditions' (Cobb *et al* 1992) or 'social practices' (Boaler 1997) that can be found in mathematics classrooms suggests that interactions are not necessarily mathematical. Such research has tended to focus on paradigmatic cases of different types of classrooms. This approach is valuable in highlighting the importance of differences between such types but it is important to recognise the way in which particular classrooms may *share* features of the different 'typical cases'. We believe that the evidence presented in this paper tends to support this view.

Analysis of the student experiences we report here reflects that their experience of secondary school mathematics has been predominantly that of exposition plus routine practice, within which mathematics essentially consists of questions to which there is a right or wrong answer. However, the students did report experiences of different types of interaction. We are encouraged that the students, in expressing their desire for greater participation, often focused on those aspects of their experience which are more commonly found in 'inquiry' classrooms (Cobb *et al* 1992). In reading about the difficulties that the students report, it is important to keep in mind that some of their comments refer to their experience of secondary mathematics teaching in general rather than to their current experience and also to acknowledge that they valued the teaching they were receiving. Their difficulties are despite this.

We choose to represent here the responses of the students themselves for a number of reasons (cf Angier and Povey 1999). Because we are interested in the affective dimension, it seemed to us sensible to work directly with the insights the students were prepared to share since, phenomenologically, they have privileged access to their own feelings. We wished also to be part of a developing tradition that seeks to listen carefully to what school students have to say, inviting them to contribute to the construction of knowledge about schooling. We share the view that

> … young people are observant, are often capable of analytic and constructive comment, and usually respond well to the responsibility, seriously entrusted to them, of helping to identify aspects of schooling that get in the way of their learning. (Rudduck *et al* 1996, p8)

The context of the research

The research findings presented here arose out of collaborative work undertaken by a student teacher (Peter) and a researcher (Mark) working together to support Peter in developing reflective aspects to his practice. They were focusing on issues to do with mistake making and teacher questioning in whole class situations. Questioning of pupils, either directly using the grammatical form of a question or by other forms of cued elicitation (Edwards and Mercer 1987), is a prevalent feature of whole class mathematics teaching. A particular interest was in exploring and developing alternatives to 'hands up' as a form of answering.

Peter's final teaching practice was in an 11-18 school in a semi-rural area on the edge of a large conurbation. The school has a comprehensive intake with respect to gender and class but is almost exclusively white. Results in external examinations are near national averages. This research reports on the views of a class of twelve-year-olds, a 'top set', consisting of 12 girls and 17 boys. The decision to ask the class about their experience of whole class interaction arose from an initial discussion that occurred during one of Peter's lessons when Mark was present.

Peter introduces the day's topic – revision of formulas for n^{th} term of a series. He asks the class to spend a moment individually thinking about the topic to see what they can remember and invites those that "have some thoughts" to put there hands up. About four or five hands are raised. He then *asks the students to discuss in pairs. Nearly all students seem to be involved in this and the discussion seems to be centred on the topic.*

He now asks again for people to put their hands up if "they have any ideas". There are now perhaps 6 or 7 hands up. Peter comments that lots more should have something to say, he indicates two girls as an example "you were saying some really interesting things". Looks a little perplexed as to why there were not more hands up. I intervene and refer to the implicit question, that "who has some thoughts" also implies "who wants to say something".

Peter picks this up and asks again for hands up but this time saying he won't ask anybody to share, nearly all put their hands up. He asks the class why the difference. One boy responds, his comments include "people don't want to make a mistake, they might look stupid" (Mark's field notes)

An interesting discussion followed which raised a number of issues and it was decided to continue the dialogue with the class about some of them.

Data collection and interpretation

A number of sources provided data for analysis. The story from the earlier lesson was re-told to the students and responses were invited. Peter devised a short questionnaire, completed individually, that focused on the students' willingness to answer questions in class. The students, working in groups, completed an exercise devised by Mark. This involved them in ranking statements about possible strategies a teacher might use after asking the class a question against three criteria: the frequency in which the situations occurred in mathematics lessons; how nervous they felt in the different situations; and how helpful the different means of responding were to their learning. The reasons for using this instrument were twofold. First, students' behaviour in whole class interactions is socially focused and the intention was to reflect this. Second, Mark and Peter wanted to explore the potentially transformative effect of such discussion in helping to develop a community of mathematical practice (Winbourne and Watson 1998).

The students' responses to these sets of data were collated. The answers to the open question in the questionnaire were analysed initially on the basis of an open-coding of meaning units and these generated some more general themes on the basis of which a summary in the form of a class letter was prepared. The results of the sorting exercise were summarised and compared. Following this, seventeen of the students were interviewed in single gender groups about the class' responses. The data collection as a whole attempted to develop a cycle of interpretation in which the pupils' initial responses were interpreted and then this interpretation was the subject of further discussion and validation by the students. Details of the analysis of the questionnaire are presented elsewhere (Boylan and Lawton 2000): here, drawing principally on data from the interviews, we lay out some of the themes which emerged.

Emergent themes

The nature of the curriculum

Much of the discussion reflected the fact that, in the eyes of the students, most whole class interactions centred on questions to which there was a correct answer, already known to the teacher, which they were also expected already to know. This interpretation of what it means to be asked a question in mathematics infuses the interviews with the students so strongly that it might be difficult to imagine teacher or student conceiving interactions in mathematics classrooms differently. The only alternative apparently considered to the right answer being required is that a wrong answer can be helpful too. Although immediately recognisable by mathematics teacher and mathematics student alike, such an outlook does not, of course, permeate the rest of the curriculum.

- Because if you take RE, there's not like no definite answers for RE questions, like - but in maths there's - most of them are definite answers, so they might not be as confident if they know that, if they get it wrong, then they're definitely wrong, kind of (boy interviewed by Peter)
- In other subjects you get it read out, you take the answers from a book, you have it written down in front of you (girl interviewed by Mark)

When, in the questionnaire, the students were asked to choose the part(s) of the lesson in which they felt most involved, the results suggested that they wanted to move beyond the sort of mathematics curriculum which had dominated their previous experience.

Part of lesson	Boys	Girls	Total
Exercises	4	6	10
Puzzles	11	9	20
Questions	1	0	1
Quizzes	0	1	1
Discussions	9	2	11
Group activities	1	0	1

These responses were explored in the interviews. Both boys and girls were positive about the effect of 'puzzles' on the curriculum. Both groups wanted the puzzles incorporated into the topics they were studying and both made a spontaneous connection between the value of 'puzzles' and the fact that they permitted, and indeed provoked, discussion.

- I think [the puzzles] they've got to - they've got to have some sort of maths in them, and they've got - you want to try and get them more related to - if you're doing like a topic, then try and get them more related to the topic - so you can put a little bit of that in so it'll help them learn …
- Yeah, 'cause that's a bit more exciting and you get to have a bit of investigation

- You want a variety - you want a variety of sort of things - so you don't want straight maths all the time - you want puzzle maths - a bit of discussion in there, so - make it easy to learn and more helpful to learn (boys interviewed by Peter)

- *Did you put 'puzzles' [in the questionnaire]?*
- I think it's mainly because, like, they can have a go at things and if they don't understand it they can, like, confer with other people and ask the teacher
- I think they enjoy it more as well
- It depends which type because, like, if you're doing one subject that you can't really have puzzles on them - so it's, like, better to do questions. But on others - like with sequences - I think it's better to do puzzles on them
- I think it's better to do it on that topic because, if you go onto something else and then you come back to it, you forget all about it (girls interviewed by Peter)

We note two things about these responses. First, the students clearly differentiated between the sort of thinking generated by 'puzzles' and what they saw as the demands of 'questions' in mathematics lessons. At the same time they found it hard to conceive of an approach to learning in which 'puzzling' was the norm and where they were actively engaged in the construction of their knowledge.

- *What would be better would be to try and help you to be able to see questions a bit more like puzzles, where you have to puzzle it out for yourself?*
- I don't understand
- I don't know
- Try and puzzle out things? You mean work things out? (boys interviewed by Mark)

Second, the girls, who had not connected 'discussion' with personal involvement when answering the questionnaire, nevertheless were motivated by the opportunity to 'confer'.

'Discussion' as part of the lesson

Because previous research had indicated that where an opportunity for talk is part of the students' experience it is welcomed (for example, Povey and Boylan 1998), we were initially surprised by the girls' relative lack of enthusiasm for 'discussion' as 'part of the lesson' in their answers to the questionnaire. However, their responses in the interviews offered a different view. Both groups of girls said that they were keen to discuss their mathematics: they were emphatic about this and returned to the assertion several times even when the interviewers intended to move on.

- *If you could imagine your maths teaching to be different, how would you make it different?*
- I think I'd make it more discussions
- I'd make it so you could, like, talk about an answer with your friends and then answer
- Talk about it before you answer (girls interviewed by Peter)

- I think it would be better if we could discuss amongst like, in a little group, and be together instead of on your own because some people just don't know answer (girl interviewed by Mark)

We believe that the girls had interpreted 'discussion', in the questionnaire, as a whole class interactive session with dialogue consisting only of public exchanges between individual students and the teacher, with the desired student response confined to offering right answers. Thus the girls' responses in the questionnaire reveal the way in which the discourse of what counts as a discussion has been constructed in their classrooms. As suggested by earlier research (for example, Boaler 1997), the discussion they valued was exploratory, seeking after shared knowledge. What is perhaps less well documented is that this was also the type of discussion described and valued by the boys.

- I prefer to get it discussed and then see what other people think and then see where they're coming from
- Yeah, 'cause if you discuss it with someone then you know that someone else is thinking along the same lines as you after your discussion
- You're more confident then
- You're more confident
- You've got it right, because more people are thinking (boys discussing with Peter)

- I think it's better to discuss it
- More people answer
- It gives you more ideas so you understand it more fully before you answer
- Another thing is, like, if you discuss, like, if you don't understand it then your friend, like, who you're sat with, knows how you learn and they can, like, explain it in a way that you'd understand it straight away (girls interviewed by Peter)

'Getting it wrong'

What many of the boys and all of the girls talked to us about was the public shame of getting an answer wrong in the question-response-evaluation context in mathematics lessons.

- 'Cos they're, like, kind of embarrassed. If they're - some people are kind of embarrassed if they get it wrong
- If you get it wrong in front of the whole group and you're - when you don't get the right answer then people think that you're totally rubbish at maths
- If you get it wrong, everyone thinks you're not very good (boys interviewed by Peter)

- I think quite a few questions are asked, but people don't like to answer if they get it wrong, 'cause there's quite a few ways you can do things - and there's quite a few answers to quite - to some questions. And they don't like getting it wrong
- I think if you don't know the answer and, like, somebody points at you straight away that makes you feel worse than if you do know it and they ask you
- Well, I wouldn't put my hand up if I thought it was wrong, or I weren't, like, totally certain (girls interviewed by Peter)

This links to the theme of security and vulnerability discussed elsewhere (Boylan and Lawton 2000). What we want to point up here is the debilitating experience, also, of private shame, felt by both boys and girls.

- You could talk to the person sitting next to you, and you and your partner could agree on an answer and then at least if you're getting it wrong someone else is getting wrong as well, so no one can take mick out of just you
- But not everyone takes the mick out of you, some people just feel embarrassed, you might just be feeling bad or something
- You feel sick and tired
- There's no one taking the mick out of them, they're just feeling bad about it (boys interviewed by Mark)
- Kids think of it in a different way because they don't like being wrong. Like most of them think, most of them like being right, but if they put their hand up and get something wrong then they don't like it. Not, like, they don't like other people seeing them get it wrong, but they don't like it themselves, because it makes them feel as though they don't know anything (girl interviewed by Peter)

Unfortunately, there is not room here to discuss two of the gendered aspects of the students' responses. First that of public shame, experienced by girls much more than boys as making participation not worth the risk. Second that of 'shouting out' by some of the boys, experienced by the girls as making participation not worth the battle and as unhelpful by many of the boys (see Zevenbergen 2000 for an important discussion about this issue related to class). Nor is there room to discuss the students' unhappiness with the 'top set' experience and its the associated pace (Boaler 1997; Boaler, Wiliam and Brown 2000) which surfaced strongly despite being outside the researchers' agenda. But both the themes of 'discussion' and of 'getting it wrong', and also these themes to which we only have space to allude, can be seen as representing a desire on the part of the students to have different social practices in the classroom. In turn, the nature of these social practices influences and is influenced by, indeed, in part, constitutes, the nature of the mathematics.

Conclusion

With this class of students, we encountered no opposition to whole class interactive work in itself: indeed, as seen above, group activities *as such* were not experienced as engaging. When given the opportunity to discuss their experience of whole class interactions, the students selected those aspects which allowed them to participate more fully, both socially and mathematically. The students preferences and suggested changes to classroom practice point to a community of mathematical practice which gives time and opportunity for the construction of a more shared knowledge in the mathematics classroom and for a more 'spacious' pedagogy (Angier and Povey 1999).

The students could point to features of their experience that hinted at the possibility of such a pedagogy even if this was in the context of a generally transmissive orientation. This indicates the way in which small changes of practice can begin to create the space for greater participation and for a different pedagogy to be developed even within such an orientation. For example, the idea of discussing with someone next to you before making a public contribution, considered in the sorting exercise, re-emerged strongly in the interviews. This is a relatively simple practice to arrange in mathematics classrooms. It does not, in itself, make a major change to the epistemology of the classroom but, in a small way, it recognises the legitimacy of mathematical authority for oneself and other learners. These students also say that it means they would be more likely to participate.

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SO THAT'S WHAT A CENTIMETRE LOOKS LIKE: STUDENTS' UNDERSTANDINGS OF LINEAR UNITS

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In this paper the results of a set of tasks designed to investigate student understanding of linear measurement units and the process of measuring length are presented. By Grade 5 the majority of the students were able to use informal paper-clip units to measure length and to identify linear units. However, few students in Grades 1 to 4 showed an understanding of the linear nature of units when they were asked to show a centimetre unit length in a variety of contexts. The results indicate that teachers need to identify units explicitly when they are teaching measurement because many students do not seem to have abstracted this concept in grades where it was assumed they had done so.

INTRODUCTION

In 1976 Carpenter reviewed the research on students' learning of measurement, based largely on the work of Piaget. Carpenter questioned how students might benefit from the information that had been gained because, in his opinion, training on specific concepts of conservation and transitivity seemed to be less important than training in measurement itself. At that time he referred to the lack of direct research relating the results of research to the measurement curriculum. He pointed out that "although research has identified levels of development of measurement concepts and rough age approximations for the development of certain operations, it is not immediately clear what implications this has for the curriculum" (Carpenter, 1976, p.71). Other researchers have queried the idea that concepts such as transitivity and conservation must be learned prior to measurement. Nunes, Light, & Mason (1991) have argued that measuring activities themselves assist in the understanding of transitivity and conservation.

Since that time there have been a number of fundamental studies of how students learn measurement concepts, in particular, Battista, Clements, Arnoff, Battista, K., & Borrow (1998); Hart, Johnson, Brown, Dickson, & Clarkson (1989); Outhred & Mitchelmore (2000); Wilson & Osborne (1988); Wilson & Rowland (1992). The emphasis of these studies is an understanding of how students interpret measurement concepts linked to the mathematics curriculum. However, not only curriculum knowledge is important. Hiebert (1984) has commented that "many children do not connect the mathematical concepts and skills they possess with the symbols and rules they are taught in school" (p. 498). He points out that if learning is to be applicable then students need to connect classroom and real-life experiences with the formal mathematical abstractions.

The first topic primary-school students usually encounter is linear measurement, and their knowledge of length provides the basis for the later development of area and volume concepts, as well as understanding of measurement scales which are essential for mass, time and temperature. A useful way of examining the way children think about linear measurement is to use Hiebert's (1986) distinction between the formal symbols, skills and procedures (procedural knowledge) and the intuitions and ideas about how mathematics works (conceptual knowledge). He states that the critical connection between procedural and conceptual knowledge is required when students have to know 'how the system works' to solve tasks or problems.

An example of procedural and conceptual knowledge in measurement would be the scale on a ruler. A ruler involves symbols (marks representing the beginning and end of each unit linked with a numeric scale, as well as shorter marks representing subdivisions) and procedures for use (aligning the ruler with the object to be measured and reading off the scale). These involve procedural knowledge. By Grade 5 almost all students can measure the lengths of objects using a ruler, that is they have mastered the "form" of ruler use but they do not understand its construction (Bragg & Outhred, 2000). Making an accurate ruler to measure in informal units (say, paper clips) would seem to involve an "understanding" of how the measurement process works in Hiebert's use of the term.

Hiebert suggests that "Many of children's observed difficulties can be described as a failure to link the understandings they already have with the symbols and rules they are expected to learn. Even though teachers illustrate the symbols and operations with pictures and objects, many children still have trouble establishing important links" (1984, p. 501). An understanding of measurement units would seem to be fundamental to establishing links across different measurement topics. The aim of this paper is to investigate the growth of students' knowledge of linear units across the primary school years. The paper reports the results of five tasks from a larger study of the development of children's understanding of linear measurement.

METHODOLOGY

The study was cross-sectional; 120 students from Grades 1-5 (aged 6-10 years) were selected from three state primary schools in a medium to low socio-economic area of Sydney. Each class teacher selected six students: one girl and one boy considered 'above average', 'average', and 'below average' in terms of mathematical concepts. However, in one school twelve students were selected from each grade. The first researcher interviewed individual students towards the end of the school year (September-November). Thus, they had been exposed to a large part of the measurement program for their grades. The interview tasks were designed to elicit information about the students' understanding of length measurement. Five interview tasks, a subset of the larger study, were used to determine what the students understood about the linear nature of units of measure using both informal and formal units. An understanding of measurement units would seem to be a fundamental aspect of learning to measure. Work with informal units is used to help students become familiar with important properties of units and how lengths may be measured and compared (Campbell, 1990). The subset of tasks (see Table 1) was used to determine how students might represent linear units that are formalised on portable tools such as rulers and tape measures.

The first task (Task 1) required students to apply their knowledge of informal units to construct a ruler, using the length of the paper clips as the unit. The second task (Task 2) was used to establish if students could measure with informal units. Tasks 3, 4 and 5 required students to identify and represent units of linear measure (centimetres). In Tasks 3 and 4 students were asked to either mark the centimetre unit on a printed ruler or on a washable plastic 1cm cube. In Task 5 students were shown a picture of a gesture commonly used to indicate a centimetre (thumb and forefinger opposed to show a gap of approximately one centimetre) and to mark what one centimetre would look like if you could see it.

The tasks were presented in the same order to all students. The paper clip items (Tasks 1 and 2) were presented first followed in order by Tasks 3, 4 and 5. However, the five tasks were separated by other tasks not listed.

Table 1 The five tasks involving linear units.

Task	Description	Knowledge
1	Make a ruler using paper clips as the unit of measure (students were given a long rectangular strip of light cardboard).	A scale can be constructed by iterating a unit and marking each endpoint. These marks can be associated with numbers.
2	Use 2 paper clips to measure a line 28cm long (noting the fractional unit).	A length can be measured by iterating a constant-size unit with no gaps or overlaps. Fractional units may result.
3	Count 5 sea horses shown on a card and state what the '5' represents. Then explain what '5' on a ruler represents and identify a single unit.	Linear units are separated by marks. A numeric scale aligned with the marks gives the number of linear units from the origin.
4	Draw the linear unit on a picture depicting a familiar representation of a centimetre: thumb and forefinger placed 1cm apart.	Identification of the linear unit in a pictorial representation.
5	State which part of a 1cm cube (a 'short') is used when measuring a length.	The length of an object gives the measurement unit (its area and volume are not relevant).

RESULTS AND DISCUSSION

The results for Tasks 1 and 2 involving constructing a paper clip ruler and measuring with paper clips are presented in Figure 1 for each grade level. There were 24 students at each grade level.

The results for Task 1 (see Figure 1) show a gradual increase in the construction of an accurate paper clip ruler. No Grade 1 students and only a few Grade 2 (21%) students were successful but, by Grade 5, 75% of the students constructed an accurate ruler. Students were not successful for the following reasons: they used the paper clips as unit markers; they did not maintain a constant-size unit; or they used an arbitrary unit length. There was a very clear distinction between those students who successfully used a paper-clip length as the unit of measure and those who used them as markers. Successful students were observed to mark each unit length carefully and then add the correct numeral.

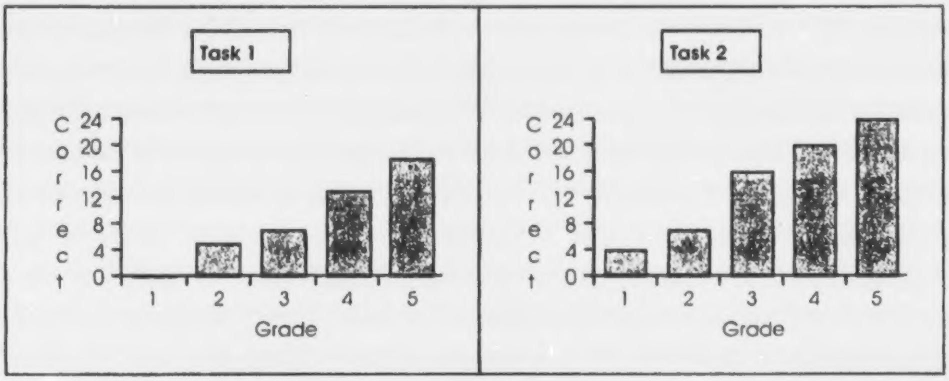


Figure 1 Number of correct responses for each grade for the tasks involving informal units (Task 1 and Task 2)

The graph for Task 2 (see Figure 1) shows an increase from Grade 1 to Grade 5 with the greatest change between Grade 2 and Grade 3, when there is an emphasis on teaching length measurement. Only a small number of Grade 1 students could measure a length by iterating paper clips and indicate that the result involved a fraction of a paper clip, but by Grade 5 all students could successfully complete this task.

However, when the results of Tasks 1 and 2 are compared with Tasks 3 to 5, it is evident that being able to measure with informal units and to construct an accurate paper clip ruler are not sufficient to show that students understand the linear nature of the units. The results for Tasks 3 to 5 (see Figure 2) indicate that until Grade 5, very few children could show what a centimetre would look like on a ruler (Task 3), between a picture of an opposed finger and thumb (Task 4) or on a cube (or "short") (Task 5). Their responses showed that, in the case of the centimetre on the ruler students represented the unit either as a space (e.g., by placing their finger on the space) or as a feature of the ruler (e.g., as marks).

Daniel's (Grade 4) remark was typical of many students who had a 'spatial' view: "It's these spaces here, you just count them". His finger fitted the 1cm space on the ruler. He, like many others, coloured in the space between the printed fingers (Task 4) and said that you counted the face of the 'short' (Task 5).

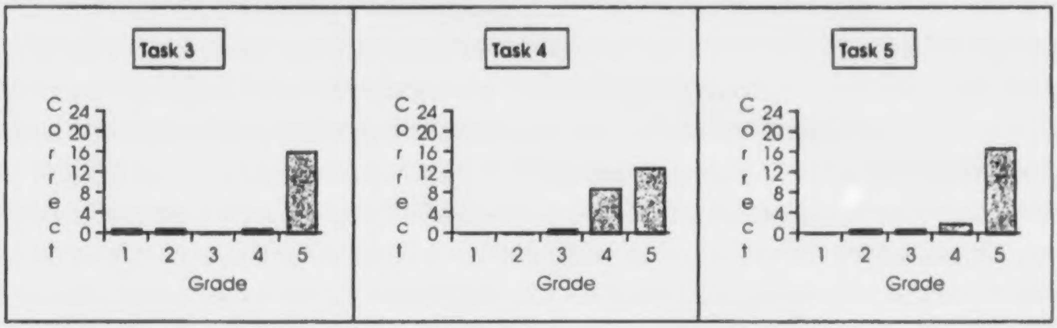


Figure 2 Number of correct responses for each grade for the tasks involving identification of the linear unit (Task 3, Task 4 and Task 5)

For Task 3, in which students were shown a picture of five seahorses and asked how many. If the answer given was five, the interviewer prompted "five what?" Then the student was asked what the "five" on a ruler represented and if the response was 5 cm, the student was asked to show one centimetre. A number of students represented centimetres as a mark or as marks perpendicular to the length. These students would either say that the measure (5cm) was "Where the line ended", or they would indicate that the 5 centimetres were the five marks: Comments similar to "The lines on the ruler point to the numbers you need." were very common. A large proportion of students (38%) from Grades 3 to 5 made observations such as "...there's nothing at the edge of a ruler anyway, that's where you rule lines." Similar misunderstandings were reflected in the responses to Task 5. In Grades 1 to 4 students usually counted the cubes to measure lengths and most said it was "the flat part" that was used to measure lines. These results suggest that many students have not abstracted the concept of a centimetre as a linear unit from their experiences of measuring length.

Students' different representations of a centimetre unit in Task 4 are shown in Figure 3. The percentage of students who gave each response is shown in Figure 3. There appeared to be four main forms of representations: unrelated to length, area representations; ruler-like representations; and linear representations. Eduardo's response cannot be interpreted as he has transformed the finger-thumb opposition into a "C' cue with a small m inside to remind him of "cm". There is not indication that he has a linear unit in mind when this representation is shown.

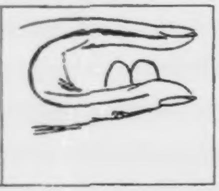
The area representations highlight why a response of pointing to a space on a ruler is not sufficient to assume that a student has abstracted the idea of a linear unit. Older students who made this error were more likely to draw a 1cm square whereas the younger ones were more likely to colour in the whole space. Such students may have a mental representation of a two-dimensional unit in the space between the marks. The confusion with a centimetre cube is evident in the second and third examples (2(b) and 2(c)).

The first two ruler-like representations involve marks to indicate a scale (3(a)), and numbers to indicate a scale (3(b)) between finger and thumb rather than a single 1cm unit. In 3(c) students draw a ruler perhaps because they link rulers and centimetres but they cannot isolate one unit whereas in 3(d) they seem to be indicating the marks delineating a unit (similar to a ruler) but the unit itself could be two-dimensional.

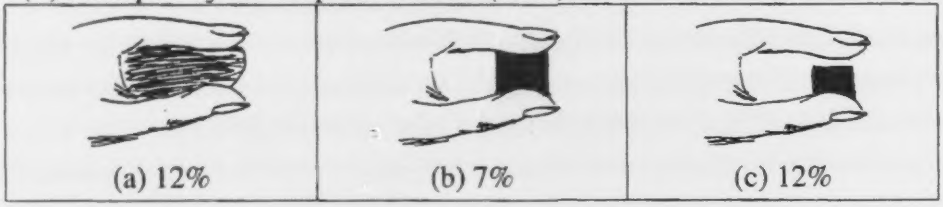
The first and second linear representations (4(a) and 4(b)) suggest that students have no idea of the part of the diagram that is meaningful. Similar responses were found in Task 5 in which students did not know what part of the cube was relevant when measuring length. Only 19% of students in the sample, almost all from Grade 4 and Grade 5, were able to draw an accurate representation of the linear unit. This inability to represent a single linear unit was also found in the other two tasks, the sea horses and the cube.

(1) Unrelated representation (3%):

"I learned how to remember a centimetre this way 'cause I got mixed up with a metre." Eduardo, Year 4.

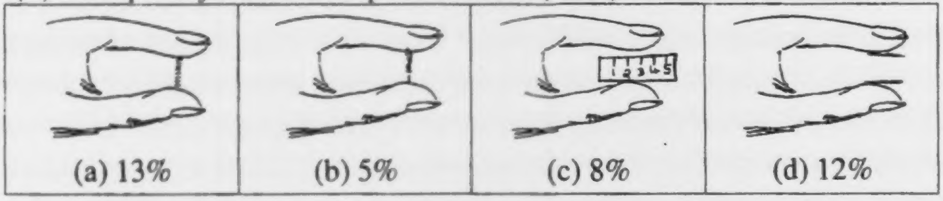


(2) Examples of area representations (31%):



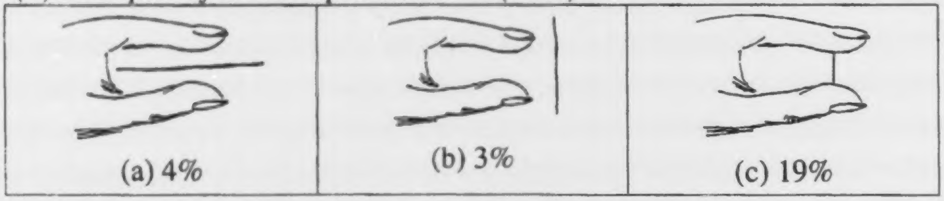
(a) 12% (b) 7% (c) 12%

(3) Examples of ruler-like representations (38%):



(a) 13% (b) 5% (c) 8% (d) 12%

(4) Examples of linear representations (25%):



(a) 4% (b) 3% (c) 19%

Figure 3 Categories of responses to Task 4 and the percentages for each response.

CONCLUSION

According to Kamii & Clark (1997) student performance on National Assessment of Education Performance (NAEP) items remains disappointing. The results of this study suggest possible reasons why performance on measurement items may be disappointing. Although these students were able to manipulate informal units to measure lengths, there is little evidence to show that they have constructed an understanding of the linear nature of the units of measure until Grade 5. While students in Grades 3 and 4 would have had many opportunities to measure using both informal and formal units, few of them were able to identify a cm unit on a ruler, on a cm cube or on a drawing showing a thumb and finger 1cm apart.

Learning about measuring and the identification of units of measure is very complex (Campbell, 1990) and the dimensionality of the units contributes to this complexity. The analysis of students' drawings of the centimetre unit on a drawing showing a thumb and finger indicated that the majority of the students in Grades 1 to 4 had constructed either a 'spatial' or a 'ruler-feature' concept of a centimetre. Teachers may be unaware of the multiplicity of representations that students construct for this "convention" of showing the size of a centimetre unit.

These results support the theories of Hiebert (1990, 1986, 1984) and Skemp (1979) that many students are learning the "procedures" of mathematics but not the understanding of, or relationships among, the fundamental concepts, linear measurement units in this particular case. Paper and pencil tests and exercises with informal units often assess only student knowledge of routines and procedures and do not reveal if students possess an understanding of units of measure, that length may be represented by a line and that the units of measure are also linear. For example, use of cubes to measure lines may contribute to students' confusion unless teachers make explicit the part of the cube being used as a unit.

The results from this study have shown that, in spite of their facility with informal units, the majority of students have not constructed a clear representation of linear units of measure. Since students rarely establish explicit and unambiguous connections (Hiebert, 1984) researchers and teachers need to work together to investigate how to teach young students to link experiences with informal units with formal measurement, especially the construction of rulers and scales. Students construct "mental images" (Shaw & Cliatt, 1989) and referents that make sense to them from classroom contexts. Incorrect or confused representations may remain undetected if assessment relies on paper and pencil questions and procedural tasks, such as ruling lines or measuring objects. While teachers are encouraged to help students 'make sense' of measurement and not rely on procedures (NCTM, 1989), there has been insufficient research to recommend to teachers how they can best assist students to understand key measurement concepts.

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PEER INTERACTIONS AND STATISTICS LEARNING

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Abstract: This research is part of the project Interaction and Knowledge, whose main aim is to study and promote peer interactions as one of the possible forms of developing pupils' socialisation and positive attitudes towards mathematics, as well as to promote their socio-cognitive development and enhance their school achievement. In this paper we analyse a case that illustrates the role of peer interactions in knowledge appropriation in a statistical task.

Introduction

The curricular reforms implemented in the last few decades have stressed the need to stop considering only aims related to contents and start taking into account the development of attitudes and values, as well as abilities and skills (Abrantes, Serrazina and Oliveira, 1999). These authors claim that "these three aspects (knowledge, abilities and attitudes) are inseparable, not just in the new tasks pupils are presented with but also in the learning process itself" (p. 22).

[1]One of the suggestions most often made in official documents regards turning to group work and giving importance to interactive processes that are present in the didactic relation (NCTM, 1991; van der Linden et al., in press). Abrantes, Serrazina and Oliveira (1999) state that "since pupils are different from one another and build different images and conceptions about the topics under study, the teacher has to value the interactions between pupils and between these and the teacher" (p. 29). This is precisely the main goal of the project *Interaction and Knowledge*: to study in detail and promote peer interactions as a way of fostering pupils' full development and their school achievement in Mathematics. At the same time, the intrinsic advantages of a greater link to reality and pupils' experiences have been highlighted, and Ponte, Matos and Abrantes (1998) have declared that "Statistics and Probabilities are essential themes that allow for a link between school mathematical knowledge and the Maths used in everyday life" (p. 170). This way, when we chose Statistics as a curricular unit for detailed studying in our project, we had in mind that it adapted particularly well to the development of studies of an interactive nature and would encourage exercising a critical and participative citizenship.

The main aim of this report is to analyse the performances of the dyad in tasks related to the concept of mean, stressing the role of peer interactions in the co-construction of solving strategies, in statistical tasks.

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Theoretical background

According to Shaughnessy (1992), twenty years ago there was hardly any research in the field of statistical education. In Portugal, despite the topics of Statistics and Probability being a part of the curricula since the 80s, only in the 90s did they begin to be taught by most elementary-level teachers (Ponte, Matos and Abrantes, 1998). The aims of Statistics contents include aspects such as: developing communicational abilities, autonomy and solidarity (including showing a critical and rigorous spirit; trust in one's reasonings; approaching new situations with interest and initiative; assessing situations and making decisions) or the capacity to use quantitative methods to analyse real-life situations. Reaching this kind of aim is not compatible with teacher centred teaching, neither with solving routine activities, where pupils only have to turn to instrumental knowledge, such as using formulas and algorithms, without needing to interpret the proposed situation (Batanero, 1998, 2000; Carvalho and César, 2000a; César and Silva de Sousa, 2000; Ng and Wong, 1999).

Putting into practice the suggestions of the current curricula and educational policy documents means creating novel tasks, promoting horizontal interactions (pupil/pupil) and not just vertical interactions (teacher/pupil), being able to explore pupils' reasonings and response strategies, posing stimulating questions that increase their involvement in the tasks. Thus, the didactic contract or the experimental contract established with the pupils or subjects of a certain research is of the utmost importance. These are the (generally implicit) rules that legitimise many of the mutual expectations of the various social partners involved in an interactive process, with which the appropriation of knowledge and the mobilisation of relational competencies are supposed to exist. Therefore, implementing novel classroom practices or experimental activities also involves defining contracts that are novel themselves and that encourage peer interactions (César, 2000a, 2000b, 2000c; Grossen and Py, 1997; Schubauer-Leoni and Perret-Clermont, 1997).

Several investigations have discussed the role of social interactions in cognitive development and in the promotion of pupils' school achievement, namely in Mathematics (César, 2000b; Perret-Clermont and Nicolet, 1988; Schubauer-Leoni and Perret-Clermont, 1997). In Portugal, the first studies by César (1994) were an attempt to find answers to the challenges brought forth by teachers who wanted to know how to form efficient groups in their daily practices in order to improve pupils' mathematical knowledge. These studies were contextualised and they allowed us to understand the mechanisms related to peer interactions and their role in the promotion of pupils' socio-cognitive development and knowledge appropriation. In recent studies (Carvalho and César, 2000c) we showed that working in peers was an effective way of promoting pupils' cognitive development and better statistical performances. But in order to understand how interactive processes contribute to these progresses we need to undertake an in-depth analysis of the interaction itself.

The contributions of the Vygoskian theory (1962, 1978) were essential in order to understand the difference between actual development and a potential one. This difference leads to the notion of Zone of Proximal Development (ZPD) which is one of the most fruitful constructs arising from this theory and one of the most explored in educational settings (Allal and Ducrey, 2000; Moll, 1990). But in order to work collaboratively pupils need to construct an intersubjectivity (Werstch, 1991) that allows for exchanging ideas and knowledge, i.e., to be able to follow each other's solving strategies as well as to manage every aspect related to the relational context.

Method

The project *Interaction and Knowledge* is divided into two different levels: 1) - A micro-analysis level, in which we studied different types of peers, their interactions, the tasks we propose, the mistakes they make and the progress that peer interactions are able to generate in statistical contents (Carvalho and César, 2000a, 2000b, 2000c, in press; César, 2000a); 2) - An action research level, in which some mathematics teachers implemented peer interactions as a daily practice during a school year (César, 1998a, 1998b, 2000a, 2000c; César and Torres, 1998; César and Silva de Sousa, 2000). The data we are going to present are from level 1.

Subjects

The sample was formed by 136 dyads. Subjects attended the 7th grade in two public schools near Lisbon. Their ages were between 11 and 15 years (Average=12,5 and Sd=0,8). The case we are going to analyse is representative of most dyads.

Instruments

The statistical tasks used in peer work sessions were "unusual" ones according to teachers' statements and to our previous observations of their classes (Carvalho and César, 2000c; César, 2000a). They corresponded to more open and innovative tasks.

Procedure

In this research pupils had three sessions in which we promoted collaborative work. Pupils were grouped in dyads and the interactions were audio taped only in the second session of peer work. The episode we are going to analyse is related to one of the tasks presented in the second session. The working instructions that were given asked them to discuss before writing their answers and to explain everything they thought. Each dyad only had an answering sheet.

Results

The task that pupils were solving in these episodes is the following one: *"The mean of four numbers is 25. Three of those numbers are 15, 25 and 50. Which is the missing number?"*

Case 1 - Co-constructing statistical knowledge in an asymmetric dyad

14 A - First we must sum up 15 + 25 + 50.
15 C - 15 + 25 + 50 is 90 [She uses the calculator]. If this was the mean of 90, 90 dividing by three would be 30.
16 A – Let's try with 94 divided by four. It must be an integer number. So, adding four doesn't make it right.
17 C - 90 plus what? 6? Wait a moment, I'll do it. [She uses the calculator] 25 isn't right either
18 A - So, how do you calculate the mean? When you calculate the mean you sum up all this [she points to the values] and you divide by a number.
19 C - How many they are?
20 A – 4. So, what you should do is this, 15 + 25 + 50 + X and divide by 4 which equals 25.
21 C - How much is the X?
22 A – It's the result of 15 + 25 + 50 + X divided by four which was 25.
23 C – It's 6.
24 A - It can't be. In order to solve this [equation] you have to do 25 times 4.
25 C - Wait a moment!
26 A - Let me do it. 25 times 4 is 100, eh! Delete it [C is doing the computations with the calculator]. Now try to do 100 dividing by four to see whether... [C finishes her computations and she agrees moving her head] it's 25. Then, it has to be 10. It's 10.
27 C - 15 + 25 + 50 + 10 is 100.
28 A - Now, if you do 100 dividing by four, which is the total of the data. Do it: 100 dividing by 4 is...
29 C - 100 dividing by four is 25 [She writes down]
30 A - Equals four, how dumb! Equals 10, because 25 times 4 is 100 and then... If we had 100 to 90 there's 10 left. Let's move on to the next one.

In this episode A.'s leadership is clear although, according to the rules of the experimental contract we established, she always explains everything she is thinking to C. and takes care to check if C. is following her solving procedures for the proposed task. So, C.'s first phrase is to show A. she knows how to calculate the mean. However, this phrase does not show a solving strategy for the problem, for C. just says what the mean of the 3 numbers on the sheet would be. The fact that classroom practices usually stick to applying algorithms and procedures (Batanero, 2000; Carvalho and César, 2000a; Shaughnessy, 1992; Skemp, 1979) may explain why this pupil associates solving this problem with merely applying an algorithm, regardless of this leading or not to the solution of the problem. This way, C. would dominate what Skemp (1979) calls instrumental knowledge but, if she worked individually, she wouldn't manage to reach a relational knowledge.

As the mean is 30 instead of 25, A. realises that the missing number mustn't be very high. So she decides to start applying a trial-and-error strategy, which César (1994) considers to adapt well to problems of a medium complexity, but in this case her attempt fails from the start. However, we can see that despite C. having the calculator, it is A. who controls the credibility of the results: *"It must be an integer number"*. So she is the one who understands, through the result on the calculator, that her hypothesis isn't the result they want, for the number is not an integer.

C., who is having more calculation difficulties, doesn't want to leave the solution of the problem to A. alone. Therefore repeatedly she tells her *"Wait a moment"*, which not only shows that she acknowledges that she works at a slower rhythm than her peer but also that work should be carried out in dyads and she intends to contribute to it, however slow she is. Curiously, the several temporal expressions present in this dialogue have a well defined role: for C., they mean to tell A. that she needs more time to think, so A. doesn't answer just yet. So they serve to put the brake on the speed of the solution. For A., who starts several times by saying *"Now..."*, they serve to accelerate the rhythm, make suggestions or even give orders, so C. can collaborate but not put the task solution at risk. This is also visible in the last phrase of the episode, when A. wraps up the task solution, adding, *"Let's move on to the next one"*.

On the other hand, since C. has less knowledge that is necessary for the task solution, the pauses she imposes are also a way of negotiating – in the relational context – the appropriation of the knowledge in question in the task solution. This probably explains how C. progressed, between pre- and post-test (see Carvalho and César, 2000c for more details) in cognitive terms and in terms of her level of performance in statistical tasks. Besides, this kind of attitude also shows she understood and stood by the proposed experimental contract, since it suggested they only write down an answer when both agreed with it and when both had collaborated in the task solution.

These rules of the didactic contract give rise to a few words of a pedagogic nature, on behalf of the more competent peers: *"So, how do you calculate the mean? When you calculate the mean you sum up all this* [she pointed out to the values] *and you divide by a number"*. This way, we may state that the rules of the experimental contract play a very important role in the type of interaction that pupils establish, as several authors stress (César, 1994; Schubauer-Leoni and Perret-Clermont, 1997).

As soon as A. realises that through trial and error she is not reaching the result she wants, she drops this tactic and changes to an algebraic strategy, typical of pupils who already have access to formal reasoning (César, 1994; Perret-Clermont and Nicolet, 1988). She easily verbalises the translation of the problem on the sheet in an equation: *"15 + 25 + 50 + X dividing by 4 equals 25"*, which means A. does not only master Statistics contents. In this part of the interaction the difference between the operatory level of each of the subjects is clear: for A., the value X really represents an unknown quantity and does not need to be immediately replaced with the value it represents; C., who is on a less advanced operatory level, needs to ask *"How much is the X?"*, for she needs to materialise this value. Actually, it is funny that C. seems to adopt A.'s initial strategy for a moment, for she still tries to see whether X can be replaced with 6, before calculating its value by solving the equation A. formulated.

Another very interesting point is the understanding A. gains through C.'s difficulties. When she realises C. has difficulty in reasoning in formal terms, she abandons the equation solution that had been her second suggestion and chooses an arithmetic strategy: 25 times 4 is 100; the previous numbers had summed 90. So, the missing number is 10. This way A. does not drop the leadership of the problem solution but adapts her speed and solving strategies to the difficulties revealed by C. They are able to construct something that Werstch (1991) calls an intersubjectivity, whereby both contribute to find an adequate solving strategy. This capacity to simultaneously deal with the cognitive as well as the relational aspects of the task is what enables them to work in their ZPD, which, according to Vygotsky (1962, 1978), promotes cognitive development. This is what happened to the less competent peer, who progressed to performances that are typical of the formal stage.

Regarding the relational aspect for a moment more, it seems important to us that there is an opposition between A.'s patience and the type of vocabulary she uses when C. has difficulties or makes a mistake, and the way she reacts when she makes a mistake herself. Only one expression with a negative connotation is used in this last case: *"How dumb!"*. This means that A. has a true concern in supporting C., in contributing so that she gains confidence in her competencies and mobilises them more easily. When it comes to judging herself, A. becomes a harsher evaluator, probably because she doesn't feel her competency threatened by the fact.

It seems clear to us that for either one of the elements of this pair, it is the fact that they work collaboratively and with certain rules of the experimental contract that allows them to develop a relational knowledge. The fact that they have to explain everything they do, discuss several solution hypotheses, experiment with several possible strategies, makes the level of appropriate statistical knowledge more complex, for they discuss the inherent context of the proposed task between each other, enriching the meaning they are capable of giving it.

As for statistical performances, in both cases they had better results in the post-test than in the pre-test, which is in accordance with results reached by several pupils, namely in Portuguese investigations (Carvalho and César, 2000c, in press; César 1998a, 1998b, 2000a, 2000c; César and Torres, 1998). Contrary to what Vygotsky (1962, 1978) believed, it is not just the less competent peer who benefits from the interactive process. Peer interactions have shown to be a powerful mediator in knowledge appropriation and in the mobilisation of competencies on behalf of the pupils. Therefore, when we promote peer interactions in the Maths class we are facilitating pupils' full development and their school achievement in this subject, for on a post-test level, which is already solved individually, subjects are often capable of using solving and reasoning strategies that we saw them developing for the first time during the collaborative working sessions.

Statistical tasks adapt particularly well to collaborative work, as long as teachers use open tasks that stimulate the use of varied strategies and the development of a critical spirit (Carvalho and César, 2000a). In this case, more positive attitudes towards Maths are also promoted, besides contributing to the development of an academic self-esteem on behalf of the pupils who usually have difficulty in reaching good performances. Thus, as we see in the example we presented, pupils show their intent in solving tasks instead of abandoning or rejecting them. Hearing and talking to a colleague forces them to express their doubts and solving processes clearly, and this contributes to them being able to appropriate more knowledge and do so in a relational way instead of a merely instrumental way (Skemp, 1979). This is in conformity with the aims suggested in the educational policy documents.

Final Remarks

Attributing a meaning is a fundamental step in solving a mathematical task and for knowledge appropriation and the mobilisation of competencies. As Vygotsky (1962, 1978) stressed, pupils need to de-contextualise and re-contextualise knowledge so that it goes from being external and social to being internal and personal. Facilitating the attribution of meaning is essential in order to promote school achievement and numeracy, so the nature of the tasks we propose to pupils must be taken into account. Besides this, social interactions, namely peer interactions, play a truly relevant role in the facilitation of meaning attribution, so the implementation of a novel didactic or experimental contract is also a very important element to take into account if we intend to reach the aims of current curricula and educational policy documents.

In order to promote school achievement in Maths, namely in Statistics contents, we need to facilitate the possibility of pupils going from an instrumental knowledge to a relational knowledge. But for this to become a reality and not just an intention, we need to learn to observe pupils' solving strategies and reasonings in detail. Only this way can we, as researchers and teachers, promote practices that allow pupils to work in their zone of proximal development, as several authors suggest.

The recommendations of several official documents point to the need to value social interactions in the Maths classroom, particularly if we consider, as Abrantes, Serrazina and Oliveira (1999) state, that "Maths education may contribute, in a significant and irreplaceable way, to help pupils become individuals who are not dependent but, on the contrary, competent, critical and confident in the essential aspects in which their lives relate to Maths" (p. 18). However, we feel this aim will only be fully reached if we know how to implement rich and fruitful social interactions in the classroom.

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ARITHMETIC AND ALGEBRA, CONTINUITY OR COGNITIVE BREAK? THE CASE OF FRANCESCA

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***Abstract**. Taking the 'operational/structural' perspective, as introduced by Sfard, this paper analyses the passage from computing with numbers and computing with letters. The discussion is based on a case study, taken from long term research project, still in progress. A key aspect, characterizing the transition from the two types of computation, will be highlighted: the change of role of operation properties.*

Introduction

High school Algebra activities are often characterized by the use of symbolic manipulation. According to the Italian tradition, symbolic manipulation constitute a basic element of secondary school curriculum and in school practice. Usually, students are introduced to the manipulation of algebraic expressions only after experiencing large amounts of computation of numerical expressions: therefore the problem of transition from numerical to algebraic computation arises. Analysing the connection between Algebra and Arithmetic, Lee & Wheeler (1989) showed that the relationship between calculation with numbers and calculations with letters is not so direct and transparent. They argue that, in spite of the use of common operation signs, the activities of writing and manipulating expressions in algebra and in arithmetic are quite different.

In this paper we shall point out and discuss some aspects characterizing the difference between calculating in Algebra and in Arithmetic; the following analysis will be carried out in terms of Sfard's *operational-structural* theory (Sfard, 1994).

Both the necessity and the difficulty of achieving such a duality is clearly expressed by the author:

"The formula, with its operational aspect (it contains 'prompts' for actions in form of operators) must be also interpreted as the product of the process it represents."

"[…] our intuition rebels against the operation – structural duality of algebraic symbols, at least initially."

(Sfard, 1994, p.199)

The operational character of pupils' conceptions related to algebraic formula and expressions tends to persist; at the same time, although symbolic manipulations of algebraic expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397).

This paper aims at analysing the relationship between the two levels of computation:

computation with numbers and computation with letters. We will argue that, contrary to what books and teachers usually state, the transition from computing with numbers and computing with letters is not so smooth and in fact, it may present a cognitive break: as suggested by Francesca: *"Our teacher says that with letters it [computing] is the same as with numbers, but to me it doesn't look the same, it looks very different [It: non mi sembra la stessa cosa]"*.

Methodology

The results we are going to discuss are part of a long term project (Cerulli & Mariotti, 2000) concerning the introduction of pupils (aged 14-15 years) to algebra and in particular to symbolic manipulation. A 9th grade class was split into two separate groups, one following the project, the other following a traditional approach to algebra. Comparison between the groups is interesting because the pupils have the very similar school experiences: apart from the class of mathematics, and they share the same courses for all the other subjects.

The following discussion concerns only some results related to the exemplary case of Francesca, a medium-high level student who attended the traditional course.

1 Before the algebra course

At the beginning of the teaching experiment, before splitting the class, a test was submitted to the students. Some answers given by Francesca are analysed. The first item of the test concerned the correctness of some equalities between numerical expressions.

> **T1** Observe the following statements, for each of them explain why you think it is correct or why you think it is wrong.

Let us consider the following answers given by Francesca:

T1.2 $17 + (6 + 9) = (17 + 6) + 9$

This statement is correct because of the associative property.

T1 .5 $8 + 9 \cdot (3 + 2) - 17 = 8 + 27 + 18 - 17$

This is not right because one can't get rid of the brackets and compute 9•3 and 9•2 and then add [the terms] because the result changes.

T1. 8 $3 + 6 \cdot 73 + 6 \cdot 8 + 13 = 3 + 6 \cdot (73 + 8) + 13$

It is right because adding few numbers, which are multiplied by the same number, or multiplying them by the previously defined [common] number the equivalence remains unchanged.

Unlike other students, Francesca seems to tackle the problem within a structural, instead of a operational approach (Sfard, 1994). Both aspects actually seem to be present: on the one hand the operations properties are considered rules which determine whether or not it is possible to pass from one expression to another ("*...keeps the equivalence unchanged*"); on the other hand, they are considered equivalent relationships between computing procedures. Answers to items 5 and 8 are clearly in contrast but both justifications have the same nature: Francesca is so strongly convinced about the acceptability of the transformation rule, that she performs no computations to check the equality statements.

Thus the rules known by Francesca are instructions which make it possible to move from one expression to another, from one computing procedure to another; the sign "=" is interpreted according to a fixed direction (from left to right) and this peculiarity may affect the acceptability of reversible transformations of algebraic expressions.

In conclusion, as concerns the operations properties, both the structural and procedural aspects can be found in Francesca's answers, but they do not seem to be stable and merged together.

2 After a traditional algebra course

After one year of activities within a traditional framework concerning algebra and symbolic manipulation, a number of interviews were designed, aimed at investigating the relationship between computation with numbers and computation with letters. In other words, we were interested in studying the evolution of the conception of "computing, taking into account the fact that in Italian a unique word "calcolo") includes both symbolic and numerical computations.

2.1 What is the meaning of "calcolo"?

When asked what she intends by the word "*calcolo*", Francesca refers to the primitive model of computation related to the four operations, natural numbers and eventually fractions; the meaning of "computing" ("calcolare") is *"finding something unknown [qualcosa che non si conosce]"*, she says. Changes occur when letters are introduced; consider the following excerpt of Francesca's interview.

30.A. Listen, and after these computations… are there any others?

31.F. With letters

32.A. With letters they are even more difficult, aren't they?

33.F. *(Laughs)* Yes.

34.A. Let's write a computation *(calcolo)* with letters, but a very difficult one …

35.F. (*She thinks and writes)* Mmm …something…how was it? *(while producing the expression she tries to remember some "prodotti notevoli"*[1]*, then she writes a cube)* Yes this one *(the cube)* that I could never work out, then the other one with fractions *(she writes the fraction line and denominator)*

$$\frac{((a+b)\cdot(a-b))^4\cdot(a-b+2)^3}{(a+b)\cdot(a-b)}$$

The construction of the example seems to be inspired by a strong model of computing with letters: the expression is obtained combining various "prodotti notevoli"; within this model, success in computing depends on whether one knows/remembers the formula associated with computing chunks (35).

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[1] A "prodotto notevole" is a standard equivalence statement used to speed up computations with letters, an example is $(a+b)\circ(a-b)=a^2-b^2$.

In the case of Francesca, this model contrasts with the original model of computing numerical expressions, based on the idea of "finding something", i.e. finding a result.

The common aspects shared by the two models are evident. Furthermore, the teacher states that they "are the same thing", but Francesca in convinced that computing with letters is quite different from computing with numbers, and she says:

64.F. Well, I know that…also the teacher always tells me that computing with letters is the same thing as computing with numbers, but to me it is not the same. Because if I am given 10+3 whilst…[if you give me] a•b+c+d I get stuck… *(laughs)* I can't work it out. With numbers we are back to something real, well … for me numbers are not real, but they are still more real than letters in mathematics

65.A. O.K. so the teacher said that it is the same.

66.F. Yes.

67.A. And you say "to me it is not the same", let's start from this point. Why isn't it not the same for you? That's what I am interested in.

68.F. Because!…because if you do 10+13 times, 25+3, you find it *[the results]*, but if you do a+b you can't do it, if you have to do it, for example *(she writes a numerical*

$$(10 + 13) \cdot (25 \cancel{+} - 3)$$
$$(a + b) \cdot (a - b)$$

expression and a literal expression)…here we put a minus *(she changes 25+3 into 25-3)*, if you do this, you get a number, but if you do that, you find "a square minus b square" *(she writes the "result" of the "prodotto notevole").*

$$(10 + 13) \cdot (25 \cancel{+} - 3)$$
$$(a + b) \cdot (a - b) = a^2 - b^2$$

Francesca's words clearly show that the meaning of numerical computation is consolidated and based on concrete models, whereas *"to find something"* has a specific stable meaning, but this meaning cannot fit the case of letters. Francesca knows quite well how to cope with algebraic expressions, and knows how to transform them using standard formulas, but she still can't accept the supposed similarity between the two ways of manipulating expressions: the model of computing with letters is not derived from an evolution of the model related to numerical computations. As a consequence she feels a break between the two situations; in other terms we can recognise a break between the two meanings of "calcolo". the next part of the interview shows clearly where such a break occurs.

2.2 Computation with numbers and computation with letters

Francesca tries to compare the two kinds of computations, and in order to express her uneasiness she starts to compute the expressions she has produced. In the numerical case, she first computes the sums in brackets, then multiplies the numbers obtained and finally obtains the result which is 506. In the case of the expression with letters, Francesca has already produced a result "a^2-b^2", thus she makes explicit the intermediate computation steps, previously skipped: she multiplies the two sums in brackets term by term, and finally sums the similar terms. At this point the interviewer asks:

74.Did you perform the same operations *(with numbers and with letters)*?

75.F. No I didn't

76.A. in this case and in that case *(she points to the two expressions)*?

77.F. No…well, yes, I multiplied, here *[expression with letters]* I multiplied everything, while here first I have to add, first I calculate the parts in brackets

78.A. And here, *(pointing to the numerical expression)* could you multiply everything or couldn't you? Is it forbidden?

79.F. You mean multiply 10 times 25, as I did here *[in the case of letters]*?

80.F. I never tried to!

81.A. Fine; it's ok that you never tried; do you think you would get 506? I mean do you think you would get the same result…or not?

82.F. I don't know…I don't think so…I don't know. *(she doesn't seem to be sure at all, she is curious, and starts to compute)*…250…13•25…

83.F. Aha! It is the same!

84.A. Is this a surprise for you?

85.F. Yes!*(laughs)*

$$\begin{array}{r} 250 - 30 + 325 - 39 = 545 - \\ \underline{39} \\ 506 \end{array}$$

86.A. Well, you never tried: is it true?

87. *(She nods)*

88.A. And now are you convinced…that it would be always the same, if you had done …because this is a specific case, you might have been lucky…do you think it would always work, or not?

89.I don't know. How can I …? I didn't know the rule that if you do so you always get the same result, it is not as if it were an axiom that tells you that it is always the same, I thought it was just a case, and not a rule.

90.A. Bah!, you wouldn't trust it. So, here, this way to compute *(points to the literal expression)* why… are you sure that this is correct?

91.F. Yes I do

92.A. And why do you trust it here?

93.F. Because I was taught to do so *(laughs)*

94.A. Because you were taught, and does that make any sense?

95.F. The sense is that of not writing such a long computation, and to write only those two numbers *(points to a^2-b^2)*, but there is…

It is surprising to see how astonished Francesca is when she realizes that the result obtained by applying the rules of symbolic manipulation to a numerical expression is the same as that obtained previously by computing sums and multiplications. Furthermore it is surprising that, even after having verified such a phenomenon by executing computations, she is still not convinced of it's generality. It looks as if no link has been established between the two kinds of computations and the acceptability of the rules is strongly influenced by school practice: when pupils are required to transform an algebraic expression into another one, the validity of the transformation depends on the external control of the teacher ("I was taught to do so" 93). Nevertheless, Francesca is able to establish a link between numerical and algebraic expressions:

96A. […] what is the relationship between the expression $(a+b)\cdot(a-b)$ and the expression $a^2 - b^2$? What is the relationship…

[…]

97.F. The res*[ult]*…well, if you put numbers instead of a and b, if you do this with 3 and 2, you get a number which is equal to…if you perform the other computations, there…

98.A. If I put numbers I get that result.

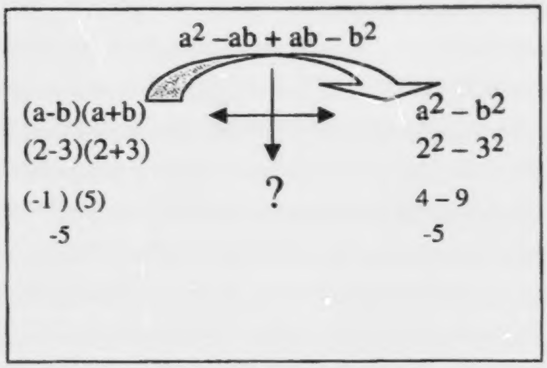
99.F. Yes

100. A. So, why don't you trust the fact that if you had a numerical expression it wouldn't…

101. F. I don't know…because I was never told…thus *(she smiles embarrassed)* …well if I think it would have, well… if it was that, well… anyway everyone tends to shorten *[computations]* and do things as quickly as possible, but then I don't understand anything anymore…

Francesca knows that two expressions are equivalent when they have the same value if letters are substituted with numbers; but this equivalence relationship, in terms of "values of the expressions", is conceived considering the two expressions as autonomous entities. The equivalence relationship of this type does not concern the symbolic manipulation which transforms one expression into another. In Francesca's view, two algebraic expressions [$\mathbf{(a-b)(a+b)}$ and $\mathbf{a^2-b^2}$] are two completely independent calculation procedures, which can be accomplished only by substituting letters with numbers. This shows that Francesca is conceiving the two expression according to a dual meaning (Sfard, 1994): both as calculation procedure and as two single entities that can be compared. Nevertheless she seems not to have related the two kinds of computation: "computing with numbers and "computing with letters"

This represents a rupture between the two meanings of computing ("calcolo"): the two procedures (for number on the one hand and for letters on the other) follow different rules which in Francesca's view have nothing in common. As a consequence, it is possible to accept the idea of equivalence in terms of values of expressions, but not to accept or believe that calculating with numbers "is the same as" calculating with letters.



3. Properties of the operations as instruments for symbolic manipulation

The case of Francesca is particularly interesting because it clearly shows the complexity of the relationship between the two meanings of 'computing' ("calcolo"). It shows that it is possible to access some key aspects of this relationship, such as a structural and operational conception of operation properties and algebraic expressions, equivalence between computing procedures, but still lack (CONTROLLARE) a comprehensive meaning of computing including both the case of numbers and that of letters.

As a matter of fact, grasping the link between computing with numbers and with letters, requires a radical change of perspective, of which the operation properties are the core.

According to Sfard's hypothesis, when computing with algebraic expressions a new operational level must be achieved, but this must be achieved without breaking the link with the previous one. The analysis of Francesca's case shows that not only must the reification of an expression be accomplished (expressions can be acted upon as new objects), not only must the structural level be consolidated (equivalence between expressions must be stated in terms of their values), but also a relation between the two 'computing procedures' ("calcoli") must be constructed explicitly.

The key-point is that properties of the operations have to become *rules of transformation*, i.e. "instruments" of computation, and in order to do so, they must assume a dual meaning (structural and operational): properties state the basic equivalence relations and function as instruments for symbolic manipulation, i.e. instruments by means of which any symbolic transformation is derived.

Within the numerical context, operation properties do not play an operative role; they simply express the equivalence of computing procedures, but they are not necessary, and thus not usually employed for computation.

Within the algebraic context, operation properties must assume an operative role and must become the instruments for transforming expressions.

Such a change of role is not made explicit in school practice and focusing of attention on memorisation of particular shortcut procedures such as algebraic formulas ("prodotti notevoli") may definitely hide it.. In conclusion, it seems reasonable to take the hypothesis that this change of role becomes a first goal in introducing pupils to symbolic manipulation.

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UNDERSTANDING HIGH SCHOOL MATHEMATICS TEACHER GROWTH

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This paper reports on a study of the professional growth of four inservice high school mathematics teachers who made fundamental changes to their teaching over their teaching career. The analysis focused on the nature of the teachers' belief structure for mathematics and the relationship between it and changes in their teaching. The findings indicated that belief structure played a key role in when and how changes occurred in the participants' teaching in terms of creating pedagogical tensions and as generative metaphor, which were important to facilitate the generation of new perceptions, explanations and behaviours in their teaching. The findings highlight the possible significance of consciously attending to these factors to assist mathematics teachers in achieving desired changes and choices in their teaching.

Background

Change seems to be a significant challenge for mathematics teachers. Even teachers who are interested in change do not necessarily succeed at making substantive or fundamental shifts in their teaching. In the last decade, the mathematics education literature has reflected a growing emphasis on beliefs as playing a key role on if, when or how change occurs because of their apparent relationship to behaviour. Ernest (1989), for e.g., argued that beliefs are a primary regulator for mathematics teachers' behaviours in the classroom. Although it is not clear that beliefs by themselves can account for mathematics teachers' classroom behaviours, studies do suggest that there is a strong and important influence (e.g., Cooney et al 1998; Chapman, 1997; Lloyd and Wilson, 1998; Pehkonen, 1994; Raymond, 1997; Thompson, 1992). The relationship to change, for the most part, has been deduced from such studies and others which have demonstrated or implied that shifts in beliefs accompanied shifts in teaching for inservice teachers participating in innovative approaches to professional development (e.g., Chapman, 1999; Cobbs et al, 1990; Simon & Schifter, 1991). The literature, however, seems to lack studies with an explicit focus on the relationship between belief structure and inservice mathematics teachers' growth. This relationship deserves attention given the ongoing importance of understanding the mathematics teacher and helping them to make fundamental changes in their teaching to reflect current reform recommendations in mathematics education. This paper reports on a study that investigated this relationship for inservice high school mathematics teachers who have changed their practice on their own from a teacher-centered to a student-centered perspective. Specifically, the focus

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is on the nature of the belief structure (how the beliefs are held) for mathematics and the relationship between it and changes in their teaching over their teaching career.

Belief Structure

Green's (1971) metaphorical analysis of belief structures provides one way of interpreting the ways beliefs are held. Green described 3 dimensions of belief systems: (1) *primary or derivative* (i.e., the quasi-logical relation between beliefs), (2) *central or peripheral* (i.e., the relations between beliefs having to do with their spatial order or their psychological strength), and (3) *isolated clusters* (i.e., beliefs are held in clusters, more or less in isolation for other clusters and protected from any relationship with other sets of beliefs). Beliefs are also held evidentially or non-evidentially. Green (1971) explained that the importance of a belief to the believer is determined by whether it is psychologically central. Thus, the question of which beliefs are amenable to change may have to do not with their being primary or derivative, but with the strength with which they are held. For e.g., a teacher who holds a psychologically central belief that mathematics is a collection of facts may be less likely to change it regardless of whether it is primary or derivative. Change can also be restricted when, as Green (1971) explained, isolation of belief clusters occur to facilitate contradictory beliefs developed in contexts in which beliefs are not explicitly compared or when beliefs are held from a non-evidential perspective, a perspective immune from rational criticism.

Research Method

The participants of the study were 4 experienced (16 to 33 years) high school mathematics teachers (pseudonyms Linda, Elise, Mark, and Rose) who were known in the school system as excellent teachers. They were very articulate and open about their thinking and experiences in teaching mathematics. Data collection and analysis followed a humanistic approach (Chapman, 1999; Creswell, 1998). Data collection involved open-ended interviews, role-play, and classroom observations. The interviews focused on paradigmatic and narrative accounts (Bruner, 1986) of the teachers' past, present, and possible future teaching behaviours and their thinking in relation to mathematics pedagogy. Paradigmatic accounts, triggered in a variety of ways, highlighted the teachers' theories about a situation (e.g., problem solving). Narrative accounts highlighted the teachers' lived experiences and consisted of stories that described the experiences as they occurred, e.g., stories of lessons they taught that (i) were memorable, (ii) they liked, and (iii) they did not like. Classroom observations focused on recording what the teachers said and did and to identify scenarios for the teacher to talk about or role-play. All interviews and classroom talk were audio taped and transcribed.

The data were scrutinized for situations indicating change in teaching approach and for beliefs related to teaching mathematics. Beliefs were identified in terms of significant statements and actions that reflected, for e.g., personal judgements, intentions, expectations, and values of the participants in the context in which they were described. Belief about mathematics emerged as the most dominant belief in the teachers' story of change and was made the focus of the study. The nature of the belief structure and the relationship to the teachers' growth in teaching mathematics were determined by examining the beliefs in the contexts in which they occurred and the relationships among contexts. An abbreviated account of the findings follows.

Belief Structure for Mathematics

The dominant beliefs the participants held about mathematics were mathematics is play/game (Elise), mathematics is shared experience (Mark), and mathematics is language (Linda and Rose). The beliefs emerged from and were supported by the participants' personal experiences. For Elise and Rose, the beliefs emerged from their experience as students of mathematics. Elise explained,

> As a student in the classroom, I just thought it [math] was a blast… something that you absolutely love to do for no other reason. … I mean, I just loved the thrill of the chase. I loved proof. You know, it's the, it's a game, like it's just play, and it has a set of rules but it doesn't really have a set of rules.

Rose's experience as a student involved a lot of small-group discussions about mathematics that framed her view of mathematics as language, i.e., "something you speak, do and use to number the world."

Mark and Linda's beliefs emerged from their experiences as teachers. Mark's experience using manipulatives and small-group work when he started teaching elementary school mathematics along with his high school teaching framed his view of mathematics. He pointed out,

> Math to me is an experience. That's the way, that's how I started to see it with the elementary [school] kids.

Linda's experience working through mathematics problems in her own way in planning a mathematics course she was teaching for the first time framed her view of mathematics as language – "a tool used to understand our world."

There were two significant ways in which these beliefs were held – in terms of Green's (1971) belief structure and as metaphors. The way the participants held their beliefs about mathematics seemed to be primary, evidential and central (Green, 1971). The beliefs were primary because they were not explicitly based on other beliefs and evidential in that the teachers' lived experiences provided the primary evidence for them. Thus, they were amenable to change if these experiences changed. The beliefs seemed to be central, i.e., psychologically strong, because of the passion and conviction the participants displayed for them and their resistance to change. For example, each participant was very critical of any thinking or actions of teachers that was not consistent with her/his belief. He/she was very judgemental of her/his own teaching when in conflict with the belief and not vice versa. The belief also remained stable in that while there were extensions in its interpretation, it did not change conceptually/philosophically since it was constructed. The psychological strength of the beliefs was further validated and reinforced by current reform recommendations about mathematics adopted in the revised mathematics curriculum of the teachers' province. The humanistic perspective of mathematics embodied in the beliefs resonated positively with this orientation of the reform recommendations.

In addition to the preceding belief structure, another way in which the participants' beliefs about mathematics were held was as metaphors. Mathematics is play, experience and language can be viewed as descriptive metaphors from the teachers' perspective in that mathematics was being described and understood in terms of characteristics directly appropriate for some other domain. In the context of their stories, however, there were times when the beliefs also seemed to become generative metaphors (Schön, 1979) and helped to facilitate change as discussed later.

Relationship Between Belief Structure and Change in Teaching

The preceding structure of the participants' beliefs of mathematics played an important role in when and how changes occurred in their teaching. These roles are discussed in terms of *pedagogical tensions* and *generative metaphor*.

Pedagogical Tensions

Substantive changes in the participants' teaching were preceded by pedagogical tensions and a desire to resolve them. Both of these conditions seemed to be influenced by the psychological strength of their beliefs about mathematics. Fundamental shifts in the teachers' teaching were generally directed, often unconsciously, to eliminate or reduce tensions between their teaching and their beliefs about mathematics that created a state of disequilibrium for them. This tension was triggered by situations in their classroom experiences that made their teaching not feel right or students' learning seem to not meet their expectations. The tensions were resolved by extending the interpretation of doing mathematics without changing the primary belief of mathematics and modifying the teaching approach. Thus, belief about mathematics seemed to be held with more dominance, and had a stronger influence, than belief about teaching, in that, regardless of how the latter was held, e.g., central or peripheral, the change was to make actions reflect belief about mathematics. Elise's case will be used to illustrate these tensions and corresponding changes in teaching.

As a beginning teacher, Elise's expectation was that she would be able to teach to reflect mathematics as play even though she did not have a clear conception of what teaching looked like to realize this. This expectation was quickly smothered when Elise started her practice. She was told by her experienced colleagues that their approach (which Elise described as "stand and deliver") was the only realistic way to teach high school mathematics. There was nothing play-like about high school mathematics and she should abandon any thoughts of wanting to make it that. Not sure of what else to do, Elise reluctantly adopted her colleagues teaching approach and started to experience her first significant pedagogical tension. In order to deal with this conflict, instead of changing her belief about mathematics, she developed a position to protect it. She explained:

> You know, that that's the time when I separated mathematics from teaching mathematics. That's when … it became internalized to me that there must be a difference between teaching mathematics and doing mathematics. But those aren't the same thing and they can never be the same thing. And that to this day frustrates me because I don't want that to be the way it is.

With this ongoing desire to resolve the tension, Elise continued to think of how to make high school mathematics be play for her students. When she no longer felt under the influence of her experienced colleagues, since what she was doing lacked fun, she decided that if she added some fun activities, students might start to experience mathematics as play. These special activities, however, did not resolve the tension between what Elise believed about mathematics and how she was teaching it. They were "fun" in an isolated way and did not give her a feeling of play or a sense of the students engaging in play in terms of the mathematics being taught/learnt or mathematics in general.

After a few years, Elise realized that those "fun" activities did very little to foster her beliefs about mathematics in her classroom. They were too detached from the core content being taught and served more of a recreational purpose in the transition from one unit to the next. Elise also felt that, for the most part, the students were simply mimicking her instead of engaging in play or being problem solvers. As she focused on how to help them to become problem solvers, she eventually made a connection between game and problem solving, in particular, viewing problems as games and emphasizing the importance of strategies. She explained,

> I thought, if I'm going to be a good problem solver, I have …to think about what strategies to try. And I really firmly believe as a learner, what I need to do is look at them [problems] as a game. When I play monopoly, I know the rules but it's a dynamic, it changes. When I solve a problem, I have my strategies that colours the rules, but it's a dynamic situation, and so sometimes I use this strategy, sometimes I use that strategy, but I'm more relaxed because it's a game

Elise's interpretation of strategy included a way of thinking, seeing patterns, making connections, and reasoning and was seen as relevant to all areas of mathematics. For her, strategies were not just techniques to solve problems, but a way of viewing and learning mathematics. They were also "something you must see for yourself". This perspective of strategies provided a way for Elise to think of high school mathematics and her teaching of it differently. For her, in addition to fun, high school mathematics and doing mathematics became being and focusing on strategies, respectively, and consistent with her belief of mathematics as play/game. With this, Elise's teaching shifted from being "stand and deliver" to being more student-centered, but teacher guided. She tried to guide students to seeing strategies, e.g., looking for patterns in developing a procedure for themselves. She started to use more questioning and less telling. She followed the textbook less and selected or developed activities in which students could discover strategies through discussion. Students worked in small-groups to figure out strategies and shared them in whole-class discussions. However, Elise often intervened in the groups with questions to guide them to a strategy or led interactive, whole-class discussion to do so. This strategy-based approach created for Elise an acceptable level of harmony between her belief about mathematics as play and her teaching. The primary beliefs about mathematics remained unchanged while teaching behaviour was modified to an acceptable level of harmony.

The next significant pedagogical tension arose when Elise's view, that her approach of focusing on mathematics in terms of strategy was helping students to think for themselves, was challenged by her students' performance on the grade 12 provincial diploma examination (required for graduation in mathematics). In the mid-90's, this exam was revised to include genuine problem-solving items and Elise was surprised when even her best students did not perform at the level she expected. In trying to resolve this new tension, Elise eventually concluded that guiding students to see strategies was not enough for them to be good problem solvers. They must be able to think for themselves to be successful. Elise decided to facilitate this by getting students to think about their thinking through writing and self-questioning. She noted,

> I've focused much more the past 5 years on reflective thought for each person, each individual, and trying to not only encourage but in many ways force kids to do it. … Reflecting on what it is that you know and what does it mean to understand the remainder theorem [for e.g.] is very important. …

The reflective process Elise added to her teaching was, for her, both a strategy and a way of making sense of strategies. It also increased the level of harmony she was developing between her belief about mathematics as play/game and her teaching.

At the time of the study, Elise's teaching had evolved from a "straight stand and deliver" at the beginning of her practice to a combination of teacher-guided and teacher-facilitated situations. Her pedagogical tensions were resolved by modifying her teaching approach within the primary belief of mathematics as play/game. This belief did not change, but her understanding of it broadened (i.e., as fun, strategies, and reflection) in response to resolving the tensions and resulted in significant shifts in her teaching.

Elise's pedagogical tensions were unique to her but the underlying process of dealing with them was representative of that of the other participants. Mark, Rose and Linda had their own tensions that corresponded to significant turning points in their teaching. For example, in Mark's case, his teaching shifted from lecturing, which was in conflict with his belief of mathematics as experience, to teacher-guided/facilitated situations to correspond to his broadened interpretation of experience as communication, connection, and problem solving, each of which resulted from trying to resolve a pedagogical tension.

Generative Metaphor

While the preceding section highlighted the relationship between pedagogical tensions associated with the psychological strength of the participants' beliefs about mathematics and change, this section highlights the relationship between the metaphorical structure of these beliefs and change. One way of understanding the process of change in the participants' thinking and teaching is in terms of viewing the way the belief about mathematics was held, i.e., as play, experience and language, as a generative metaphor. Generative metaphor (Schön, 1979) or structural metaphor (Lakoff and Johnson, 1980) facilitates learning through a process that involves generating or structuring one concept in terms of another. As Schön (1979) noted,

> Metaphor refers to a certain kind of product … and to a certain kind of process, a process by which new perspectives on the world come into existence [p 254].

The latter situation refers to the generative quality of metaphor in terms of helping an individual to generate new perceptions, explanations and inventions to understand and deal with his/her world. This generative quality seemed to underlie the way in which the participants broadened their perspective of mathematics within the primary beliefs of play, experience and language and made changes to their teaching. The generative process was triggered by the pedagogical tensions as a way of resolving them. From the perspective of metaphor, these tensions occurred when the two domains of the metaphor seemed incompatible in relation to teaching high school mathematics, e.g., high school mathematics (first domain) as play (second domain) for Elise, or experience for Mark, was difficult for them to relate to their teaching. The generative process allowed for the elaboration of the interpretation of the second domain and corresponding changes in the first domain and their teaching. For continuity, Elise's case will be used as representative of the other participants' situation to illustrate the following five stages that seemed to constitute this process.

(i) *Hold belief as a metaphor:* This stage involved holding and articulating the belief about mathematics as a metaphor. Elise was able to do this at the beginning of her practice. Concrete, personal experiences were important to the creation and articulation of the metaphor.

(ii) *Experience pedagogical tension:* The generative process was triggered by these tensions described earlier.

(iii) *Elaborate assumptions/characteristics:* This stage involved becoming aware of and articulating assumptions/characteristics flowing out of the phenomenon providing the context of the metaphor (i.e., play for Elise) based on personal experience of the phenomenon. The first characteristic elaborated by Elise was fun.

(iv) *Connect two domains of metaphor:* The two domains of the metaphor were connected through the characteristics identified in stage (iii). For Elise, fun was mapped to high school mathematics.

(v) *Revise perception:* The mapping led to a revision of perception of both domains of the metaphors. Elise extended her understandings of high school mathematics, interpreting it as fun activities. Mathematics as play could now be seen in the context of high school mathematics and consequently the teaching of it. Change in Elise's teaching was then accomplished to reflect these ways of viewing mathematics.

Stages (ii) to (v) were repeated when the conflict in stage (ii) was not completely resolved or a new conflict arose. In repeating these stages, Elise generated strategies and reflection as further articulation of characteristics of play. These were accompanied by corresponding changes to her teaching. Thus, in the context of the generative process, only after further articulation of the nature of play was Elise able to expand her interpretation of high school mathematics and make substantive changes to her teaching.

Conclusion

The study suggests that the belief structure about mathematics and experiential/ concrete contexts for generating and interpreting pedagogical tensions related to the belief about mathematics are important factors that determine when and how changes in teaching occur. The belief structure in the form of metaphor could help to facilitate teachers' generation of new perceptions, explanations, and inventions in their teaching of mathematics. Thus, the study brings to light the possible importance of generative metaphors that may underlie mathematics teachers' personal story of growth and the possible significance of consciously attending to such metaphors to assist teachers in achieving desired changes and choices in their teaching.

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Developing Formal Mathematical Concepts over Time

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This paper investigates the development of university students' understanding on 'equivalence relations & partitions' over a period of time. Although these ideas are taught in the same topic, they have quite different cognitive properties. We find that, although the concept of 'relation' can be visualised, an 'equivalence relation' is more subtle. A partition, however, is more easily visualised than remembered formally. Our focus is on if and how these different properties influence students' concept development.

Introduction

Chin & Tall (2000) focused on a theory in which informal mathematics becomes formalised by introducing definitions, proving theorems and compressing formal concepts into cognitive units appropriate for powerful formal thinking. The theory was tested by a questionnaire filled in by 36 students after 6 weeks studying the formal theory of equivalence relations and partitions. It was found that:

> Less than half gave *formal* responses in terms of *definitions* or *theorems*. […] This confirms a picture in which the majority of students following a formal course at a highly rated university responded at an informal level after several weeks' experience of formalism. At the same time, two able students worked in a different way using the *compressed concept* that encompassed both equivalence relation and partition. Chin & Tall, 2000, p. 183

In this paper we follow the development over a longer time period to gain further insight into the students' constructions. We focus on fifteen students, of whom ten were tutored by the first author for an hour per week during the first two terms and on into the second year. Data was collected through audio-taping tutorials and in-depth interviews, with a second application of the questionnaire to determine long-term changes in conceptions.

An evaluation of the Foundations course by the students at its conclusion in the first year revealed that the students considered 'relations' to be the most difficult topic—a comment that had been repeated for several previous years of assessment. Summarising the perceptions of the students, the annual report commented that '*Euclid's algorithm* and *symbolic logic* were well understood, *basic set theory* and *functions* generally required extra work, but the topic on *relations* was often poorly understood.' On an average, only about 20% of students declared that they understood relations well with nearly a third of students claiming that, even after extra study, they only understood the topic poorly. It was this observation that drew us to study the topic of 'equivalence relations and partitions'. We considered that an understanding of students' difficulties in this topic that they found most problematic might shed light into wider difficulties in the understanding of formal mathematics. In particular, *why* do the students claim to have such difficulty with 'relations'? We now focus on the longer-term development from the first to second year of the course.

Analysis and comparison

The subjects are 15 second year mathematics students following a course in the highest ranked pure mathematics department in the whole of the UK. Their marks for the first year are widely distributed—three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, three between 40 to 49. They answered the same questionnaire on the topic of 'equivalence relations & partitions' that they have already learnt for about a whole year and were interviewed during the first term in their second year.

The formal definition of equivalence relation

The formal definition of equivalence relation in terms of being 'symmetric, reflexive, transitive' proves to be relatively easy for students to learn and reproduce, though the precise use of quantifiers in each part of the definition is a little more subtle. Table 1 shows that 14 out of the 15 students reproduced a definition although only 5 of these gave the full quantified definition, 4 gave the formal definition without quantifiers and 5 gave an informal response in terms of the three words 'reflexive, symmetric, transitive'.

	First Year (N=15)	Second Year (N=15)
Formal/detailed	5	9
Formal/partial	4	5
Informal/outline	5	1
Total definition	**14**	**15**
Example	0	0
Picture	0	0
Other	1	0
No response	0	0

Table 1: Responses to 'equivalence relations'

Only one student—whom we shall call 'Arthur'— did not give the formal definition in the first year. He explained later that he could not remember the definition at the time; instead he attempted to explain the notion of equivalence classes in terms of a partition:

> An equivalence relation generates a subset of elements
> that are all related to each other, and divides
> the set into partitions.

Note the imprecision of the language here, for example 'generates *a subset*' and 'divides the set into *partitions*'. Arthur obtained 50% in the end of year examination and had to resit one of his courses. Nevertheless, even he was able to give a formal detailed response in the second year. This gives us our first major piece of evidence: *all* of the students could reproduce the definition of equivalence relation; 9 out of 15 gave a complete version, the other 6 at least remembered 'reflexive, symmetric, and transitive'.

This was reflected in responses to an informal question asking if the relation 'has the same surname as' is an equivalence relation on the set of students in the class (Table 2).

		First Year (N=15)	Second Year (N=15)
Informal	Informal Definition	2	1
	Other	0	1
	No response	0	0
Formal perhaps with some informal language	Definition	12	13
	Theorem	0	0
	Partition	1	0

Table 2: Responses to the informal 'surnames' question

Both the students giving informal responses in the second year were able to give a full formal response in interview. Both thought the question too trivial to merit a detailed response in writing. For instance, John (whose response was classified as 'other') wrote:

> Since the relation is to do with "equality" of the surname, it must be an equivalence relation.

He was a talented student with a mark of 68% in the first year examination who showed his understanding to be definition-based on all his assignments and in tutorials. Thus all fifteen students were capable of a formal definition response by the second year.

Table 3 shows the responses to the following question:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$. Is this relation an equivalence relation?

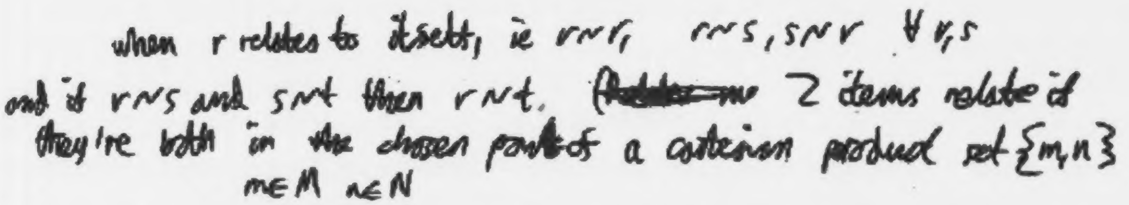
		First Year (N=15)	Second Year (N=15)
Informal	Informal Definition	3	0
	Other	1	1
	No response	0	0
Formal perhaps with some informal language	Definition	7	2
	Theorem	3	12
	Partition	1	0

Table 3: Responses to the formal 'bijection' question

This data shows that after being given a period of time to digest what they had been taught, whilst only 3 were theorem-based in the first year test, 12 are able to upgrade their understanding to the theorem-based level in the second. This is consistent with the successive move from definition-based conceptions to theorem-based conceptions over a time in which the ideas are being used formally.

The definition of equivalence relation on a set S as a subset of $S \times S$

When responding to the notion of equivalence relation, none of the selected fifteen students used the general notion of relation as a set of ordered pairs in their definition. Only one student (Nathan) in the first year alluded to the idea as follows:



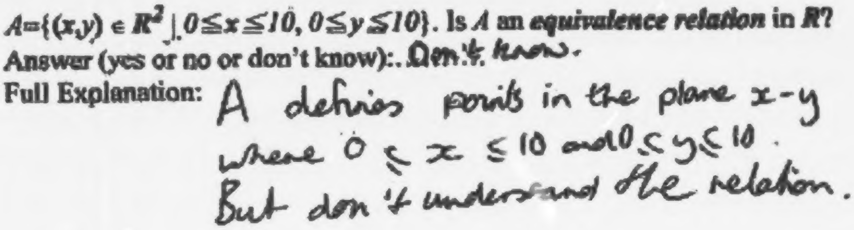
when r relates to itself, ie $r \sim r$, $r \sim s, s \sim r$ $\forall r,s$
and if $r \sim s$ and $s \sim t$ then $r \sim t$. 2 items relate if they're both in the chosen part of a cartesian product set $\{m,n\}$
$m \in M$ $n \in N$

Even here the notion is an afterthought following the definition in terms of the notation $a\rho b$, for the relation ρ rather than the notation $(a,b)\in\rho$ which was given initially in the lectures. Notice that even here Nathan used the notation $\{a,b\}$ (used in the course for *un*ordered pairs) rather than the correct notation (a,b).

In the second year, only one student (Simon, the most successful with a mark of 85%) referred to a relation as 'a subset of $A\times A$' in his response to the meaning of 'equivalence relation'. He also was the only student to give a satisfactory answer to the following:

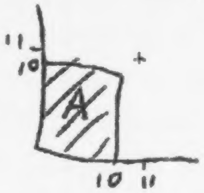
$A=\left\{(x,y)\in \boldsymbol{R}^2 \mid 0\le x\le 10,\, 0\le y\le 10\right\}$. Is A an ***equivalence relation*** on $\boldsymbol{R}$?

In the first year *no* student responded positively to this question. Several wrote explicitly that they did not understand what the question meant:



$A=\{(x,y)\in \boldsymbol{R}^2 \mid 0\leq x\leq 10,\, 0\leq y\leq 10\}$. Is A an *equivalence relation* in $\boldsymbol{R}$?
Answer (yes or no or don't know): Don't know.
Full Explanation: A defines points in the plane x-y
where $0 \leq x \leq 10$ and $0 \leq y \leq 10$.
But don't understand the relation.

In the second year, Simon responded as follows:



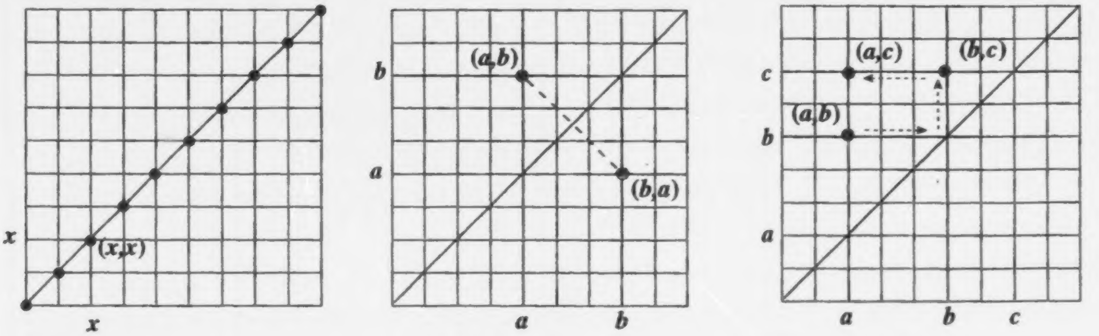
Consider $11 \in R$, as $(11,11) \notin A$,
not reflexi

He described an equivalence relation as 'a subset of $A\times A$' with reflexive, symmetric and transitive properties that can divide a set into a partition. He also offered the formal definition with all the detail. He therefore had a conception of equivalence relation and partition as a rich cognitive unit.

We therefore obtain our second major piece of evidence: *all but one* of the students did not relate the notion of relation as a set of ordered pairs with the notion of equivalence relation.

The gap between relations and equivalence relations

We see that *all* fifteen students could work with the notion of equivalence relation using the notation $a \sim b$, but only *one* evoked the notion of relation on a set S as a subset R of $S\times S$. On reflection, one can see that the notion of 'equivalence relation' on a set S does not have an easy visual image. Seen as a subset R of $S \times S$, the reflexive law can be pictured by saying that the diagonal elements (x,x) are all in R, the symmetric law can be seen in terms of reflection of the element $(a,b) \in R$ in the diagonal to also give $(b,a) \in R$, but the transitive law $(a,b), (b,c) \in R$ *implies* $(a,c) \in R$ is a little more sophisticated. (The transitive law moves horizontally from (a,b)—maintaining the second coordinate b—to the diagonal then vertically to the point (b,c), completing the rectangle to give the third point (a,c).)(Figure 1).



reflexive:
$\forall x \in S, (x,x) \in R$

symmetric:
$(a,b) \in R \Rightarrow (b,a) \in R$

transitive:
$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

Figure 1: Visual representations of the three axioms for an equivalence relation R on a set S.

The complexity of the visual representation is such that it was not taught in the course. Thus, although the notion of relation on a set S is given in terms of a subset of $S \times S$, *it is never represented as a visual picture.* In this way there is a complete dichotomy between the notion of relation (interpreted as a subset of $S \times S$) represented by pictures and the notion of *equivalence relation* which is not.

Furthermore, the topic of 'relations' also includes *order relations*. We hypothesise that the typical student will find it difficult to give a coherent overall meaning to the notion of 'relation' that encompasses both order relations and equivalence relations. Partial support for this hypothesis is the students almost total failure to respond to the equivalence relation defined as a set of ordered pairs compared with almost total success with questions using the form $a \sim b$.

After interviewing 10 of the 15 students, the authors find that these students learnt the definition of relation on a set formally as: 'a subset of the cartesian product of the set itself'. But they learnt the definition of 'equivalence relations' focused on the three properties of reflexive, symmetric and transitive. The following conversation recorded in an interview with two students (whom we name Jack and Nathan, respectively) offers some evidence. Jack and Nathan were being asked about the question in which the relation A is defined as the subset $A = \{(x,y) \in \mathbf{R}^2 \mid 0 \le x \le 10, 0 \le y \le 10\}$ of $\mathbf{R}^2$.

Jack: Sorry! I can't understand what this question means?
Interviewer: O. K. Nathan, can you understand it?
Nathan: Well, umm…(*Pondering for a while.*) No, I don't think so.
I: Can you think of the formal definition of 'relations' first?

They started trying to recall their memory of 'relations'.

N: I think it's a sort of ordered pairs, isn't it?

I: Yes. You are right. Can you say it more formally?

J: Let me think. It's ages ago, I don't think I can remember it.

I: How about you, Nathan?

N: *(Shaking his head.)*

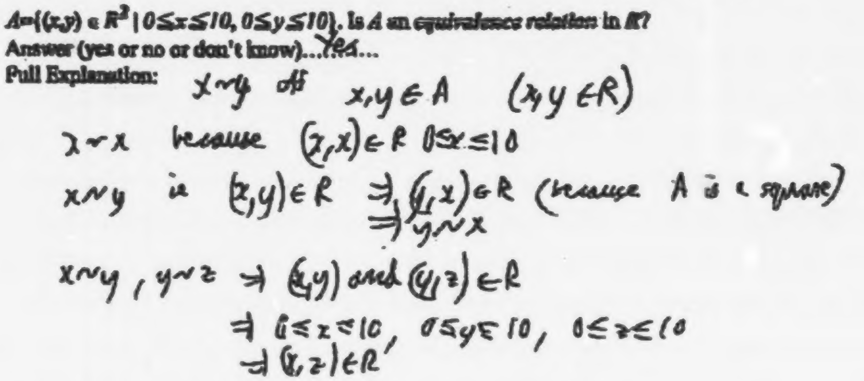
I: O.K. Let me write it down on the board.

The interviewer wrote the definition (Stewart & Tall, p.69) on the board and explained it to them.

J: Yes. I see. That should be what we learnt in the lecture a long time ago.

I: O.K. Now, can you try to answer this question again?

Nathan immediately made the whole deduction, answering 'yes' after checking the three conditions although he did not include the quantifier in 'reflexivity'. Jack still seemed confused.



$A=\{(x,y) \in R^2 \mid 0 \le x \le 10, 0 \le y \le 10\}$. Is A an equivalence relation in R?

Answer (yes or no or don't know)... Yes**...**

Full Explanation: $x \sim y$ iff $x, y \in A$ $(x, y \in R)$

$x \sim x$ because $(x,x) \in R$ $0 \le x \le 10$

$x \sim y$ ie $(x,y) \in R \Rightarrow (y,x) \in R$ (because A is a square)

$\Rightarrow y \sim x$

$x \sim y$, $y \sim z \Rightarrow (x,y)$ and $(y,z) \in R$

$\Rightarrow 0 \le x \le 10$, $0 \le y \le 10$, $0 \le z \le 10$

$\Rightarrow (x,z) \in R$

J: I still can't see how to check *A* is an equivalence relation in *R*.

I: You can understand the definition of 'relations' we just reviewed, can't you?

J: Yes. I think so.

I: 'Equivalence relation' is just a kind of 'relation' but with some more properties, isn't it?

J: Yes.

I: Just add the three properties to the definition of 'relation', then try to answer this question again.

Jack was stuck checking 'reflexivity'.

J: I'm getting confused. What's the point of checking 'reflexive'?

I: Nathan has finished his deduction. Let's have a look at his answer then I'll answer you, Jack. Do you think Nathan's answer is correct?

J: mmm…*(Pondering for a while.)* Yes, I think so.

I: O. K. Let's have a careful look at 'reflexivity'. What is the quantifier for it?

N: For all the elements in *A*?

I: What do you think, Jack?

J: Should be 'for all the elements in *R*'.

I: Do you agree with Jack, Nathan?

N: mmm…Yes. Yes. I think he's right.

I: So, do you think *A* is 'reflexive' now?

J: I see. No. Because *A* doesn't cover the whole plane, so it won't be 'reflexive'.

N: mmm… So the answer should be 'no'.

I: I think you both get the point. Now can you check if *A* is 'symmetric' or 'transitive'?

N: Yes, both of them. I think I used the wrong notations. They should be *A*, not *R*.

I: Well done! Jack, could you make a conclusion of this question?

J: You mean *A* is not an equivalence relation because it is symmetric and transitive but not reflexive.

This data seems to suggest that even students who do not have a formal concept image of a former idea (like the idea of 'relations') can still build up their understanding of the next relevant idea without having much particular difficulty (like the idea of 'equivalence relations') if the two definitions are not directly related.

Partitions

The development of the notion of partition also improved over the year (table 4).

	First Year (N=15)	Second Year (N=15)
Formal/detailed	2	8
Informal/outline	6	3
Total definition	8	11
Example	0	0
Picture	1	1
Other	4	3
No response	2	0

Table 4: Responses to 'partitions'

The number of detailed formal definitions increases from 2 to 8 and the overall definitions increase from 8 to 11. However, 4 students fail to give a definition for partition when all can give a definition for 'equivalence relation'. Looking closely at the responses reveals that the majority of students tried to use *their own language* to interpret the definition of 'partitions' so that their answers were highly varied.

Interestingly, all ten students interviewed said they had a mental picture of a partition. Nine of them thought they understood 'partitions' better than 'equivalence relations'. The exception, Jack, explained that although he could picture partitions, he still did not know the formal definition and was happier handling the formal definition of equivalence relation. Of the other nine, Arthur was typical in saying that he felt he understood 'partitions' better than 'equivalence relations' because he could visualise 'partitions' but not 'equivalence relations'.

When asked to give examples of partitions, twelve out of fifteen gave satisfactory answers. The other three revealed an interesting misconception. Jack wrote:

Write down two different *partitions* of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the *equivalence relation* that it determines.

$P_1 = \{a\}, \ P_2 = \{b,c,d\}$

$a \sim a, \ a \nsim b, a \nsim c, a \nsim d.$

$b \sim c \sim d \ .$

At first sight this may seem as if Jack has written down one correct partition. However, in interview, he explained that he thought that his two partitions were P_1 and P_2. All three of the students giving unsatisfactory responses shared the same misconception: that the term 'partition' referred to each individual subset, not to the collection of all subsets.

In this way we see that the class as a whole retain their understanding of 'equivalence relation' at the definition level and apparently shift their perception of partition to the theorem-level, whilst some are still having difficulty with the definition of partition.

Conclusion

In this paper we have been considering the development of 'equivalence relations' and 'partitions' a year after the students first met the concepts in the Foundations course. During this time they would have met the ideas in other courses and revised for the end of year examinations. We questioned why the students claimed that the notion of 'relation' was the most difficult in the whole of the Foundations course. We found that, after a year, although all 15 students could give the definition of equivalence relation using the notation $a{\sim}b$, only *one* could respond to a question where an equivalence relation was given in terms of a subset of the cartesian product. We showed that, although the notion of relation is easily visualised, the notion of equivalence relation is difficult to visualise but easy to remember as a verbal definition. We also hypothesised that the introduction of the very different notion of order relations at the same time gives little common ground amongst the examples of relation to allow a coherent link to be made between the examples and the general concept.

We note that nine out of ten students interviewed claimed that they felt they understood partition better than equivalence relation, whereas in fact their performance on the test showed that they were able to handle equivalence relation better than partition. This is accompanied by the observation that they say they can visualise a partition, but not an equivalence relation. We consider this to be consistent with the notion of 'embodied mathematics' (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000) giving a deeper human sense of meaning. Thus the development of the formal thinking characteristic of the 'rigour prefix' (Alcock & Simpson, 1999) is here underpinned by the embodied concept-image and formal concept-use in the sense of Moore (1994).

Over the year there is a general shift from 'definition-based' deduction referring specifically to the formal definition to 'theorem-based' deduction, using already proven theorems. One student clearly had the composite notion of equivalence relation and partition as a rich cognitive unit. The investigation of whether others have such a cognitive structure is more likely to arise in interview rather than standard written questions. This remains a topic of our current research.

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VALUE-LOADED ACTIVITIES IN MATHEMATICS CLASSROOM

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ABSTRACT

This paper describes a form of value-loaded activities emerged in teaching and learning of mathematical induction in which the value of pleasure is shared by an expert teacher and his students. Using case study method, values are explored with classroom observation, teacher and student interview, and questionnaire survey. Conceiving identity theory from the pedagogical aspect, values are identified in terms of the significant value statements attached to and reflected by teacher and student's pedagogical identities. In consequence, implications about the teaching of pedagogical values and the need for more research on pedagogical values are suggested.

INTRODUCTION

S_i (whole class replies): Oh! No, you are cheating ---.
S_3 (single student replies): My goodness! What if I do ---.

Why the students above would react like that? We will show in this paper that these reactions have to do with the value-loaded activity that the students engage. Goals of the school mathematics curriculum in many countries indicate explicitly desirable values, for example 'pleasure' and 'appreciation' in Australian curriculum statements (AEC, 1991). There were considerable discussions of the link between affective and cognitive elements, in particular the role of values in mathematics teaching, in the PME conferences. For example, at the 1993 conference in Japan the overall theme was "How to link affective and cognitive aspects in mathematics education", and there was also substantial exchange of implicit values in various sessions at the 1998 conference in South Africa. There is a strong concern both to question and to challenge the values currently being taught in mathematics classroom. And yet, the school curriculum following these goal-initiated values, for example the values of *happy* and *knowledge for learning* that the new Taiwanese curriculum statements "happy to learn" and "knowing how to learn" underline (ME, 2000, p.135), say little about their practice, particularly in the teaching of such values. Although there are several levels of values transmission (e.g., Bishop, 1988) but we are still lack of the relevant researches looking particularly at the form of teacher-student values interaction in mathematics classroom. To conceptualize values about mathematics and pedagogy, we have to consider carefully the question of 'What implicit values could classroom teaching convey and through what kind of teaching activities could such values be taught?'

Values have been conceived as personal experiences, objects of thought, or psychological phenomena (Frondizi, 1970), as individuals' feelings (Meinong, 1894) or objects to be desired (Scheler, 1954), as individual principles of selection and judgment (Samuel, 1937), as ideas or concepts concerning the worth of something (Swadener & Soedjadi, 1988). In this case, they are personal *preferences* concerning individuals' *standards* for considering *the importance* or *worthwhile of* something for themselves to *think* and *act*. A domain of research relevant to values is beliefs. Mathematics teachers may hold various pedagogical beliefs, which differ in forms as mathematical or pedagogical, or in levels as enacted or espoused. Nevertheless, beliefs have to do with an individual's propositions about mathematics and pedagogy, and values are more about personal principles or standards of thinking and action across such propositions. Rokeach (1973) in studying the nature of human values suggests that values are prescriptive or proscriptive beliefs wherein some means or end of action is judged to be desirable or undesirable. Allport (1961) further contends that a value is a belief upon which a person acts by preference. These arguments conceive value as a preference, a desirable mode of conduct, or a desirable end-state of existence, concerning the conception of something that is importance and worthwhile of thinking and doing for the person (Rokeach, 1973). Therefore, beliefs are more about "the nature of *propositions* about phenomena" (Bishop, 2001), however, values are likely to be the key *substances* underlying such propositions for people to think and act.

The Value In Mathematics Teaching (VIMT) project funded by the Taiwanese National Science Council (NSC, 1997-2000) aimed to: explore mathematics teachers' values about mathematics and pedagogy; examine and increase the extent to which mathematics teachers' can clarify their own pedagogical values; and investigate students' values about mathematics and mathematics learning, and the values interaction activity in the classroom between teacher and student. Based on a portion of VIMT results (Chin & Lin, 2000a; Chin & Lin, 2000b), this paper aims to describe an activity observed in the teaching and learning of mathematical induction in which the value of *pleasure* was loaded, and through which a teacher and his students interacted overtly. Other values can be found in Chin and Lin (2001, submitted)

THEORETICAL BACKGROUNDS

The Value of Pleasure in Mathematics Teaching

The analog/unconscious mathematics (Davis & Hersh, 1981) is related to the idea of intuition and feeling, in which sense making and free thought are the two important elements. It is our belief that the more efforts that we make for students to learn mathematics through understanding the better they would appreciate the value of pleasure. The statement, "The thing that is important is that doing math is fun. That's what I try to put across to the kids --- What I try to do is to tell math to kids on the basis that it's fun" (ibid, 1981, p.272), may convey "Learning mathematics is interesting and doing it is a pleasant thing" undrelining the value of *pleasure* in pursuit of having fun in the teaching and learning of mathematics. The analytic/conscious aspects conceive mathematics from the other way round as a combination of forms, rules, and proofs. This is to addresses the concept of mathematical induction as a formalized subject for learning through a step-by-step mechanical process of proofs. As a result, these effects may create the value of *un-pleasure*. In discussion of the emotional influences on learning school mathematics, Skemp (1989) proposed a model of conceptualizing affective responses such as *pleasure* and *un-pleasure* in terms of the orientations towards goal state or away from the goal state. Pleasure signal changes towards a goal state, and leaving from the goal state creates un-pleasure. The goal state here depends on the implicit learning orientations about mathematical induction that individual students stand.

Values Statement as Carrier of Values

We conceive a statement of values as the carrier that contexturalizes individual pedagogical values into a concise sentence, representing his/her core principle for thinking and action. A value carrier thus contains a set of values that the teacher endorses. A values statement, as Taylor (2000) pointed out, is a goal-directed description indicating the values by which the school intends its practices to be guided, and setting out the values the school intends to promote and which it intends to demonstrate through all aspects of its life. An example of a secondary teacher's values statement is "We want our school to be caring and Christian, disciplining, encouraging, happy" (ibid, p.157). This statement includes the values of care, discipline, encouragement and happiness that secondary school is expected to address across the curriculum. Values, in the light of this view, are situated in and entertained with the propositional statements of teacher and student conceptions about mathematics and pedagogy (e.g., Haydon, 2000).

The Pedagogical Aspect of Identity and Its Relationship to Value

Values are conceived here as a sort of individual identities concerning mathematics and pedagogy. Fereshteh's (1996) definition of teaching, as an intellectual activity requiring varied abilities in educating students certain knowledge, as guiding and evaluating students' learning processes, and as an artistic and scientific activity, seem to suggest that the underlying pedagogical values for such an activity are *intellectual* and *knowledge acquisition*, *guidance* and *evaluation of learning process*, and *artistically scientific discovery*. In pursuit of such values, a teacher is expected to play as a manager, creating and organizing the lessons, as a motivator, fulfilling the student needs, and as a professional, enjoying and developing their career. To accomplish these roles, a teacher identifies him/herself with the professional identities of a manager whose classroom teaching reveals some role specifics. This aspect of identity seems to shift its sociological nature into a pedagogical realm, reflecting how and in what ways a teacher should think and act '*as if*' he/she is a person accompanying with such identity. In an article of reporting the role of values in pedagogical content knowledge (PCK) from four experienced English and History teachers, Gudmundsdottir (1990) concluded that values, as implicit personal curricula, are integral part of teachers' excellence in teaching influencing several aspects of PCK, such as choice of pedagogical strategies and perception of the students' needs. These elements, reflecting a teacher's excellence in teaching, play an integral part of his/her pedagogical identities. In discussion of a continual development of values from early youth to old age, Erikson (1963) indicated that the value systems properly reflected those features of the development of eight identity stages. In Rokeach's (1973) researches, values are integral to self-identity that people strive to be authentic, moral beings by acting on the basis of values tied to their desired self-conceptions. Therefore, pedagogical identities seem to tie to the values of individuals.

RESEARCH METHODS

The case study method, including questionnaire survey, interviews, and classroom observations, was used as the major approach of inquiry to explore an expert teacher (Ming) and his students' pedagogical values. Ming had a master degree in mathematics and taught mathematics in a public senior high school for 21 consecutive years. We used critical teaching events as probes for post-lesson interviews. Another teacher, Yuh who taught mathematics in the same school, participated in each interview. Four teaching topics were videotaped and transcribed during 1997-1999 including mathematical induction. A senior secondary mathematics teacher acted as an independent checker to examine the reliability of the observational data. A questionnaire was used for all students, designed to uncover students' preferences on selecting two of the six problems relating to the topic taught and to collect students' reasons of doing so. First part of this questionnaire consists of six open-ended problems designed by Ming based on his core pedagogical values, and the second part has several items for students to express their agreements on each statement, concerning the reasons of doing and not doing so. The second questionnaire consisted of 20 questions, using a five-point Likert format. It asked students to express their views on each statement according to two different contexts, test scores taken for grading or not to be taken. The 6 sample students were selected according to their mathematical performance in the first questionnaire and willingness to talk, and the representative of the student class. These questionnaires were piloted and revised with Yuh's students.

RESULTS

A Value-loaded Teaching Activity

According to the school curriculum, the topic should be taught in duration of five 50 minutes lessons. Usually teachers, for example Yuh, would take less than ten minutes to introduce the format of mathematical induction, and the rest for exercises. Ming spent 15 minutes in the activity of Hanoi tower to develop student ideas of 'potential infinity'. Although the activity is well known, however, Ming re-framed it to address student manipulation and teacher-student dialogue. Two critical questions used by him to guide the student thinking were: 'Can you do it?' and 'Do you believe that if N=3 is possible then N=4 will also be possible?' Ming told the rules first followed by student manipulations and teacher-student dialogue. Finally, he showed the solution. A brief snapshot of the activity after three students manipulated was:

> Ming: Let me show how I solve it. First of all, can you do it if the number is 3?
> S_i (whole class replies): Yes, of course we can.
> Ming: If it is 4, could I pack it up as a unit and move the package from A to B?
> S_i: It is okay.
> Ming: Then, I move the fourth one to C, is it okay?
> S_i: Yes.
> Ming: Then, if I move again this package from B to C, can I do it?
> S_i: ***Oh! No, you are cheating.***
> S_3(single student replies): ***My goodness! What if I do it the way that you just showed us by packing up the case of 4 to solve the case of 5, and then to solve the case of 6, and so on?***
> Ming: Are you sure?
> S_3: Why not?
> Ming: Excellent (he smiles expressively), are you convinced that I didn't cheat you?
> S_i: Yes! There should not be any problem.
> (a recursive procedure of potential infinity through teacher-student dialogues)
> S_i: It can also be done by the same way. There will not be any problem.
> Ming: Therefore, we can do it all the way through in the same method?
> S_i: Yes, why not.
> Ming: Are you convinced that for any counting number we can always do it this way?
> S_i: Yes, we can do it by counting up.

The expressions "Oh! No, you are cheating" and "My goodness! What if ---", referring to the freshness and power of interpreting mathematical knowledge, are related to the value of *pleasure*, which will be discussed later. The subsequent dialogues, in which a recursive procedure of step-by-step reasoning format is introduced, develops meanwhile the ideas of potential infinity that the values of *infinite* and *reasoning* underlay (see Chin & Lin, 2001, submitted).

The Underlying Teacher's Pedagogical Values

The activity of Hanoi Tower was used to inculcate the value of *pleasure*. As Ming said that "the idea of 'infinite reasoning' underlying the 'empirically counting truth' conviction, should play the significant role to encourage students to do *mathematical* investigations in which *enjoyment* of knowledge are of paramount importance". In particular, he concerns the affective aspect of learning and teaching school mathematics as "I really hope that all of my students will feel happiness, enjoyment, and pleasure in their own processes of investigating mathematical knowledge in these activities. They are the affective and humanistic concerns that I have been trying very hard to express in my teaching, such as mathematical induction". Referring to the pre-lesson planning of the activity, Ming professed that "I intended to develop activities in which my students would feel that mathematics could be very interesting and they might in this case be eager to attend the subsequent lessons. The Hanoi Tower activity was just designed to initiate such student motivation through learning the concept of mathematical induction". He explained further "Most students are *not happy* in the mathematics lessons --- *they don't feel that the knowledge is useful or practical in their life*. Therefore, most of them feel *panic* and *anxiety* when learning mathematics. This is the reason I have been trying so hard to motivate them to learn mathematics through *enjoyment, pleasure, and anticipation* using investigative games or activities, and focusing on the nature of the knowledge---".

The acceptance of such a value of Ming for the students will be further examined in terms of the data collected from the student phase.

The Attainable Students' Pedagogical Values

One question in the first questionnaire, "Suppose that the concept of human is well defined, and the life in the earth has gone through about 4 billions year. Prove that there is a human being whose mother is not a human (called Genesis)", was used to examine student understanding and the ability of application from an unconscious/analog aspects and meanwhile loading specifically with the value of *pleasure*. The implicit pedagogical values that Ming intends to pass to the students are then conextualised into 6 tasks in assessment, including Genesis, for students to select and solve. It was supposed that in the process of selection that the values of students would become explicitly.

When we asked 'Why did you choose Genesis rather than other questions?' S_1 said "Because I felt that these two questions ask me to elaborate and the context of the problem make sense to me". The text and situation of Genesis was much easier for him to get access. They created motives of *pleasure* such as interesting and fun, and he was eager to solve the problem since "These are questions that I have not seen before. They are *new* for me. But, I am quite familiar with the other questions that I am not so interested. They are so *boring* because I have already known the answer of the questions and the procedure of solving them". This value encouraged curiosity and willingness for the student to solve individually, for "The questions raise my *curiosity* to solve them and I *like* to find out myself. I really like to know whether if I can solve a totally new problem like this on my own". Routine questions could not create any pleasure for him, as he said that "There is *no pleasure* for me at all to face a mathematical question which has no practical use or not realistic. I don't know what could be of interest if you have already known the answer or method of solving the question?" Another student S_2 also claimed for searching an unfamiliar question to solve, as "I would have got *bored* if a mathematical question were solved easily according to certain familiar steps. It is no fun at all for doing or answering a question like that". Therefore, it was *the pleasure of reasoning* that encouraged him to solve Genesis. Because, "I like to solve the question through my own efforts. There are lots of *pleasant* during the processes of solving such questions" and "I am now telling you that I hope all mathematical will like this, encouraging me to *think* and *reason freely* according to the ways whatever I like, fulfilling my *curiosity*, and being *full of pleasure* in the process of solving them". These statements indicate the crucial role that affect played in the mathematics learning concerning the value of *pleasure*.

Therefore, the mathematical ideas of *infinite reasoning/potential infinity* and its accompanied affective element of *pleasure* are interwoven within the process of teaching and learning of the Hanoi tower.

DISCUSSIONS

A foreseeable relationship between pedagogical values, value statements, and identities is become clear. Value statement as carrier of pedagogical values portrays teacher and student principles of evaluation on teaching incidents and learning tasks. A text like this has its syntactical structure of wordings in which certain values are embedded implicitly. The key words included in a value sentence may represent different pedagogical values. Values are in this sense represented and embodied in words or a combination of words, underlying the connected value statements to which they apply. Peoples, who agree or identify oneself with a particular value statement, are conceived as carrying the underlying fundamental elements of the statement, that is, the pedagogical values that the statement portrays. This is to conceive and analyze pedagogical values from human discourse in terms of a syntactical and psychological analysis of talks and words. It is this process of pedagogical identification that reflects a person's preferences to aspects of teaching. These preferences are in connection to the teacher and student's pedagogical identities. In the light of this, values are integral to self-identity and that teacher and student strive to be authentic (Rokeach, 1973). Australian researchers also referred values to the pedagogical aspect of personal identity as in "The values taught, whether explicitly or more like implicitly, seems to depend heavily on *one's personal set of values as a person and as a teacher*" (Bishop, Clarkson, FitzSimon, & Seah, 2000, p.148). Halstead (1996, p.5) used the concept of identity to define values, as he put it "The term values --- which act as general guides to behaviors or as points of reference in decision-making or the evaluation of beliefs or action and which are closely connected to *personal integrity* and *personal identity*".

To inform classroom practices, the practitioners need more supports on learning and constructing value-loaded activities that might be useful, and also the framework for developing and elaborating on pedagogical values for different subject matters. This is an area that Tomlinson and Quinton (1986) called "De facto implemented curriculum" and Bishop (2001) claimed "the Meso classroom level" in which means of planning and enacting values are the foci. Researches on this line are urgent. It is also important for mathematics teacher educators to try to develop and examine plausible ways of inculcating specific pedagogical values at "the Macro curriculum level" (Bishop, 2001) of pedagogical value education for mathematics teachers. Therefore, we need to know more about the processes of valuing and value clarification from both teachers and students that may in turn contribute to the development of values education curriculum for them.

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DEVELOPMENTAL VERSUS DIFFERENCE APPROACHES TO SELF-EFFICACY BELIEFS OF MATHEMATICS TEACHERS

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Abstract: The developmental approach to efficacy beliefs of mathematics teachers is based on the view that comparisons between experienced and beginner teachers at the beginning of their career should indicate no substantive differences in the structure of their efficacy beliefs towards mathematics teaching. In contrast, the difference approach states that such comparisons will often reveal major deficits in efficacy beliefs. As far as we know there are no research studies discussing the developmental and difference approaches to self-efficacy. We argue that aspects of the conceptual and methodological foundations on the theory of developmental and difference approach are important for any effort to change efficacy beliefs. A paradigm based on regression analysis is recommended as the starting point for further theoretical and methodological work in the area.

Self-efficacy influences several aspects of behavior that are important to teaching and learning. Teachers' efficacy beliefs have been related to student achievement, student motivation, teachers' adoption of innovations, teachers' management strategies, and teachers' strategies in instruction (Christou & Philippou, 1998; Cooney, Shealy, & Arvold; Kyriakides, 1998; Pajares, & Miller, 1995; Woolfolk, Rossoff, & Hoy, 1990). In these studies, with a few exceptions (Pajares, & Miller, 1995), efficacy was generally assumed to be the independent variable. In this paper, we consider teachers' efficacy beliefs as the dependent variable and we propose both a theoretical and a methodological model linking teachers' efficacy beliefs with teaching experience. The proposed theoretical model reconciles two competing conceptual strands found in the literature: The first strand assumes a developmental approach to teaching efficacy, while the second one emphasizes the differences found among teachers. The methodological model refers to a new design based on the regression analysis through which both the developmental and the difference approaches can be discussed.

THEORETICAL BACKGROUND

The Concept of Teaching Efficacy: The conceptualization of teacher efficacy is based on the theoretical framework of self-efficacy developed by Bandura (1997). Bandura (1997) defined perceived self-efficacy as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3). In the same sense, teaching efficacy, which is a form of self-efficacy beliefs, can be defined as teachers' beliefs in their abilities to organize effective teaching-learning environments and have positive effects on student learning.

Recently, Soodak and Podell (1996) found that teacher efficacy is comprised of three factors labeled as Personal Efficacy (PE), Outcome Efficacy (OE), and Teaching Efficacy (TE). PE refers to teachers' beliefs that they have the skills to bring changes in students' behavior and performance, while OE refers to the belief that, when teachers implement those skills, they can achieve desirable outcomes. The TE factor refers to teachers' beliefs that teaching in general can lead to students' successful performance overcoming influences outside the classroom which affect learning, including children's home environment.

The Developmental - Difference Theory in Teaching Efficacy: The developmental and difference theory, as proposed in the present study, has its roots in the psychology literature (Cole, 1998). One of the purposes of the present study is to apply the developmental and difference theory in the field of teaching efficacy. In the following paragraph, we first explain the terms of internal and external factors, which are used throughout the study, and then we briefly discuss the developmental and difference theory as it may apply to teaching efficacy.

The internal variables refer to the extent to which teachers feel equipped with the tools needed to teach the classroom mathematics (Ross, Cousins, & Gadalla, 1996). By internal variables we mean all those variables that are closely related to teachers' attitudes and most importantly to teachers' feelings of being well prepared. By external variables, we mean all those variables called by Ross et al. (1996) as between variables such as the teaching experience, the age, and the gender of teachers or other environmental factors as described by Bandura (1997). In this study, we examine the developmental-difference theory with respect to only one external factor, the teaching experience of teachers. However, the results and conclusions of the present study can be replicated in such a way as to include a number of other external factors.

The developmental Theory: According to Bandura (1997) efficacy beliefs among teachers may be best conceptualized as following a developmental sequence. The concept of developmental sequence, in the context of the present study, assumes that teachers in their early years of experience have the opportunity to develop a sense of efficacy as professionals in the field. It is also assumed that teachers during their career constantly develop teaching efficacy beliefs, but the basic structure of their beliefs does not substantially differ from the beliefs they demonstrate throughout the years in the profession. We advocate that the continuity in the efficacy beliefs of teachers of mathematics holds true provided that teachers are equated on internal factors. The matching of teachers on internal factors provides a more complete picture of teachers' development of self-efficacy, because the internal factors are more salient in shaping efficacy beliefs especially when teachers lack experience (Tschannen-Moran, Hoy, and Hoy, 1998). Thus, the first proposition of the developmental approach to efficacy beliefs states that if teachers of mathematics equated on internal factors have a similar pattern of teaching efficacy irrespectively of external variables. This is the notion of the similar-structure hypothesis put forward in the framework of the developmental models in conjunction with the similar sequence hypothesis. The similar sequence hypothesis predicts that experienced and inexperienced teachers pass through the same phases of teaching efficacy development, differing only in the rate at which they progress and the ultimate ceiling they attain. The similar structure hypothesis involves the view that experienced and inexperienced teachers have similar processes underlying their teaching. There is also a second proposition contained in the developmental theory and it is subsidiary to the first. This second proposition states that if there are differences between internal-matched groups of teachers with regard to teaching efficacy, then these differences are likely to relate to exogenous factors such as motivation, adjustment, personality, and other background factors associated with environmental variables.

The difference theory. In contrast to the developmental theory, the difference theory states that comparisons between teachers with different external factors, often reveal major differences in teaching efficacy, irrespective of the internal variables. The difference position, in other words, states that even when teachers of mathematics are equated on internal factors, there would be differences in their teaching efficacy. The difference approach supports the view that any pattern of deficits in teaching efficacy is related to external factors such as experience, cultural factors and gender. Much of the research provides evidence that teaching efficacy is at an inferior level in inexperienced teachers compared with experienced teachers (Sanders, Borko, & Lockar, 1993). In addition, the aim of those who support the difference model is to show that the relationships between external factors and teaching efficacy are essentially different in teachers with or without long experience.

Difference theorists may reject the assumptions of the developmental theory. They may point out that teachers with different external factors demonstrate quite different feelings of teaching efficacy. Sanders, et al. (1993), for example, claimed that inexperienced teachers perform at substantially lower levels on key measures of teaching efficacy than experienced teachers, because they have difficulties in selecting appropriate examples and activities for their students. In the same way, Saber, Cushing, and Berliner (1991) found that experience levels are crucial for explaining teachers' differences in teaching efficacy, because beginning or inexperienced teachers were not able to interpret adequately instructional strategies and hypothesize reasons for different student behaviors.

Much of the research revealed contradictory findings about experienced and inexperienced teachers' efficacy. Two patterns of results emerged: Fisrt, a number of studies indicated that teaching efficacy increases with experience (Dempo & Gibson, 1985), and thus teaching efficacy in experienced teachers is higher than in inexperienced teachers (Lin & Tsai, 2000). On the other hand, Ross et al. (1996) reported that novice teachers had strong sense of teaching efficacy. There are several reasons of this conflict. Lin and Tsai (2000) referred to different measurement tools, to cultural differences and variation in the sample groupings. Besides those justifications, there are also some important theoretical and methodological issues.

THE AIMS AND HYPOTHESES OF THE PRESENT STUDY

Comparing the means of experienced and inexperienced teachers in teaching efficacy scales is the prevalent research paradigm adopted by almost all studies purporting to make comparisons between the groups of individuals with and without long experience in teaching. In the present study a different approach was applied in discussing the developmental and difference debate in experienced and inexperienced teachers. Specifically, according to developmental theory, the relationship between teaching efficacy, and attitudes and preparatory programs is the same in both the experienced and inexperienced teachers, despite the fact that, in some cases, the inexperienced group may demonstrate a lower level on measures of teaching efficacy. On the other hand, the difference theorists may state that the relationship of teaching efficacy and internal factors of experienced teachers is substantially different from that of the inexperienced. In this respect, the following two related hypotheses were stated:

(a) The teaching efficacy of inexperienced teachers is comparable to that of experienced teachers matched on internal factors, and

(b) The relationship between teaching efficacy and internal factors is the same for both experienced and inexperienced teachers.

METHOD

Subjects and Procedure: Data were obtained from 94 secondary mathematics teachers who participated in the TIMSS-R. These teachers taught 8th graders during the school year 1998-99. Thirtyfive of them were males and 59 females. The sample was representative of the population of mathematics teachers in Cyprus with regard to experience and gender. The sample selection as well as the procedures for the questionnaire completion followed the guidelines provided by TIMSS-R.

Instruments: Data were collected using parts of the TIMSS-R teacher questionnaire, and the teaching efficacy questionnaire, which was developed by the authors. The following is a brief description of the variables used in the present study for measuring the internal factors and the teaching efficacy of teachers.

Internal Factors: In the present study we considered as internal factors the attitudes of teachers towards mathematics, their content knowledge and their pedagogical content knowledge. To measure teachers' **attitudes** 9 items were selected from

TIMSS-R teacher questionnaire. These items elicit information about teachers' conceptions of mathematics on the nature of mathematics, and the on process of teaching and learning mathematics. The **content** knowledge preparation of mathematics teachers was also represented by the total score of teachers' responses to 12 items in which they were asked to indicate the extent to which they feel well prepared to teach the curriculum content of mathematics in the 8th class. The **pedagogical** knowledge was measured using 20 items from the TIMSS questionnaire which provide information about the way teachers organize their classes, the type of questions and exercises they assign to students, the way they deal with students' problems, the type of homework they assign, and the weight teachers give to different types of assessment. The measures of **teaching efficacy** were obtained through a questionnaire specifically designed for the purposes of the present study. Respondents used a 6-point agree/disagree scale to respond to 13 statements which measured TE, 11 which measured PE, and 10 which measured OE. Three composite scores (TE, PE, and OE) were produced by adding the scores of the individual statements comprising each one. TE, PE and OE were the dependent variables for the subsequent analysis.

RESULTS

The main assumption of the study was that experienced and inexperienced teachers matched on internal factors such as attitudes towards mathematics, content apprehension and pedagogical preparation do not differ substantially on their teaching efficacy beliefs. Therefore, it is important, first, to present the situation of teachers on these matched tasks, and then their feelings about TE, PE and OE. To this end, we conducted a repeated measures analysis, which showed that there were no statistically significant differences between the two groups on the tasks of internal factors (attitudes, content knowledge and pedagogical preparation). This result leads to the conclusion that both the experienced and inexperienced teachers, involved in the study, were matched on all internal tasks under consideration. The matching of both groups of teachers on the internal factors can be explained by the fact that mathematics teachers in Cyprus form a homogeneous group with respect to their qualifications, since the great majority of them are graduates of Greek universities, which mostly follow the same programs.

Following developmental theory's assumptions, we would expect that inexperienced teachers when matched to teachers of equivalent internal factor levels should perform equally well as experienced teachers on efficacy tasks. The results showed, however, that teachers in the two groups, although matched on internal factors, performed equally well on two of the three efficacy tasks, i.e. on the PE and OE efficacy beliefs but differed significantly on the TE. Non-experienced teachers ($\overline{X}=15.36$) seem to believe that they are more successful on TE than experienced teachers ($\overline{X}$=14.28). The latter difference is not sufficient to provide support for either the developmental or the difference theory.

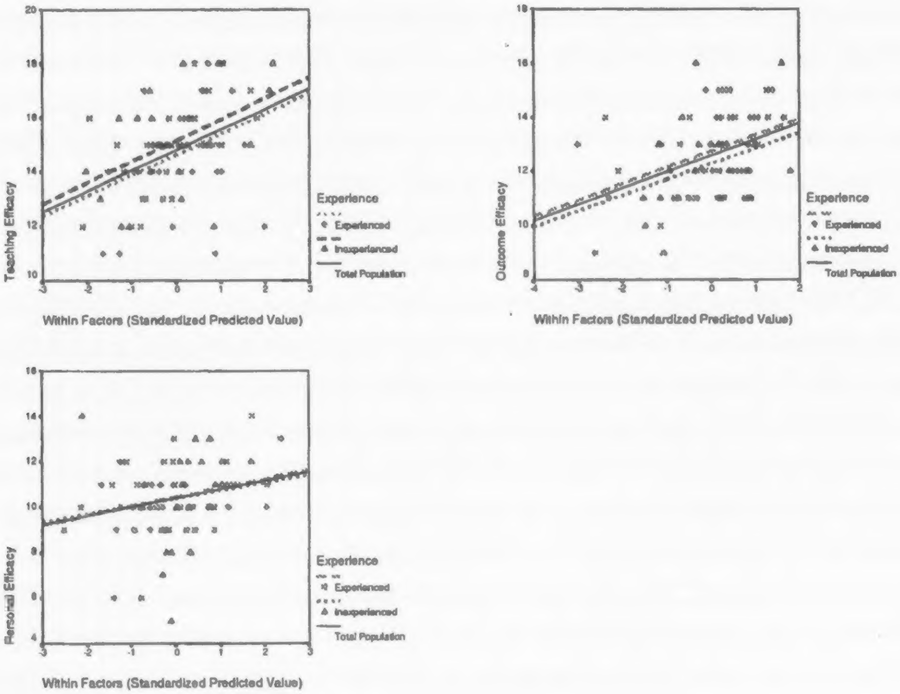

Figure 1: The Regression of TE, OE, and PE of Experienced and Inexperienced Teachers on Internal Factors.

What is of greater importance is the relationship between efficacy and internal factors and not the disparity in the mean levels of TE, PE or OE, which in some cases can be explained by differences in other variables such as motivation, school system and climate, organizational structures, etc. According to difference models, the relationships between efficacy and internal factors are essentially different in the two groups. The developmental theory posits that the relationship of efficacy and internal factors in experienced and non-experienced teachers is not different from that observed for teachers in general. Therefore, the second hypothesis of this study refers to these relationships, which are explored through the regression analysis.

According to the design, the efficacy effect is a linear combination of parameter values of internal factors. To test this model multiple regression was used. Figure 1 presents the plots of the observed and predicted values for both groups of teachers. The regressions of efficacy beliefs on internal factors for teachers as a whole and the two groups (experienced and non-experienced) separately are indicated by the straight bold line and the dotted lines, respectively. As can be seen from Figure 1, the regression lines seem to coincide in the case of PE, while in the case of OE the regression lines of the two groups are almost parallel and very close to the line for total sample of teachers. The regression line of TE shows that non-experienced teachers reported higher feelings of TE than experienced teachers, reaffirming the results of multivariate analysis. This result is not in contrast with the notion of development and thus, it can not be considered as supporting the difference model. However, in order to provide support to the developmental theory in this case, we need to examine whether the regression lines are homogeneous for both groups in TE, PE, and OE. The t-values for the difference between the line slopes of the two groups of teachers were quite smaller than the critical values in the t-distribution, leading to the conclusion that the slopes do not differ significantly. Thus the pattern of efficacy beliefs is homogeneous in experienced and non experienced teachers even in the case of TE where the means of the two groups of teachers differed significantly.

CONCLUSIONS

In the present study we advocated for a different methodological approach to conceptual issues of teaching efficacy in the context of a new theoretical framework, that of developmental and difference theory. At present most of the studies placed excessive reliance on testing for group differences between experienced and inexperienced teachers' teaching efficacy. This study suggests a shift from group means to relationships among internal factors and teaching efficacy at specified experience levels. The main view expressed in this paper is that the line of regression should be the test for the developmental or difference models. If discrepant relationships between teaching efficacy and internal factors were apparent in comparisons between groups designated experienced and non experienced teachers, then support for the difference theory would be indicated. In the present study the comparisons of the slopes of the lines of experienced and inexperienced teachers demonstrated that TE, PE and OE beliefs develop in a similar continuous way, and thus the data provide support for the developmental theory.

The developmental theory of teaching efficacy implies that experience does not alone constitute a decisive factor that influences teaching efficacy. The role of experience is moderated by internal factors. Thus, the development of TE, PE, and OE was similar in both groups of teachers. However, differences between beginning and experienced teachers may exist, but the emphasis is on the relationships among the factors that contribute to teacher efficacy. Thus in the present study we first equated teachers on the internal factors, and we advocated that when teachers are matched on internal factors then the structure of teaching efficacy follows in an equivalent manner.

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The Role of Mathematical Beliefs in the Problem Solving Actions of College Algebra Students

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This paper reports on the results of a study of the beliefs of College Algebra students. Subjects came from College Algebra classes at two universities in the southern United States. A total of 115 subjects participated in the study. Data sources included both a mathematical beliefs and attitudes survey instrument and on-going individual interviews conducted with 25 of the students. Drawing from the episodes of two students, Brad and Carrie, the analysis demonstrates and explains how the students' mathematical beliefs about formal algebraic concepts influence and sustain their problem solving actions.

Introduction. Among four-year universities, the number of students required to enroll in College Algebra classes forms a critical mass that presents unique challenges for the mathematics faculty whose mission it is to provide quality instruction. The large numbers of College Algebra students enrolled in these classes can be traced back to the 1970s, when remedial enrollments increased 72% as non-science academic programs such as Nursing and Business looked to mathematics departments to provide the necessary mathematics preparation for their students (Leitzel, 1987). In the most recent survey of the mathematics preparation of incoming students, the National Center for Education Statistics reported that 34% of entering freshmen at two-year public colleges were required to take remedial coursework in mathematics as were 18% of the freshmen enrolling at public four-year institutions (NCES, 1995). Other sources have placed similar figures much higher (Watkins, 1993). Because College Algebra serves as the core mathematics requirement for many majors, both universities and community colleges are looking for innovative ways to address the needs of College Algebra students.

The research conducted on the College Algebra population includes studies which surveyed the mathematical beliefs of these students (see for example the work of Peskoff, 1998), studies which have documented the fragmented conceptual understandings that many of these students possess (Carlson (1997), and studies which examined the effectiveness of specific instructional strategies (Underwood Gregg and Yackel, 2000; Yackel and Underwood, 1998). However, there have been no studies examining how the mathematical beliefs and conceptions of these students influence the ways they conceptualize mathematical situations and pose mathematics problems to solve. The role of mathematical beliefs in the evolution of mathematical activity needs to be documented and explored.

Purpose and Theory. The purpose of the study was to examine the beliefs and conceptions of College Algebra students, with the view that their mathematical conceptions and beliefs interact to influence their cognitive actions in mathematical learning situations. Drawing from the work of Cooney, Shealy, and Arvold (1998), we focus on the learner's beliefs as mental structures which aid his/her interpretations in mathematical situations. According to Schoenfeld (1985), the mathematical beliefs of students help constitute their "mathematical world view" (Schoenfeld, 1985, p. 157), and hence play a crucial role in the ways they "see" the mathematical problems they face. This view is compatible with Vergnaud's (1984) notion that exists a formal connection between the learner's mathematical beliefs and conceptual actions; he asserted that problem solvers often demonstrate their "mathematical beliefs-in-action" as they solve problems, and that these beliefs serve them as conceptual models upon which they can develop successful solution strategies (1984, p.7). Given these theoretical underpinnings, the study examined the role played by the students' mathematical beliefs in the evolution of their mathematical problem solving activity.

Methods. Subjects came from College Algebra classes at two universities in the southern United States. A total of 115 subjects completed a mathematics beliefs and attitudes survey developed by Yackel (1984). While the survey would yield a snapshot of sorts of the students' mathematical beliefs, we wanted to observe the reasoning activity of individual students as they solved mathematics problems. Hence, 25 of the students participated in a series of individual teaching interviews, which occurred bi-weekly and lasted about 40 minutes each. Each interview included approximately 20 minutes where the students solved algebra tasks given by the researchers; during the remaining time, the students introduced their own problems and questions.

Analysis. The analysis proceeded as follows. First, the survey of beliefs and attitudes was compared with students' activities in the interviews. Next, the interview data were examined through protocol analysis. The video-taped recordings were examined to identify instances where significant conceptual structuring activity appeared to occur. This enabled the researchers to focus on episodes of novel activity, and make inferences about the constructive role played by the subject's mathematical beliefs in the evolution of their conceptual knowledge. In addition to the video protocols, transcripts of the videos, paper-and-pencil records, the researchers' field notes, and the subjects' written tests were examined and used to develop case studies.

Given the space limitations here, we will only mention highlights of the analysis of the survey data, and then devote the remainder of the paper to episodes from the student interviews. Briefly, our survey results are consistent with what other researchers have found regarding the nature of mathematical beliefs and its impact on performance (Frank, 1986; Sackur and Drouhard, 1997; Schoenfeld, 1985). For example, 90% of the students demonstrating high-level achievement in the classes demonstrated flexible mathematical beliefs, viewing mathematics as a tool of their reasoning that is supposed to make sense to them, and that the teacher's way of solving math problems represents only one of many possible solutions. In contrast, 88% of the students demonstrating low-level achievement in the classes appeared to have more rigid beliefs about mathematics, viewing mathematics as a collection of rules and tricks, where the teacher determines what is correct and the student's goal is to imitate the actions of the teacher.

In this paper we focus on the mathematical activity of two subjects by examining episodes that illustrate the significant interplay between the students' beliefs, their conceptions, and their demonstration of mathematical structure through their problem-solving activities. We will discuss these episodes in terms of 1) how the students conceived of and interpreted their problems initially; 2) the complexities of their mathematical ideas; and 3) how they worked through the dilemmas and difficulties they faced as they solved their problems.

An Interview with Brad: Brad was a first-year Business major who had taken College Algebra the previous semester and earned a grade of D. He was repeating the class, a practice common among College Algebra students, because he needed a grade of C to satisfy his academic major.

During the first interview, Brad worked a series of tasks that involved simplifying radicals and applying the laws of exponents. After completing the tasks, Brad asked a question from the current homework on radicals. Brad had tried to simplify the expression $2\sqrt{50}+12\sqrt{8}$ using the laws of radicals, and he was concerned that his answer, **34**, did not agree with the answer given in the book, $34\sqrt{2}$. The interviewer asked Brad to re-work the problem at the blackboard.

> **Brad**: *I've worked it out twice but I didn't get the answer that's in the back of the book. First thing I do is look at radicals and see if I can simplify anything, just to drop one of the radicals. And um ..., you can't break these down into terms that can't so ... 50 will break down into 2 and 25, which are both perfect squares (sic), so that's what I went ahead and did.* ***(Writes*** $2\sqrt{25 \cdot 2}$***)*** *Some people like to break them up I'll keep them together. And 8 is not a perfect square either, but I know that 2 and 4 are (sic), which are factors of 8, so I went ahead and wrote that down* ***(Writes*** $12\sqrt{2 \cdot 4}$***) (Re-writes entire expression)***

$$2\sqrt{50}+12\sqrt{8}=2\sqrt{25\cdot 2}+12\sqrt{2\cdot 4}$$

Brad's retrospective reporting of how he tried to solve the problem demonstrated an overall understanding of the task -- that he could both recapitulate and monitor his prior activity in an objective manner. In addition, while Brad invoked an appropriate strategy, his problem involved making sense of a discrepancy between his answer and that given in the back of the book.

> **Brad**: *And from here I just go ahead and take the square root of this 25, which would bring the 5 out front, which would leave me ...2·5 and go ahead and bring the2 out which would be 1, right ? ... or now it'd just be a2, right ?...* ***(stares in space, rolls eyes)****... and then plus 12 then ... that that'll just be1 and bring out a 4, which will be 2, and we multiply by 2, ... 2 x 5 will be 10, and 12 x 2 will be 24, which will leave you with 34, but that's not what the book got.*

Brad's solution is summarized below (#1-4). Step #3 includes Brad's erroneous action, $\sqrt{2}=1$).

Brad's Solution

1. $2\sqrt{50}+12\sqrt{8}$
2. $2\sqrt{25*2}+12\sqrt{2*4}$
3. $2x5x1+12x1x2$
4. $10+24=34$

Brad's hesitation in asserting that he could simplify $\sqrt{2}=1$ ("go ahead and bring the **2** out which would be **1**, right ?") indicated that he was becoming aware of the probable source of his problem, that $\sqrt{2}$ may not simplify to 1 as he had previously thought.

Brad: *The book got* $34\sqrt{2}$. *I can't figure where* $\sqrt{2}$ *is ?*

Interviewer: *It looks awfully close, only thing is the* $\sqrt{2}$ *there in the answer. Why don't you look back at an earlier step and see if there's some place where there could've been a* $\sqrt{2}$ *and maybe it got lost in the shuffle when you reduced things. Where do you think the* $\sqrt{2}$ *might be ?*

Brad: *(reflects) ... There and there.* ***(points to*** $\sqrt{25*2}$ ***and*** $12\sqrt{2*4}$ *)*

Interviewer: *So, what did you do at that point in theprocess ?*

Brad: *I just took the square root of25, which was 5 and the square root of2, ...* ***(long reflection here)*** *... that's not perfect!..., yes, it's perfect, ... yeah for some reason, I cannot ...* ***(realizes he has a problem here, but tries to work it out)***

Interviewer: *So the question is, is* $\sqrt{2}$ *perfect ?*

Brad: *... no, it's not, is it. You know what I was getting confused with ? ... is because that* ***(writes*** $\sqrt[2]{2}$*) and for some reason I thought I could cancel* ***(cancels 2s in expression*** $\sqrt[2]{2}$ ***,e.g.,*** $\sqrt[2]{2}$*). Maybe* *that's* *where I got lost, that has to be it, because there's no other place.*

Interviewer: *So why don't you fix it from this point on ?*

Brad: *OK, so just go from this line ?* ***(goes back to his board work and starts with*** $2\sqrt{25*2}+12\sqrt{2*4}$*) Um.* $2*5\sqrt{2}$. *Still gonna keep the* $\sqrt{2}$ *and it's gonna be plus 12, um* $\sqrt{4}$ *is 2, still have* $\sqrt{2}$ *there. Then we go ahead and multiply that to be* $\sqrt{2}$, $5*2\sqrt{2}$ *plus* $12*2\sqrt{2}$, *that stays there.* ***(writes*** $10\sqrt{2}+24\sqrt{2}$*).*

Brad: **(several seconds of reflection)** *I guess this is just like 10X + 24X, the X stays the same and you just go ahead and bring down the radical. Then 24 and 10 is 34 (writes $34\sqrt{2}$) O.K. That's what I was doing. That's the kind of mental lapse I'll have, that right there. ... that's crazy, for some reason it didn't register with me on the homework And that's the kind of crazy thing I do ... crazy little careless mistakes like that. It kills me on the test. I usually catch it on the homework, I checked it twice.*

We believe that Brad's episode is noteworthy for the following reasons. First, Brad was able to distance himself from his prior activity and objectively review, monitor, and then report results to the interviewer. College Algebra students are seldom able to engage in such retrospective analysis of their actions. That Brad was able to demonstrate such a grasp over his actions indicates both the robust nature of his conceptions (his knowledge of what he needed to do) and the strength of his convictions about how these types of problems are to be solved. He systematically set about to simplify the radicals (#2-3) and never wavered from his belief that his overall reasoning was sound -- he knew what he needed to do to solve the problem with the radicals and could carry out and evaluate the efficacy of his actions. Second, Brad's inability to self-diagnose and correct his erroneous idea about cancellation of radicals ($\sqrt[2]{2}=1$), suggests that his misunderstandings were deep-rooted within his flow of continuous action. While Brad could "see" an overall structure of appropriate solution activity to carry out, he had great difficulty isolating the source of error even after repeated attempts. It was only with the intervention of the interviewer's questions that Brad became aware of the error and set about to correct his solution accordingly.

An Interview with Carrie: Carrie was a 2nd year student whose performance in the class was consistently in the B to upper C range. Carrie's responses on the beliefs survey indicated that she believed mathematics to be difficult because in order for one to be successful solving problems, one must remember many rules and procedures. She indicated that she thought mathematics was important for many careers but that she personally took mathematics courses only because they were required. She also indicated that she thought some people were naturally better at mathematics than others but she strongly disagreed when asked if mathematical ability was determined by gender. During the latter part of the first interview, Carrie introduced a rather difficult complex fraction problem from homework that had puzzled her. Her solution is summarized below as a series of simplifications (#1-5) of the original problem.

Carrie's Solution

$$\begin{array}{ccccccccc} 1. & & 2. & & 3. & & 4. & & 5. \\ \dfrac{m-\dfrac{1}{m^2-4}}{\dfrac{1}{m+2}} & = & \dfrac{m-\dfrac{1}{(m-2)(m+2}}{\dfrac{1}{m+2}} & = & \dfrac{\dfrac{m(m-2)(m+2)-1}{(m-2)(m+2)}}{m+2} & = & \dfrac{\dfrac{m^3-4m-1}{(m-2)((m+2)}}{\dfrac{1}{m+2}} & = & \dfrac{m^3-4m-1}{m-2} \end{array}$$

Summary of activity. Carrie began her work on this problem by factoring **m^2-4** (#2). Immediately thereafter, Carrie came to her first major decision - was this a division problem? Initially, Carrie stated that she did not know what to do with the denominator, **1/(m+2)**. In describing her source of indecision, it appeared that the numerator posed the more immediate problem for her. Carrie stated that she wanted to work on the numerator using the least common denominator which she identified very quickly as **(m+2)(m-2)**. While Carrie had some difficulty describing what she wanted to do, her actions indicated that she understood the process for combining rational expressions. She correctly combined the terms in the numerator (#3); however, she also altered the denominator from **1/(m+2)** to **m+2**. The interviewer intervened with a question and the subsequent episode served as a second major decision point for Carrie as she solved her problem.

Interviewer: *How did you get this in the denominator* ***(points to m+2)****?*

Carrie: *Do I apply the same LCD to this part or do I do it separately? Basically I get the LCD which is that* ***[points to (m-2)(m+2)]*** *and so all it is going to be is 1. (m+2), it's already there, so it's like..one.*

Interviewer: *You keep saying one, I'm not sure what you mean.*
[The interviewer inferred that Carrie was mentally dividing out (m+2)]*.*

Carrie*: (pause) O.K. see my LCD for this part, 1/(m+2)?*

Interviewer*: Yes, what are you going to do with it?*

Carrie: *(pause) Here it is simplified?*

Interviewer: *Yes.*

Carrie: *Should I leave it as it is?*

Interviewer*: Yes. What should you do now?*

Carrie*: Well, you don't want to mark it out* ***(Indicates cancellation in the numerator)****. So I want to multiply it out.*

The interviewer's interpretation of Carrie's activity was that she had confused the two common methods for simplifying complex fractions and was trying to apply both methods simultaneously. In her first attempt (#3), Carrie appeared to be trying to mentally multiply both the numerator and denominator with the LCD, **(m+2)(m-2)**. In the subsequent attempt, she considered the denominator separately and determined that her simplification of the denominator was incorrect. Carrie then returned to the numerator and addressed the issue of how to multiply **m(m-2)(m+2)**. She noted that she understood how to apply the "**FOIL** method"[1] but she wasn't sure if she should multiply by the **m** first.

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[1] The **FOIL** method refers to a memory device used by algebra students in the U.S. to remind them how to mutliply together a pair of binomials -- First, Outer, Inner Last.

After the interviewer suggested that she could multiply in any order that she wished, Carrie wrote: $\mathbf{(m^3 - 4m - 1)/(m-2)(m+2)}$ (#4) and then mused as to how this answer should be written in relation to the rest of the problem. At this time, Carrie reached the third major decision point of her problem solving process, when she again reflected as to the kind of problem she was faced with. She immediately declared that it was a division problem and began to work on it using the invert and multiply method. Carrie hesitated for a moment as she considered whether or not she should try to factor $\mathbf{m^3 - 4m - 1}$. After deciding against factoring, she divided out the common factor of $\mathbf{m+2}$ and wrote her final answer, $\mathbf{m^3- 4m - 1/m-2}$.

Carrie's activity indicated that she had possession of some basic mathematical tools that many students at this level have not yet mastered. Carrie could factor polynomials, combine rational expressions, and simplify algebraic expressions. However, on the basis of her survey responses and interview data, we claim that her "mathematical world view" (Schoenfeld, 1985, p. 157) is procedurally based. For example, in utilizing her rules to simplify the complex fraction, she demonstrated solution activity that ultimately led to results that did not make sense to her. In order for her to resolve the confusion regarding what she perceived as similar solution paths, she was unable to mentally coordinate the two methods, one against the other, and determine which one to apply. Rather, she needed to choose one of the paths and physically carry out the process. She appeared unable to mentally carry out a process and evaluate the results it would yield. Finally, we noted that while Carrie immersed herself within the problem, she sometimes became lost inside the details of specific sub-tasks. We contend that one reason for this is that Carrie does not see mathematics as a world of connected mathematical ideas. Survey data revealed that she believes that "mathematics consists of many unrelated topics" (Yackel, 1984).

Conclusions. We posit that the experiences of Brad and Carrie are somewhat typical of College Algebra students. Such students enter college with a collection of mathematical rules, procedures and rigid expectations concerning what it means to do mathematics. As a result, the mental structures they invoke to help them organize and direct their mathematical actions are often fragmented. For example, memorizing the definition of a linear equation may help students recognize when they have a linear equation; however, it does not ensure that they will be able to mentally reflect upon, critically examine, and choose from among potential solution strategies. While memorizing definitions and rules is an important part of learning mathematics, it is not sufficient for the development of such reflective activity. For students such as Brad and Carrie, instructional practices that merely review and reinforce procedural tasks are not likely to benefit their mathematical development. Rather, these students need to face mathematical tasks that present dilemmas for them, the resolution of which contributes to their evolving awareness of algebraic concepts and, hence, to their evolving mathematical knowledge. Our continued work in this area is directed at developing instructional activities of this nature.

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Composition of Geometric Figures[1]

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The purpose of this research is to chart the mathematical actions-on-objects young children use to compose geometric shapes. We designed a hypothesized learning trajectory based on previous research and an instrument to assess levels of this trajectory. We tested both the trajectory and the instrument with extensive case studies of 60 children, ages 3 to 7. Our research reveals that children move through levels of thinking in the composition and decomposition of 2-D figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes.

The ability to define, use, and visualize the effects of composing (putting together) and decomposing (taking apart) geometric forms is a major conceptual field and set of competencies in the domain of geometry. This domain is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (Clements, Battista, Sarama, & Swaminathan, 1997; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988). There is empirical support that this type of composition corresponds with, and supports, children's ability to compose and decompose numbers (Clements, Sarama, Battista, & Swaminathan, 1996). Although there is limited research on children's thinking about geometric composition, there is a lack of research detailing specific learning trajectories. The purpose of this research is to chart the mathematical actions-on-objects young children use to compose geometric shapes.

The genesis of the study was in observations we made of children using *Shapes* software (Sarama, Clements, & Vukelic, 1996) to compose shapes. *Shapes* is a computer manipulative, a software version of pattern blocks, that extends what children can do with these shapes. Children create as many copies of each shape as they want and use computer tools to move, combine (compose and decompose) and duplicate these shapes to make pictures and designs and to solve problems. We noticed that several of our case-study students followed a similar progression in choosing and combining shapes (e.g., rhombi or equilateral triangles) to make another shape (e.g., to fill a hexagonal frame). At first, children merely appreciated the relationship between pattern blocks, how one pattern block could be made using other pattern blocks, but their efforts to fill a hexagonal frame with other pattern blocks was by trial-and-error. Second, they could fill the hexagon with 2 trapezoids. Then they followed a sequence of filling the hexagon with 6 triangles, the trapezoid with 3 triangles, the trapezoid with 1 rhombus and 1 triangle, and the hexagon with 3 rhombi. To ascertain whether this sequence was a valid indicator of developing competencies in composing shapes, we conducted a series of studies; we report on one here.

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Theoretical Framework

Our theoretical assumption is that to solve composition tasks such as ours effectively and efficiently, children must build an image of a shape, then match that image to the goal shape by superposition (of both components and shapes), performing mental rotations as necessary to match these images. We wished to ascertain what features made certain composition tasks more or less difficult. In brief, the literature indicates that possibilities include: a horizontal or vertical side in the frame or component shape (Fisher, 1978; Hemphill, 1987; Ibbotson & Bryant, 1976), few component shapes (Vurpillot, 1976), symmetric components (Bremner & Taylor, 1982; Vurpillot, 1976), components which are symmetric halves of the frame (Clements & Battista, 1992), components that match a maximum number of the frames components (sides and angles), and presence or lack of mental rotation (Kail, Pellegrino, & Carter, 1980; Presmeg, 1991; Rosser, 1994; Shepard & Metzler, 1971). It is important to note that we are drawing indirect inferences from most of these studies in which the actual tasks were related to Piagetian horizontality and verticality tasks or disembedding tasks, not composition tasks in which we were interested.

We created a hypothetical learning trajectory from the existing research on shape composition, including our own research and intuitions generated from our work with children (Clements, in press). Hypothesized learning trajectories (Cobb & McClain, in press; Gravemeijer, 1999; Simon, 1995) ideally include "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). Unlike other approaches (Gravemeijer, 1994), we believe that existing research should be a primary means of constructing the first draft of these learning trajectories (which may, in turn, ameliorate the difficulty many development teams appear to have incorporating the research of others). The following levels constitute our hypothesized learning trajectory for the composition of shapes.

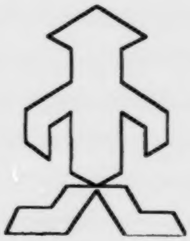
1. Pre-Composer. Manipulates shapes as individuals, but is unable to combine them to compose a larger shape.

2. Piece Assembler. Similar to step 1, but can concatenate shapes to form pictures. In free-form "make a picture" tasks, for example, each shape used represents a unique role, or function in the picture. Can fill simple frames using trial and error (Mansfield & Scott, 1990; Sales, 1994). Uses turns or flips to do so, but again by trial and error; cannot use motions to see shapes from different perspectives (Sarama et al., 1996). Thus, children at steps 1 and 2 view shapes only as wholes and see no geometric relationship between shapes or between parts of shapes (i.e., a property of the shape).
3. Picture Maker. Can concatenate shapes to form pictures in which several shapes play a single role, but uses trial and error and does not anticipate creation of a new *geometric shape*. Chooses shapes using gestalt configuration or one component such as side length (Sarama et al., 1996). If several sides of the existing arrangement form a partial boundary of a shape (instantiating a schema for it), the child can find and place that shape. If such cues are not present, the child matches by a side length. The child may attempt to match corners, but does not possess angle as a quantitative entity, so will try to match shapes into corners of existing arrangements in which their angles do not fit. Rotating and flipping are used, usually by trial-and-error, to try different arrangements (a "picking and discarding" strategy). Thus, can complete a frame that suggests that placement of the individual shapes but in which several shapes together may play a single semantic role in the picture.
4. Shape Composer. Combines shapes to make new shapes or fill frames, with growing intentionality and anticipation ("I know what will fit"). Chooses shapes using angles as well as side lengths. Eventually considers several alternative shapes with angles equal to the existing arrangement. Rotation and flipping are used intentionally (and mentally, i.e., with anticipation) to select and place shapes (Sarama et al., 1996). Can fill complex frames (Sales, 1994) or cover regions (Mansfield & Scott, 1990). Imagery and systematicity grow within this and the next levels. In summary, there is intentionality and anticipation, based on shapes' attributes, and thus, the child has imagery of the component shapes, although imagery of the composite shape develops within this level (and throughout the next levels).
5. Substitution Composer. Deliberately forms composite units of shapes (Clements et al., 1997) and recognizes and uses substitution relationships among these shapes (e.g., two pattern block trapezoids can make a hexagon).
6. Shape Composite Iterator. Constructs and operates on composite units intentionally. Can continue a pattern of shapes that leads to a "good covering," but without coordinating units of units.

We had two research goals, to evaluate (a) the geometric composing instrument and (b) the validity of the hypothesized levels of thinking in the domain of composing geometric figures.

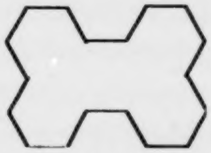
Method

Based on this model of students' learning, we created a instrument to measure each of the first five levels of thinking. The following are examples of two items on the instrument.



On the first, children are given pattern blocks and a frame of a "man" and asked to "Use pattern blocks to fill this puzzle." We categorized children as follows. *Pre-Composer:* Cannot match even well-defined, simple frame, such as the "feet." *Piece Assembler:* Can fill simple frames (e.g., "feet" only) using trial and error. *Picture Maker:* Fills frames with trial-and-error, matching shapes by boundary or matching side lengths.

Shape Composer: Completes entire frame with deliberate choices of shapes; to do so, matches configurations, sides, or angles. *Substitution Composer:* Deliberately replaces a group of shapes (e.g., two triangles) with one shape (e.g., blue rhombus) or vice versa. *Shape Composite Iterator.* Deliberately, systematically iterates a composite group of shapes to fill a region.



A second example asks children, in three separate questions, to determine how many yellow hexagons, red trapezoids, and green triangles they would need the cover the puzzle(given a limited number of the latter shapes).

We categorized children for the *Piece Assembler* to *Shape Composer* levels in ways similar to the first example, but this item was designed especially to target *Substitution Composer:*The deliberate recognition and use of the relationships between the hexagons, the trapezoids, and the triangles (e.g. 2 trapezoids = 1 hexagon).

Participants were 60 children from 4 classrooms selected at random from all children who completed a human subjects permission letter. All children were interviewed individually by one of two graduate research assistants following a protocol for administering the composition instrument. The researchers also asked questions of children, as in a clinical interview, whenever they believed that such questions would clarify the nature of children's thinking.

Each session was videotaped. These tapes were partially transcribed, coded, and analyzed, both to complete the scoring of the instrument for each child and to identify additional themes.

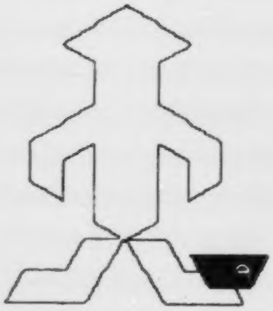
We compared the results of applying the scoring rubric to the qualitative analysis of each child's response to each item with the intent of determining, for each item, if (a) the item elicited the types of thinking we wished to observe and (b) the scoring rubric accurately encapsulated the type of thinking that the qualitative analysis revealed. We then used the results of both the scoring rubric and the qualitative analyses to determine whether (a) items designed to measure the same level of thinking elicited similar responses, providing information as to the reliability of the items and the coherence of the hypothesized levels of thinking, (b) the levels form an invariant sequence. For example, each student's scores were entered into a spreadsheet divided into categories based on the hypothesized trajectories; each item was classified according to the level it was designed to measure. The resultant spreadsheet was examined visually to answer the research questions. In addition, the proportion of items providing evidence of attainment of each level of thinking were computed. This allowed us to examine the percentage of children whose scores followed a pattern consistent with the hypothesized trajectory (e.g., if half of the items indicated thinking at level *n* , more than half should reliably indicate mastery of thinking at level *n – 1*, etc.). Qualitative analyses were also used to ascertain whether the levels evince "incorporation"; that is, if thinking and actions of an earlier level are incorporated in the next level. Thus, we were assessing the main developmental criteria of constancy and integration (across a period, there is a type of thinking that forms an integrated whole), invariant sequence, and incorporation (Steffe & Cobb, 1988).

Findings and Discussion

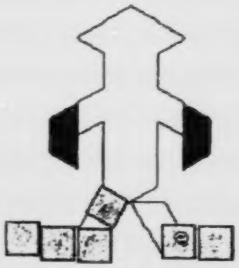
Findings generally supported the hypothesis that children demonstrate the various levels of thinking when given tasks involving the composition and decomposition of 2-D figures, and that older children, and those with previous experience in geometry, tend to evince higher levels of thinking.

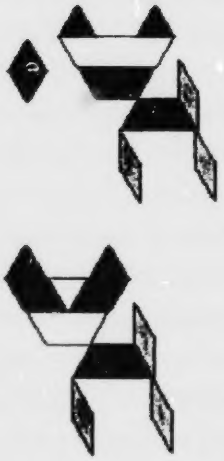
The most intensive work was in the qualitative analysis of children's responses. We found that the levels of thinking could be reliably differentiated, and that children could be reliably assigned to a level of development (including those in the process of developing the next level). Here we can discuss only a few examples.

Mary, a preschool (4.3 years) child, exhibited actions that typified the behaviors of our early level of composition: the Piece Assembler. When Mary began working on the above described puzzle man, she tried (correctly) to place a trapezoid in the foot as shown. The trapezoid was 180° opposite of the orientation needed to fill the frame and Mary, through her minor rotations in each direction, was unable to arrive at the requisite orientation and thus rejected the piece. Moments later, she uses the trapezoid to fill the arm, this time successfully rotating the shape to match the frame.



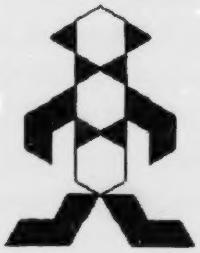
After similarly filling the other arm, Mary returned to the legs and concatenated 4 squares to (incorrectly) cover one leg, and two to cover the other before deciding to move on to another item. Using squares inappropriately shows that Mary was not attending to angle, a behavior typical of all of the children categorized as Piece Assemblers.





Kevin, a grade 1 student (6.0 years) demonstrated the actions used frequently by children at the Picture Maker level. The "picking and discarding" strategy that typifies this level can be observed repeatedly as Kevin attempts to fill the dog puzzle item. As Kevin tries to fill the puzzle, shapes are selected and "tried out" for a fit through placement and manipulation of the shape directly on the puzzle; there is a notable lack of the construction of a mental image of the shape and its relationship to the puzzle frame. The accompanying figure displays the puzzle nearly completed. Kevin attempted to fit a rhombus into the open space, then a square, and eventually, unable to fill the open frame, he rejected the arrangement and cleared away the shapes in the head. He then placed a trapezoid along the left side of the head and similarly on the right, thereby creating two simple frames that did allow him to complete the puzzle.

The Picture Maker level of thinking evinced by Kevin precedes the Shape Composer level which is exemplified in the work of Alice, a grade 2 (7.4 years) student. In this level we no longer observe the random selection of pieces, but rather deliberate selections are made as the child creates a mental image of how the shape may fill the frame. In addition, the process of completing a puzzle often becomes systematized.



As Alice worked on the puzzle man she first placed a trapezoid on one arm, then the other, followed by a rhombus on each arm. Similarly, Alice carefully considered how to fill the leg as she looked back and forth from the shapes to the puzzle prior to making her selection. Once she filled one leg with two trapezoids, she was able to think of the concatenated pieces as a whole and simply duplicated the process on the other leg. She filled the body of the puzzle man in a similarly systematic way with the finished puzzle reflecting this nicely.

Across all 60 children, examination of the items indicating attainment of each level similarly confirmed the hypotheses. If a child evinced a level of thinking on one item, they were more likely than not to attain it on the other items measuring that level. Most exceptions involved the highest level the child had attained; many children, unsurprisingly, were in the process of developing that level of thinking, so that scores were mixed. With few exceptions, once a higher level was reached, children had mastered the vast majority of items at each lower level. Quantitative summaries supported these conclusions. Computing the percentage of children whose scores followed a pattern consistent with the hypothesized trajectory, we found that 84% of the children followed the pattern exactly. Of the 16% that did not, all but one broke the pattern in the same way: they scored slightly higher on the Substitution Composer than the Shape Composer level. Older children, and those with previous experience in geometry, tend to evince higher levels of thinking. Total scores for PreK, K, 1, and 2 were 3.28, 9.91, 12.3, and 12.5.

Conclusions and Future Research

Our research reveals that children move through levels in the composition and decomposition of 2-D figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes.

The next phase of this research is to evaluate the usefulness of the present findings for instruction and to assess children longitudinally in teaching experiments. We have created a sequence of activities aligned with the learning trajectory and will engage children from preschool to second grade in these activities, charting their development through the learning trajectory

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LISTENING: A CASE STUDY OF TEACHER CHANGE

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The data for this study is taken from a current research project [1] looking into the development in year 7 students (aged 11-12) of a 'need for algebra' (Brown and Coles, 1999) in four teacher's secondary classrooms in the UK. For this case study I introduce the notions of evaluative, interpretive and transformative listening, (adapted from Davis, 1996), to analyse three transcripts taken from the lessons of one teacher on the project. The project design and case study were informed by ideas of enactivist research (Varela, 1999, Reid, 1996, Brown and Coles, 1999, 2000). A significant change occurred in Teacher A's classroom, as shown in the transcripts, and the listening of both students and teacher became transformative. There is evidence that specific teaching strategies were linked to this change in listening and that once the change occurred the students started asking their own questions within the mathematics.

BACKGROUND

In the summary of findings (Coles, 2000) from a one year teacher-research grant (awarded by the UK's Teacher Training Agency (TTA)) I identified teaching strategies that were effective in establishing a 'need for algebra' (Brown and Coles 1999) in a year 7 class (students aged 11-12 years) whom I taught. 'Algebraic activity' in this project was interpreted as being synonymous with 'thinking mathematically' (see Brown and Coles, 1999). Evidence for students finding a 'need for algebra' was that they were able to ask their own questions about complex mathematical situations and structure their approach to working on these questions.

The results of the TTA research formed part of the background to a current research project [1], funded by the Economic and Social Research Council (ESRC). This project involved three other teachers, who had all been part of a steering group on the TTA research, and wanted to work at developing a 'need for algebra' in their own year 7 classes (the first year of secondary school in the UK).

Since a 'need for algebra' was linked to students asking their own questions, whole class discussions in which students developed these questions were seen by all the teachers on the project as being a vital component of their lessons. If discussions amongst a whole class (around twenty six students for each teacher) are to be effective in allowing students to develop their own ideas, then the quality of listening of the students is a key factor. Before presenting a case study of the changing practice in Teacher A's classroom in relation to the listening that took place, I need to set out what I mean by listening and by different types of listening.

LISTENING AND HEARING

The dictionary definitions of listening and hearing are as follows:

> hearing is; 'the action of the faculty or sense by which sound is perceived ... the action of listening ... knowledge by being told'

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listening is; 'the action or act of listening ... to hear attentively ... to give ear to ... to pay attention to ... to make an effort to hear something' (Little et al, 1973)

There is an overlap in these definitions in that both can be used to mean 'the action of listening'. The different aspects of the definitions of listening all share this active component; *'to give ear to'*, *'to pay attention to'* and in each phrase I take listening to involve an act of will or decision on the part of the listener *'to make an effort ...'*. This sense of listening involving the will is echoed both in research in psychology: *'Listening is a process that is triggered by our attention.'* (Rost, 1994 p.2) and in mathematics education: *'The act of listening ... requires a full and conscious effort to tune into the how and the what of the students' idea'* (Wassermann, quoted in Nicol, 1999 p.57).

The definitions of hearing, in contrast, I take to refer to two different phenomena. The *'faculty ... by which sound is received'* (Little et al, 1973) seems to refer to the mechanical aspect of perceiving sound. However, the last definition of hearing in the quotation above: *'knowledge by being told'* (ibid) does not fit with this 'mechanical' meaning. *'Knowledge by being told'* implies that when I hear, something happens internally. I may be attending to whether what was said agreed or conflicted with my previous knowledge or whether what I heard extends the ideas I previously held.

There is an implied distinction here between a listening that is active but where no connection is felt with what is said and times where there is a connection made and where the hearer is changed by what they hear. I have found this distinction useful in thinking about classroom discussions but, in analysing lesson dialogue, I needed a finer grained and observable categorisation. The definitions that follow are based on Davis' (1996) notions of the evaluative, interpretive and hermeneutic listening of teachers, which I adapt for analysing the listening of students as well as teachers.

THREE FORMS OF LISTENING

(1) Evaluative listening

If a teacher is listening in an evaluative manner then they will characteristically have a *'detached, evaluative stance'* (Davis, 1996 p.52) and they will deviate *'little from intended plans'* (ibid). For such a teacher: *student contributions are judged as either right or wrong ... listening is primarily the responsibility of the learner'* (ibid). The teacher makes assumptions based on a supposed *'knowledge of the other's subjectivity'* (ibid) or rather the assumption is the students have knowledge of the teacher's subjectivity - hence it is the student's responsibility to listen and learn from the unproblematic access they will thus have to the teacher's thinking.

If students or teacher are listening in an evaluative manner then they would see what others say in terms of right or wrong, and see listening as the others' responsibility. This is indicated by, for example, someone responding immediately to another's suggestion with a judgement that it is incorrect (or correct).

(2) Interpretive listening

Interpretive listening is characterised by an awareness of the *'fallibility of the sense being made'* (Davis, 1996 p.53). If I hear someone while listening in an interpretive manner then along with whatever connection I make, or any idea that arises, or whatever meaning I take from the words, I am aware that this may not be the connection, idea or meaning the speaker intended. There is a recognition that listening requires: *'an active interpretation - a sort of reaching out rather than taking in'* (ibid). A response might offer feedback to the speaker not by evaluating what is said but e.g. by offering an interpretation and asking for clarification.

(3) Transformative listening

What distinguishes transformative listening from the previous category, interpretive listening, is that the interpretive listener is still 'standing back' from the speaker. There is an attempt to interpret and make sense of what the speaker says, but always from the point of view of the listener.

When I listen in a transformative mode, then as well as an awareness that what I hear may not be what the speaker intended (characteristic of the hearing of interpretive listening) I am open to the interrogation of assumptions I am making, e.g. that allow me to believe communication is possible at all.

I have again drawn on Davis' (1996) categories of listening. He defines his third form of listening (which he labelled 'hermeneutic') as:

> ... an imaginative participation in the formation and the transformation of experience through an ongoing interrogation of the taken-for-granted and the prejudices that frame these perceptions and actions. (Davis, 1996 p.53)

The notion of the *'transformation of experience'* links this form of listening to traditions of Buddhist mindfulness, in which knowledge is seen as *'equivocal'* and *'open to question or revision'* (Claxton, 1997 p.219).

Evidence of transformative listening and mindfulness in a classroom includes a willingness to alter ideas in a discussion, to engage in dialogue, to entertain other points of view, and hold them as valid, independent of whether they are accepted or not. If a student makes a connection to a previous piece of work or links something that has been said before, this would indicate the transformation of experience, the re-structuring of categories. Similarly, if a student creates a new categorisation, this indicates a mindful attention to what is happening: the seeing of *'a new world'* (Thera, 1996 p.32). This sense of re-structuring previous categories or ideas, seeing a *'new world'* is indicative of learning.

CASE STUDY - TEACHER A

With these distinctions I have been able to analyse the listening in teachers' classrooms across the project. After a brief description of the methodology and methods of the study, I present three transcripts (see Appendix 1) to illustrate how the listening changed in Teacher A's classroom over a period of four months.

Methodology

There are four researchers on the ESRC project (one of whom is myself), each responsible for a different strand of analysis (e.g. teaching strategies, algebra).

The whole project design has been informed by ideas of enactivist research (Varela, 1999, Reid, 1996, Brown and Coles, 1999, 2000) and a key component of the research process has been that we take multiple views of a wide range of data. This is ensured by the different strands of the researchers. We will often look at one piece of data, e.g. a short piece of a videotape of a lesson, and discuss what we see from each of our perspectives.

We also tell stories of the changes that are happening over time for the students, teachers and researchers on the project. The three transcripts that I use in this paper are part of a story about learning and about teacher change. All four researchers have written about an expanded version of the last transcript (Brown et al, 2000) weaving a different story to the one I present here.

There is no sense of there being a 'best' theory for our work or, for example, of the perspective of listening in this paper being 'better' than a previous analysis of the same data. An explicit part of the project is that we see *'research about learning as a form of learning'* (Reid, 1996 p.208). From an enactivist viewpoint learning is the telling of multiple stories and the awareness of ever finer grained distinctions.

Methods used for this case study

There were four teachers on the project who were videotaped in each of the six half-terms that make up an academic year. The camera was fixed at the back of the classroom - focused on the board but with around half the students in view. The data for this study is taken entirely from the videotapes of one teacher, Teacher A (TA). I was looking at times during the lesson of whole class discussion, i.e. when there was a single conversation occurring in the room. I initially watched the videotapes and noted - at 5 second intervals - whether a student or the teacher was speaking. This record helped me identify times when students responded directly to each other or when there was significant interaction between teacher and students. I then transcribed those sections of dialogue from the video recording. I chose Teacher A for the study because, of the four teachers on the project, there was the clearest evidence of a change in listening on the videotapes of his lessons.

Analysis

The dialogue in Transcript 1 shows evidence of evaluative listening. After the comments of both S_1 and S_2, Teacher A says *'they do'* thus evaluating and confirming the students' contributions. S_3's comment is greeted with a *'thank you'* which the other comments were not, suggesting to me that this is the comment that the teacher wanted (although the comment is unclear, from Teacher A's response I interpret S_3 as saying something about the first and last digits of the three numbers under consideration). Further evidence for the teacher having a pre-given idea of what he wanted the students to say is that having started with the general question: *'Any comments about those three numbers'*, Teacher A then asks: *'what can you tell me about the first and the last?'*. Having started with an open question, since the students were not offering what was wanted, the teacher directs their attention to a specific aspect of the problem.

It seems possible here to pick out sentences and analyse them using the categories of listening. However, in viewing more videotapes this rapidly became problematic. In looking at transcripts of sections of dialogue to decide what type of listening was being displayed I needed, in most cases, to take into account the wider context of what was happening in the lesson. For example, in a different lesson a student said to his neighbour: *'You are wrong'*. On the surface this seems typical of evaluative listening. However if this comment was the start of an interaction in which the students began to explore their differences, the listening would be interpretive or transformative. It therefore made more sense to characterise whole lessons or sections of lessons as evaluative, interpretive, etc.

In fact, when I analysed longer sections of the lesson transcribed above the listening was more interpretive. In general Teacher A does not evaluate the students' contributions as right or wrong. However, the task for the students is to fit their comments and suggestions to the teacher's plan. Teacher A interprets the students' comments and gives feedback in relation to the idea he has chosen to focus upon.

I believe the listening in Transcript 2 moves from interpretive to transformative. A student makes a suggestion: *'It's got six lines of symmetry'*, which is dealt with in a different manner to the ones just before. Rather than continuing the interpretive listening pattern of repeating each student's contribution and asking for other comments, Teacher A says: *'Where's your lines of symmetry then?'*. The teacher cannot know where S_1's lines of symmetry are, hence he is genuinely involved in making meaning of the comment.

Teacher A then asks for the rest of the class' opinion: *'Who thinks it's a line of symmetry? Hands up'*. After S_5's comment, Teacher A gets an A4 piece of paper and starts folding it the ways S_5 and then other students suggest. The teacher responds directly to suggestions from students. The task for the class (in this case deciding what is a line of symmetry and how many there are on a rectangle) emerges from the interaction of students and teacher. I read Teacher A's comment at the start of the transcript: *'right, we're talking symmetry'* - which was said with a slightly higher tone of voice, as further evidence that he had not anticipated dealing with issues of symmetry. There is a feel of collaboration and participation in the dialogue - characteristic of transformative listening.

The participatory nature of discussion is even more evident in Transcript 3 (taken from later in the same lesson as Transcript 2) in which the listening is also transformative. The teacher here is not running the discussion (e.g. by posing questions for the students to respond to). It is the students who are asking questions: '*What about 100?*', '*What would just a straight line be?*'. Students are now talking directly to each other and extending each other's ideas e.g. '*S_3: And a quarter times 48 is twelve*'.

The transcripts provide evidence that there was a significant change in the listening in Teacher A's classroom. The listening in videotapes of lessons up to Transcript 2 was interpretive or evaluative and in all later videotaped discussion the listening was transformative, so the change appears to have been a lasting one.

TEACHING STRATEGIES

It is beyond the scope of this paper to deal with what factors have contributed to the change in listening in Teacher A's classroom, however it is striking that there are a number of teaching strategies in evidence in Transcript 2 (and later discussions) that were not being used in Transcript 1. These strategies include:

- *the teacher asking a question they do not know the answer to.* Teacher A says: *'Where's your line of symmetry then?'* Having made this comment there is immediately the possibility for other students to engage with S_1 in dialogue.

- *responding to students' suggestions.* There is evidence of this particularly in the sequence when Teacher A gets a piece of paper and starts folding it.

- *asking for feedback from the whole class.* Teacher A asks for *'Hands up'* in response to the question *'Who thinks it's a line of symmetry then?'*. Feedback from this response allows the teacher to use the next strategy.

- *asking a student to explain their idea to the class.*

These strategies can all be seen as 'slowing down and opening up discussion'. They are strategies that encourage and allow different students to engage in dialogue with each other. In Transcripts 2 and 3 over a quarter of the class speak in a period of a few minutes. Another way of characterising the strategies is that they all depend on the teacher's contingency upon the responses of the students. It is important to note that this does not imply the teacher will do anything the students suggest but only that students' voices *can* be heard and *can* play a part in the creation of the lesson focus.

There is evidence from other teacher's lessons on the project of the teaching strategies above being used during times of transformative listening.

It is striking that in Transcript 3 it is not the teacher who is 'asking a question they do not know the answer to', or 'responding to students' suggestions', but the students themselves. It seems that students are taking over some of the roles in discussion previously performed by the teacher - a culture of transformative listening is becoming established in the classroom. In Transcript 3, for the first time on any of

Teacher A's videotapes, students raise their own questions, which they could work on, related to the mathematical activity.

CONCLUSION

The description in this paper of different types of listening has provided a tool for analysing classroom dialogue. The evidence of this study is that teaching strategies based on the teacher's contingency to the responses of the students allow the opening up and slowing down of class discussions, which seems necessary for the development of transformative listening. There is evidence that in discussions in which the listening of students and teacher is transformative, students exhibit behaviour associated with these teaching strategies in responding to each other. In such discussions, there is the opportunity for students to ask and work on their own questions, which for the ESRC project is linked to them having a 'need for algebra'.

[1] 'Developing algebraic activity in a 'community of inquirers'' Economic and Social Research Council (ESRC) project reference R000223044, Laurinda Brown, Rosamund Sutherland, Jan Winter, Alf Coles. Contact: Laurinda.Brown@bris.ac.uk.

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APPENDIX 1

[NB The numbering of students in each transcript is done independently.]

Transcript 1: September 1999

TA: *Any comments about those three numbers?* [The numbers referred to are: 92101, 29810, 54321]

S_1: They all have two in them.

TA: *They all have two in them* [pause] *they do* [pause] *anything else?*

S_2: They all have one in them.

TA: *They do* [Two more students offer suggestions, which Teacher A responds to.]

TA: *Now remember what we were saying ... when we were looking at four digits we were comparing the first and the last, we were comparing the two middle ones. What can you tell me about the first and the last with those ones ... what can you tell me about the first and the last?*

S_3: [unclear]

TA: *Thank you S_3: nine is bigger than one, two is bigger than zero, five is bigger than one.*

Transcript 2: March 2000

S_6: It's got four sides

TA: *It's got four sides, okay, very good, anything else?*

S_7: It's got four equal angles

TA: *Four equal angles, yes*

S_1: It's got six lines of symmetry

TA: *Six lines of symmetry, right, we're talking symmetry. Where's your lines of symmetry then?*

...

S_1: Across the right hand top corner to the bottom left hand corner

TA: *This is a line of symmetry?* [TA holds up a ruler along a diagonal of the rectangle] [pause] *he's unsure. Who thinks it's a line of symmetry? Hands up* [pause] *a couple of you.* [pause] *Who thinks it's not a line of symmetry?* [lots of hands go up] *Oooh, okay, S_3, convince those that think it is why is it not a line of symmetry do you think?*

S_3: You can only have diagonals in a square

TA: *Oh right, okay*

S_4: Or a circle

TA: *Why is that one not a line of symmetry though? S_5*

S_5: Well, if you get like a A4 paper, that's a rectangle, you can fold it diagonally so that it goes all [unclear]

Transcript 3: March 2000

TA: *Excellent. Oh, lovely. Well done.* [Students applaud] *So, 3 times 4 is 12, 2 times 6 is twelve, 1 times 12 is twelve and a half times 24 is also 12..*

S_3: And a quarter times 48 is twelve

TA: *And a quarter times 48 ...*

S_3: And an eighth times ...

S_4: Three quarters.

TA: *And an eighth times ...*

S: I'm not saying.

S: You can actually go on.

TA: *... We could carry on forever couldn't we?*

Ss: What about 100? How could you draw it though?

TA: *Well, it would be a sixth of a unit. Very small.*

S: If you drew it really big so one square was 6

S: Sir, what would just a straight line be?

AN INVESTIGATION OF PRESERVICE ELEMENTARY TEACHERS' SOLUTION PROCESSES TO PROBLEMATIC STORY PROBLEMS INVOLVING DIVISION OF FRACTIONS AND THEIR INTERPRETATIONS OF SOLUTIONS

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In this study, we examine 68 preservice elementary teachers' solution processes to a problematic story problem involving division of fractions. The problem was problematic in the sense that the numerical answer to the division does not necessarily provides the appropriate solution to the problem, at least if one takes into consideration the realities of the situation embedded in the context of the word problem. It was found that the sample of prospective elementary teachers did not always base their responses on realistic considerations of the context situation. In fact, only 19 (28%) of the participants' responses contained a realistic solution to the given problem. In addition, only 4 (6%) participants provided an explanation for their solutions. None of the participants made any comments about the problematic nature of the problem. It was also found that other factors contributed, at least in part, to students' unrealistic solutions: inappropriate mathematical models and mistakes on the execution of the mathematical procedures.

Story problems play a fundamental role in school mathematics because of several reasons. First, they intent to offer examples of where mathematics can be applied in "real-life" contexts. Second, they offer students opportunities to make connections between mathematics procedures or formulas and real-world situations. Third, they offer opportunities for students to practice mathematical procedures and formulas. However, in many cases the "stereotyped" nature (Nesher, 1980) of word problems leads students to apply mathematical procedures in an instrumental way, without realizing the inappropriateness of their actions. As an example, consider the following problem: What will be the temperature of water in a container if you pour 1 jug of water at 80°F and 1 jug of water at 40°F into it? (Nesher, 1980, p. 46). Verschaffel, De Corte, and Lasure (1994) used this item with 75 fifth graders in Flanders. They reported that only 13 (17%) students provided a realistic response to the problem. Other studies (e.g., Contreras, 2000; Reusser & Stebler, 1997; Silver, Shapiro, & Deutsch, 1993) further illustrate that many students fail to activate their real-world knowledge or to use realistic considerations to solve problematic word problems.

This study is part of a larger project whose main purpose is to understand prospective teachers' knowledge of the meaning of quotients and reminders when solving story problems involving division. The purpose of the present paper is threefold. First, we examine prospective elementary teachers' solution processes and their use (or lack of) of realistic considerations to solve problematic story problems involving division of fractions. Second, we examine the extent to which prospective elementary teachers explicitly interpret the solution to mathematical procedures. Third, we also examine the extent to which the participants' solutions support or refute a referential-and-semantic-processing model proposed by previous research (Silver et al., 1992, 1993).

THEORETICAL AND EMPIRICAL BACKGROUND

Word problems also provide students with experiences about the process of mathematization, especially mathematical modeling. We define mathematization as the representation of aspects of reality by means of mathematical procedures. The process of mathematization involves the application of mathematical ideas and procedures to solve real-world problems. We will adopt Silver et al.'s (1992, 1993) model of mathematization for the story problems presented to the participants of the present study. Figure 1 displays Silver and colleagues' model.

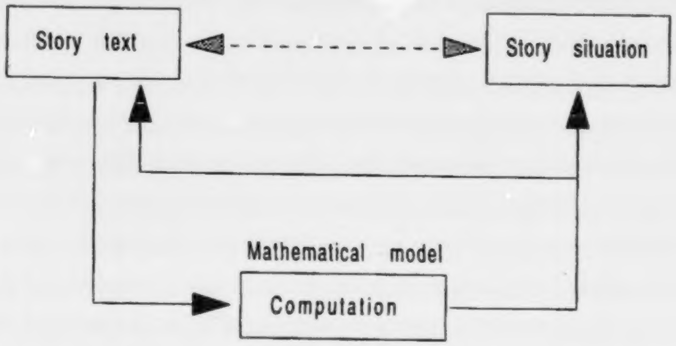

Figure 1: Silver et al.'s (1992, 1993) referential-and-semantic-processing model

According to this model, students' understanding of the structure of the problem might enhance their abilities to represent its solution through a mathematical operation or procedure. That is, students would map from the story text (the problem) to an appropriate mathematical model. Students then perform the required computations. Next, students should interpret the results of the computation using their real-world knowledge about the story text or story situation in the "real world." In other words, students need to map the computational result back to the story problem or to the implied "real-world" situation. If students fail to perform any of those three activities of the process of mathematization, they may not be able to provide a realistic response to the problem posed. While Silver et al. (1992, 1993) focused on students' lack of mapping from the numerical answer of the mathematical model to either the story problem or story situation, we hypothesized that students' range of solutions to the problem could be explained more completely by paying attention not only to the semantic processing feature of the model (mapping from the computation of the mathematical model to the story text or story situation) but also to the mapping from the story text to the mathematical model and to the execution of the mathematical procedures. Our hypothesis was based mainly on our teaching experience that suggests that problems involving division fractions are cognitively more complex than division of whole numbers. Our hypothesis turn out to be reasonable. We will refer to the Silver et al.'s model as *referential-and-semantic-processing model* to distinguish it from their emphasis on the semantic feature of the model.

The story problem examined in the present study involve division of fractions in which the quotient does not represent the solution to the problem, at least if one takes into account some realistic considerations of the story context or story situation in the "real world". Such problems are called problematic division problems. One of the first sources to document students' solutions to problematic problems involving division is the Third National Assessment of Educational Progress (Carpenter, Lindquist, Matthews, & Silver, 1983). It was reported that only about 24% of the students who took such test gave a correct solution to the problem: "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?" Others studies (e.g., Cai & Silver, 1995; Ruwisch, 1999; Silver, Mukhopadhyay, & Gabriele, 1992; Silver, Shapiro, and Deutsch, 1993) have investigated students' solutions processes and responses to problematic division problems. In their study, Silver, Shapiro, and Deutsch (1993) examined 195 middle school students' solution processes and their interpretation of solutions to a division problem involving remainders. The problem read as follows

> The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

The researchers administered three versions of this problem. The different versions differed in the amount of people going to the game (532, 540, and 554) so that the decimal part of the numerical quotient were less than .5, .5, and greater than .5. The researchers had hypothesized that the size of the remainder could influence students' responses. They reported that over 60% of the students who used an appropriate procedure to find the solution executed their procedures correctly, but only 43% of the total number of participants provided 14 as their solution to the problem. The researchers also found that the size of the reminder did not influence students' responses. The authors concluded that the semantic feature of their model (Fig. 1) provides a solid explanation about students' lack of sense making or "suspension of sense-making" (Schoenfeld, 1991) when solving problematic story problems. In this paper we extend Silver and colleagues' research to a new content domain (fractions) and to a different population (preservice elementary teachers).

METHOD

Sixty eight prospective teachers participated in the study. A paper-and-pencil test was constructed consisting of five items involving division of fractions and decimals. As stated above, our original purpose was to examine prospective elementary teachers' understanding of the meaning of quotients and remainders of division with fractions or division with decimals. In this paper we report the results of the first problem which was stated as follows,

> Lida is making muffins that require $\frac{3}{8}$ of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make? (*The muffins problem*)

The prospective elementary teachers were enrolled in two sections of a mathematics course for elementary majors at a southeastern state university in the USA. The test was given to students during regular class and they were told orally that they would have enough time to complete it. Students were not allowed to use calculators. Written instructions asked students to explain each of their solutions and to write down any questions, comments, or concerns that they might have about each problem.

ANALYSIS AND RESULTS

Students' written responses to the problem were examined to detect four features of their solution processes (a) mathematical model, (b) execution of the procedures, (c) solution to the problem, and (d) explicit interpretation of the solution to the mathematical model. The mathematical model refers to the mathematical operation or procedures that students used to represent and obtain the solution to the problem. Execution of the procedure referred to the set of steps or actions that students took to obtain the solution represented with a mathematical operation or procedure. The solution to the problem referred to the solution that students provided to the story problem presented to them. Finally, the explicit interpretation of the solution provided by the mathematical model referred to the explanation that students provided to justify their solution to the word problem. Table 1 presents the distribution of the mathematical operations or procedures (mathematical model) that students used to represent and obtain the solution to the word problem.

Table 1: Distribution of the mathematical models

Procedure	Frequency		Percentage	
Appropriate procedures (Total)		47		69%
Division	38		56%	
Proportion	4		6%	
Multiplication	2		3%	
Unit rate and multiplication	2		3%	
Others	1		1.5%	
Inappropriate procedures (Total)		20		29.5%
Multiplication	12		17.5%	
Reverse division	3		4.5%	
Others	5		7.5%	
No responses provided		1		1.5%

A procedure was judged appropriate if it could potentially produce the correct solution to the problem. As indicated in Table 1, less than 70% of the students used an appropriate procedure. In particular, nearly 6% used a proportion, about 3% used multiplication, and nearly 3% used the concept of unit rate and then multiplied by 10. Table 1 also shows that about 29.5% of the students used procedures that were judged as inappropriate. Specifically, about 17.5% multiplied the two given number, 4.5% used reverse division (3/8 ÷10), and nearly 7.5% used other procedures such as estimation, incorrect concrete models, etc. Only the appropriate procedures were examined for correctness of execution. Out of the 47 students who set up an appropriate procedure, 40 executed the procedure correctly, 5 incorrectly, and 2 did not show work. As we can see, only forty students were able to set up an appropriate procedure, execute it correctly, and showed work.

Table 2 exhibits a categorization of students' solutions and the percentage of the students providing each solution. A solution was coded as realistic (RS) if it was 26, 27 or any other solution with an appropriate justification. Nearly 28% of the solutions were categorized as RSs. A solution was categorized as reasonable unrealistic solution (RUS) if it was the result of an appropriate procedure executed correctly and it was different from 26. About 37% of the solutions were categorized as RUSs. A solution was categorized as incorrect reasonable unrealistic solution (nearly 6%) if it was the result of an appropriate procedure executed incorrectly. Finally, underestimated solutions close to 3 (3 and 3 3/4) or 23 (23.1 and 23) and overestimated solutions (28, 35, 375) were classified as unreasonable unrealistic responses (about 28%).

Table 2: Distribution of students' solutions

Category and numerical solution	Number of students		Percentage of students	
Realistic solutions		19		28%
26	17		25%	
27	1		1.5%	
26 2/3 or 26	1		1.5%	
Reasonable unrealistic solutions		25		37%
26 2/3	15		22%	
26.6_	3		4.5%	
26.6	3		4.5%	
Other solutions	4		6%	
Incorrect reasonable unrealistic solutions		4		6%
Unreasonable unrealistic responses		19		28%
No response		1		1.5%
Total		68		100%

We also examined students' interpretations of the solutions to their procedures. An interpretation was defined as the written explanations provided by the students to justify the solution to the problem based on the execution of the procedure. An appropriate interpretation was one in which the student gave a meaningful interpretation to both the integer and fractional parts of the solution produced by the mathematical model. Only two students provided appropriate interpretations. An interpretation was judged inappropriate when the student's interpretation of either the integer or fractional part of the solution to the mathematical model was incorrect. Two students gave inappropriate interpretations. For example, one of them said that "Lida can make 26 muffins with 2/3 cup of flour left over." The other student wrote a similar statement. Sixty three (about 93%) students did not provide an interpretation to the solution produced by the mathematical model. In particular, 10 students wrote procedural explanations that described the procedure followed to get the answer to the problem, 36 reported a solution to the problem equal to the solution produced by the mathematical model (and hence they did not probably see the need to justify their solutions), and 17 did not explain why their solution to the problem was different from the solution produced by the mathematical model.

Finally, we also gather evidence to support or refute aspects of the referential-and-semantic-processing model proposed by Silver et al. (1992, 1993). About 78% of the responses provided direct evidence to support the model. Specifically, nearly 6% of the responses provided an appropriate justification to support a realistic solution and 72% contained information that explained, at least in part, why some students failed to obtain a realistic solution. The 6% of the realistic solutions supporting the model contained the numerical solution of 26 and also an appropriate interpretation of 26. Out of the 72% of the unrealistic solutions, about 26.5% contained an inappropriate mathematical model, 4.5% a procedure executed incorrectly, and about 39.5% contained a reasonable unrealistic solution that was the result of an appropriate procedure executed correctly but there was not an interpretation of the result produced by the mathematical model.

About 16% of the responses contained indirect evidence to support the referential-and-semantic-processing model. These responses contained the solution of 26 to the problem but failed to include an appropriate interpretation or justification about why the solution to the problem was different from the result produced by the mathematical model. Nearly 6% of the responses were judged as containing counter-evidence for the model. These responses contained the realistic solution of 26 but also contained flaws either on the selection of an appropriate mathematical model or on the execution of the mathematical model. In other words, those students managed to obtain the realistic solution without a correct mathematization of the problem.

To examine the influence of students' lack of semantic processing on their final solutions to the given problem, we focused on the 48 responses that contained realistic solutions or reasonable unrealistic solutions. That is, responses that contained or potentially could have contained realistic solutions. We found that 29 (about 60.5%) of those responses were unrealistic because students failed to map from the computation produced by the mathematical model to the story problem or story situation in the "real world".

DISCUSSION

One of the major goals of this study was to better understand some of the contributing factors to prospective elementary teachers' solutions to a problematic story problem involving division of fractions. Silver et al.'s (1992, 1993) referential-and-semantic-processing model provided with some solid explanations of students' failure to offer a realistic solution to the problem. The model includes three important actions that successful problem solvers might take to assure a realistic solution: set up an appropriate mathematical model, correctly execute the procedural features or steps called upon by the mathematical model, and interpret the numerical solution produced by the mathematical model. About 72% of the solutions were categorized as unsuccessful solutions. In particular, nearly 1.5% of the responses contained no work, about 26.5% involved an inappropriate mathematical operation, nearly 4.5% had flaws on the execution of the procedure, and about 39.5% did not include a semantic interpretation of the result produced by the mathematical model. About 28% of the responses contained a successful (realistic) solution to the problem. In particular, only about 6% of the responses explicitly included a semantic interpretation to the numerical answer produced by the mathematical model, and over 16% reported a realistic solution without a written explanation or justification for their response. However, nearly 6% of the responses included a realistic response but contained either inappropriate mathematical models or flaws in the execution of procedures. This finding suggests that some students sometimes use non-mathematical means to solve mathematical problems. These responses were considered as evidence to refute the referential-and-semantic processing model. The other responses were judged as supporting either directly or indirectly the referential-and-semantic processing model.

Although the majority of prospective elementary teachers set up an appropriate procedure to solve the given problem, a high percentage of students failed to do so. In fact, about 30% of the students set up the problem incorrectly. The most common inappropriate procedure was multiplication (10 x 3/8). Given that the students had already posed and solved word problems involving multiplication and division of fractions, this finding seems somewhat unexpected and discouraging. On one side, instruction did not have the desirable effect; on the other, some of those prospective teachers might teach fractions without having a good understanding of modeling situations for which division or multiplication of fraction is an appropriate operation. As to execution of procedures, prospective elementary teachers had, in general, little difficulty in performing the computation procedures successfully.

We recognize that the muffins problem is problematic since it potentially admits multiple solutions or interpretations depending on the context but that was the whole point of the problem. The same division expression (in this case 10 ÷ 3/8) can represent different problem situations. We asked students to solve the muffins problem because we wanted to confront them with a problematic problem involving division of fractions for which the solution could be either the integer part of the quotient (26) or one more than the integer part of the quotient (27) depending on aspects of the situational context. In the muffins problem, we could argue that a realistic solution is 26 because each muffins requires 3/8 of a cup of flour and there is not enough flour to make another one. However, in another situational context some students might argue that Lida could make another muffin which would use 1/4 (2/3 of 3/8) of a cup of flour. In this case the answer would be 27 muffins. While we did not expect all students to provide realistic reactions or comments to the problematic situations, it was surprising to find that only 4 (about 6%) students provided an explicit explanation for their solution, even though they were asked to explain their answers and to write down any questions or concerns that they might have about each problem. We suspect that a factor that contributed to a great extent to students' failure to react realistically to the problems was students' impoverished experiences with standard problems in which the solution can be obtained by the straightforward application of one or more simple arithmetic operations with the given numbers. Further empirical research is needed to enhance our understanding of prospective elementary teachers' lack of semantic interpretation of solutions to problematic word problems.

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THE CONCEPT OF DEFINITE INTEGRAL: COORDINATION OF TWO SCHEMAS

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This is a report of a study examining students' understanding of the concept of definite integral. Using APOS, a specific framework for research and curriculum development in collegiate mathematics education, as a guide in investigation, we analyze and interpret students' responses to the interview questions. The analyses of the interviews and the results of other studies indicate that the coordination between the visual schema of the Riemann sum and the schema of the limit of the numerical sequence is necessary for developing a good understanding of the concept of definite integral. Consequently, we give suggestions for didactic and curricular changes when teaching the concept.

In communicating mathematics to students, the professor presents concepts as a combination of (a) his or her understanding (b) the understanding established by the mathematical community, and (c) the pedagogical values the professor incorporates into his or her teaching. Communication however is a two-way process, and hence an important question arises: does the communication initiated by the professor achieve its objective? In particular, in the realm of the definite integral, do students indeed think along the lines defined by the classroom instruction? Is there a lack of understanding of some key concepts? Do modifications need to be made to our communication, our sequencing of topics that lead to the definite integral? What can we incorporate into our teaching that we learn from our current students thinking?

The definite integral is the composition of two distinct constructions: (1) the geometrical one, ultimately based on[1]e principle of exhaustion, and (2) the numerical one of the infinite converging sequences and their limits. The problems encountered in the understanding of the sequences and their limits are quite widely investigated, the understanding of the definite integral has received much less attention, while the combination of sequences and limits leading upto the definite integral has received the least attention. Because of the dependence of the definite integral on the notion of the limit of the sequence, one can suspect that the problems in understanding the limit of a sequence will create difficulties in the understanding of the definite integral as well. One of the main goals of our paper is to demonstrate how this suspicion bears out. Thus, we report in this article, our findings of student understanding of the Riemann sum as revealed in the interviews, the need for a more intensive treatment of sequences and limits of sequences prior to the treatment of Riemann sums and the justification for this suggestion.

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[1] Our appreciation goes to Ed Dubinsky for his helpful suggestions during numerous conversations.

As stated earlier, inspite of being one of the essential concepts developed during the first two semesters of college mathematics, there are comparatively few articles investigating the understanding of the definite integral by students. One of the most interesting works is the article by Orton who points out students' difficulties in using the limit process for their understanding of the concept of the definite integral. The analysis of our data expands and builds upon Orton's findings, presents the evidence of students' successes and difficulties and offers suggestions concerning significant changes in the instruction design.

Literature review

Two articles that deal with pedagogy as relates to the definite integral are by Orton [1983], and Davis and Vinner [1986]. Articles that shed light on student understanding of sequences and limits of sequences and functions are by Sierpinska [1987], Williams [1991], and Cornu[1992].

One of the interesting tools of analysis in mathematics education is that of a cognitive obstacle. According to Cornu [1992] there are the following types of cognitive obstacles: genetic, didactical and epistemological. A cognitive obstacle helps to identify the difficulties encountered by students in the learning process, and to determine appropriate strategies for teaching. Genetic and psychological obstacles arise as a result of the personal development of the student, didactical obstacles arise as a result of the nature of teaching and epistemological obstacles arise as a result of the nature of mathematical concepts themselves. It is equally interesting that according to Cornu, the concept of the limit has been fraught with several major epistemological obstacles in the course of history of which we mention two that are of significance to our work: a) the failure to link the geometry with numbers; and b) The difficulties with the last term, can one reach it or not?

Davis and Vinner [1986] report on a special 2-year Calculus course, in which they included a brief treatment of sequences and limits of sequences. Sierpinska's [1987] study was intended to explore the possibilities of elaborating didactical situations that would help students overcome epistemological obstacles related to limits. She lists several epistemological obstacles encountered by students, related to limits, obstacles that maybe due to a lack of rigor, due to incomplete induction, due to notions that limits only amount to approximating, etc. Orton [1983] reports on a study investigating students' understanding of integration and differentiation. He thinks it would be very unlikely that the introduction of integration can be made easy.

He feels that the topic of limits is one of the most neglected topics at the school level, which in turn does not help the introduction of integration.

Framework

The study reported in this article, used APOS [Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K., 1996], a specific framework for research and curriculum development in collegiate mathematics education, as a guide to investigating students' understanding of the definite integral. The premise of this framework is that educators can develop knowledge about students' learning of mathematical concepts by going through a cycle of theoretical analysis, instructional treatment, and observations and assessment of student learning. The initial step in our approach, which we refer to as theoretical analysis, is to hypothesize about mental constructions that a student might make when learning a specific mathematical concept. We refer to this structured set of mental constructions as a *genetic decomposition*. The researcher's own understanding of mathematics along with her or his learning and teaching experiences are the most important components for this step of the framework. Subsequent iterations of the framework lead to an evolving genetic decomposition of the concept and an instructional treatment.

The present study is at the point of analyzing data collected during and after the instructional treatment in the first iteration of the teaching cycle. In the process of analyzing data and answering our research questions, we describe our observations in terms of actions, processes and objects [Asiala, M. et. al., 1996]. Figure 1 shown below represents the initial genetic decomposition of definite integral that was hypothesized by the instructor/researcher and used in developing instructional treatment and interview questions. This genetic decomposition assumed that students would have an object level understanding of functions, partitions, and would develop their understanding of Riemann sums by the usual approach of an action, process and object level progression. It ended by assuming that students would apply a limit schema to obtain a number. It was not clear at that point how this particular limit schema would enfold, and how it would affect the development of the entire Riemann sum schema. Thus, it seems that this particular development in the instruction would have to be determined by the enfolding illustrated by the students.

1. Object level of function
2. Object level of partition
3. Action on a function and a partition.
   Construct one Riemann sum of one function with one partition.
4. Process conception of Riemann sum.
   Coordinate the process of a function and the process of a partition via the Riemann sum formula

5. Object conception of Riemann sum
    - Encapsulate 4.
    - Variations of the sum (Left, Right, Trap, Mid)
    - Dependence on n.
6. Action on Riemann sum
    - Compare with an area or a solution to a differential equation.
    - This is done on a vague, pictorial, intuitive level.
    - Improve the approximation.
7. Process on Riemann sum
    - Interiorization of 6.
8. Apply limit schema to obtain a number.
    - At this point, very few students will have a strong limit schema so it is unclear how the concept of definite integral will grow for them. It needs study.

Figure 1: Initial genetic decomposition of definite integral

The setting and data gathering

Data for this research were collected during the fall semester of 1992. The participants were 32 engineering, science and mathematics students who had, during the previous year, taken two semesters of single variable calculus at a large mid-western university. The interview consisted of 10 questions about the concept of integral. On the average, each interview lasted for approximately one hour. The interviews were audio taped and tapes were transcribed by paid student aides.

In this paper we attempt to answer the following three research questions: What is the relationship between the preliminary genetic decomposition and the students' mental constructions of the definite integral? What are the mental constructions that were not made by students? What should the modified genetic decomposition be to accommodate for the possibility of making the required mental constructions?

To answer the above research questions, we will analyze the responses to Questions 4, 6, 7, 9, and 10 from the interviews. Analysis of the remaining items will be presented in other studies. Below are the interview questions on which this study is based.

Interview Question 4. What is the mathematical meaning of $\int_{-3}^{-1}\frac{1}{x^2}dx$.

Interview Question 6. Suppose that an object moves in a straight line at a velocity which is a function of time, $v = V(t)$. Write a formula for the net distance which the object moves starting at time $t = t_0$ and ending at time $t = t_1$.

Interview Question 7. Explain why your formula gives the distance.

Interview Question 9. Suppose now that you have a region S in space which is a body of density ρ which has a different value at different points in the region. Write a formula for the mass of this body.

Interview Question 10. Explain why your formula gives the mass.

Initially the students were given the opportunity to answer each question without prompting. Based on their response the interviewer asked additional questions or provided hints or clarification. The interviewer encouraged any kind of student's response (verbal, written, graphical) that might help to explain her or his ideas.

Data Analysis

In the excerpts of student interviews, we see instances of student thinking that follow the steps outlined in the genetic decomposition, viz., in achieving an object level of function and partition, an action and process level of Riemann sum. The next step students would need to make is to have an object level understanding of the Riemann sum. A student with an object level understanding of the Riemann sums should be able to talk and think about a Riemann sum on a partition of size 2, 3, 4…n and be able to realize that the size of the partition needs necessarily to be finite. However, it is in demonstrating an object level of the Riemann sum that we notice difficulties that are principally related to the limit concept in the following two ways:

1. The limit of the Riemann sum is seen as the infinite sum of the rectangles of small width.
2. The limit of the Riemann sum is seen as the sum of lines, i.e., as the infinite sum of rectangles of zero width. (i.e., rather than the limit of the sum of the areas of n rectangles, students state it as the sum of the limit of the areas of the rectangles.)

The student Kenard reveals the first difficulty. When questioned about what is being done to the Riemann sums in order to make it basically equivalent to the definite integral, he states:

> S: *For instance if you are using Riemann sums, you take infinity number of rectangles, you will get almost exactly the same thing because there won't be as much error.*

Jernau when questioned about how he reconciles the two different meanings mentioned earlier, states

> S: *Right. Um... well, the Riemann sum breaks this up into n, an infinite number of rectangles. And, it's difficult to use the theory behind it. It is difficult for me.*

Dadgaron reveals the second difficulty in response to the question of how one goes from the Riemann sum to make it equal 2/3, he replies

S: *By making these rectangles infinitesimally small...smaller and smaller, I mean almost until they are a line they are a unit... and then you are just adding up these units and like, the smaller this empty area is the more exact the estimation until you get to a point where there is no empty space to be accounted for and that will give you an exact number.*

In both of the above difficulties demonstrated by students, they (students) sense a correct need for fitting their intuitive tools, be it rectangles or lines, in the region under the curve, which can be pictorially seen as the area under the curve, however, they are unable to connect that area to the numerical sequence of partial sums. The best example of the absence of connection to the concept of the sequence is the following fragment of an A student (Jasarx) :

S: *Um, well the integral is, is um, basically the sum of* $f(x_i)$ *for i in this case* $-3 < i < -1$ *– not necessarily integers but every number, every single point between –3 and –1 is going to have an f(x) value...*

I: *Okay.*

S: *...and if you add all of these together...*

I: *Um-hum.*

S: *...you should get the area under the curve, ...*

Clearly the student sees area as all the parallel lines contained in it, but in the next sentence where he notices his mistake in the formula he wrote, he immediately moves to the Riemann approach, without however anywhere indicating the presence of the sequence of which that limit is the limit of.

S: *... and the integral is just, um, actually it's* $f(x_{i+1})$ *- , okay it's f(x) times x(i+1)-x(i). Okay so now you have got, you've got a height which is f(x) times i, which is somewhere between x(i) and x(i+1).*

I: *Um-hum.*

S: *And so that would be your height and then your width would be, or your base would be x(i+1) – x(i), and so if you multiply those together you're gonna get some kind of little area...*

I: *Um-hum.*

S: *...within that section. So if you sum up all of the little areas between –3 and –1 you should get a certain value. Now when you take the integral of that same function, it's still f(x*) if you want to say that, and the x(i+1) - x(i) the integral makes it get --- takes the limit as, takes the limit of that sum as x(i+1) – x(i) goes to zero.*

One might then suspect that an integrated approach which from the start correlates the pictorial representation with the numerical sequence of partial Riemann sums, would provide an answer to students difficulties. Such an integrated approach was investigated by Orton. In his work, Orton identified several key ingredients of the conceptual (or structural) problems students have with the definite integral. One of them was the difficulty "students have with the power of the limiting process in mathematics" and in particular in calculus. The majority of students, who in one of Orton's investigational tasks, obtained initial 5 terms of the sequence Riemann Sums - which were understood as the approximations to the area under the curve $f(x) = x^2$, on the interval [0, 1], appeared to grasp that this sequence consisted of better and better approximations and that it was possible to continue improving them. They were asked whether that sequence could be used to obtain the exact area under the curve. However, he states that the students "quite rightly pointed out that such a procedure would never produce the correct answer, and were unable to state that the limit would provide the answer." We see then that such an integrated approach still does not solve the students' difficulty, it has the effect of allowing students to construct the first few terms of the sequence without being able to see how the sequence could ever converge to the number which is the exact area under the curve.

This is in direct opposition to our students who can see the limit without having access to the sequence that converges to the limit.

Both difficulties appear to be grounded in the same inability to separate the concept of a limit from the last term of the sequence. Our group influenced by the pictorial image of the area would like to take the infinite sum which fits under the given curve, Orton's group on the other hand which also sees the area under the curve as the last term to reach, cannot get to it due to the infinite number of steps required to reach there. Therefore it seems essential to take the precise e-n mathematical definition of the limit of a sequence as a base foundation on which to build the notion of the limit of a sequence and of the definite integral as the limit of partial sums, which bypasses the issue of reaching or not reaching the limit and instead focuses on a process of approach [Sierpinska].

Refining theory with pedagogical implications

Based on the data analysis we suggest that the preliminary genetic decomposition (Figure 1) should be modified as follows:

2. Object level of sequence
9. Schema of the limit of a sequence
   The distance between the term and the limit of the sequence;
   The notion of the measure of the distance
10. Schema of the Riemann sum
11. Coordination between the schema of the Riemann sum and the schema of the limit of the sequence

The numbers in front of the new items added to the initial genetic decomposition denote the position of the entries in the initial genetic decomposition illustrated in Figure 1. We wish to emphasize that the requirement of an 'object level of sequences' is a major change from the existing curriculum in the order of the concepts necessary to understand the definite integral as a limit of the Riemann sum. Currently, in a typical calculus course, the topic of sequences is studied in detail after the concept of the definite integral is studied extensively. Our proposed genetic decomposition requires a certain rearrangement of the order of the topics, with other emphasis as suggested above.

Our article emphasizes the pictorial understanding of the limit of the sequence of Riemann sums while Orton's emphasized understanding of the limit of a numerical sequence. The results of both studies point to difficulties by students when only one of them is seen separately. We conclude that there must be a coordination between the visual schema of the Riemann sum and the schema of the limit of the sequence. The source of students' difficulties can well be in the didactic and curriculum which a) doesn't develop the connection between the two in the design of the curriculum, and b) does not propose an alternative to the second by eliminating the precise defintion eps-N of the limit of the sequence. This definition, the Weierstarss definition of the limit of the sequence was created especially to bypass the problem of reaching or not reaching the limit, as Sierpinska points out. Hence the absence of the instruction of the formal definition is leaving our students on the pre-modern level.

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Jennifer's Journey: Seeing and Remembering Mathematical Connections in a Pre-service Elementary Teachers Course

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We detail one young woman's journey in a 16 week mathematics content course for pre-service teachers. We analyse her initial efforts at problem solving in groups, and her ability to see and recall explicitly connections that were inherent in the instructor's conceptions of the problems. Using her writing during the course, we compare her memories with those of other students of differing achievement, and infer that she made remarkable strides in coming to terms with the language of and ideas of mathematics, and understanding how engaging with those can help her be a better teacher.

Correct answers and mathematical connections

Pre-service elementary teachers want to get mathematical answers right. They want to know which formulas to use, and how to get the correct answer. A typical comment about their perceptions of mathematics on entering a mathematics content class is: "Coming into this class, I was under the impression that finding a formula to solve a problem was, in reality, the answer to the problem."

This is where many of them stop in their understanding of mathematics. A strict utilitarian perspective often limits their mathematical vision. We are concerned when pre-service teachers figure the answer to $9+3\frac{5}{8}$ by reducing the problem to one of "how many eighths?" We are frustrated when they justify their answer that there are 16 ways of building towers of height 4 using blocks of 2 colors with the statement: "I know I have found all possible towers because every other group in the classroom got the exact same answer as us."

Changing what students *value* in mathematics is frequently a much harder challenge than teaching them mathematical procedures and application of formulas. We need an antidote to a severely procedural orientation to mathematics focused on 'correct answers' that prospective teachers have learned to value above all. How can we explicitly emphasize connections, and assist students to construct relationships between parts of mathematics that they see as different?

We addressed this issue with a class of pre-service elementary teachers at Harper College, Illinois (www.harper.cc.il.us), during the Fall 2000 semester. The two authors co-taught the first three weeks of a 16 week course on mathematical content for pre-service elementary teachers. Differences between this and previous classes of the second author are in the explicit and intensive focus on building connections in the early part of the course. The emphasis in the first three weeks of the course was on making connections between different combinatorial problems and on multiple ways of interpreting answers.

Theoretical background

We wanted to know whether the problems we set could promote the formation of useful long-term mathematical memories. We followed the model of Davis, Hill and Smith (2000) in assisting students to make their implicit, procedural memories declarative (Squire, 1994). The latter are memories we are capable of expressing in words, drawings or gestures. They are to be distinguished from *implicit*, or non-declarative, memories that assist us to carry out routine procedures and habits. There are three major types of declarative memories relevant to mathematics. Two of these are familiar from everyday memory, whilst the third is more commonly seen in its full form in mathematics and science.

Episodic memory is the system of memory that allows us to explicitly recall events in time or place in which we were personally involved. (Tulving, 1983; Tulving & Craik, 2000, and references). *Semantic memory* is the memory system that deals with our knowledge of facts and concepts, including names and terms of language. (Tulving, 1972, p. 386; Tulving, 1983; Tulving & Craik, 2000, and references). *Explanative memory* is that part of declarative memory dealing with explanations for facts. Davis, Hill, Simpson, & Smith (2000) present a case that explanative memory is a separate memory system, linked to, but different from episodic or semantic memory.

Most psychological studies of memory are oriented to memory for language. Studies in memory for mathematics are much less common. A semantic memory such as '*Paris is the capital of France*' has quite different content to one such as '*the number of prime numbers less than n is asymptotically n/log(n)*'. The first is a linguistic convention, the other expresses a deep, non-obvious fact. Our experience with student mathematical writing and verbal recall suggests that there are, at least, the following distinctions in memory for mathematical facts:

(1) Memories of labels, customs, and conventions. For example: *A prime number is a whole number with exactly 2 factors*. This sets up '*prime number*" as a conventional term. We refer to these memories as *semantic labels*.

(2) Factual memories of things sensed, or done. For example: *The proportion of prime numbers less than 500 is 19%*. One might recall this as a fact from having done a series of calculations: the recollection is of the fact, not the episode of calculation. We refer to these as *semantic actions*.

(3) Memories of things believed. For example: *There are infinitely many prime numbers: one recalls this from a book on number theory*. We refer to these as *semantic beliefs*.

(4) Memories of explanations. For example: *A proof that there are infinitely many prime numbers: one recalls an explanation*. We refer to these as *explanative memories*.

Method

We set a number of connected problems in the first three weeks of the course. These were specifically designed to set up strong episodic memories as a result of students discussing their solutions in class. For example, after students had attempted the problem of finding how many towers of heights 4 and 5 they could build using blocks of 2 colors, they were shown, and discussed, a video clip of three grade 4 students attempting the same problem. This problem and its connections with algebra, which we utilized, has been reported on by Maher & Speiser (1997). For a detailed description of these, and other problems set in the course, see: www.soton.ac.uk/~plr199/algebra.html

The combinatorial problems we set for the students in the first three weeks were connected in *our* minds: they all deal with different aspects and representations of a systematic counting problem related to binary choices. We focused on students' written expressions of memories of the course. The reason for this is that long-term declarative memories are mediated by protein formation, following gene expression, to stimulate novel neuronal connections (Squire and Kandell, 2000). The relevance of this neurological fact is that long-term memory formation is an energetic, committed process for an individual. Long-term memories - certainly those sustained over two months - are therefore a good indicator of what a student values.

Students worked on the problems in groups. After completing the sequence of problems, they explained connections as a homework exercise. We asked them to write reflectively after each of the combinatorial problem sessions, and re-writes were encouraged. Opportunities for making connections with their earlier work were provided during the semester in questions on three group and two individual exams students hadn't seen previously. Students also wrote mid-term and end-of course self-evaluations. Twenty-two students began the course and nineteen completed. Their writings provided us with a great deal of data for analysis. We present a preliminary analysis of some of that data by focussing on the development of one student: Jennifer. For fuller details of these and other student's written statements see: www.soton.ac.uk/~plr199/algebra.html

Results and analysis

We begin by placing Jennifer in the class in terms of her initial and final test scores. Table 1 shows the initial and final test results for Jennifer and two other selected students as well as the *shift* statistic, defined, as follows:

$$shift = (final\ test\ \% - initial\ test\ \%)/(100\text{-}initial\ test\ \%)$$

We interpret *shift* as how much a student has moved form their initial test result to their final test result. Of course, a student who has a relatively high initial test score does not have as much room for improvement as a student with a low initial test score. The other two students - Allison and Rebecca - we use to compare with Jennifer were chosen to be representative of students with a middle and low shift value, respectively.

(a)

Student	Initial	Final	Shift
Jennifer	63	92	0.78
Allison	13	64	0.59
Rebecca	63	75	0.32

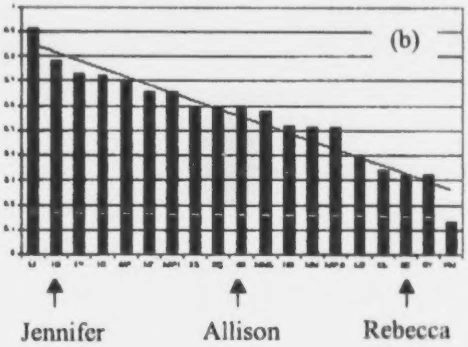

Table 1: (a) Initial and final test scores for the 3 students. (b) Distribution of the *shift* statistic for the whole class. Note that *shift* scales linearly with rank: $r^2 = 0.93$

Building towers and the grade 4 video clip

Jennifer focussed on building towers of height 4 by swapping colors, a strategy used commonly by pre-service elementary teachers. They refer to this as "being systematic": when challenged as to why they have all possible towers they commonly reply that they systematically swapped colors. Every group in the current study used this strategy.

> "We began with combinations of 3 red and 1 blue. We then altered the placing of the one blue cube which resulted in 4 different combinations. Second, we alternated the colors and made 4 combinations of 3 blue and 1 red cube, which gave us 4 combinations. Third, we made two groups of cubes, alternating between red and blue with a total of 4 in each tower. ... After building several towers, we realized that each tower had an opposite."

Working in her group she came up with a formula based on opposites that did not, however, extend to the case of towers of height 5

> "... if you multiplied the number of blocks in a tower by the number of options, you would have the number of combinations possible. Then you would double that number because each combination has an opposite: Ex: 4 high x 2 = 8; 8 x 2 = 16. However, this formula does not hold up for 5-high towers."

Jennifer realized that the number of towers had something to do with doubling. She gave no reason or explanation, other than empirical evidence, as to why this might be so:

> "It appears that you are doubling the possibilities when adding a cube to each tower. The formula we discussed in class appears to make sense where when you add a level to each tower, the possibilities double. 1 level = 2 possible; 2 levels = 4 possible; 3 high = 8 possible; 4 high = 16 possible; 5 high = 32 possible, etc."

She recognized some system in the grade 4 students' explanations, but she did not relate this precisely to her group's approach and she was – mistakenly – under the impression that she solved the problem as Stephanie did in the video clip:

> "... I now realize that there are several patterns and options to solving this type of problem. Each of us in class recognized patterns, but not one formula could clearly explain or define our cases. When building my towers, I looked for patterns similar to the way Stephanie did. My pattern differed in that I grouped my towers by building towers with one blue, and then built those with one red. I continually built towers and followed them with their opposites."

After building towers and watching the grade 4 students argue why they found all towers of a given height Jennifer stuck to her belief that building opposites is a key to systematically building towers. From our perspective the pre-service teachers were uniformly unsystematic in their attempts to build towers and to explain why they had built all possible and not repeated any. Jennifer was not alone in expressing the sentiment that since they did not know how to tackle this problem "mathematically" they would approach it through common sense:

> "Instead of looking at it as a math problem, I was looking at it as a building exercise. I first attempted the problem by guessing and testing. Tony and I first attempted the problem of four high by creating combinations of four that would design an obvious pattern."

Seeing and valuing connections

Jennifer did not immediately see connections between the problems set in the first 3 weeks of the course. Some of those connections she learned about through class discussions, following insights of other students. At the time of writing these reflections, however, she was able to articulate a common vision of "algebra" in all the problems. Not the algebra she initially imagined, namely 2^n as the formula for the number of towers of height n, but algebra based on multiple interpretations. The algebraic expansion worksheet showed $(a+b)^2 = a^2 + 2ab + b^2$ and asked students to similarly expand $(a+b)^3$ and $(a+b)^4$. Only one student in the class (not Jennifer) could do this problem. The tunnels problem was to figure how many ways there are to run through a series of 4 tunnels if each could be black or white.

> "The algebraic expansion worksheet threw everyone off at first. We really were not sure how it related to the first three exercises. What we did not see was that the "towers" were actually algebraic expansion. If two different color cubes can make 16 different towers four high, how do you mathematically write this out? Answer: 2^4. Let's say that the cubes are the colors black and white...then the formula would be written $(w + b)^2$. This is how exercise one and

> exercise four relate. Tunnel travel led us to a new discovery. A student can look at the problem and sketch the different possibilities just as he/she did with the tower building exercise or they can apply the algebraic expansion $(w + b)^4$ where w = white; b = black, 4 = number of tunnels and $(w+b)^4 = w^4 + 4w^3b + 6w^2b^2 + 4wb^3 + b^4$."

Jennifer was able to use her insights to help her solve two further problems: (a) how many pizzas can be made from 8 toppings, and (b) how many towers of a given height can be built using at most 3 colors?

> (a) "This situation is similar to the former exercises of building towers, the committee vote exercise, the grid walk problem, and the tunnel exercise with Mork. The "'with" or "without" question resembles the two color combination for the tower building exercises, the "yes" or "no" vote of the committee members, the "up" or "right" direction for the grid walk and the alternating pattern of the tunnels. ... The "with" or "without" strongly indicated powers of two as in the tower exercise. We extrapolated this to apply to the Pascalini's dilemma, so we figured that $2^8 = 256$, therefore, there are 256 combinations for pizza made of 8 toppings."

> (b) "$3^4 = 81$, n = # of cubes high; x = # of color choices; formula: x^n There are 81 towers that can be built. This is similar to white and black (2 colors) as we did in class. In class we built towers of four and five high with the combination of two colors (two choices).We also worked on committee votes of YES or NO (two choices). ... This problem also relates to the pizza problem. Instead of 8 choices of toppings you would use 3.They differ in their number of choices."

Jennifer valued the insights she gained by seeing connections. At the conclusion of the course she articulated a different vision of mathematics:

> "When I joined this class in August, I thought of math as a series of formulas, each of which should be followed in order to find an "answer". It was working on the tower building investigation and traveling through tunnels that I discovered how each relates... My original approach to the tower building revealed that instead of looking at the small picture (i.e., What do I really have in front of me? What is it I'm trying to solve?), I just dove in expecting multiple patterns. When our class finally concluded that the towers, tunnels, grids and Pascal's Triangle were all about "choices", everything seemed to fall into place. ... my perspective of mathematics changed over this semester. The changes occurred due to learning that my mathematical understanding was instrumental and not relational. I had to re-learn basic math in order to eventually teach it to children."

Memory types

Table 2, below shows the number of different types of memory statements made by the 7 students for whom we currently have transcribed data. For these students the shift statistic correlates moderately well with the total number of semantic statements (semantic action + semantic belief + semantic label; $r^2 = 0.77$, $p < 0.0001$). Whether this correlation holds more generally we do not yet know.

Memory type	Jennifer	Jim	Amy	Shannon	Allison	Michelle	Rebecca
Episodic	80	14	26	17	49	25	36
Semantic action	90	78	127	102	85	83	46
Semantic belief	55	15	2	6	24	25	22
Semantic label	31	23	23	12	3	6	4
Explanative	0	0	5	0	5	0	4
TOTAL	256	130	183	137	166	139	112

Table 2: The number of statements according to memory type in 7 student's written work. Note that the classification of memory types has not yet been subject to marker reliability.

For this group of students the number of semantic labels correlates almost as well with the shift statistic: $r^2 = 0.71$, $p < 0.02$. Recall that semantic labels are memories of conventional facts: their mathematical depth is negligible. Some examples given by Jennifer are listed below. Bear in mind that these statements may also contain connotations of other types of memory (episodic, for example).

- "All problems assigned present two choices or a binomial." Jennifer illustrates here that she knows the meaning of he conventional term '*binomial*'.
- "...place value as we know it today is also known as the Hindu-Arabic numeration system." She shows that she knows another conventional name for the place value system.
- A number is considered a factor of another when it can divide that number without a remainder. This shows that Jennifer has a meaning for the term '*factor*s.
- We used proof by exhaustion when working with a finite set of numbers; listing all of the possible cases. Here she is able to explain '*proof by exhaustion*' in other terms.

Examples such as these are significant: they show that Jennifer is coming to terms with the language of mathematics, that she is able to interpret and use conventional mathematical terms. They show, we believe, that she has accepted her entry into the mathematical community and now feels part of it; perhaps a small part, but a part nonetheless. Compare this with part of her final written reflection, at the end of the course:

"One issue I have always had problems with in mathematics is definitions. I can physically work through a math problem, but to try to put my efforts into words is a challenge. *Definitions in mathematics play a vital role in building a solid base of one's knowledge and abilities*. It is the basis of your criteria. The mistakes our class made in defining even numbers were (a) we assumed that we were working in base ten and (b) we tried to define even numbers by using the word "even."! If definitions are the base of our mathematical foundation, then algorithms are the brick in the bridge of our mathematical path...an algorithm is a systematic procedure that one follows to find the answer to a computation."(Our italics).

Conclusion

Jennifer made a significant change in her understanding of mathematics. She began, as many pre-service elementary teachers do, expecting to apply formulas and get correct answers in order to be "mathematical". By the end of the 16 week course Jennifer expressed a different view of mathematics: one that she herself characterized as more relational. She established manifold long-term memories of mathematics: factual, episodic, and relating to the conventional use of mathematical language. Her tests score improved from not satisfactory to excellent.

How important were the experiences of the first 3 weeks in setting Jennifer on a path to seeing and valuing connections, and establishing lasting useful mathematical memories? In her words:

> "I feel this was the most productive experience I have ever had in my educational career. I deeply feel that I will be a better educator because of it."

The beautiful phrase: s"If definitions are the base of our mathematical foundation, then algorithms are the brick in the bridge of our mathematical path, " is a sharp illustration of how well she assimilated the mathematical experiences of the semester, and how these assisted in deepening her understanding of and competence in mathematics.

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SECONDARY SCHOOL PUPILS' IMPROPER PROPORTIONAL REASONING: AN IN-DEPTH STUDY OF THE NATURE AND PERSISTENCE OF PUPILS' ERRORS

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Abstract. *This paper describes an in-depth investigation (through individual interviews) of the problem-solving processes of 12–16-year old pupils who improperly apply linear models when solving problems involving lengths, areas and volumes of similar plane figures and solids. The results showed that the linear model was used in a spontaneous and self-confident way by almost all pupils, and that these pupils were almost insensible to the confrontation with conflicting data. Furthermore, it was shown that the poor results were due to a bad understanding of the principles governing the enlargement (or reduction) of geometrical figures, to pupils' inadaptive beliefs about and attitudes towards mathematical problem-solving, and to their poor use of heuristics and of metacognitive strategies.*

1. Theoretical and empirical background

Pupils' tendency to apply proportional or linear reasoning in non-proportional problem situations was already exemplarily described in several mathematical domains, such as elementary arithmetic, algebra, probability and geometry. Best-known in the domain of geometry and measurement is pupils' improper application of linearity in problems about the relationships between the lengths and the area and/or volume of similarly enlarged or reduced figures. In the NCTM Standards, for instance, it is stated that "... most students in grades 5–8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled." (NCTM, 1989, pp. 114–115). Gaining insight in the quadratic, respectively, cubic growth rates of areas and volumes, appears to be a slow and difficult process, and, therefore, it deserves our close attention, both from a phenomenological and a didactical point of view. According to Freudenthal (1983, p. 401), "this principle deserves (...) priority above algorithmic computations and applications of formulae because it deepens the insight and the rich context in the naive, scientific and social reality where it operates."

Recently, several studies have shown that – in the context of enlargements or reductions of plane figures and solids – the "illusion of proportionality" (or linearity) is a widespread and almost irresistible tendency among pupils (see, e.g., De Bock, Verschaffel & Janssens, 1998; De Bock, Verschaffel, Janssens & Claes, 2000). In these studies, large groups of 12–16-year old pupils were administered (under different experimental conditions) a written test consisting of proportional and non-proportional word problems about lengths, areas and/or volumes of different types of regular and irregular figures. The majority of the pupils in these studies failed on the non-proportional problems because of their alarmingly strong tendency to apply proportional reasoning "everywhere". Even with considerable support (such as the provision of drawings, of metacognitive stimuli in the form of an introductory item accompanied with both a correct and an incorrect solution, or embedding the problems in an authentic problem context), only very few pupils appeared to make the shift to the correct non-proportional reasoning.

Despite our rather extensive knowledge about the phenomenon of the "illusion of linearity" in this domain, the research method used so far, namely administering a collective test of large groups of pupils under different experimental conditions, did not yield adequate information on the *problem-solving processes* underlying improper proportional responses. This is one of the main reasons that the data could not provide a satisfying answer to the question *why and how* so many pupils fell into the "proportionality trap". Therefore, we made a shift in our methodology by having exploratory in-depth interviews with individual pupils who fall into the "proportionality trap".

2. Method

To obtain in-depth information about pupils' problem-solving processes, semi-standardized individual interviews were performed with eighteen 12–13-year olds and twenty-three 15–16-year olds. During **Phase 1** of the interview, each pupil had to solve one non-proportional word problem from a set of problems involving irregular plane figures or solids. Previous research had shown that the vast majority of pupils from these age groups solves these problems in a proportional way. Below, we give two examples of such problems.

Problem with irregular plane figure	**Problem with irregular solid**
A publicity painter needs 5 ml paint to make a drawing of a 40 cm high Santa Claus on a store window. How much paint does he need to make a drawing of a Santa Claus with the same shape, but a height of 120 cm?	In a perfume store, bottles of "Eau Fraîche" are sold. The bottles have a height of 8 cm and contain 10 cl perfume. In the store window, a publicity bottle is shown with the same shape, but enlarged, and also filled with "Eau Fraîche". This bottle has a height of 24 cm. How much perfume will this large bottle contain?

By asking each pupil to "think aloud" (Ginsburg, Kossan, Schwartz & Swanson, 1982) while solving the problem, we could retrieve (parts of) the solution process. The interviewer also asked well-specified questions to make (parts of) the reasoning process more transparent. The pupil was asked how (s)he exactly calculated the answer, why (s)he thought his or her answer was correct, and how sure (s)he was about the correctness (using a five-point scale from "certainly wrong" to "certainly correct").

In **Phase 2**, we tried to raise a first, weak form of cognitive conflict by confronting the proportional reasoning pupils with a fictitious frequency table of the answers given by a group of peers. There were two major answer categories in this fictitious frequency table: 41% of the peers gave an incorrect, linear answer (i.e., respectively 15 ml and 30 cl in the examples given above), but another 41% gave the correct, non-linear answer (i.e., respectively, 45 ml and 270 cl). Then, the pupil was asked to re-evaluate his or her own initial answer.

If the pupil dit not change his or her answer, a stronger conflict was elicited in **Phase 3** by giving the argumentation of a fictitious peer from the 41% who answered the problem correctly (e.g. in the first example: "one pupil told me that if the Santa Claus becomes three times as high but keeps the same shape, not only his height is multiplied by 3, also the *width* has to be multiplied by 3, so that you have to multiply by 9"). Again, the pupil was asked to re-evaluate his or her own answer. Pupils who had exchanged their original linear answer for the correct non-linear one in Phase 2 or 3, were interrogated at the end of the interview about the origin of their initial wrong answer.

3. Results

The table below presents the (cumulative) number of pupils who chose the correct answer in each phase. The tendency to give a linear answer was strongly present in both age groups. All pupils spontaneously gave the wrong linear answer. Even in Phase 2, *only one* pupil realised that his original answer was wrong. In Phase 3, finally, only another nine pupils changed their answer to the correct one, so that in sum only ten pupils (24%) accepted the non-linear answer as the correct one. Most of these pupils belonged to the older age group. We now will look at each phase in more detail.

Age group	*N*	Phase 1	Phase 2	Phase 3
12–13-year	18	0	0	3
15–16-year	23	0	1	7

Phase 1: Solving the word problem

The mean response time after the first confrontation with the word problem was one minute. During this minute, none of the pupils made a drawing or any other kind of external representation involving more than the three given numbers. They mainly read and re-read the problem, wrote down the numerical data, and performed calculations on these numbers to obtain the answer. Hereby, thirty-seven pupils calculated the ratio of the given lengths of the two figures, and thought that the ratios of the *areas* or the *volumes* (or their indirect parameters: paint or content) should be the same. Four pupils applied formally the so-called "rule of three": first calculating "the amount of paint (or the content of the bottle) needed for 1 cm", and then multiplying the result with the height of the large figure. The use of one of these strategies led all pupils to an improper linear answer, as expected on the basis of earlier studies by De Bock et al. (1998, 2000).

Most pupils had great difficulties explaining why their method was the correct one. After insisting, most pupils (a) referred to the fact that their solution is the most logical one, (b) explained that they solved the problem as they had learned to do at school or (c) simply repeated how they carried out their computations. These superficial answers seem to indicate that pupils do not spontaneously check whether a model is applicable in a given situation or not. Pupils do not seem to have clear arguments justifying its use, or do not realise there are other possible models too. It seems that the linear model is used in an implicit, routine and mindless way.

Despite this difficulty in justifying the correctness of their answers, most pupils were very or quite sure they *were* correct. Twenty-one pupils said their answer was "certainly correct", fifteen said it was "probably correct" and only five "had no idea". The alternatives "probably wrong" and "certainly wrong" were never chosen. If pupils had reasons to doubt about their solutions, these reasons were mainly superficial and general (e.g., "word problems are difficult", "I am not good in math", "I might have made a calculation mistake"). None of the argumentations expressed any doubt about the correctness of the applied model.

Phase 2: Reactions to a weak form of cognitive conflict

For the majority of the pupils, the confrontation with the frequency table really induced a cognitive conflict: they started wondering where the alternative frequently-chosen answer could come from. However, the search for its origin was done – again – in a very superficial way. The single pupil who made a drawing of the small and large figure in this phase, chose the correct answer, but all other pupils limited themselves to "randomly" trying out several combinations with the basic arithmetical operations (+, –, ×, :) and the given numbers to obtain the other answer, regardless of their contextual meaning (e.g., some pupils added 40 cm and 5 cl to obtain the alternative answer 45). Only in very rare cases, we observed pupils re-evaluating their own strategy, searching for the meaning of the alternative answer or representing the problem. The superficial strategies used by most pupils were not very helpful. All pupils (except for one of the oldest group) persisted in their original linear solution.

A further deduction from the fact that the pupils did not immediately change their incorrect answer is that the mistake was not simply caused by an underestimation of the difficulty level of the word problem. In the latter case, the correct solution (strategy) would have been a sufficient scaffold to choose the correct answer.

Phase 3: Reactions to a strong form of cognitive conflict

In the third phase, another nine pupils (three of the youngest and six of the oldest age group) changed their incorrect answer into the correct one. Apparently, the given argumentation provided them the insight that, to maintain the same shape a figure has to be enlarged in all dimensions. The nine pupils who changed their answer were asked to explain why they originally gave the wrong linear solution. Their explanations referred to the fact that (1) they did not solve the problem in a reflective manner but immediately (a pupil called it "instinctively") started calculating or (2) they had made no real mental representation of the problem, but just were fixating on the formulation of the word problem (which only referred to the height).

The thirty-one pupils who still chose to withhold their original answer after the argumentation of the fictitious peer, made serious efforts to justify their choice. Their reactions were diverse, but can be grouped into three different categories (each covering about one third of the reactions).

A first group of pupils justified their answer by referring to the implicit rules for solving school mathematics word problems. Often a simplistic view was shown, assuming that all word problems can be solved using simple mathematical calculations, and that real-world knowledge should not be involved in the solution process. Some examples are: "I think you don't have to use such a complex solution to solve a word problem", "you have to calculate only with the data that are given", "if they wanted you to calculate the width too, they should have explicated that in the problem statement".

A second group of pupils violated the mathematical principles relevant to this problem. The first principle that is ignored or not understood is that *if a figure is enlarged (or reduced) but maintains its shape, all dimensions (height, width and depth) are enlarged (or reduced) by the same factor*. Some pupils reacted that "if you only know the height, you can't know the width", "height and with are not that much related to each other", or "the width and depth will change too, but you cannot know how much", and used these arguments to simply *ignore* that the width (and depth) also change and determine the solution. The second principle pupils seemed to struggle with is that *if the linear measurements of a figure are enlarged (or reduced) by a factor k, its area is enlarged (or reduced) by k^2 (and its volume by k^3)*. Typical examples are: "the width is already incorporated in the small one, so it isn't necessary to calculate it in for the big one again", "the Santa Claus is not a spatial object where you have to calculate the volume, it is flat; consequently, only the height plays a role", "I think 270 cl actually is quite a lot".

After being confronted with the argumentation for the correct solution, a third group of pupils tried to give an alternative interpretation to the word problem. As already said earlier, most pupils did not construct any mental representation about the problem before the third stage of the interview. Once they were confronted with an argumentation rejecting their answer, many pupils looked for an alternative interpretation in which their incorrect solution still would make sense: "the figure is stretched, only the height changes and the rest remains the same", "if you make it higher, that doesn't mean it becomes wider", "it says *with the same shape*, so it only is a higher one, not wider or deeper". We cannot absolutely exclude that some pupils may have held this alternative interpretation of the problem situation already earlier during the interview, but our data indicate that they form only a small minority. Moreover, when the interviewer confronted pupils with the concrete consequences of their alternative interpretation (by means of a drawing or a description) most pupils admitted the strangeness of it (e.g. a very high but narrow Santa Claus, a copy of a perfume bottle that isn't really a copy with the same design). Analogous defensive reactions of students, who try to withhold an original erroneous answer, even if they realise this answer is untenable, have also been observed in other studies. E.g. Verschaffel, De Corte and Vierstraete report that pupils "tirelessly came up with contextual considerations in which their unrealistic response would still hold. (...) These far-fetched context-based considerations were (...) only made during the whole class discussions by pupils who became aware their group had answered the problem in an uncritical, stereotyped manner" (Verschaffel, De Corte & Vierstraete, 1997, p. 595).

4. Conclusions and discussion

The interviews provided a lot of information about the actual process of problem-solving from pupils falling into the "proportionality trap" and the mechanisms behind it.

First of all, several possible causes were *rejected* by the research data. From the collective tests (De Bock et al., 1998, 2000), it was impossible to find out (1) whether pupils gave the wrong linear answer reluctantly by lack of a better alternative or (2) whether pupils gave the wrong linear answer because of the expectation that the test would contain routine tasks only. We think both possible explanations can be refuted, the first one because most pupils declared to be sure about their initial incorrect answer, the second one since the confrontation with the correct solution (even with an accompanying explanation) was not sufficient to make them change their answer.

Second, there is a parallel between the problem-solving processes observed in the first phase of the interview and the "intuitive rules" theory developed by Tirosh and Stavy (1999). These authors claim that there are some common, intuitive rules that come in action when students solve problems in mathematics and sciences. These rules appear to be self-evident (i.e. true without a need for further justification), receive great confidence, and are persistent despite formal learning. All these characteristics seem to apply to the incorrect reasonings of the interviewed pupils too. More specifically, Tirosh and Stavy have distinguished two schemes that (whether correct or not) frequently come in action in an intuitive way: "Same A – same B" (while in fact $A_1 = A_2$, but

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sometimes $B_1 \neq B_2$) and "More A – more B" (while in fact $A_1 < A_2$, but sometimes $B_1 \geq B_2$). In our case, pupils seem to apply first the "More A – more B" rule (which is correct for this problem: the *more* height, the *more* area/volume). The mistake happens, however, during the intuitive quantification when applying the "Same A – same B" rule: pupils reason that the figures share the *same* shape, so all measures (length, area and/or volume) enlarge by the *same* factor. This is illustrated in the following quotations: "I knew it was enlarged, but not how much, so I calculated 180 : 60 and then I knew the multiplier", "because the picture becomes larger, you need more paint, so you have to multiply by 3", "it has the same shape, but is enlarged, so you have to multiply the content by the same number". The specific connection between the "illusion of linearity" and the "intuitive rules" theory certainly needs further investigation.

Third, we found that many pupils (as well the younger as the older ones) struggled with the principles behind the enlargement of figures/objects and the relationship between length and area/volume. They had already learned these principles in school, but nevertheless they seemed to have a bad or weak understanding of them, or at least they were not able to apply them correctly. Further research should determine whether the struggling really is a *cause* of the mistakes or rather that pupils *post hoc* violate the principles in a self-defensive attempt to save their original answer. The same goes for the alternative interpretations some pupils gave to the word problem after they heard the interpretation for the correct answer. Some pupils may have had it in advance, but most of them had made no real representation of the problem until the third phase.

We want to argue that the described findings are also related to the fact (supported by a vast amount of research, see, e.g., Verschaffel, Greer & De Corte, 2000; Wyndhamn & Säljö, 1997) that many pupils have inadaptive beliefs and attitudes towards mathematical problem-solving, and have a poor use of heuristics and metacognitive strategies. The intuitive reasoning in the first phase and the small impact of the conflict in the second and third phase only could occur because the pupils approached the word problem in a superficial way, only looking at the numbers without making a clear and realistic problem representation, assuming that all application problems can be solved with some simple mathematical operations on the given numbers, and without any control of the correctness of their answer afterwards. Further research will have to evidence if stimulating realistic modelling in pupils has a beneficial impact on overcoming the "illusion of linearity"[1].

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[1] We found some evidence for this supposition since we asked several pupils – after the interviews were finished – to make a drawing for the word problem. At that moment, nearly all of them then really "discovered" that a figure with the same shape must enlarge in all dimensions, so that the area/volume enlarge by a larger factor.

A final remark concerns the educational value of the cognitive conflict to enhance pupils' metacognitive awareness or to provoke cognitive change (see, e.g., Forman & Cazden, 1985). Our experiences show that it is very difficult to induce an effective cognitive conflict in pupils if they have no minimum metacognitive awareness about their problem solving process.

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DO INTUITIVE RULES HAVE PREDICTIVE POWER?
A REPLICATION AND ELABORATION STUDY ON THE IMPACT OF 'MORE *A*–MORE *B*' AND 'SAME *A*–SAME *B*'

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Abstract. *In recent years, the intuitive rules theory has received growing attention in the mathematics and science education research community because of its capacity to explain and predict various kinds of responses of students to a wide variety of tasks from scientifically different content domains. Two major intuitive rules show up in comparison tasks: 'More A–more B' and 'Same A–same B'. Although these two rules have been extensively studied by Israeli researchers, the relation between these rules and their gradual impact on students from countries with different traditions in mathematics and science education require further research. In this paper, we describe two consecutive studies aimed at re-examining the predictive power of 'More A–more B' and 'Same A–same B' in situations wherein both rules lead to different erroneous judgements.*

Theoretical and empirical background

Building on the pioneering work of Fischbein (1987) about the role of intuitive thinking in mathematics and science, Israeli researchers recently established the intuitive rules theory (Stavy & Tirosh, 2000; Tirosh & Stavy, 1999a; 1999b). According to this theory, students' responses to various conceptually unrelated tasks in mathematics and science can be explained and predicted by several common, intuitive rules that are activated by some salient, external features of a task. Two such rules show up in comparison tasks. The first is called 'More *A*–more *B*': students comparing two objects differing with respect to a certain salient quantity A ($A_1 > A_2$), intuitively argue with respect to another quantity B that $B_1 > B_2$. In many everyday life and scientific situations, such an intuitive reaction leads to conclusions that are accurate (e.g. squares with a larger perimeter have a larger area), but students of different ages strongly tend to use this rule also beyond its range of applicability (e.g. plane figures with a larger perimeter always have a larger area). The second rule is called 'Same *A*–same *B*': students comparing two objects being equal with respect to a certain salient quantity A ($A_1 = A_2$), intuitively argue with respect to another quantity B that $B_1 = B_2$. Clearly, this intuitive rule can also lead to both correct (e.g. squares with the same perimeter have the same area) and incorrect responses (e.g. plane figures with the same perimeter always have the same area). Stavy and Tirosh provided a variety of examples of students' erroneous and correct responses that are in line with these intuitive rules. Furthermore, they collected an impressive amount of research data demonstrating how strongly students of different age groups and study streams are influenced by these basic rules.

Although the intuitive rules theory is widely known in the international mathematics education community nowadays, replications of the results obtained by Stavy and Tirosh in other countries with different cultural and educational traditions, are quite rare. Furthermore, we need more convincing evidence that the reasoning *process* of students who respond 'in line with' a given intuitive rule, is actually affected by that rule. Especially for older students, who can generate a particular answer by simply applying an intuitive rule, but also by relying on a more advanced mathematical notion or strategy learned at school, this point needs to be clarified. Finally, connections between these intuitive rules and other misconceptions and systematic errors known from the mathematics education research literature (see, e.g., Davis, 1989; De Bock, Verschaffel & Janssens, 1998), has to be examined more thoroughly.

Study 1

We set up a first study to arrive at a better understanding of the nature and predictive power of the intuitive rules 'More *A*–more *B*' and 'Same *A*–same *B*'. More specifically, we aimed at providing answers to the following research questions: (1) do Flemish students react in a similar way as Israeli students to comparison tasks eliciting the intuitive rules, (2) do students' explanations contain explicit references to 'More *A*–more *B*' or 'Same *A*–same *B*', and (3) do their answers and/or explanations yield evidence for other erroneous strategies or misconceptions known form the literature. Because both 'More *A*–more *B*' and 'Same *A*–same *B*' were targeted, we looked for tasks wherein both rules lead to a different but inaccurate answer.

Seventy students of grade 10 (15–16-year olds) of a Flemish secondary school participated in Study 1. All participants were divided in three equivalent subgroups and each subgroup was confronted with two problems. All problems were variants of a geometrical problem about the ratio between the surface area and volume of differently-sized cubes, – a task introduced by Livne (1996), and mentioned in several publications of Tirsoh and Stavy (see, e.g., Stavy & Tirosh, 2000; Tirosh & Stavy, 1999a) (see Figure 1). The first group (Group I) received Livne's original problem about cubes and its two-dimensional variant about the ratio perimeter/area of differently-sized squares. The second group (Group II) was confronted with conceptually the same two problems, but formulated with spheres and circles. Also in the third group (Group III), students received two mathematically equivalent problems, but formulated with irregular two- and three-dimensional figures (maps and bottles, respectively). The problems were

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formulated in a multiple-choice format with three alternatives. One alternative proposed an incorrect answer in line with the rule 'More *A*–more *B*' (the *larger* figure has the *larger* ratio). A second (also incorrect) alternative was in line with 'Same *A*–same *B*' ('*same* shape, *same* ratio'). The third alternative, which did not involve one of these two intuitive rules, was the correct one. Students were not only instructed to select one of the three alternatives (larger than, equal to, or smaller than), but also to explain their choice. As in Livne's study, drawings of the geometrical figures were given.

Cube problem. Consider two differently-sized cubes. Is the ratio between the surface area and volume of Cube 1 smaller than/equal to/or larger than/the ratio between the surface area and volume of Cube 2? Explain your answer.

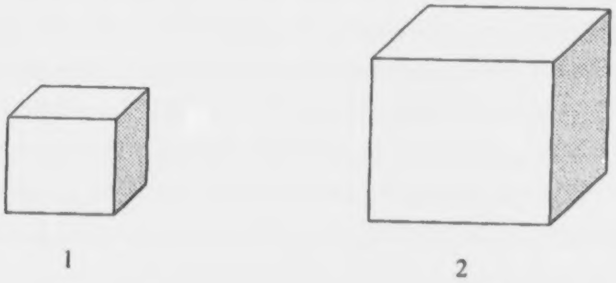

Figure 1. Task by Livne (1996)

Livne (1996), who confronted biology majors in grade 10, 11, and 12 (15–18-year old students) with this problem, reported the following results: (1) Substantial percentages of students in grades 10, 11, and 12 (41%, 45%, and 55% respectively) incorrectly argued that the ratio surface area/volume is the same in both cubes. Typical explanations were: 'Cube 1 and Cube 2 have the *same geometrical shape*, hence *the ratio* surface area to volume is *the same* regardless of their size'; 'The surface area and the volume in Cube 1 are proportionally smaller than in Cube 2 and therefore the ratio is constant'. (2) Only 24%, 19%, and 24% of the students in grades 10, 11, and 12 respectively claimed that the ratio surface area/volume is larger in Cube 2 than in Cube 1.

Table I gives an overview of the results of our study.

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Groups		I $N = 22$	II $N = 24$	III $N = 24$	Total $N = 70$
Plane figures	'Same *A*–same *B*'	36	41	50	43
	'More *A*–more *B*'	27	37	4	23
Solids	'Same *A*–same *B*'	36	17	38	30
	'More *A*–more *B*'	23	71	10	36

Table 1. Distribution (in %) of the answers of the three groups of students in Study 1

These quantitative data do not reveal a clear trend: the student answers seem to be more or less uniformly distributed. For plane figures, most students tended to select the alternative in line with 'Same *A*–same *B*', but this tendency was not confirmed for the solids. With respect to the geometrical figure involved, most correct answers were given for the items about squares and cubes (Groups I) and for the items about irregular figures (Group III) (39% and 49%, respectively); least correct answers were given for the items about circles and spheres (Group II) (17%). Remarkably, 43% of the students selected a different answer for task 1 than for task 2. Probably, as a result of the 'experimental contract' (Greer, 1997), many students expected different answers for two tasks given in the same test.

A qualitative analysis revealed that 24% of the explanations contained an explicit reference to proportional reasoning. Although 'k times A, k times B' (with $k > 1$) can be seen as a straightforward quantification of 'More *A*–more *B*', students' response sheets contained no empirical evidence that students' misuse of proportionality was the direct result of a prior 'More *A*–more *B*' judgement. Above all, this finding confirms students' tendency to apply proportional reasoning 'everywhere' (see, e.g., De Bock, Verschaffel & Janssens, 1998), – a misconception that seems to be affected by students' instructional histories rather than by their general intuitions. Only 3% of the students referred explicitly to shape or shape similarity and not a single student reportedly reasoned that 'the larger figure must have the larger surface/volume ratio'. Apparently, Flemish students do not *readily* verbalise the 'Same *A*–same *B*' or 'More *A*–more *B*' rule. If these rules affected students' reasoning, it seems they were not aware of it.

In conclusion, the strong 'Same *A*–same *B*'-tendency reported by Livne (1996), was not confirmed for the different geometrical shapes included in our study. With respect to the tasks involving solids, the global percentage of responses in line with 'Same *A*–same *B*' was even below chance level! This finding seems to question the predictive power of 'Same *A*–same *B*' for that kind of geometrical problems. But also students' explanations, especially those explicitly referring to shape or to shape-similarity, qualified as being 'typical' in the Livne-study, proved to be quite rare in our first study.

Study 2

Because (1) the quantitative data of Study 1, especially the 'total scores' did not show a convincing trend and (2) very few of students' explanations explicitly referred to the intuitive rules that, according to the theory of Stavy and Tirosh, would have affected their (erroneous) answers, we decided to set up a follow-up study on a larger scale.

In this study, fifty-eight, fifty-eight and fifty-six students of, respectively, grades 10, 11, and 12 were confronted with a written multiple-choice test consisting of five problems from different mathematical subdomains. As in Study 1, each problem was accompanied by three alternative responses. One alternative proposed an incorrect answer in line with the rule 'Same *A*–same *B*'. A second alternative (not necessarily in this order) was also incorrect, but was in line with 'More *A*–more *B*'. The third alternative, which did not involve one of these two intuitive rules, was the correct one. Students were instructed to explain their choice on their answer sheet. They were given 50 minutes to complete the test and were not allowed to use calculators.

Two of the five test items functioned as 'anchor items', allowing a straightforward comparison of our results with those obtained in Israel: the Cube problem that was already used in Study 1 (cf. Figure 1) and the 'Carmel problem' (see, e.g., Tirosh & Stavy, 1999b; Stavy & Tirosh, 2000) given in Figure 2. We completed these two items with three self-made tasks, which are also listed in Figure 2, together with an indication how the distinct alternatives are linked to the intuitive rules.

Carmel problem. The Carmel family has two children, and the Levin family has four children. Is the probability that the Carmels have one son and one daughter larger than/equal to/smaller than/ the probability that the Levins have two sons and two daughters?

> 'Equal to' is in line with the rule 'Same *A*–same *B*' ('*same* ratios ($\frac{2}{4} = \frac{1}{2} = \frac{1}{2}$), *same* probability').
>
> 'Smaller than' is in line with 'More *A*–more *B*': the larger family has a larger probability.

Cake problem (inspired by Lin, 1991). On Wednesday afternoon, Els and her father are baking together a cake. Both cakes have the same height. The diameter of Els' cake is 15 cm, the diameter of the cake of her father is 30 cm. Els adds 150 gram sugar to her batter, her father adds 300 gram sugar to his batter. Is the cake of her father less sweet than/as sweet as/or sweeter than/ the cake of Els?
(This problem was accompanied by drawings of the two cakes, having a cylinder form.)

> 'As sweet as' is in line with the rule 'Same *A*–same *B*' ('*same* ratio ($\frac{15}{150} = \frac{30}{300}$), *same* sweetness').
>
> 'Sweeter than' is in line with 'More *A*–more *B*': the cake with more sugar is sweeter.

Root problem. Is $\sqrt{10}$ larger than/equal to/or smaller than $\sqrt[3]{15}$?

'Equal to' is in line with 'Same A–same B' ('*same* ratio ($\frac{2}{10} = \frac{3}{15}$), *same* expression').

'Smaller than' is in line with 'More A–more B': $15 > 10$ and/or $3 > 2$.

Physics problem. The gravitational pull on a mass m at a distance s metres form the centre of the earth is given by a formula of the form: $F = c\,\frac{m}{s^2}$ (in which c is a constant).

Suppose a mass A is twice as heavy as a mass B and the distance of A to the centre of the earth is twice the distance of B to the centre of the earth. Is the gravitational pull on mass B smaller than/larger than/or equal to the gravitational pull on mass A?

'Equal to' is in line with 'Same A–same B' ('*same* ratio ($\frac{m_A}{s_A} = \frac{m_B}{s_B}$), *same* gravitational pull').

'Smaller than' is in line with 'More A–more B': $m_A > m_B$ and/or $s_A > s_B$.

Figure 2. The four new tasks from Study 2 (the fifth problem was given in Figure 1)

Table II gives an overview of the results.

Task		Grade			
		10 $N = 58$	11 $N = 58$	12 $N = 56$	Total $N = 172$
Cube problem	'Same A–same B'	40	43	39	41
	'More A–more B'	16	20	16	17
Carmel problem	'Same A–same B'	33	24	36	31
	'More A–more B'	3	9	5	6
Cake problem	'Same A–same B'	48	41	43	44
	'More A–more B'	19	10	9	13
Root problem	'Same A–same B'	5	9	7	7
	'More A–more B'	24	17	14	19
Physics problem	'Same A–same B'	48	31	23	34
	'More A–more B'	5	19	13	12

Table II. Distribution (in %) of the answers by grade and by problem in Study 2

For five problems and the three grades together, 55% of the students selected the correct alternative. Responses in line with 'Same A–same B' and 'More A–more B' were below chance level (31% and 13%, respectively). For the five tasks together, the percentages of correct responses slightly increased with age: 52%, 55%, and 59% correct responses in grades 10, 11, and 12). The Root, the Carmel, and the Physics problem proved to be the easiest tasks (for the three grades together, respectively, 74%, 63%, and 54% correct responses). For these problems, the percentages of responses in line with each of the intuitive rules were not above chance level for all three grades. For the Cake and the Cube problem, the percentages of correct responses and those in line with 'Same A–

same *B*' were more or less the same (between 41% and 44%), whereas the number of responses in line with 'More *A*–more *B*' were far below chance level.

Although it is problematic to compare test results form different countries on the basis of a rather small and not carefully matched sample of students, it seems that students in Flanders are less affected by the 'Same *A*–same *B*' intuitive rule than their Israeli peers. For the Cube problem, 41%, 45%, and 55% of Israeli students in, respectively, grade 10, 11, and 12 responded in line with this rule; in Flanders we found (slightly) lower percentages in all grades, respectively, 40%, 43%, and 39%. For the Carmel problem, the results in both countries diverge completely. In Israel, where about forty students in each grade were confronted with this task, an increasing percentage of students (57%, 50%, and 62% in grade 10, 11, and 12, respectively) responded in line with 'Same *A*–same *B*', incorrectly arguing 'the ratio is the same, therefore the probability is the same' (see, e.g., Tirosh & Stavy, 1999b); in Flanders, such an increasing trend could not be established and the corresponding percentages were much lower, respectively, 33%, 24%, and 36%. Correspondingly, the percentages of correct responses in Flanders (64%, 67%, and 59% for grades 10, 11, and 12) were significantly higher than in Israel (24%, 42%, and 30%). This last finding does, however, not necessarily mean that the Flemish students displayed a higher quality of probabilistic reasoning. A closer look at the response sheets of the Flemish students revealed that the majority selected the correct answer on the basis of another *erroneous* reasoning process! Some students misinterpreted the statements 'one son, one daughter' and 'two sons, two daughters' as, respectively, 'first a son, then a daughter' and 'first two sons, then two daughters', which leads to, respectively, the probabilities of 1/4 and 1/16. However, the majority of the participants that selected the correct response proved to suffer highly from the so-called *equiprobability bias* (see, e.g., Lecoutre, 1992), according to which all events were thought as equally likely. So, in the Carmel family, there are three possible events ('two sons', 'one son, one daughter', and 'two daughters'), each having a probability of 1/3; in the Levin family, there are five possibilities ('four sons', 'three sons, one daughter', 'two sons, two daughters', 'one son, three daughters', and 'four daughters'), each having a probability of 1/5.

Discussion

The results of two Flemish studies on the intuitive rules 'Same *A*–same *B*' and 'More *A*–more *B*' did not provide a strong confirmation of the predictive power of these intuitive rules. In both studies, percentages of responses in line with these rules never exceeded 50% and were mostly below chance level. In the explanations accompanying their answers, students seldom referred explicitly to these rules.

In our view, two points need further investigation. First, a more detailed and systematic picture of problem types and characteristics that elicit the intuitive rules is required. In this picture, there should be room for intercultural differences. Indeed, it seems that the intuitive rules and how they affect students' mathematical problem solving compete with alternative conceptions and strategies – both correct and incorrect ones – that are shaped by culturally bound instructional traditions. Second, there is a need of more fine-grained, process-oriented research that tries to unravel in which stage of the problem-solving process intuitive rules show up, and to detect under what internal and external conditions this intuitive thinking is replaced by more advanced mathematical reasoning. A possible research approach could be to administer individual interviews in which a student has to respond instantly to a comparison situation, and is then encouraged to reconsider his initial judgement.

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INTERACTIVE LEARNING AND MATHEMATICAL LEVEL RAISING: A MULTIPLE ANALYSIS OF LEARNING EVENTS

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Abstract

From different perspectives we have analysed an episode of two children working together on a mathematical task. Integration of our analyses brings to the fore authentic dilemmas and paradoxes, which are also experienced by students and teachers during collaborative work. We will present some examples.

Introduction

In the past, analyses of student learning have considered learning from cognitive, social or motivational aspects in isolation. Now in contrast to this earlier view, an interest exists in developing a "multidimensional framework for understanding mathematical learning" (Pintrich, 2000, p. 221). This is an approach that we have taken in collaborating in analyses of an episode of two children working together on a mathematical task.[i] We will first describe the different perspectives we use in research and then explain our recent work--a multiple analysis of learning events (Dekker, Elshout-Mohr, & Wood, in press). Although we differ in our perspectives, a common goal for research is to develop ways to describe events in students learning as they are occurring in classroom situations. In this case learning is defined as conceptual understanding and mathematical level raising.

THEORETICAL ORIENTATIONS

A Process Model for Mathematical Level Raising

Dekker and Elshout-Mohr (1996, 1998) developed a process model for interaction and mathematical level raising based on empirical analysis of students learning working in small groups while solving mathematical problems (Dekker, 1991). The problems are specifically developed to stimulate mathematical level raising among students (cf., Dekker & Elshout-Mohr, 1999). In the process model three types of activities are incorporated; they are *key activities* in the learning process; *regulating activities*; and *mental activities*. Each of these is further discussed below.

Key Activities

Dekker and Elshout-Mohr (1996, 1998) deem four key activities are primary conceptual level raising. Whenever these key activities occur, we speak of learning events. The key activities are:

- *to show one's work*
- *to explain one's work*
- *to justify one's work*
- *to reconstruct one's work*

The process model is presented in Table 1 with the key activities indicated in bold print. Dekker and Elshout-Mohr (1996, 1998) assume that the conjunction of these activities leads to the desired type of learning (i.e. conceptual level raising). The four key activities have the following characteristics:

a) They can be demonstrated by students who work individually, but more so by students who communicate with each other during their work;

b) They are easily observed;

c) They have a function in the learning process and contribute to mathematical understanding;

d) They can be influenced by didactic factors, such as the nature of problems and by teaching actions.

Regulating Activities

The process model further captures regulating activities, which elicit key activities, and mental activities, which concur with key activities. Table 1 shows the regulating activities identified in italics. Students asking for explanations, justifications, or transformations are described as regulating activities. For instance, when students are working together and show each other what they are doing and thinking, they may become aware that different participants have different knowledge about a central concept, and they may start thinking about these differences. They may also ask each other to show and explain their work, or to justify or transform it. In principle, a student who works alone can perform all the key activities, but it takes a great deal of self-regulation. However, by communicating with other students, the key activities will take place in a more natural way. Therefore, the process model also contains interactive and communicative activities that we call regulating activities.

Mental Activities

While key activities and regulating activities can be observed externally, mental activities occur which are not observable. Mental activities involve those activities which 'go along' with key activities. For instance, *to show one's work* includes the mental activity of becoming aware of one's own work. It has the effect that a focus on task-progress is temporarily replaced by taking a look at the work from the outside. *To explain one's work* means that one has to think about one's own work. It leads to elaboration of one's task-related conceptual knowledge. Attempts *to justify one's work* may include reinforcing prior knowledge or questioning it, and a prerequisite *to reconstruct one's work* is to criticise one's own work. Table 1 shows the process model of these mental activities in standard print.

Table 1

A Process Model for Interactive learning and Mathematical Level Raising

A and B are working on the same mathematical problem. Their work is different.

A is working		B is working
A asks B to show his work	*what are you doing?* *what have you got?*	*B asks A to show her work*
A becomes aware of her own work		B becomes aware of his own work
A shows her own work	**I am doing this...** **I have got this...**	**B shows his own work**
A becomes aware of B's work		B becomes aware of A's work
A asks B to explain his work	*why are you doing that?* *how did you get that?*	*B asks A to explain her work*
A thinks about her own work		B thinks about his own work
A explains her own work	**I'm doing this, because...** **I have got this, because...**	**B explains his own work**
A thinks about B's work		B thinks about A's work
A criticises B's work	*but that's wrong, because...*	*B criticises A's work*
A thinks about B's criticism		B thinks about A's criticism
A justifies her own work	**I thought it was right, because...**	**B justifies his own work**
A thinks about her justification		B thinks about his justification
A criticises her own work	oh no, it isn't right, because...	B criticises his own work
A reconstructs her own work	**I'll better do it like this...**	**B reconstructs his own work**

Note: **Bold: key activities**
Standard: mental activities
Italic: regulating activities

Social Interaction and Learning Mathematics

The theoretical perspective taken by Wood (1996, 1999) in analysis of student learning is to consider the very social nature of children's learning and the fact that rich social interactions with others substantially contribute to children's opportunities for learning. Therefore, Wood claims that there is a need to consider an analysis of educational settings that attends to the social cognitive processes involved in learning as well as the cognitive processes (Wood & Turner-Vorbeck, in press).

This perspective is influenced by the view of Bruner (1990) and others that children need to adapt to a social existence and to develop a system of shared meanings in order to participate as members of their culture. Sociologists interested in the human need to adapt to social existence and develop a system of shared meanings provide insights into the importance of social structure in the lives of humans. Both Garfinkel (1967) and Goffman (1959) contend that the social structures in everyday life consist of normative patterns of interaction and discourse. Once established, these patterns become the reliable routines found in interactive situations. Individuals, when they participate, come to anticipate certain behaviours for themselves and for others so that much of what happens "goes without saying" (Garfinkel, 1967).

Several researchers argue that the social structures that are created in the classroom influence the 'mathematics' a child learns. They claim that the everyday patterns of interaction and the norms that are constituted contribute to children's beliefs about the nature of mathematical knowledge and the ways in which one learns and uses mathematics in everyday life (e.g., Carraher, Carraher, & Schliemann, 1985; Cobb, Wood, Yackel, and McNeal, 1992). In order to understand the children's mathematical learning we need to examine the social situations that teachers establish with their students. The norms (expectations for self and others behaviour) underlie the social interaction that reveals the 'practice' of mathematics in the classroom.

Moreover, it is thought that learning that is conceptual can benefit considerably from dialogue and collaboration with others (Yackel, Cobb, & Wood, 1991). Although social norms are initiated and established by the classroom teacher, the children's ability and commitment to adhere to the shared expectations is equally important. Kieran and Dreyfus (1998) provide evidence that the social cognitive facility of negotiation of meaning influences individual learning.

Using a qualitative research paradigm, drawing on microenthnographic procedures developed by Voigt (1990), analysis of learning situations is conducted using a line-by-line examination of the dialogue and interaction. This provides detailed description of the events that occur. Through this process interpretation can be made of the meanings held by students during collaborative problem solving classroom situations.

DATA SOURCE AND ANALYSIS

The multiple analysis is best explained in contrast to the approach that we followed to construct the above-mentioned process model. In the process model, we incorporate three types of activities: key activities in the learning process, regulating activities, and mental activities. The key activities are given the central place in the process model, whereas other activities are merely presented in so far as they are directly connected to the key activities. Thus, employment of the process model produced a coherent description of learning events in terms of the elements of the process model. In addition, a focus on social interaction provides information on the social conditions for learning including the influence of the teacher. The aim of the analysis is to reveal how the social norms established in the class affect the collaborative work of students and how these expectations actually provide the space for learning.

Towards Multiple Analyses and Integration

In the multiple analysis approach, however, we did not give priority to the elements of the process model or social interaction. Instead, we began by performing three separate analyses on one protocol of a student collaborative session wherein two primary-aged (8 year old) students worked together to solve a mathematical task.[ii] The mathematical task was developed to encourage students' conceptual understanding of multiplication beyond their intuitive notions of multiplication as repeated addition.

The first analysis of the episode was guided by theory about the role of key activities in achieving mathematical level raising. The second was guided by theory about social cognitive processes and the role of social interaction in learning, and the third was guided by theory about the role of time on task in learning outcomes. Once completed, the results of the three analyses were integrated in ways that allowed each perspective to be represented.

In the integrative stage, which we are currently in the process of conducting, preliminary findings reveal the complexity of multiple effects on students' learning activities. For instance, an activity that is evaluated positively from a social perspective on learning mathematics does not necessarily contribute to the occurrence of key activities, nor is it necessarily evaluated positively from a time on task perspective on learning. In the presentation, examples will be given that show how the methodology of multiple analyses brings to the fore authentic dilemmas and paradoxes that are also experienced by students and teachers during collaborative work in classrooms. One example is the fact that the social norm of collaboration leads to a lowering of level of one of the students, which conflicts with the aim of level raising. We will discuss this example in detail.

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[i] It was in 1988 at ICME 6 in Budapest where we first met. Dekker (1988) gave a presentation about her classroom observations. Inspired by the ideas of Freudenthal (1978) about learning of mathematics in small heterogeneous groups and the Van Hiele's (1986) level theory, Dekker showed learning materials specifically developed for mathematical level raising and observations of small groups working with those materials. Wood was in the audience and expressed similar research interests. It was nine years later at the CIEAEM 49 in Setúbal where we met again, both giving plenary lectures on interactions in the mathematics classroom. Wood's focus was on the role of the teacher and the influence of the social norms on the learning opportunities for students (Wood, 1998). Together with Elshout-Mohr and Pijls, Dekker presented a process model for the analysis of interaction and mathematical level raising (Dekker, Elshout-Mohr, & Pijls, 1998). We again discussed our common interest albeit with different perspectives and decided to conduct a joint analysis of a classroom event.

[ii] The episode is from the project, Recreating Teaching Mathematics in the Elementary School, supported by the National Science Foundation under award RED 9254939. All opinions expressed are those of the authors.

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PUPIL PARTICIPATION IN 'INTERACTIVE WHOLE CLASS TEACHING'

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One particular result of recent policy and practice changes in mathematics teaching in English primary (elementary) schools, is that pupils are increasingly involved in 'interactive whole class teaching'. Longitudinal case study data are informing our examination of the different ways that pupils engage in such sessions and manage their participation. Drawing on observations of three children we develop the argument, that within whole class sessions, pupils are becoming skilled at presenting themselves in ways that enable them to appear to be engaged with the mathematics in the way that the teacher expects them to be, while in actual fact they are engaged in other ways and for reasons other than interest in the mathematics.

INTRODUCTION

The English National Numeracy Strategy (NNS), officially introduced from the September 1999 has begun to affect the teaching of primary mathematics profoundly. It is rare now to find primary mathematics lessons that do not follow the form and content set out in the NNS document 'Framework for Teaching Mathematics'

A central tenet of the NNS is the need for increased emphasis on 'interactive whole class teaching'. All mathematics lessons are expected to begin with a ten minute 'oral and mental' starter, where the whole class joins in activities that often either require everyone to show answers (through, for example, use of digit cards or individual white boards) or have the expectation that individuals may be called upon to respond to questions posed. The end of each lesson is also expected to have a whole class 'plenary' which may also involve question and answer interactions about the lesson.

Our observation of some 150 lessons over the past year indicate that these interactive whole class sessions have become the norm. Moreover as recommended, the main part of lessons, between 'starter' and 'plenary', also frequently including a substantial element of 'public' answering of questions in order to prepare for a period of group or individual work. Indeed, the use of questioning is promoted as a major teaching tactic with resources, equipment and 'exemplary' videos supplied to school to encourage this. There is also strong encouragement for lessons to have 'pace' and much of the whole class work that we have observed places emphasis on speed as well as correctness.

Thus a strong 'performative' element, viz. being seen to take part and being able to produce correct answers to closed questions and appropriate answers to questions inviting explanation is entering into English primary mathematics lessons. The strategies that children develop in order to be seen to participate in such sessions are likely to affect learning outcomes and is the focus of this paper.

THEORETICAL BACKGROUND

Our theoretical starting point for examining the sort of learning that might arise through such whole class sessions is analysis of pupils 'participation in sociocultural activities' (Rogoff, 1994). Working together in the whole class 'mental and oral starter' provides a microcosm 'community of practice' (Rogoff, 1994). Similarly, Coffield (1999), discussing post 16 education and arguing for a social theory of learning sees learning as

> located in social participation and dialogue as well as the heads of individuals; and it shifts the focus from a concentration on individual cognitive processes to the social relationships and arrangements which shape, for instance, positive and negative 'learner identities... (p. 493)

But while all pupils are, in a sense, participating collectively in such sessions, in that the teachers set up activities to involve everyone and monitor participation, the ways in which particular pupils manage their involvement and what motivates them to engage with the activities will vary from individual to individual. A shift in attention away from the individual to the collective might also be accompanied by decreased focus on innate ability. As Claxton (1999) points out, the ability to learn in a flexible way in our current age of uncertainty needs to emphasise the importance of engagement rather than 'ability'. But, as the examples that follow demonstrate, the reasons that children engage with activities may be far removed from enthusiasm to engage with the mathematics.

Rogoff (1994) argues that 'adult run models of instruction' problematise children's conceptual engagement. Pollard and Triggs, et al. (2000) provide an example of this in their case study of four to seven year olds found that:

> children had only a vague idea of teachers' instructional objectives. Rather than engaging in some synergetic process between teacher and pupil to extend existing understanding, most children were simply concerned to do what they needed to do to avoid being embarrassed or told off or having to do the work again. We found that children felt pressured by classroom constraints to develop task engagement. (p. 302-303)

Pollard also finds that it is 'necessary to facilitate emotional engagement as well as intellectual challenge' (Pollard with Filer, 1996), an issue explored in detail by Goleman (1996). Part of the emotional engagement will rest upon maintaining a successful 'presentation of self' (Goffman 1959) and may well be a 'necessary precondition of stable engagement with learning' (Pollard with Filer, ibid. p310.)

DATASOURCES

The Leverhulme Numeracy Research Programme (LNRP) is a longitudinal study of the teaching and learning of numeracy *investigating factors leading to low attainment in primary (elementary) numeracy in English schools, and testing out ways of raising attainment*. Two cohorts of children, one starting in Reception (four and five year-olds) and one in Year 4 (eight and nine year-olds), are being tracked through five years of schooling. Pupil data are being collected at several different levels ranging from large scale, twice yearly, assessments on each of the two cohorts (some 1700 children in each cohort) through to detailed case studies of six children from each cohort in five schools, each of whom are observed for two weeks of mathematics teaching each year of the programme (Brown, Denvir, Rhodes, Askew, Wiliam, & Ranson, 2000). While the large scale assessment data provides insights into pupil progress at a general level, the case study data enables us to gain insights into how individual children respond to specific teaching. To explore more deeply our notions of participation we draw on data for three case study children now aged seven years: George, Meg and Oscar.

Through this case study data we seek to elaborate the notion of participation and, in particular, to ask:

> how do pupils present themselves during whole class sessions?
>
> what motivates pupils' to take part?
>
> is it fruitful in terms of mathematical learning?

MEG

Episode 1.

As part of a whole class session, the teacher is working on halving numbers. Each child has an individual white board and marker pen with which to display answers.

Teacher: Half of 36?

Meg starts to lift her board up to show the teacher. She has written '15', but before she shows it she notices that others around her have '18'. She quickly changes it; the teacher does not notice and says, 'Well done, Meg.'

Teacher: Half of 72

Meg puts on an act. She takes the top off her pen, pushes it back again and looks puzzled. She appears to be counting - her lips are moving but it is not clear what she is saying. She turns round and sees what George has written then turns back again and wrinkles her face (as if to say, 'I'm concentrating hard'). Then she looks around at several boards and see what answer others have got. Next she closes her eyes and screws up her face. After a time her face lights up as if she's just made a big discovery and she writes down '36'.

Episode 2.

The teacher is using a counting stick (metre length rod, with ten divisions but no number marked) to count on from zero in 10s, 5s, 2s going up to 100, 50 or 20 respectively. The children each have a number fan to show their answers. From the way that Meg looks at the rod and nods her head, it seems that she relies a lot with the higher multiples on counting from zero (as opposed to, say, knowing that when counting in 5s the other end is 50, so the ninth mark must designate 45). She is often still searching for the two digits on her fan with which to show her answer when the teacher has moved on to the next question.

After two counting on in 10s questions (where Meg was not quick enough to show her answer) the teacher changes to counting in 2s. She points to the 8th division and asks for its value.

Meg, again, repeats her nodding and looking at the divisions from zero, notices that the boy sitting next to her has set his fan to show 16. She stops counting on and puts out 16.

The teacher then points to the 9th division. Meg nodding and counting from zero, puts out 18 on her fan, the teacher asks her how she got the answer.

Meg: You count in ones to nine and then go backwards and then its like double again.

Teacher: Meg is using what we did last week, like doubling and halving.

While it is possible that Meg was multiplying by 2 her actions suggested otherwise. Once she had counted along to the number she went straight to showing it on her fan, that is that she arrived at the answer by counting on in twos. There was little suggestion that she was carrying out any operation on a number such as counting along to nine and doubling it. If she had realised that she could get an answer quicker by doubling it is not clear why she talked about "counting in ones to 9" or "going back".

It seems that Meg is not trying to explain her method but only striving to take part in the 'game' of providing an explanation. Time and again, we have observed Meg produce post hoc explanations which do not match what she did but are sometimes not even mathematically correct (add on 9 by adding on 10 and taking off 6). She can do it with great conviction, and even present it in [illegible] that covers up the nonsense.

It is not that she is not capable of invoking a learning orientation. On those occasions where she has been encouraged to slow down and think about the mathematics rather than investing her energy to convince others that she knows it all her delight at succeeding is palpable. She often resists admitting that she might need help on. In another incident when she was attempting to shade one quarter of various rectilinear shapes drawn in her book she protested that questions from the researcher were making her terribly confused, rather than saying that she wasn't sure about the work. But when asked if she would welcome some help she looked both pleased and interested, listened carefully and seemed to take on board intelligently the suggestions offered.

What motivates Meg when she is relating to the teacher, here and in other examples, is her status. Throughout the four years that we have been observing Meg, her teachers say she is able, hardworking and reliable. Meg strives to continue to appear like this to the teacher. In relation to other children, Meg behaves differently, enjoying having power and some control over them. In one incident, having been entrusted with a set of cards for a fraction game for her group, she insisted they all sit still and quiet while she, playing 'teacher', took her time choosing who she would allow to set them out.

GEORGE

Episode 1

The teacher and the class are playing a game where the teacher has a hidden shape and the children have to ask questions with a yes/no answer to figure out what it is.

Teacher: It's a shape. You're going to have to guess what shape. It might be a solid 3-d shape or a 2-d shape. Put your hands down (they've already got their hands up, presumably to guess) and I'll tell you some clues first.

Teacher: It's got, 6 faces, all square

George: (immediately, calling out, not loudly but still quite clearly): cube

Teacher: It's got 8 vertices, 12 edges which are all the same length.

George: (Putting hand up this time) Is it a cube?

Teacher repeats all this information. There are now 6 hands up.

Teacher: Lenny?

Lenny: (who is sitting next to George) A cube.

Teacher: See if that is right.

She pulls out the cube from under the puppet's body.

George: You gave too many clues.

Episode 2

The class are all on the mat, playing the guess the number game: the teacher has a glove puppet that is hiding a numeral and nods yes or no to children's questions.

George takes a lead role in this. He listens to the other information and uses this to frame his questions. Questions he asks include:

Is it a three digit number?
Is it under 300?
Is the second digit number 3?

Is it 444 (this shortly after someone has asked if all three digits are even (perhaps George is interpreting 'even' as all the same?) and knowing that it is between 400 and 450)

Somebody gives the correct answer of 428

George: Someone already said that.

In these examples George is engaged with the mathematics and is seeking to engage with the teacher and the class by using his mathematical insight. His motivation to engage comes from his interest in the mathematics as well as a desire to maintain his position as a clever, articulate boy. He appears to dislike behaving in a way that would make him appear the same as all the other children and frequently stands out by being seen to be doing something different. It is also possible that some of his responses are a challenge to the teacher to keep up with him, or to provoke more serious intellectual challenges being offered to him. Pollard (1996, p. 311) quotes a 1984 study by Doyle and Carter which observed how pupils and teacher collude to reduce the intellectual level of tasks by making them more routine. Perhaps George's habitual behaviour could be a response to his frustration at such routinisation.

OSCAR

Episode 1

The teacher is asking children to double and halve various numbers. Each time they write down their answers on a white board and show it to the teacher.

Oscar gets all of the answers correct. I can't see how he is doing them, whether he is doing them by himself or copying.

Teacher: Half of 36?

Teacher: Oscar how did you get 18?

Oscar: I don't know.

The teacher asks someone else.

Later in the same lesson the teacher is using a disk of card folded into four segments to stand for a cake:

Teacher: This one is cut into 4, into quarters . X comes along and eats a quarter. How much is left? Write it down on your boards.

Oscar writes 1/3

Teacher sees the researcher looking and looks herself.

She says, ' Oscar, How many is it cut into?'

Oscar: 'Umm, three, .., no four.'

Teacher: 4. So that's the number at the bottom. The other number goes at the top.

Oscar changes his 1/3 to 3/4.

Episode 2

Oscar's contributions to the whole class guess the number game.

Oscar: Is it lower than 100?

George Is it a three digit number?

Yes

Teacher to Oscar: So is it lower than a hundred?

Oscar says no and shakes his head.

Oscar: Is it above 400?

Oscar: Is it below 450?

The teacher, after the number has been found, picks up on this and says that the questions were good until they knew that the number was between 400 and 450, and asks what might have further asked.

Oscar: Is it above 410?

Teacher Or is it between 410 and 430?

As children leave the carpeted area to go to text book tasks, Oscar tell me he is in blue group and that he and George are best at maths in that group and ahead even of Harry.

Unlike George, Oscar seems to like being fairly unobtrusive in the classroom and keeps a low profile, offering 'safe' answers and, unlike Meg, sticking with 'Don't know' rather than risking an incorrect response when asked to describe his strategy. Initially he was identified by his teacher as 'average' in mathematical attainment. He used to work quite slowly, taking his time, capable and proficient. Now he works in the same group as George, identified as higher attaining throughout. George and Oscar now spend time together as a pair both inside and outside the classroom. The friendship with George is very important to Oscar and this maybe the reason for the culture of speed and competitiveness which is creeping into his work and which prompts him to fall back on getting the answers from George. His desire to maintain his position in the class as George's friend seems to compete with his inclinations to work slowly and steadily.

DISCUSSION

As Pollard (1996) notes, children who are 'the most 'effective learners' are likely to be those children who can manage their classroom identities so that they derive support from both their teacher and other pupils' (p. 310). The examples presented here show how children are in different ways simultaneously both learning and establishing or protecting their identities in the public forum of class 'question and answer' activity.

These examples support the idea that when children participate in whole class interactive teaching in mathematics they may not be participating in the mathematical thinking which is intended. This arises from other motivations than the desire to learn.

The emphasis on whole class interactive teaching does seem to be connected with the notion that all children should participate in shared discussions of mathematical ideas. A major concern is that the strong "performative" element referred to above prompts children to adopt classroom behaviours which mitigate against them developing good habits as learners. Pollard (2000 p.293) in discussing how children develop a "learning disposition" quotes the New Labour election manifesto of 1997, *Because Britain Deserves Better:*

> Primary schools are the key to mastering the basics and developing in every child an eagerness to learn throughout life.

Certainly behaviours that are identified here especially Meg's need to maintain her image and George's experience of inadequate challenge seem unlikely to foster that "eagerness to learn".

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A STUDY OF CHILDREN'S VISUAL IMAGERY IN SOLVING PROBLEMS WITH FRACTIONS [1]

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We describe some contrasting aspects of the internal, visual imagistic representations of two elementary-school children, inferred from fine-grained analysis of their problem solving during a pair of carefully scripted, videotaped task-based clinical interviews 16 months apart. The problems involve conceptual interpretation and use of fractions. We look for and find concrete/pictorial, pattern, and dynamic imagery, based on set (discrete quantity) and spatial extent (continuous quantity) models, and memory images of formal notation. We also infer internal operations on and transformations of imagery, and discuss our interpretations.

This study considers in some detail the internal, visual imagistic representations of two children as they solve some problems involving rational numbers (fractions). We acknowledge at the outset important limitations to the methodology of inferring internal representational configurations from observed external statements, behaviors, or productions. Internal representation involves ambiguity, and inferences about it entail context-dependent interpretations. In this exploratory and descriptive "status study," our goal is not (yet) to commit to a definitive, reliable or generalizable coding scheme, but to make as explicit as possible the bases for our inferences, improving task-based interview methodology as we explore individual children's imagery.

Various researchers investigate and discuss imagistic representation generally, and visual imagery in particular, as a fundamental system of cognitive representation for mathematical problem solving (Bishop, 1989; English, 1997; Goldin, 1982, 1987, 1998; Goldin & Kaput, 1996; Owens, 1993; Presmeg, 1985, 1986, 1998; Thomas & Mulligan, 1995). We shall focus here on concrete/pictorial imagery, pattern imagery, memory images of symbolic notation, dynamic imagery, and mental operations on or transformations of images. These characteristics are not mutually exclusive; e.g., pattern imagery may be dynamic, and transformed *via* mental operations.

We use definitions (consistent with those of Owens and of Presmeg) as follows. *Concrete/pictorial imagery* is spoken of or gestured at as if it were a picture or a physical object. The image may be given a name referencing what it resembles in real life. Presmeg (1985) found pictorial imagery to be the predominant form during problem solving; Owens also found it to occur frequently, especially during the orientation phase of problem solving and again as a kind of retrospective, global check. In *pattern imagery* the image abstracts and/or generalizes mathematical relationships. Presmeg (1985, p.175) notes "such imagery may be vague or vivid but its essential feature is that it is pattern-like and stripped of concrete details." Krutetskii (1976) calls pattern imagery a graphic scheme where the problem's terms and relations are represented in a visually schematic way. *Memory images of symbolic notation* refer to the visualization of formal mathematical expressions, as Presmeg reports students who describe "seeing" a mental picture of a formula and "reading" the image. In *dynamic imagery* there is active movement or change in all or part of the image (Thomas & Mulligan, 1995). *Mental operations or transformations* refer here to visualized purposeful acts by the imager that modify or transform the image, suggesting but not requiring dynamic imagery (as they can also be visualized as a succession of static images). Presmeg also uses the term "kinaesthetic imagery" because of associated physical actions; Owens' term is "action imagery."

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As our study of imagery is centered in the domain of fractions (Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, & Post, 1988), we consider the children's imagery in relation to various possible conceptual models: e.g. discrete (set) *vs.* continuous quantity or spatial extent (area or volume, sometimes termed "region") models; part/whole *vs.* comparison or ratio relationships; mental operations such as partition. We also focus on understandings of fractional symbolic notation through visual imagery.

Research questions and design of the study

The following questions motivated our analysis. (1) *Inferring internal imagery.* Can we create explicit criteria for inferring from hand gestures, body movements, paper and pen or marker diagrams, pictures, and charts, and physical manipulations of concrete materials, the five characteristics of visual imagery discussed above? Do we find consistency among these in the individual child? (2) *Relation between internal and external imagistic representation.* What relationship can be discerned between the child's internal visual representational capability and her facility in constructing external imagistic representations? How does the use of paper and pencil or marker to draw pictures, diagrams, and/or create charts, or the use of concrete manipulative materials, assist in or influence the visual imagery used in the problem solving? (3) *Relation between internal visual imagery and strategy.* Heuristic strategies of interest include drawing diagrams, working backward, solving simpler or similar problems, guess and check, subproblem decomposition, etc. How does the child's internal imagery interact with or influence strategy use? (4) *Relation between internal imagery and inference of mathematical patterns.* How is the child's facility in constructing or recognizing a mathematical pattern influenced by her imagery. Which imagery characteristics are important here? (5) *Internal imagistic representation and* *mathematical conceptual development.* Is there a relation between the child's internal imagery and the development of her conception of a fraction over time?

In a longitudinal study of individual children's mathematical development, whose design has been described in greater detail elsewhere (Goldin, DeBellis, DeWindt-King, Passantino, & Zang, 1993; Goldin, 1997), 22 children ages 8 to 10 years at the outset were selected to participated in five highly structured task-based interviews over three school years. Interviews #2 and #5 involved non-routine problems about and with fractions. A broad, cognitive analysis with respect to fraction representations and strategies of all 20 children who participated in these two interviews has been completed (Passantino, 1997; see also Goldin & Passantino, 1996). The present report is part of a study focusing in greater depth on the visual imagery of a cross-sectional subset of four children. Below we discuss brief excerpts from interviews with two of them, Marcia and Londa, inferring some contrasting representational characteristics.

The interview scripts were developed at Rutgers by a team of experienced teachers that included the clinicians, and revised after rehearsal and pilot testing to incorporate many different contingencies. Free problem solving is encouraged throughout--after posing a question, the clinician allows for *spontaneous* response before questioning to explore the child's answer or request an external representation. If the child is not engaged or seems at impasse, planned suggestions/hints are offered. The clinicians are not to impose their methods or correct or confirm the child's solution, as it is not the goal here to teach rational number concepts. At certain points, decided in advance and described in the interview script, the children are guided toward particular understandings essential for subsequent questions to be meaningful. Two video-cameras operated simultaneously during each interview, one focusing on the child's work and the other showing the interaction of the clinician with the child. Outlined below are elements of the two interview scripts needed for the discussion; the complete scripts are in Passantino (1997), or available from the authors.

In task-based interview #2, materials placed before the child include a pad, pencil, markers, and red and black chips (checkers). Some preliminary, non-mathematical questions precede a sequence of mathematical questions posed by the clinician. For *each* one, the follow-up includes, as appropriate, "Can you help me understand that better?" "Why?" and/or "Are there any other ways to take one half (one third)?" [The first such questions are:] ♦ "When you think of one half, what comes to mind?" ♦ "When you think of one third, what comes to mind?" ♦ "Suppose you had twelve apples. How would you take one half?" ♦ "... one third?" ♦ [Cutouts are presented in succession: a square, a circle, a 6-petal flower. For each the child is asked:] "Here is a shape. How would you take one half?" ♦ "... one third?" ♦ ... ♦ [Part way through:] "What was on your mind when you were answering the questions up to this point?" and if no image is described, "Did you have a picture in your mind while you were answering any of the questions?" with follow-up. ♦ [The interview continues:] "Can you write the fraction one half?" "What does this fraction mean to you?" (similarly for one third) ♦ ... ♦ The clinician then goes on with more complicated exploratory activities that involve taking one half and one third of an array, and visualizing the cutting of a cube into fractional parts. At the end of the interview retrospective questions again address the child's visual imagery.

Task-based interview #5 re-explores specific topics from interview #2, and extends to further rational number problems. Materials at the outset include red and white chips, paper circles, squares, and triangles, markers, pencil and paper, a calculator, a ruler, ribbon, scissors, and other items specific to later questions. The follow-up for each question includes, as appropriate, "Why?" "Why not?" "Can you show me what you mean?" "Can you show me [using] the materials?" [The main questions begin:] ♦ "When you think of a fraction, what comes to mind?" ♦ [Bold-face printed expressions in vertical format for five fractions, 1/2, 1/3, 2/3, 3/4, and 4/6, are presented on a sheet of paper:] "What fractions do you see here?" ♦ "Can you explain ... what one of these fractions means?" ♦ "Why is it written this way?" ♦ ... ♦ [Several main questions later, the clinician asks:] "Imagine a big birthday cake shaped like a rectangle. Can you imagine what it looks like?" ♦ "Describe what it looks like." ♦ "Now imagine that there are 12 people coming to the birthday party and they each want a piece of cake. Your job is to cut the cake so that each person gets the same size piece. How will you cut the cake?" ♦ ... ♦ "Are there any other ways to cut it?" ♦ "Now think about the icing. Suppose the cake has icing on the top and side." ♦ ... ♦ At the end of the interview further retrospective questions address visual imagery.

Observations and inferences: Londa

At the time of interview #2 Londa was 9 yrs. 8 mos. old, in the 4th grade of a school in a low-income, urban district. Her spontaneous representations of "one half" and "one third" are verbal: "*a cookie, split in half*" "*... and then when you split it in half, it's two ... Two smaller pieces,*" and "*Half of our classroom, kids in our classroom,*" "*... we have 19 kids in our classroom so half of that would be 9 ... no, we have 18, half of that would be 9,*" followed by "*box cut in threes ... because I think a box would be easy to picture,*" "*one-third of an orange,*" and "*one-third of an apple. Really one-third of anything.*" All these phrases suggest 3-dimensional, concrete/pictorial imagistic representation, drawn from familiar real-life contexts, with interpretation of the fraction based on a mental operation of partition of a whole into equal parts. All but the second provide a "spatial extent" or continuous quantity model. The "kids in our classroom" can be characterized as a "set" or discrete quantity model. Asked to take one-half of 12 apples, she replies, "*Split all of the apples in half.*" Showing with checkers, "*Well, take 12 of these*" [takes 6 red, 6 black] "*I would split them all in half. So then there'd be 24*" [she counts the 12 checkers, assigning a value of two to each] "*'Cause you cut this in half this would be two ... four, six, eight*" [she continues to 24]. When asked to take one-third, "*Cut them all into threes*" [she counts by threes with the 12 checkers, to 36]. Londa never explicitly states what would be one-third. Her external representations to this point suggest mental operations visualized through a succession of static, 3-dimensional concrete/pictorial image-configurations, using exclusively a continuous quantity model for "fraction."

She does not come back to a "set" model; having acted to transform her image from 12 apples into 24 pieces, she does not seem to treat the latter as equivalent elements but retains the pairing derived from the apple at which they originated.

Asked to write the fractions one-half and one-third, Londa does so correctly in a vertical format, symbolic notation. Interpreting what she wrote, she explains, *"This* [the 2] *means that you have two parts, and this* [the 1] *being one whole. One whole and two parts."* She explains one-third as *"one whole, three parts."* [Clinician:] "Is there any other way you can think of what those fractions mean?" [Londa:] *"Either that or three wholes and one part."* She explains, *"if I had one cookie, two cookies, three cookies"* [draws three circles], *"and I take one cut, part out of each of them"* [makes a small wedge at the top of each circle]. Londa translates consistently but with some instability among verbal expressions, pictorial images with mental operations, and memory images of symbolic notation. Her interpretation of the numerator as representing the whole and the denominator as the number of parts is consistent with her first answers, and with her method of taking one-half and one-third of twelve apples. She does not show evidence of a ratio or comparison model for fractions.

At the time of interview #5 Londa was 11 yrs. 0 mos. old, in the 5th grade in the same school. Her initial statement of what comes to mind for a fraction is, *"I think about how the denominator means that it's the whole and the numerator means how many parts are out of it."* ... *"Like two thirds, the three is the denominator, that means the whole thing; and two means how many, um, two pieces out of the three whole."* [Clinician:] "Could you show me using some of these materials perhaps?" [Londa:] *"Okay, this is two,* [picks up two yellow cutout circles] *this is the two-thirds, and then you have three on the bottom, you have three all together* [puts a third circle with the others] *and then you have this is two* [indicates the two circles] *and then you have one left over* [indicates the third circle]. *So, well, like, if you say two-thirds minus, um, one-third you have two-third, no, two-thirds minus one-third you have one third and that's it."* [without further action with the circles] Londa's first words suggest a memory image of symbolic notation related directly to part-whole imagery. Her use of the circles makes no further reference to "pieces of the whole," but suggests a possible relation to a set model, even the genesis of a ratio model, but with additive imagery. She evidences a memory image of an algorithmic procedure for fractions.

Later, asked to visualize the rectangular birthday cake, Londa describes it, explains that *"Since it's a rectangle you can divide it into equal parts,"* shows its height with the ruler perpendicular to the table, *"It's about two inches thick,"* and draws a rectangle with two rows and six columns to show exactly how to cut it in 12 equal pieces. Asked if there are other ways, she replies, *"I don't think so because that's the way we cut my birthday cake. I,* [shakes her head] *I don't think so."* Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she (correctly) explains *"Just the four, because ... you can't put icing on the inside of a cake ..."* and *"one-third, well, four-twelfths you can reduce it, four-twelfths can be reduced by four and it'd be one-third."* Her verbal and external pictorial representations of how she cut the cake, and her way of determining the number of pieces with icing on exactly two sides and the top, suggest static, internal 3-dimensional, real-life pictorial imagery, with operations of partition and counting closely embedded in the context. Londa uses her imagistic part-whole model and an internal algorithm to obtain the desired fraction of the pieces.

Observations and inferences: Marcia

At the time of interview #2 Marcia was 10 yrs. 4 mos. old, in the 5th grade in a small, lower middle income school district. Her spontaneous, verbal representations of "one half" and "one third" are *"a half of a circle," "a half of a triangle,"* and *"a third of a circle."* These suggest possible internal pattern or pictorial imagery with a region model for the fraction. Asked about the 12 apples, Marcia replies, *"Count 'em. Well if you know there's 12, and 6 ... and since 6 and 6 is 12, take 6 of them if you want half."* [arranging 12 red checkers in two rows of 6, and counting them] *"... you know that 6 and 6 is 12 you could just take half of them away and you would have half."* [She pushes the 6 checkers from the top row into a casual irregular group. Clinician:] "Why is it one half?" [Marcia:] *"Because, well, if you place it like this and you add the 12 to your 6 here"* [she arranges the two rows of 6 again, indicating that taking the bottom row of checkers is equivalent to taking the top row, and counts] *"... it's the same amount the other way."* For one-third of twelve apples Marcia replies, *"if you knew 4 times 3 was 12, you could take 4 away and you would know it was a third ... because 4 and 4 is 8 and then another 4 is 12."* [pushes 4 checkers to the right, then separates the remaining 8 into two groups of 4, maintaining the row and column structure] Marcia's external representations suggest memory images of notation, and additive and multiplicative structures for part-whole relation in a "set" model. She focuses on the numbers of apples, doing some mental computation; importantly, her physical manipulation of the checkers suggests an internal pattern image that includes operational one-to-one correspondence between items in parallel rows.

Later Marcia correctly writes the fractions one-half and one-third in vertical-format symbolic notation, explaining, *"well, I can tell that it's one-half because like this is a two* [points to denominator] *and it's a one* [points to numerator] *and if you add one and one it would equal two,"* and *"if you could times one times three or one and one and one is three so it would be one-third"* Her repeated addition of the numerator to reach the denominator value is suggestive of her earlier pattern imagery.

At the time of interview #5 Marcia was 11 yrs. 8 mos. old, in the 6th grade in the same community. She now describes a fraction in quite general terms, *"like a part of a whole of something"* [writes the fraction three-fifths] and explains, *"like three-fifths it's like the five-fifths is the whole, and it's like that's part of the whole of whatever the thing is."* To us this suggests not just internal memory images of notation, but also a level of abstraction that includes a pattern image of a fraction as an operator.

Marcia initially responds to the birthday cake question verbally, kinesthetically, and with a drawing: *"Well, um, it's like this* [tries to indicate with her hands how the cake would look, then draws a triangle] *like, a, uh, oh a rectangle"* [draws a long, thin vertical rectangle]. After some dialogue she measures and redraws a horizontal rectangle 4 inches long. "*... and like you would divide the four into half ... which is at two* [draws a vertical line at 2 inches, dividing the rectangle in half] *and then you would divide each of these into halves* [draws vertical lines dividing each section in half] ... *so you'd have four pieces ... and then you divide these, which is at the half an inch, 'cause the other one was at the inch to divide ... and then you have to divide these which is at a fourth of an inch"* [counts 16 pieces]. ... *"But there's more than twelve so ..."* Prompted to redraw freehand, Marcia suggests, *"it would depend on like how big the cake is 'cause like if the cake is, like, um, 24 inches long then you would have to divide them each into two inches"* ... *"So, it depends how long the cake is."* Further encouraged to redraw freehand, Marcia again considers halving, "*... there would be 16 ... since there's 8 now ... so it would double it so like you ...*" [Clinician:] "Yeah, could you do it?" [Marcia:] "*... would have to divide the cake into thirds"* [draws a long horizontal rectangle, with vertical lines dividing it in thirds] "*... and then divide these so then there's 6* [draws vertical lines dividing each of the 3 sections in half] *and then you divide these into half* [draws additional vertical lines dividing each of the six sections in half] *which would make that 12 pieces."* Marcia does not think of other ways to cut the birthday cake into 12 equal pieces. Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she answers (correctly), *"there would be 10, 'cause there, it would be on the top here* [points to interior of the rectangle] *and then it would be on this side and this side* [points to the upper and lower horizontal edges] *but you can't count these sides* [points to the right and left vertical edges] *'cause there would be three since there's the edges."* She continues, *"10 out of 12, or five-sixths," explaining, "you divide the 10 and the 12 by two and you get five-sixths."* Space permits no further detail, but from this and other evidence we infer internal visual and kinesthetic pattern imagery based on linear and region models, interacting with the "trial and evaluate" halving strategy. We think her linear model stems from her measurement process.

Comparison and conclusion

We infer a close relation between the children's internal imagery and their conceptual development of fraction, based not only on the presented excerpts but on our analyses of the full interviews. Londa's imagery is predominantly pictorial, centered in familiar concrete objects. She consistently connects her memory images of symbolic notation with just a part/whole conceptual model for fractions, relating each problem directly to a real-life embodiment. Her use of pattern imagery, which might help her abstract and connect different ideas or metaphors of fraction, is not well developed. In contrast, Marcia evidences predominantly pattern imagery, connected with memory images of symbolic notation. She uses part/whole, operator, additive, multiplicative, and linear conceptual models for fractions, and is not "stuck" in particular contexts. Her frequent pattern imagery and her wider range of conceptual models suggest that for her symbolic notation is more flexibly representational of her imagery. Both children show consistent visual imagery characteristics from the first interview to the second, with evidence of developing memory images of symbolic notation and school-taught algorithms between the two interviews.

While initial verbal responses usually allow us to infer some imagery, the planned prompts for additional external representational forms yield evidence of different internal imagery, not easily discernible from the purely verbal descriptions. Throughout the interviews Londa's and Marcia's drawn pictures and physical manipulations of concrete materials suggest concrete/pictorial and/or pattern imagery; their hand gestures and body movements suggest dynamic imagery and/or mental operations; and their notations suggest memory images of symbols and algorithmic operations.

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Difficulties Confronting Young Children Undertaking Investigations

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Abstract

One approach to providing a mathematically rich curriculum is to involve young children in mathematical investigations in which they engage in the exploration of meaningful problems, and problem posing. However, there is limited research on how teachers can facilitate young children's learning through investigations. This study explored the difficulties seven-to-eight year old students experienced when they began an investigatory program. We present examples of specific difficulties students confronted in conceptualising and conducting investigations, as well as general difficulties that they experienced which hindered their investigations, such as limited observation skills. Our contention is that mathematical investigations can enhance young children's learning provided that their difficulties are addressed.

Background

The importance of providing students with opportunities to work as mathematicians has held credence for at least the past three decades (e.g., Papert, 1972; Wells, 1985) and has recently been strongly advocated (National Council of Teachers of Mathematics, 1998, 1999). As mathematical investigations are central to the work of mathematicians (e.g., Hoffman, 1998), they are fundamental to children's work as young mathematicians (Baroody & Coslick, 1998; Wells, 1985). Investigations are defined as "something more than solving the problem" in which "there will be questions to ask as well as questions to answer" and that require "speculation and conjecture" coupled with opportunities "to test out ideas and to convince others of their validity" (Jaworski, 1986, p. 3). The cognitive activities fundamental to investigations are consistent with those advocated in reform classrooms. In these mathematics classrooms, students should raise questions, pose and solve problems, participate in constructive dialogue and debate, and explain, clarify, and revise their mathematical ideas and problem constructions (Baroody & Coslick, 1998; Bowers, Cobb, & McClain, 1999; English, 1998). Thus, mathematical investigations are ideal for implementing these practices and supporting the child-centred approach that underpins reform initiatives (Borasi, 1992; Jaworski, 1994; Shifter, 1996).

However, there is an urgent need for classroom-based research that addresses the teaching and learning issues associated with young children undertaking investigations. While classroom-based research on the teaching and learning of children conducting mathematical investigations exists for the upper primary years (e.g., Oliveira, Segurado, da Ponte, & Cunha, 1997), there is limited research in the early primary years. The existing research on young children implementing investigations tends to focus on the implementation of an investigation with an individual child (e.g., Juraschek & Evans, 1997), and hence, provides scant guidance for implementing investigations with a class. As engaging young children in investigations requires teachers to teach mathematics in new and different ways (Baroody & Coslick, 1998; Skinner, 1999; Taber, 1998), such research is fundamental to the reform vision for improving student achievement (Hiebert, 1999).

Ideally, classroom investigations should parallel real life problems and provide children with opportunities to apply their basic knowledge (Holding, 1991). Such problems might involve making decisions that are influenced by aesthetics, economics, pragmatism or safety. Associated tasks that may involve observation, collecting data, seeking patterns and relationships, characterise original thinking in mathematics and provide authentic circumstances for conjecture, logical thinking and proof, all of which are cornerstones of authentic mathematics (e.g., Greenes, 1996).

Although investigatory tasks for young children need to be commensurate with their interests, experiences, and mathematical capacity, the tasks needs to be relatively challenging to have cognitive benefit (Lappan & Briars, 1995; Stein, Grover, & Henningsen, 1996). Due to the cognitive demand that occurs when students are engaged in challenging tasks, teachers may scaffold students' problem solving by simplifying tasks or providing hints (e.g., Rosenshine & Meister, 1992). However, even with young children, scaffolding should be used judiciously because when a teacher takes over the challenging aspects of the task, it becomes routinized (Stein et al., 1996) and the cognitive value of the task is reduced (Henningsen & Stein, 1997). Routine investigations in which the "investigation degenerates into an algorithm" have limited cognitive value (Roper, 1999). Thus, while teachers may initially pose and guide children's investigations (Baroody & Coslick, 1998), children should ultimately develop and implement their own solution plans (Brahier, Kelly, & Swihart, 1999), and pose investigations (Rowan & Bourne, 1994).

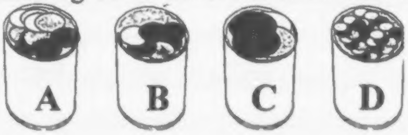
A key consideration for facilitating learning is teachers' pedagogical content knowledge, which includes an understanding of students' difficulties (Carpenter, Fennema, & Franke, 1996). In this paper, we report on some of the difficulties that confronted young children when they began a program of investigations. This is part of a larger project exploring how young children in the early years of primary school engage in mathematical investigations.

Design and Methods

The research adopts an exploratory case study design (Yin, 1994) in which a teaching experiment was conducted with the goal of supporting the development of investigatory abilities in young children. This study was implemented in class in which one of the researchers (CMD) assumed the role of the teacher while the other researchers provided feedback as a non-participant observer (JJW) and "critical friend" (LDE). Twenty-seven seven to eight-year-old students were selected for the investigations program on the basis of their interest and strength in mathematics from four class groups within the same school. Students worked as a "class group" and received 90 minutes weekly of investigatory activities over a 14-week period.

This paper reports on the initial five-week phase of the program, which was implemented in the early part of the school year. In this phase, students worked on a series of mathematical investigations involving Smarties[1] (Table 1). The first three investigations were teacher-initiated, although questions posed by students during these investigations were followed up. The fourth investigation was a student-initiated task, which the students undertook with a partner. These investigations are described in detail elsewhere (Diezmann, Watters, & English, 2001).

Table 1. Overview of the Smartie Investigations

Investigation 1 (I-1): How many Smarties in the can? Students were asked to investigate the numerical contents of small, white, translucent, sealed (film) canisters that had been filled with Smarties. Pairs of students were provided with a few Smarties, an empty can and a filled, sealed can. Students had access to a range of common tools, such as kitchen scales, balance scales, rulers, calculators, and magnifying glasses.
Investigation 2 (I-2): Smartie Cans Students were asked to explore and predict the numerical contents of a series of Smartie Cans that varied in fullness and contained different sizes of Smarties. This task was designed to develop students' skills of observing, predicting, collecting and analysing data, and reasoning. Additionally, this task provided a rich environment for developing an understanding of volume and size relationships. 
Investigation 3 (I-3): Distribution of Smartie Colours The students were each given a small packet of Smarties to explore the distribution of colours. This involved representing the number of each colour Smartie on a table and a graph, answering questions about these representations, and comparing their results with other students.
Investigation 4 (I-4): Independent Smartie Investigation The students were given support to identify investigable questions about Smarties. (E.g., What is the most popular coloured Smartie?). Their findings were presented as pages for a class book about Smartie facts. Students had access to various common-place resource materials.

The case study database comprised video and audio records, classroom artefacts, the teacher's lesson plans and reflections, and notes by the research team. Four video cameras recorded events during each lesson supplemented by audio taping of selected individual group interactions. A research assistant made focussed observations that involved an ongoing record of the interactions of particular children. The data also included children's written work. After each lesson, the video-tapes were reviewed and salient events discussed among the researchers. These discussions permitted the team to analyse behaviours, develop conjectures, and plan strategies. Summaries of discussions were compiled as a diary containing descriptions of events, hypotheses, and reflections about teaching and learning. For this paper, the data were analysed to identify the range of difficulties encountered by students when they were introduced to investigations. Only examples of these difficulties are presented due to space limitations.

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[1] "Smarties" are sweets similar to "M & M's" and "Beanies".

Selected Results

In this initial phase of five weeks, the students encountered two types of difficulties. These were Investigation-Specific Difficulties and General Difficulties that impact on other school work in which students engage apart from investigations.

Investigation-Specific Difficulties

Students experienced a range of difficulties when engaged in teacher-initiated and self-initiated investigations. Four examples of these difficulties follow.

1. A lack of understanding of the problem under investigation. For example, in I-4 (Investigation 4) Melissa was asked to explain how to investigate the most popular colour Smartie. Her response suggests that she interpreted the term "popular" to mean the most frequently occurring item instead of a consumer preference. Other students interpreted the term similarly.

> I think you would open all the Smartie jars you had and then, and then put the colours into groups say, purple, yellow pink and different colours and when you are finished putting them into groups well you count them up and (find) … the colour that has the highest number.

2. Failure to link the findings of an investigation to the answer to the problem. For example, in I-1 students used a variety of tools, including rulers, scales, calculators, and magnifying glasses to investigate "How many Smarties in the Smartie can?" Though they generally used these tools proficiently, most students did not use their measurements in producing their answers. Catherine's response is typical of students' responses: "We used the ruler to measure the Smartie Can to see how many Smarties there were." A couple of students, for example, Caroline, provided further information: "We used the ruler for the height. The height was four and a half centimetres". Robert was the only student to explicitly link his measurement to his answer. He and his partner weighed a can containing four Smarties and the full can. They then determined how many partially-filled cans were equivalent in mass to the full can. Finally, they multiplied the result of their calculation by four, as there were four Smarties in the partially-filled can. However, they failed to realise that the mass of the partially-filled can should have only been included once in their calculations. They never attempted to weigh an empty can.

We used the scales to measure the can with four Smarties to see how many it weighed to help us find out the answer to the problem...Well, if we weighed the "four can" then we could multiply the four can on the calculator.

3. *Difficulty posing a problem to investigate.* For example, in I-4 Jason was unable to identify a problem to investigate. He wrote, "Different Smarties go down the slide" (Figure 1). The Smartie slide was a cardboard construction that was used for measuring the speed of Smarties as they travelled down the slide. However, while some students were unable to spontaneously pose their own problems in Phase 1 of the program, other students, such as Tim, clearly articulated a problem: "How long does it take different types of Smarties to go down the Smartie slide?

4. *A lack of prerequisite mathematical knowledge to complete an investigation of interest.* For example, in I-4 Robert's initial problem was "What is the chance of the first Smartie out of the box being your favourite colour?" Although Robert recorded the outcomes of his trials and identified the colours that were drawn the most and least times, he was unable to proceed further with his exploration of "chance". He chose not to present this relatively sophisticated investigation to the class, but presented a simpler investigation designed by his partner. Robert's sense of self-efficacy might have been diminished because he was unable to complete the initial investigation to his satisfaction. This is an instance of where an investigation requires more advanced mathematics and intervention by the teacher.

It is not surprising that students experienced difficulties with teacher-initiated or self-initiated investigations, as they were novices. The teacher provided students with support to overcome investigation-specific difficulties and addressed these difficulties in subsequent lessons. None of these difficulties was considered sufficiently insurmountable to obviate the benefits of an investigations program.

Figure 1. Slide.

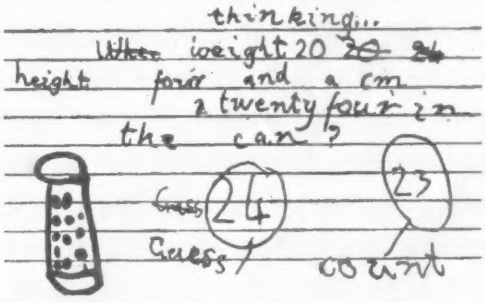

Figure 2. Inappropriate calculation.

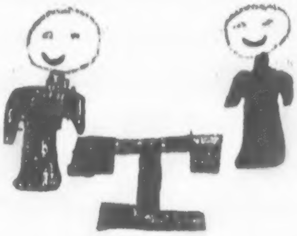

Figure 3. Using scales

General Difficulties

In addition to investigation-specific difficulties, students experienced various general difficulties that hindered their investigations. Six of these difficulties are outlined here. The first three difficulties relate to mathematics and the latter three difficulties relate to communication and representation.

1. A failure to detect critical differences. For example, in I-2 students were unable to explain the discrepancy between their prediction and the actual count for the partially filled Can B. After prompting to compare the full Can A with Can B, Robert picked up the cans and explained, "Well this one here (Can B) it's not as full as this one (Can A)".

2. A lack of an understanding of what can be added. For example, in I-1 Leanne calculated the number of Smarties in her can to be 24 by combining the can's mass of "20" with the can's height of " four" (Figure 2).

3. Difficulty in identifying how to use a tool for a particular purpose. For example, in I-4, Leanne and Libby encountered difficulty in trying to weigh a single giant Smartie on kitchen scales and balance scales. This difficulty was overcome by prompting the students to weigh more than one giant Smartie (Figure 3).

4. Difficulty conveying ideas clearly orally or in writing or in a drawing. Throughout Phase 1, students were frequently asked to clarify and elaborate on their oral and written responses. Additionally, drawing did not appear to be a regular feature of their mathematical thinking or communication. Even when students were instructed to include drawings in their reports, some students failed to complete a drawing or their drawing lacked adequate detail.

5. Difficulty using common mathematical representations. For example, in I-3 many students needed considerable support to produce a simple table and bar graph.

6. A lack of understanding of the correspondence between objects and their symbolic and pictorial representations. For example, in I-3 some students had difficulty understanding that their count of a particular colour Smartie could be written on the table beside the corresponding colour and could also be represented on the bar graph.

These general difficulties highlight the range of knowledge or skills that students utilise in undertaking investigations. Hence, teachers may need to provide individualized and differential support to address particular difficulties that hinder students' investigatory work. While problematic, these difficulties provide invaluable opportunities for learning within a meaningful context.

Conclusion

The results indicate that young students are capable of planning and implementing investigations but they encounter a range of difficulties in the process. Knowledge of specific difficulties experienced by students enables the teacher to structure an investigations program to pre-empt and address likely difficulties, and provide students with opportunities for success on challenging tasks. Knowledge of the general difficulties that impact on students' capacity to engage effectively in investigations assists the teacher to determine the preparedness of particular students for investigatory work, and the type of support they may require to successfully engage in investigations. General difficulties experienced by students also provide teachers with an insight into the students' capacity to apply previously learnt knowledge or skills within a new and challenging context.

Engaging young students in investigations requires that teachers reconsider their understanding of the nature of mathematics and how mathematics is learnt. Mathematical investigations are one of the few classroom mathematics activities in the early years that require high-level thinking and task commitment. However, investigations provide students with the satisfaction of successfully completing a challenging task and being able to identify and investigate their own problems. Hence, the time and effort invested by teachers in planning and supporting children's investigatory work can yield worthwhile cognitive and motivational dividends.

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A MODELING PERSPECTIVE ON STUDENTS' LEARNING THROUGH DATA ANALYSIS

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A modeling approach to problem solving shifts the focus of the learning activity from finding an answer to a particular question to creating a system of relationships that is generalizable and re-usable. In this research paper, we discuss the nature of tasks that can be used to elicit the development of such systems. We present the findings from one classroom-based case study of Australian children and a summary of findings from all U.S. and Australian classes in our studies. Student reasoning about the relationships between and among quantities and their application in related situations is discussed. The case study suggests that students were able to create generalizable and re-usable systems (or models) for selecting and ranking data.

Introduction

Data analysis is increasingly recognized as an important topic within school mathematics and has gained an increasingly visible role in the K-12 curriculum (English, Charles, & Cudmore, 2000; Greer, 2000). The justification of such significance is generally made by an appeal to the usefulness of such skills in everyday life as well as in a wide range of work-related settings. The analysis of data as part of school mathematics is sometimes justified on the basis of contexts which are motivating and of interest to students. In this research, the context is seen as providing a site that is rich for the sense-making activities of learners and that holds the potential for the development of more generalized understandings across a range of contexts. In this paper, we discuss the results of an analysis of a sequence of modeling activities in which middle-school students investigated non-routine problem situations where the core mathematical ideas focused on the creation of ranked quantities, operations and transformations on those ranks, and, finally, the generation of relationships between and among quantities to define explanatory and predictive relationships.

Theoretical Framework

A modeling approach to the teaching and learning of mathematics focuses on the mathematization of realistic situations that are meaningful to the learner. The emphasis on modeling involves three important shifts in the approach to teaching and learning mathematics: (1) the nature of the quantities and operations that are useful, (2) using contexts that will elicit the creation of useful systems (or models), and (3) developing and refining such models in ways that are generalizable. We briefly discuss each of these aspects in turn.

The quantities and operations that are needed to mathematize realistic situations often go beyond what is usually taught in school mathematics (namely quantities such as counts, measurements, ratios, rates and proportions, shapes, and the four basic operations of arithmetic). Often in realistic situations, the kinds of quantities that are needed include accumulations, probabilities, frequencies, ranks, and vectors. The operations needed include sorting, weighting, organizing, selecting, and transforming entire data sets rather than single, isolated data points. In solving typical school "word problems," students generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematized for the student. The student's goal is to unmask the mathematics by mapping the problem information in such a way that an answer can be produced using familiar quantities and basic operations. In modeling tasks, the student's goal is to make sense of the situation so that s/he can mathematize it in ways that are meaningful to her/him. This will involve a cyclic process of selecting relevant quantities, creating meaningful representations, and defining operations that may lead to new quantities (Lesh & Doerr, in press).

A modeling approach to problem situations explicitly uses meaningful contexts that elicit the creation of useful systems (or models). Modeling begins with the *elicitation* stage, which confronts students with the need to develop a model to describe, explain and predict the behavior of an experienced system. Principles for designing such model-eliciting problems have been put forward by Lesh and colleagues and are described elsewhere (e.g., Lesh, Hoover & Kelly, 1992). Models are systems of elements, operations, relationships and rules that can be used to describe, explain or predict the behavior of some other experienced system. We are particularly interested in those models in which the underlying structure is of mathematical interest. The sequence of data analysis problems (described in more detail below) provide us with a setting in which we can examine the development of students' interpretations of the problem situation, their reasoning about relevant elements of the system, their selection of quantities, operations, and representations, and their multiple cycles of interpretation.

Engaging in this kind of model building is not seen as finding a solution to a given problem but rather as developing generalizations that a learner can use and re-use to find solutions (Bransford, Zech, Schwartz & The Cognition and Technology Group at Vanderbilt University, 1996; Doerr, 1997). To this end, we argue that the students need multiple experiences that will provide them with opportunities to explore the mathematical constructs, to apply their system in new settings, and to extend their model in new ways. Each of these stages of the model development process includes multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and re-constructed by the learner, ordinarily interacting with other learners. This view of student's conceptual development through modeling is shaped by earlier research that posits a non-linear, cyclic approach to model building (Doerr, 1997). Generalizing and re-using models are central activities in a modeling approach to learning mathematics. Thus, a modeling perspective leads to the design of an instructional sequence of activities that begins by engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs in other problem situations leading to a generalizable system (or model) that can be used in a range of contexts.

Description of the Study

The model development sequence (described briefly below and elaborated at http://soeweb.syr.edu/mathed/HMDproject/Main.html) was designed by the first author and has been investigated in several middle-school classrooms (students aged 12- 13) in the United States. The sequence was later revised by the second author for use in Australian classrooms with students aged 11-12 years. The Australian children addressed here were from a grade 6 class who participated in the sequence of model development activities described below. The activities were implemented in the children's classroom each fortnight over a period of 11 weeks. The children worked in small groups of 3-4 in sessions of approx. 90 minutes duration.

Description of the Modeling Tasks

The overall sequence of model development activities consists of four problem situations centered on the core mathematical ideas of ranking, weighting ranks, and selecting ranked quantities. Since the problem situation is focused on ranking, of necessity the students analyze and transform entire data sets or meaningful portions thereof, rather than single data points. The students had no specific formal exposure or instruction on these ideas prior to the unit. Rather, the unit was designed so that the students could readily engage in meaningful ways with the problem situation and could create, use and modify quantities (e.g., ranks) in ways that would be meaningful to them and in ways that could be shared, generalized, and re-used in new situations. The sequence of problem situations was designed to be completed by a small group of students, thus providing a social setting for the negotiation of conjectures and justifications and for the clarification of explanations. In addition to the sharing which would take place within the small groups, discussion of specific systems created by students provided a forum that could potentially lead to the sharing of either common or multiple systems within the whole class. Each of these tasks is described briefly.

The sequence begins with the Sneakers Problem, which was designed to elicit the notion of ranking in the first place and the multiplicity of factors that would lead to the need for selecting based on ranks. This is followed by two tasks, namely, the Weather Problem and the Summer Camp Problem, which were designed to explore and extend the constructs elicited in the Sneakers Problem. The final task in the sequence, the Crime Problem, provides the opportunity for students to apply the constructs developed in the earlier tasks, to symbolize the systems they have developed, and to refine their notions about weighting factors.

In the Sneakers Problem, the students encounter the notion of multiple factors that could be used in developing a rating system for purchasing sneakers and the notion that not all factors are equally important to all people. Students were asked "What factors are important to you in buying a pair of sneakers?" This generated a list of factors where not all factors were equally important to the students; the students then worked in small groups to determine how to use these factors in deciding which pair of sneakers to purchase. This resulted in different group rankings of the factors. The teacher then posed the problem of how to create a single set of factors that represents the view of the whole class; in other words, the group ranks needed to be aggregated into a single class ranking.

The second problem, the Weather Problem, was designed to extend, explore and refine the idea of ranking, sorting, selecting, and using quantitative and non-quantitative data. The context of the problem is a travel agency that provides a re-location service for clients who specify certain climatic factors that govern their choices of possible destinations. Students are given a table of data on the climatic conditions of various cities, together with letters from two clients (extracts shown below). The students' task is to develop a generalised rating system for comparing the climates in different places, and to write a letter to the travel agency recommending the first and second best, and worst cities for each client. The students are to explain to the agency how their rating system works and why it is a good one.

Climatic Information

City	Clear Days	Days below 15°	Days above 30°	Average yearly rainfall (mm)
Sydney	85	12	15	1220.4
Alice Springs	195	40	169	274.5
Hobart	36	184	6	516.3

Letter from client: *Dear Global Travel, My wife and I are retiring in several months and would like to relocate in a warm and sunny area. We don't care if there is a lot of rain and we definitely don't want to be too cold. What are some cities we should consider living in? Sincerely, Mr & Mrs Johnson.*

The Summer Camp Problem (designed by Lesh and colleagues) is a structural analog to the Weather Problem and asks the students to create a generalized system for selecting teams for track and field events in a summer recreational program. In the Crime Problem, students are presented with crime rates on both violent and property crimes for various cities. They are asked to devise a system that can be used to determine if a particular city (in the present instance, Ipswich) on the list of cities is "safe enough" or warrants increased expenditures for the police budget. Intended in the mathematization of this data set is the notion that although violent crimes against persons have lower overall rates, they would be considered more significant than property crimes.

Data Sources and Analysis

Data sources from both the Australian and American classes included audio- and videotapes of the students' responses to each of the tasks, together with their work sheets and final reports detailing their models and how they developed the models. Transcripts were analyzed for evidence of students' mathematical development over the course of the tasks. Field notes and audio-tapes were analyzed for the varying strategies that were developed across classroom settings. The researchers shared their data analyses across the two sites and the analysis is on-going.

Results and Discussion

First, we consider the development of one group of 3 children from the Australian classes (Ben, Erin, and Rose). The group began the Sneaker Problem by listing factors using non-mathematical arguments and decisions that reflected their own experiences in buying sneakers ("*I think style then size*;"commented Rose, to which Erin replied, "*But what if it doesn't fit them*?"). Ben soon realized that they needed to be more objective ("*It doesn't matter*!"). This informal negotiating strategy was gradually replaced by a more mathematical one when the group faced the problem of aggregating several group lists into one list. They used a frequency-based strategy (described later) in combining the lists until they faced a difficulty with ranking two of the factors (brand and grip). The girls tried to negotiate here on a subjective basis but Ben rejected this approach and began the shift from subjective negotiating to mathematizing the elements in the lists as mathematical quantities in their own right. He totalled the ranks of each of the two problematic factors and then re-ranked their totals. The group reverted to their earlier frequency-based strategy, however, to complete their aggregation of the lists. Nevertheless, it was interesting to observe Ben's comments in the subsequent whole-class sharing session. After one group explained, "*We put grip* [before brand] *because there's two grips there and there's only one brand at* 6" [the student was referring to the sixth place on her list]), Ben asked, "*What if it was the opposite? What if grip was where brand was, and brand was where grip was. Would you choose grip or brand?*" Some children responded in a subjective way ("*Use your own opinion*"), while one child suggested adding the ranks across the lists and finding the average (although his group had not done so). Ben explained how he solved this dilemma: "*I had a bit of trouble so I added up all the brands and got 27 and when I added up grip I got 25, and 25 means it's, like .... grip before brand.")*.

On the Weather Problem, which the students completed two weeks after the Sneaker Problem, Ben's group displayed two major mathematical developments. At the beginning of the Weather Problem, the group appeared focused on finding a generalized system. Rose and Erin recommended using an averaging system, which they had developed independently on a small analogous task at the end of the Sneaker Problem session (this task involved working with data from a McDonalds' survey of people's top 5 reasons for coming to McDonalds. Students had to develop a master list that ordered the reasons from most important to least important.). Ben wanted to create his own system, however, his initial ideas were rather unsophisticated ("*Why don't we take each city and say what we think of it, like we could say it's pretty cold, pretty hot…*"). The girls kept reminding Ben that they had to find a generalized system ("*Method, method, because it has to work for all of them.*"). After some negotiation, the group started to develop their system by ranking cities according to the number of clear days. Rose stated that they should "*rank from lowest to highest and then you average it.*" The group then proceeded to add the data in one of the columns, namely, the number of clear days, and then divided by 9 (there were 9 cities listed in the table). Following Ben's comment, "*If we wanted a perfect city, we should go about getting all the averages for a start,*" the group found the average of each of the remaining climatic factors in the table of data. After deciding that, if the clients "*want a hot one and it's above the average that should mean it's hot*" (Ben) and "*yeah, if it's below it's cold*" (Erin), the group proceeded to draw a table to record those cities that were above or below the average for each climatic factor. In trying to refine the group's explanation of their system, Ben stressed, "*If people want a hot day, you find a city that, hang on, hang on, just listen, you find a city that has over 118 days over 30 degrees,*" and "*you wouldn't look for cities that are the closest to the average.*" In returning to the clients' letters, Ben directed the group to list the cities that were above/below the average according to what the client requested. In scanning their list for cities that met the criteria, the group became somewhat bogged down, until Ben decided that the use of an elimination strategy might help: "*Write down all nine cities and then cross out the ones we don't think it is and go right down to the last three.*" As the group was doing so, Rose was working at a "meta-level," re-focusing the group on the goal of their efforts: "*I'm going to re-read the question again…we don't need to look at every single one..we don't really have to worry about rain.....they definitely don't like cold days so we definitely don't want to be cold.*"

The group used the same "mean-split" system in working the Crime Problem. They found the average of the violent data of all cities listed in the given table and then repeated this for the property data. The group compared these two overall averages with Ipswich's rate for property and its rate for violence to make a decision on whether Ipswich is a safe place to live. They then decided to make a more fine-grained analysis by considering the individual components of property and violence for Ipswich, and explained, "*We added down every one of them* [the columns of data] *and murder is about average, and assault's better than average, and robbery's better than average, and burglary is better than average. All of them is better than average. But the overall trend* (for Ipswich) *is up* (the 5-yearly trends for each of the cities were listed in the table). Ben added, "*We found Ipswich is good against other cities but bad against itself.*" With minimal intervention, Ben's group had developed a generalizable and re-useable system for selecting and ranking data; this sophisticated system was developed across three different problem contexts.

We now report the range of systems that we have found across all of the classes in our studies. In analyzing the aggregation of the ranked sneaker data, we have identified five distinct systems that students have developed. Similar to the development shown by the small group discussed above, these systems are further developed by the students as they progress through the modeling sequence. Here, we are summarizing just the variations in models that occur in the first task. In general, we have found that at least three of these systems will emerge in most classrooms.

The first system to determine an aggregated class rank is a frequency-based strategy. The factor with the greatest number of one's becomes the first ranked and so on for the second and third factors. The students were easily able to identify the highest and the lowest ranking factors by this system as well. What became problematic with this system was the selection and identification of a rank for the factors in the middle; this was the problem encountered by Ben's group, above. These middle factors were often assigned ranks using an estimation strategy that yielded a "close enough" rank from the perspective of the students. An alternative strategy, such as what Ben proposed, that is occasionally used is to use successive pairwise comparisons. So if two factors are competing for the same rank on the list, these two factors are compared in terms of the relative order in which they appear on all the original lists. The factor with the greater number of higher rankings is then ranked higher. We note that this notion of pairwise comparisons is mathematically powerful and useful. However, it does not become a strategy that persists across the sequence of tasks. Finally, we note that the difficulty in resolving the problematic issue of the middle ranks often precipitates a rejection of this model and a move to one of the other systems below. Hence, in most cases, this frequency-based model is an early way of thinking about the problem that is later abandoned as the students strive to accommodate all the data into their system.

A second system developed by students is to arithmetically average the ranks of each factor and then to explicitly re-rank those averages. Significantly, this system indicates a shift from seeing the ranks as labels or as positions on a list to seeing them as quantities that can be operated on by averaging. The factors now have explicit numerical values, rather than implicit places in a list. In turn, the newly constructed quantities of the average rank are assigned explicit numerical ranks. In one instance, we found that the occurrence of a tie in ranks led to a refinement of the system. The tied factors were assigned a common rank, and the next rank was skipped. At the time, we were surprised at this solution, since to the best of our knowledge students had not had any formal instruction in methods to resolve ties. In subsequent reflection, we surmised that the students may have had experience with this solution from sporting events in which this solution is a common strategy.

A third system developed by students is to simply total the ranks of each factor and then to explicitly re-rank those totals. This system reflects the same shift in thinking about the ranks as does the "averaging" system above. However, it would appear that the students are focusing on the meaning of the quantities in a slightly different way. Occasionally students will express the total as "the number of points" attributed to the factor.

A fourth system that is devised explicitly ranks the initial data so that a factor that is "highest" on the list (i.e. is ranked number one) receives the greatest number of "points." Similarly a factor that is "lowest" on the list is given the least number of "points," namely one. We see the students in this case explicitly shifting from a label or position on a list to a rank quantity ("points") that measures the relative merits of the factor. These ranks are then summed (similarly to the third system) and these totals are re-ranked in reverse order so as to keep the "best" factor in the first position, with the greatest number of points.

A fifth system used to determine an aggregated class rank simply ignores the group data and resorts to a “voting” strategy. This occurred in a classroom where the discussion of how to combine the lists was initiated not in the small groups, but in a whole class discussion. Hence, a salient feature of the ranks became the values given by each individual class member, rather than by the groups. Some students argued that this strategy was “fairer” than using the group data since it took into account the opinion of everyone in the class.

Concluding Points

The above five systems reflect the thinking of various groups of students about the task of aggregating data that are in ranked lists. These are the initial models that are elicited by the context of the task and importantly, are later developed by the students with minimal intervention from the teacher. As we saw in our case study, Ben’s group progressed from an initial frequency-based and at times, subjective strategy, to mathematizing elements in lists by totalling ranks, and then to a sophisticated “mean-split” system that was generalizable across tasks.

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UNPACKING A CASE STUDY: UNDERSTANDING TEACHER EDUCATORS AS THEY UNDERSTAND THEIR PRE-SERVICE SECONDARY TEACHERS

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The challenges facing those who seek to prepare mathematics teachers are well established in the literature. Most of the research to date has focused on the perceptions and understandings of pre-service teachers, but not on the perceptions and understandings of teacher educators. In this study, we explore how teacher educators understand their pre-service teachers as these pre-service teachers attempt to make sense of teaching through the investigation of a multimedia case study of practice. We found that in using the case study, teacher educators elicited pre-service teachers' thinking about the complexities of the teacher's role in small group work, about the limits of their abilities to extend lesson ideas, and about the value of revealing teacher's reflections.

Introduction

Mathematics teacher educators face several practical problems in their work with pre-service teachers: finding sufficient high quality classrooms for placements, developing robust understandings of mathematics among pre-service teachers, supporting their reflections on students' mathematical thinking, and developing images of practice that go beyond "telling clearly." The number and intricacy of theories attempting to model mathematics teaching has increased substantially over the last two decades (Koehler & Grouws, 1992). However, it is only recently (e.g., Cooney, 1999; Cooney, Shealy & Arvold, 1998; Simon, 1995) that theories of teacher development are beginning to emerge that go beyond theories that describe models of effective teaching to theories that explain the nature and the development of teachers' knowledge.

Such theoretical work appears to hold promise for informing the practical work of teacher educators. Yet much of the current research focuses primarily on pre-service teachers' mathematical content knowledge, their beliefs, and their pedagogical content knowledge. There is very little research on the teacher educators who work with these pre-service teachers. This study intends to contribute to the emerging theories of teacher development by examining the work of teacher educators as they understand the development of their pre-service secondary teachers. In this study, we investigated the nature of the implementation of a classroom-based, multimedia case study that was used by teacher educators as part of the professional preparation program of pre-service teachers. We seek to understand how teacher educators understand their pre-service teachers as these pre-service teachers attempt to make sense of teaching through the investigation of a multimedia case study of practice.

Theoretical Framework

Our theoretical framework is grounded in two areas of research related to teachers' professional development. The first area involves research regarding the complexity of classroom environments while the second involves research regarding the use of case studies in supporting the development of effective practices within such complex settings. The complexity of classroom environments presents particular problems for the novice teacher whose limited experience and knowledge make it difficult to effectively observe the complexity of interactions that occur, often with great rapidity, in a typical classroom. Beginning teachers are often concerned with issues of classroom management and the planning of lessons as important priorities. They may not focus on the subtleties of understanding student thinking or the nuances of facilitating group work and class discussion or the tradeoffs inherent in teachers' decisions.

The complexity of these interrelated issues is addressed by Simon (1995) as he describes mathematics teaching as the cyclic interrelationships of teacher knowledge, thinking, decision making, and mathematical activities, all of which are influenced by the teacher's understanding and evolving hypotheses about students' learning. Cooney (1999) notes the importance of examining the contexts through which teachers develop and use their knowledge. He states that "whatever lens we use to describe teachers' knowledge, that lens must account for the way in which knowledge is held and the ability of the teacher to use that knowledge in a reflective, adaptive way" (p. 171). Teacher educators are faced with the challenging task of simultaneously meeting the practical needs and concerns of their pre-service teachers while supporting their professional development along lines that will deepen their mathematical knowledge, develop their understanding of children's reasoning, and enhance their ability to reflect upon their decisions and actions in the classroom.

The second perspective that is brought to this study is the notion of the case study as a vehicle for understanding complex practices such as law, medicine, engineering, and (more recently) education (Barnett, 1998; Merseth, 1991). In developing approaches for using case studies in the professional preparation and development of teachers, broad and varying appeals are made to the potential of case studies. These appeals range from the potential for providing paradigmatic exemplars of practice to providing a means of understanding theoretical principles, while bridging the gap between theory and practice (Sykes & Bird, 1992). Others have suggested that the primary purpose of case studies is to support the development of critical analysis and informed decision making. A key characteristic of a case study is that it is embedded in the context of teaching (and schooling) with all its concomitant complexity, ambiguity, and incomplete information. As Feltovich, Spiro and Coulson (1997) have argued, the knowledge base of teaching is an ill-structured domain and as such is best learned by a criss-crossing of the landscape through the study of cases of practice. It is precisely within the complex, ambiguous, and partially understood context of practice that teachers have to make reasoned judgments and decisions for action. Learning through cases studies, it is argued, promotes teachers' understanding of the complexities of practice and of the need to become more analytical about the data of classroom practice (Wassermann, 1993). It is this paradigm—that case studies are a site for reflection and analysis by those preparing to become teachers—that guides the analysis in this study. We wish to examine how teacher educators understand the thinking of pre-service teachers as they reflect on and analyze a case study of practice.

Methodology and Data Analysis

This qualitative case study is part of a larger research project on the use of multimedia case studies by mathematics teacher educators to support the professional growth and development of pre-service secondary mathematics teachers. The teacher educators participating in this study were all experienced faculty, one at a small college and two at mid-sized universities. For purposes of anonymity, all results are reported in a single gender. The pre-service teachers were graduate and undergraduate students in the final stages of their preparation for full-time student teaching. Most were involved in classroom observations and had some limited teaching experiences. All of the participants in this study had a copy of the CD-ROM "Ranking Data to Make Decisions: The Case of the Sneakers Purchase" (Bowers, Doerr, Masingila & McClain, 1999). These multimedia materials were intended to capture the artifacts of practice in a 7th grade class of 23 students in an urban public school over the course of a one and a half day lesson.

The purpose of the case study lesson was to engage middle school students in the collaborative analysis of real world data in an effort to make a mathematically viable group decision about the most important factors to consider when purchasing a pair of sneakers. The problem first involved identifying criteria that might be used when deciding on a pair of sneakers to be purchased. The students were then asked to rank order the criteria from most important to the least important. After each of the six groups ranked the list of eight criteria, students were challenged to develop a system to aggregate all of the groups' lists into one final, ordered list of criteria. The case study teacher then called upon groups of students to present their ranking systems to the class. The case study materials included background information on the school, the teacher's lesson plans, the teacher's anticipations of the lesson and her reflections after the lesson, video of the whole class discussions and small group interactions, a scrolling transcript that was linked to the video, copies of student work, and related mathematical activities.

The teacher educators had a facilitator's guide that suggested some ways that the case study could be used and contained a set of discussion questions that organized the issues of teaching and learning mathematics around four themes: planning, facilitating group work and whole class discussion, understanding student thinking, and mathematical content and context. These four themes were intended to guide and support the reflection and analysis by the pre-service teachers without overly constraining or dictating how the case study would be used by the teacher educators. Each teacher educator used the materials for a minimum of three to four class hours over at least a two week period.

Both the teacher educators and the pre-service teachers completed questionnaires that were designed to understand (a) how the case study was used and the goals of the teacher educator, (b) the background and experiences of the pre-service teachers, and (c) the salient issues in the case study for the teacher educator and the pre-service teachers. In addition, we conducted semi-structured interviews with the teacher educators to probe the issues that were raised through the use of the case study materials and to better understand how the teacher educator perceived the relationship of those issues to the professional growth and development of the pre-service teachers. The pre-service teachers completed several written assignments based on study questions from the facilitator's guide and an essay on the characteristics of effective teaching. The questionnaires, interviews, assignments and essays constituted the data corpus for this study.

The analysis of this data was conducted in three phases, using inductive qualitative methods. In the first phase, we coded the responses of the teacher educator and pre-service teachers to the open-ended questions on their respective questionnaires (e.g., “For me, the most valuable part(s) of this case study investigation was . . .”) to identify emerging themes for the pre-service teachers. In the second phase, we analyzed the students’ written assignments and essays for instances of these themes and other possible themes that may not have been addressed in the responses on the questionnaires. We also analyzed the teacher educators’ interviews, seeking elaborations and instances of these themes and identifying key issues from the teacher educators’ perspectives. In the third phase of our analysis, we compiled profiles on the use of the case study by each of the teacher educators and identified those themes and issues that were critical for the teacher educator or for the pre-service teacher. These profiles then became the basis for interpreting each perspective in light of the analysis done in the first two stages.

Results

We report the results of our analysis in terms of the issues that emerged from the use of the case study by each of the teacher educators. In the first teacher educator's investigations with the multimedia case study, she was very open-ended in her approach to the case study. She described her pre-service teachers as very "autonomous" learners and simply directed them to "use it how you want to but if I were you I wouldn’t watch the follow up reflection yet, today at all, until later when we’re using this." This teacher educator gave her students these materials as a case for them to investigate from their own perspectives, with only the suggestion that they refrain from looking at the case study teacher's reflections until after they have discussed the lesson in the case. Two central issues emerged for this group of pre-service teachers. First, the pre-service teachers expressed concerns over the case-study teacher's interactions with one of the small groups of students. The pre-service teachers felt that the teacher did not understand what the group was doing and needed to spend more time with that group. They felt that the teacher did not probe deeply enough into their thinking, as she did with some other groups. Furthermore, they found that this lack of probing was a mismatch with the teacher's professed belief in the value of listening to students in order to move the lesson forward.

The second issue that emerged for this group of pre-service teachers was related to the judgements that they were forming about the case teacher's decisions. These pre-service teachers became somewhat more tentative in their judgements about the teacher as their analysis of the case progressed. They wanted to see all the other groups, not just the three selected for the case study; they wanted to know more about what happens next; they expressed a need to have "the whole picture." They noted that "she has so much to do" and that "it is hard to run a classroom in that way." The teacher educator saw these responses in terms of how the pre-service students were beginning to "see the complexities" of the classroom.

The second teacher educator intended to "emphasize the value and importance of anticipating students' thinking" with her pre-service teachers. In this teacher educator's class, the two central issues that emerged were related to anticipating children's thinking and on the interactions between the teacher and the students. For the first issue, the teacher educator directly encountered the difficulties that pre-service teachers have in anticipating children's thinking. The teacher educator had spent a class session focusing on the mathematics of the case study and found that her pre-service teachers had difficulty "thinking about how else middle school kids might solve" this problem. Later, since the case study was only a one and a half day lesson, the teacher educator involved her students in developing the mathematical lessons that would follow, in order to bring out the ideas of "weighted averages." The teacher educator felt that the mathematics of this concept was significant content that her pre-service teachers needed to understand both mathematically and pedagogically. The activity of developing a next lesson revealed to the teacher educator the pre-service teachers' "beliefs in action." She found that they would suggest, for example, that the lessons needed to mix different modes of instruction or types of assessment, but that they could not articulate why this should be done. The pre-service teachers could not articulate "the criteria that they should use to decide what kinds of activities they should use" to develop the mathematical ideas. The teacher educator reflected that she had not anticipated how revealing of the pre-service teacher's thinking this activity would be, that it was very valuable for her to see them engage in this type of planning, and that she was then thinking about how to bring this in as a component in the methods course.

The second issue for these pre-service teachers and their teacher educator was the focus on the interactions between the case-study teacher and the seventh grade students. The same incident of the teacher's interactions with one group emerged as significant. The pre-service teachers sensed that the case-study teacher was more passive in her actions with that group and that she did not pursue or continue with questions. This led the pre-service teachers to speculate as to why that might be. As with the first teacher educator, this group was not able to come to any definitive conclusion as to the reasons why the case study teacher acted as she did. This then led to a discussion of "what role the teacher might have in the interacting with the students, what kind of information maybe teachers should be keeping in mind and in relationship to her lesson plan." In this way, the teacher educator acted to encourage a level of analysis of the role and the thinking of the case study teacher among her pre-service teachers thereby supporting their development as practitioners who will analyze and reflect upon their decisions.

In the third teacher educator's classroom, the first issue that emerged was that the pre-service teachers were struck by the reflections of the case study teacher. The teacher educator noted that her pre-service teachers found the reflections "very focused and purposeful." One of the pre-service teachers observed that the case study "gave me a more general but paradoxically more specific idea of what goes into effective teaching practices." The teacher reflections in other video-based materials that this teacher educator had used were "generic comments" and "almost in the order of endorsements." In contrast, the reflections in this case study showed reflective, careful planning and the teacher's thinking about what she thought her students were doing. The teacher educator commented: "We teach them [pre-service teachers] about teaching, but we don't actually show them teachers practicing, reflecting, and evaluating." In other words, the artifacts of the teacher's plans and her reflections went beyond what the pre-service teachers had found in other video materials, observation experiences, and readings. For the pre-service teachers, the reflections appeared to provide an important link to the relationship between the planning and anticipation and the actual outcomes of a lesson. As one pre-service teacher wrote: "She provided insights on what she had expected and what actually happened." Another observed that the reflections "let you know what the teacher was trying to do and what the teacher accomplished. It gave you insights into the lesson." Analyzing the teachers' reflections supported the pre-service teachers in understanding how the case study teacher's planning affected the actual lesson.

The second issue that emerged for this teacher educator and her pre-service teachers focused on the complexities of managing small group instruction. The teacher educator expressed the pre-service teachers' recognition of the complexities and their desire to learn more in one student's comment that "I'm going to try to find somebody that knows how to do that so I can watch it more because it looks pretty complicated." The teacher educator found that the case study helped her to reveal the role of the teacher in interacting with small groups. The pre-service teachers came to recognize that the teacher's interactions were purposeful and that this as an explicit role of the teacher: "They sensed that that was a very purposeful thing on the part of the teacher that was made possible because of the careful preparation in which the teacher thought about what she wanted the students to do and thought about the kinds of expectations." At the same time, the teacher educator found that the pre-service teachers were "anxious [because] they didn't know enough yet to do anything more than appreciate it [the teacher's role]." The teacher educator felt that her students had a greater appreciation of the teacher's role, but since they didn't know enough yet to construct a plan and implement it themselves, many hoped to work in their student teaching with a teacher who did.

In addition to these two issues, the teacher educator reported on the distinctive quality of engagement of the pre-service teachers with the multimedia materials. In addition to the immediacy and the reality of the context, the teacher educator commented that the case study had "incredible depth" and noted that "it takes a long time to unpack it as information and then you have to repackage it into your own thinking for professional training." She anticipated having even better results with her students now that she has experienced the depth of the case study.

Discussion and Conclusions

In each of these three cases, the teacher educators and their pre-service teachers focused on elements in the teacher-student interactions in the group situations. With the first group of pre-service teachers, this was accompanied by more tentative judgments about teaching. With the second group, the pre-service teachers began to analyze why the case teacher might be interacting with the students in this particular way. The shift towards tentativeness in judgment on the part of the pre-service teachers and the awareness that their perspective is limited suggest that these pre-service teachers are beginning to appreciate the complexity and difficulty in understanding classroom interactions. With the third group, the pre-service teachers focused explicitly on the complexity of the role of the teacher and the relationship between her plans and purposes and her subsequent interactions with students. The teacher educator in this instance seemed satisfied that an important step had been taken by her students in appreciating the planning, knowledge and skills that enter into effective interactions with groups. However, she also recognized, as did her pre-service teachers, that the analysis of the case study alone had not prepared them to plan and implement effective group interactions. Overall, we see these responses to the case study materials as leading to an appreciation of the difficulties of a teacher's interactions with groups of students and insights into the complexities of the classroom.

The second group of pre-service teachers and their teacher educator focused on the mathematics of the case. The teacher educator found that the pre-service teachers had difficulty in anticipating middle school student's responses and that their own planning for follow up actions was revealing of the limits of their beliefs about teaching. In this instance, the pre-service teachers were not able to move beyond their own actions to articulate rationales for pedagogical strategies. The third group of pre-service teachers and their teacher educator focused on the case-study teacher's reflections. The case study teacher appeared to provide reflections that were more focused and purposeful to the pre-service teachers, moving beyond general reflections. The analysis of the case study supported the pre-service teachers in the recognition and perhaps valuing of reflections tied to specific teaching. The case study teacher's reflection revealed the link between the teacher's plans and anticipations and the actual outcomes of the lesson.

We make no claims that the analysis of the reflections helped the pre-service teachers become more reflective in their own subsequent practice. Rather, we argue that these instances of analyzing and reflecting on the practice of the case study teacher provided an opportunity for the teacher educator to support the development of pre-service teachers' reasoning about the complexities of practice. The analyses on the part of the pre-service teachers provided the teacher educators with insights into their students' thinking. The case study materials appear to be effective as a resource that provides a site for pre-service teachers to analyze practice while revealing the strengths and limitations of their analyses to teacher educators.

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THE CONSTRUCTION OF ABSTRACT KNOWLEDGE IN INTERACTION

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We take abstraction to be an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. Abstraction is thus a context dependent process. During group work, peer interaction is an important component of the context. In a previous publication, we proposed a model for processes of abstraction. The model is operational in that its components are observable epistemic actions. Here we use the model to analyse an interview with a pair of grade seven boys collaboratively constructing an algebraic proof. The analysis of the interview reveals subtle links between the abstraction process and the peer interaction.

Abstraction is a central process in learning mathematics; however, it is notoriously difficult to observe. In a previous paper (Hershkowitz, Schwarz & Dreyfus, 2001, referred to below as HSD), we have proposed a model for abstraction that is operational in the sense that its components are three observable epistemic actions. The model considers abstraction as a process occurring in context. Although our outlook is theoretical, our thinking about abstraction has emerged from the analysis of experimental data from the CompuMath curriculum development project (Hershkowitz & Schwarz, 1997). Group work in a computer-rich environment is a prominent feature of CompuMath and thus part of the context for the processes of abstraction we observe. In this paper, we briefly review our definition and model of abstraction (see HSD for a more detailed description), and illustrate them by means of processes of abstraction by an interacting pair of students working on an algebra problem with a spreadsheet.

Mathematics educators have proposed that abstraction consist in focusing on some distinguished properties and relationships of a set of objects rather than on the objects themselves. Abstraction is thus a process of decontextualization. According to Davydov (1972/1990), on the other hand, abstraction starts from an initial, undeveloped form and ends with a consistent and elaborate final form. Similarly, Ohlsson and Lehtinen (1997) see the cognitive mechanism of abstraction as the assembly of existing ideas into more complex ones. Noss and Hoyles (1996) go even further. They situate abstraction in relation to the conceptual resources students have at their disposal and see it as attuning practices from previous contexts to new ones. Therefore, according to Noss and Hoyles, students do not detach from concrete referents at all. Leaning on ideas of these and other authors, we define abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The use of the term activity in our definition of abstraction is intentional. The term is directly borrowed from Activity Theory (Leont'ev, 1981) and emphasises that actions occur in a social and historical context. The reorganisation of knowledge is achieved by means of actions on mental (or material) objects. Such reorganisation is called vertical (Treffers and Goffree, 1985), if new connections are established, thus integrating the knowledge and making it more profound.

According to this definition, abstraction is not an objective, universal process but depends strongly on context, on the history of the participants, on their interactions, and on artefacts available to them. As abstraction is an activity consisting of actions, our research included the identification of actions involved in abstraction. We focussed on epistemic actions, that is actions relating to the acquisition of knowledge (Pontecorvo & Girardet, 1993). In many social contexts, such as small group problem solving, participants' verbalisations may attest to epistemic actions thus making them observable. The three epistemic actions we identified as related to processes of abstraction are Recognising, Building-With and Constructing, or RBC.

Constructing is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new structure to which the participants become acquainted. *Recognising* a familiar mathematical structure occurs when a student realises that the structure is inherent in a given mathematical situation. The process of recognising involves appeal to an outcome of a previous action and expressing that it is similar (by analogy), or that it fits (by specialisation). *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. The same task may thus lead to building-with by one student but to constructing by another, depending on the student's personal history, and more specifically on whether or not the required artefacts are at the student's disposal. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity: In building-with structures, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, it is indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct.

The three epistemic actions are the elements of a model, called the dynamically nested RBC model of abstraction. According to this model, constructing incorporates the other two epistemic actions in such a way that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. The genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; (c) the consolidation of the abstract entity through repeated recognition of the new structure and building-with it in further activities with increasing ease. We have argued in HSD that this model fits the genesis of abstract scientific concepts acquired in activities designed for the special purpose of learning. In such activities the participants create a new structure that gives a different perspective on previous knowledge. The model is operational: It allows the researchers to identify processes of abstraction by observing the epistemic actions and the manner in which they are nested within each other.

In the remainder of this paper, we will illustrate the model and its use for studying processes of abstraction by an interacting pair of students. For this purpose, we focus on a pair of grade 7 boys who will be identified as Yo and Ra, or collectively as Yo&Ra. These students' CompuMath algebra curriculum consisted of a sequence of activities, most of them with a spreadsheet, in which they learned to use algebra and the spreadsheet to express generality. On the other hand, they were not asked to justify general properties by using algebraic manipulation. The students usually worked in pairs. In an interview situation, Yo&Ra were presented with an activity that presented a definite potential for abstraction to them. The activity was designed for students from whom the use of algebra for proving properties could possibly be expected but who had never actually done it. The activity was intended to lead students into a situation, in which they felt the need to justify a property whose proof requires algebraic manipulation. Students were asked to investigate properties of rectangles of same type as the following ones:

7	13
9	15

3	9
5	11

After creating (in the spreadsheet) a 'seal' that generates such rectangles upon input of any number into the upper left cell, and after discovering and investigating properties of such rectangles, students' attention was drawn to the difference between the products of the diagonals. When they claimed that this difference equals 12 for all such rectangles (the diagonal product property or DPP), they were asked to justify their claim. The easiest way of justifying the DPP is to use algebraic manipulation and compare $X(X+8)$ (the expression for the main diagonal) to $(X+6)(X+2)$ (the expression for the secondary diagonal). While reorganising their knowledge so as to arrive at a proof of the DPP, this activity presented two opportunities for abstraction to the students. The first such opportunity is the construction of the extended distributive law $(a+b)(c+d)=ac+ad+bc+bd$: The students had never used that law yet but needed it in the present activity to transform the expression for the secondary diagonal. The second opportunity for abstraction more global in that it concerns the entire proof task, namely the establishment of the general perspective that algebra can serve as a tool for the justification of general properties.

In order to study abstraction by interacting pairs of students one needs to study simultaneous cognitive and social processes. For this purpose, we videotaped and transcribed Yo&Ra's work on the seals activity and then carried out two analyses of the interview protocols, one that analyses the cognition and one that analyses the interaction. Our aim was not to give precedence to either of these analyses, but to carry them out as independently from each other as feasible, and then to compare the resulting patterns.

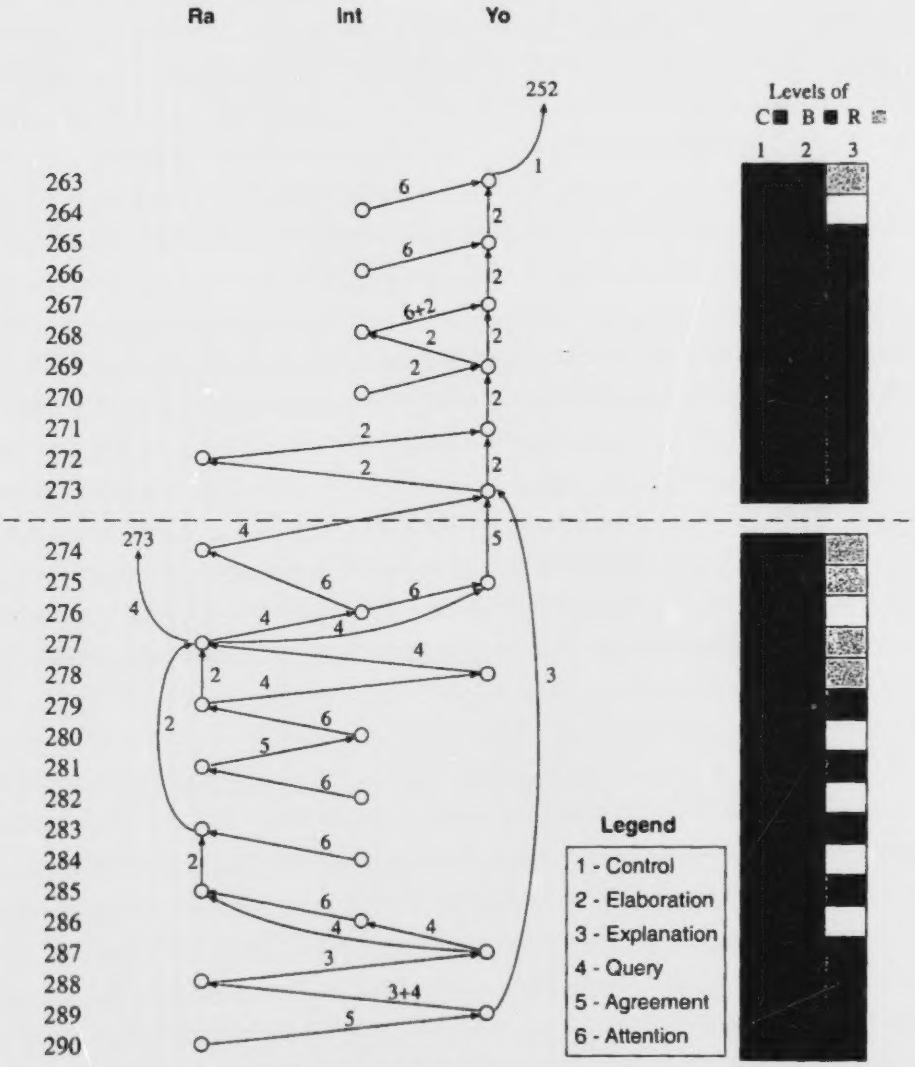
We first produced a coarse segmentation of the protocol into cognitive segments. Next, we proceeded to two independent analyses. On the one hand, we identified interaction patterns between the peers. We categorised the students' conversational moves according to their function into six categories to be discussed below. On the other hand, we closely followed the methodology we used in HSD for the identification of the students' epistemic actions. For pairs of students, this identification poses problems, which were not present in the previous study. Because of the subjectivity of the epistemic actions, what is constructing for one student may be building-with or recognising for the other. We classified such cases as constructing for the pair.

Yo&Ra began by generalising the pattern of the seal so that the spreadsheet produces the entire seal from the left upper cell. When asked to find properties of the seal, they found many simple ones and some more complex ones. In particular, they found the DPP and verified it inductively by substituting numbers in the left upper cell of the generalised seal. When asked to justify the DPP, they produced, after some difficulties, the algebraic expressions X(X+8) and (X+2)(X+6) for the diagonal products. They were unable to progress further because they faced the algebraic obstacle of comparing these expressions. The interviewer's suggestion to end the interview triggered the segment Y263-R290, transcribed below, and three more segments, which together constitute a justification of the DPP.

Yo263 *Oh, I know: You can do the distributive law here.*
Int264 *Yes.*
Yo265 *X times 8 plus X times X ...*
Int266 *Yes.*
Yo267 *Here one can do ...*
Int268 *Here I'll write X times 8 ...*
Yo269 *Yes, plus ...*
Int270 *Plus X times X, yes?*
Yo271 *Now, here one can do X times X ...*
Ra272 *...plus X times*
Yo273 *... plus X times 6 plus 2 times X plus 2 times 6.*
Ra274 *Why? You are using too many factors!*
Yo275 *No, it's OK.*
Int276 *Yes* [writes], *don't you agree?*
Ra277 *No, there are too many factors!*
Yo278 *It's correct!*
Ra279 *We already used the ..., I don't understand what the 8 is for, first of all*
Int280 *This?*
Ra281 *Yes!*
Int282 *Ah, yes!*
Ra283 *We have X times X ...*
Int284 *Yes!*
Ra285 *plus X times 6, and then the normal continuation will be X times 2.*

Int286 *X times 2, OK!*
Yo287 *Why X times 2?*
Ra288 *Because I go according to how you do it, then this is X times X, X times 6, X times 2...*
Yo289 *No, this is X plus 2. Look, you do, this is the distributive law, and you do X times X plus X times 6; now you pass to the 2; 2 times X and 2 times 6.*
Ra290 *Ah, logical! I got it.*

Because of space limitations, we analyse only this segment in the present paper, although some of the conclusions we will draw will be partially based on other segments as well. Our main tools of analysis are presented in the following figure:


Ra
Int
Yo
252
Levels of
C ■ B ■ R ▨
1 2 3
263
264
265
266
267
268
269
270
271
272
273
274
275
276
277
278
279
280
281
282
283
284
285
286
287
288
289
290
273
1
2
3
4
5
6
6+2
3+4
Legend
1 - Control
2 - Elaboration
3 - Explanation
4 - Query
5 - Agreement
6 - Attention

The analysis will be presented in three stages: Interaction patterns, epistemic actions, and relationships between them.

Interaction patterns: At the beginning of the interview Yo and Ra both have the mathematical curiosity and the drive to complete the mathematical activity. In addition, each of them is conscious about his mathematical potential, and likes it to be seen by the other and by the interviewer and even the future video observers. At the same time each of them is quite aware of his friend's mathematical ability. Both students are assertive and try to convince the other from time to time.

Details of the interaction patterns in this segment are coded in the central part of Figure 1. They show an interaction pattern that is very typical for Yo&Ra. Although the "seen interaction" indicates mostly individual work done by each student, the conclusions are at the end agreed on by both of them. A thorough analysis suggests a quite intensive type of interaction between them: In 263, Yo makes a proposal (category 1) that he elaborates step by step (category 2), while Ra is silent but follows very carefully the proposal and the elaboration. Then (274) Ra starts his elaboration of the same proposal by opposing it (category 4). This opposition expresses his struggle to understand the elaboration, which Yo just completed. Only then Ra elaborates it at his own speed. He completes this process by expressing his agreement with the process (290). During Ra's elaboration Yo is silent. Each of them has the need to elaborate a given idea in his own way. By remaining silent for a while, each student respects his peer's need. But both of them reach a point where they refer explicitly to the other's elaboration and criticise it, until they reach an agreement (category 5). This agreement appears within a collaborative explanation (category 3) of the original claim (Y263, as elaborated in Y273).

Epistemic actions of abstraction: In HSD we showed that constructing is a combination of the three epistemic actions where recognising actions are nested in the two others, and building-with actions are nested in constructing actions. Here, we show that the nested model is even more intricate. The constructing action may be quite long and contain shorter segments, which themselves are constructing actions. For example, in the case of Yo&Ra the main constructing action (which we will name C_1) occurs along the entire justification process, and includes the construction of the extended distributive law (which we will name C_2, see the right side of the Figure). C_2 is nested in C_1. Building-with and recognising actions are nested in C_1 as well as in C_2. In other words, constructing actions, like building-with and recognising actions, may be nested in a more global constructing action.

The Figure (right hand side) describes the model for our segment. Level 1 represents the whole process of C_1 (only part of it can be seen in this segment), in which the epistemic actions of level 2 are nested. In the present segment, all level 2 actions are constructing actions (C_2). In other segments, building-with and recognising actions also occur at level 2.

The construction of the extended distributive law starts with Yo's breakthrough suggestion to use the distributive law. The students both recognise the structure of the expression X(X+8) as appropriate to apply the (simple) distributive law. Yo then builds-with the elements of (X+6)(X+2) and obtains the correct expansion (Y273); in other words, he constructs the expanded distributive law. Although he makes no explicit reference to the simple law, we surmise that the immediately preceding use of the simple law has guided him in building the more complex expression. Now Ra takes over. After being momentarily confused by the many addends, he goes through the same constructing process Yo went through, step by step building up the expanded law with the elements of the expression (X+6)(X+2). This lower level (C_2) construction does not occur in the void but as a crucial part of the justification of the DPP. It is therefore nested in the higher-level (C_1) construction of the algebraic justification of the DPP.

From the beginning of their struggle to construct the DPP justification, the students are at the level of the C_1 construction. On this C_1 level their progress is controlled and monitored by their awareness and their need to accomplish the DPP justification. During this process, they face algebraic obstacles, which are quite unfamiliar to them. Overcoming these obstacles necessitates the construction of new mathematical structures, which are the C_2 level constructions. These C_2 constructions are controlled only indirectly by the motive of the C_1 construction. The students enter these C_2 level "adventures" without any knowledge about the needed mathematical structures, and they have to discover as well as to construct them. The C_2 constructions thus make the C_1 level into a deep holistic construction, which goes beyond the specific construction of the DPP justification, and in which the constructions of unfamiliar algebraic structures are nested. In this sense C_1 is an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure, which fits our definition of abstraction.

Relationships between epistemic actions and interaction patterns: The Yo&Ra interview is a case of collaboration between the two students. This collaboration finds its expression in the students' cognitive RBC actions on one hand and in their pattern of interaction on the other hand. The RBC flow and the flow of the interaction patterns are developing in parallel. There are no clear causal relationships between the two of them. It rather seems that both of them are different "indications" of the single collaborative process revealed in the interview. Our understanding of this process is dependent on our understanding of both, the RBC flow and the interaction patterns, as well as the relationships between them. In the following we will try to throw some light on these relationships.

Globally, Yo&Ra share the activity, because they share the motives of searching for the mathematical properties of the "seals" and of justifying these properties; they also share the justification processes themselves, as well as their conclusions. We claim that the students constructed a global structure of meaning for algebraic justification (the C_1 action, which can be seen only partially in this paper). From the interaction perspective, there are long-range control (category 1) and explanation (category 3) arrows that can be considered as the "glue" that ties this justification process together. The diagram of the entire process reveals that these long-range interaction arrows are connected to the beginning and/or end of the main cognitive segments. The part of the diagram in the figure of this paper shows a category 1 arrow emanating from the beginning of the segment to an earlier segment, and a category 3 arrow concluding this segment.

Hence, long-range interactions occur between statements that are milestones in the RBC flow. In this sense the interaction pattern has nesting characteristics similar to the RBC flow, where various patterns of interaction are nested in the overall global collaboration. And the cutting edges of the interaction patterns are those that at the same time define the different segments of the RBC flow. In other words, the cognitive segmentation we started out from fits the interaction as well.

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THE CONCEPT OF PARAMETER IN A COMPUTER ALGEBRA ENVIRONMENT

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Abstract

The study that is presented here concerns the learning of algebra in a computer algebra environment and, more specific, the concept of parameter. Students of 14 – 15 years old used a TI-89 symbolic calculator during a five week period. They studied the parameter in different roles such as placeholder, changing quantity and generalizer. The results indicate that using parameters in the computer algebra environment requires a clear view on the roles of the different letters. Also, the reification of a formula seems to be important for an appropriate instrumentation.

The research project

The study presented here is part of an ongoing research project called 'The learning of algebra in a computer algebra environment'. The general research question of this project is:

> How can the use of computer algebra promote the insight in algebraic operations and concepts?

This question is specified in two sub-questions:

1. How can the use of a computer algebra system contribute to a higher level understanding of parameters as they appear in algebraic expressions and functions?
2. How does instrumentation of computer algebra take place and what is the role of the relation between machine technique and mathematical conception?

Why parameters?

A reconsideration of algebra education is currently taking place in the Netherlands as well as in many other countries. The transition from the informal, reality-bound algebra at the lower secondary level (students of 12 – 15 years old) to the more formal and abstract algebraic skills that are required at the upper secondary level is difficult for many students in the Netherlands.

The conception of the parameter can be a suitable topic to try to bridge this gap. Variables and parameters are in the heart of algebra. The parameter is an 'extra' variable in an algebraic expression or function that generalizes over a class of expressions, over a family of functions, over a sheaf of graphs. The parameter can be considered as a meta-variable: the a in $y = a.x + b$ can play the same roles as an 'ordinary' variable, such as placeholder, unknown or changing quantity, but it acts on a higher level than is the case for a variable. For example, a change of the parameter value does not affect one single point locally, but the complete graph globally. The different roles of the variable are resurfaced, but now at a higher level, and the generic function becomes the object of study. The concept of parameter, therefore, is adequate for enhancing the abstraction of concrete situations, so that the more formal and general algebraic representation can become a natural part of the students' mathematical world.

Why computer algebra?

Taking into account the affordances of technology in general, and the algebraic capacities of computer algebra in particular, it seems obvious to use a computer algebra system (CAS) for the purpose mentioned above. It can serve as a powerful and open algebra environment that allows students to concentrate on the concepts and the problem solving strategy. We conjecture that performing procedures in the computer algebra environment contributes to the development of insight in algebraic operations and concepts such as substitution and the distinction of the different roles of letters. Using the machine, the students don't have to worry about the calculations and this may enhance a more global conception of the problem solving procedures.

On the other hand, however, computer algebra can be demanding in its use. Guin and Trouche (1999, p. 205) pointed out that an adequate use of computer algebra tools requires making explicit the different roles of the letters to a further extent than is the case for paper and pencil work. This kind of explicitness that computer algebra demands is a burden but in the mean time it can stimulate the student to handle the operations more consciously.

Furthermore, from the perspective of Realistic Mathematics Education the integration of computer algebra is not a trivial matter. In Drijvers (2000) the issue is raised whether the development of informal strategies and the process of vertical mathematization, that are so important in this educational theory (e. g. see Gravemeijer, 1994), are stimulated in a computer algebra environment. As far as the informal strategies are concerned, it seems that the computer algebra environment does not support them. For vertical mathematization the CAS seems more appropriate.

Previous research and theoretical framework

Much research has been done into the concept of variable. Usiskin (1988) classified the different roles of variables as unknown, indefinite, generalized number, dynamical variable and parameter.

Not so many studies have been published on the learning of the parameter. Bloedy-Vinner (1994) stresses the hierarchy of substitution and the implicit quantifier structure that is often involved while using parameters. These 'hidden quantifiers' also are described by Furinghetti and Paola (1994), who state that parameters are conceptually more difficult than variables.

Several studies have been devoted to the use of computers for the learning of the concept of variable. Graham and Thomas (2000) successfully used the graphing calculator to stress the placeholder-role of the variable. Boers-Van Oosterum (1990) also claimed to improve the conception of the variable by using different software packages. Brown (1998) used a computer algebra environment for generalization of patterns, for solving equations step by step and for solving number problems. None of these studies used information and communications technology tools (ICT-tools) for the learning of the concept of the parameter, as is done in the project described here.

An important part of the theoretical framework of this study is the theory of the *instrumentation* of ICT-tools. Following Guin and Trouche (1999) and Lagrange (1999), we consider the development of instrumentation schemes as a crucial and non-trivial step in the appropriation of an ICT-tool. In the acquisition of such schemes, technical skills and mathematical conceptions are interwoven. Examples of this complex relationship can be found in Drijvers & Van Herwaarden (in press).

A second part of the theoretical framework concerns the dual character of mathematical concepts, that have both a procedural and a structural aspect. Sfard (1991) uses the word *reification* for the gradual development of a process becoming an object. In the function concept, for example, the process of calculating function values may develop into the image of a function as an object that is represented by a formula or a graph. Sfard and Linchevski (1994) elaborated this for the case of algebra. Close to the idea of reification is the encapsulation that is described by Dubinsky (1991). Dubinsky states that encapsulation of processes into objects is an important step in reflective abstraction. He suggests that performing processes using a computer may stimulate its encapsulation. As a third means of representing the bilateral nature of mathematical entities we mention the procept that has been developed by Gray and Tall (1994). 'Procept' is a contamination of process and concept. The authors stress the flexibility that learners of mathematics need in order to be able to deal with the ambiguity of mathematical notations. In 3+5, the + may be an invitation to perform the process of addition, whereas in $a+b$ the + is only a symbol that defines the object 'the sum of a and b'.

It is our conviction that the theories of reification, encapsulation and procept are very relevant to the learning of algebra and to the instrumentation of computer algebra tools. For the students, the mathematical entities in such an environment may tend to have a structural character, whereas the processes are more distant to the objects than is the case with work using the traditional paper-and-pencil.

Research design and methodology

As a research paradigm, the developmental research method was used (see Gravemeijer, 1994). According to this methodology, the researcher tries to develop (local) instruction theories by means of constructing and developing thought experiments and educational experiments in the classroom situation. This involves a cumulative process of consideration and testing.

The research method was mainly qualitative. The most important data consisted of audio recordings and field notes of classroom observations, audio recordings of mini-interviews with students, and written work of the students. The data were analysed by coding the classroom incidents and solution methods according to previously defined categories. While doing so the most dominant categories came up clearly.

Development of the didactical scenario

A conceptual analysis of the phenomenon parameter led to the identification of three essential steps in the learning trajectory: the parameter as a placeholder, as a changing quantity and as a generalizer. The table below summarizes these steps. Also, it indicates by means of what kind of activities the students are supposed to pass to the next phase and how computer algebra supports these activities.

parameter role	*a in y = ax+b*	*graphic model*
placeholder	*a* contains specific values, one by one	one graph, that can be replaced by another
changing quantity, 'sliding' parameter	*a* walks through a set dynamically	'comic' of the dynamic graph
generalizer, 'family' parameter	*a* represents a set, generalizes over situations	a sheaf of graphs

student activity	*CAS function*
systematic variation of parameter values	solve equations substitute animate graphs
generalization of situations and solutions	graph sheafs solve parametric equations

Classroom experiment: aim and situation

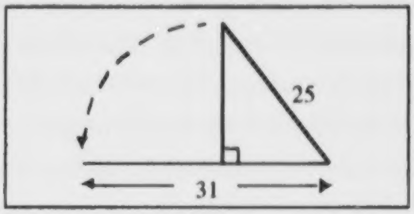
The aims of the classroom experiment were to investigate if the students' conception of parameter would develop according to the didactical scenario and if computer algebra serves as an aid for this. Also, we were interested in the process of instrumentation of the computer algebra tool with respect to the different roles of the letters involved.

The classroom experiment took place during a five week period in the spring of the year 2000. The subjects were 50 students of 14 – 15 years old, divided into two classes. The students were high achieving in general but not specifically in mathematics. As computer algebra tool the TI-89 symbolic calculator was used at school as well as at home. Each class had four mathematics lessons of 45 minutes each week. Because the students had no previous experience with technology such as graphing calculators, using this type of handheld technology was really new to them. Their knowledge of formal algebra was quite limited. For example, the general solution of a quadratic equation had not been taught so far, so the help of the symbolic calculator was needed in case such an equation was encountered.

Episodes of student behaviour

Classroom observations indicate that it was important that students are aware of the different roles of the letters, especially if there are parameters in the equations. We saw that some students found it natural to use parameters to generalize a relation or procedure, whereas others seemed to be confused by several letters in one expression, each having a different role. The differences between the students were considerably, as is shown in the following episodes, that concern the phase when students use the computer algebra environment to solve (systems of) parametric equations.

John and Rob work at the following assignment:



> *The two right-angle edges of a rectangular triangle together have a length of 31 units. The hypotenuse is 25 units long.*
> *a. How long is each of the right-angle edges?*
> *b. Solve this problem in case the total length of the two edges is 35 instead of 31.*
> *c. Solve the problem in general, that is without the given values of 31 and 25.*

At question **c** John and Rob wrote down in their notebooks:

$$a^2 + o^2 = p^2$$

$$o + a = s$$

$$o = s - a$$

$$a = s - o$$

Then they entered into the TI-89:

$$\text{solve}(o^2 + a^2 = p^2 \mid o = s - \ a, o).$$

The wrong letter at the end. The response of the machine was:

$$0 = -s^2 + 2.a.s - 2.a^2 + p^2$$

The boys corrected this by solving this a second time, now with respect to a:

$$\text{solve}(0 = -s^2 + 2.a.s - 2.a^2 + p^2, a)$$

This time John explained the choice for the letter a at the end:

"You want to know the a".

This way they found the solution for a expressed in the parameters s and p.

John and Rob introduced the parameters themselves and generalized the problem solving strategy of the concrete cases of questions ***a*** and ***b*** without difficulties. Their solution schema did not seem to be confused by the presence of the parameters. However, there were some instrumentation problems at the start that may be related to a limited consciousness of the roles of the different letters.

For others, however, the presence of parameters was an extra complication, as illustrates the following observation of Sandra. The assignment was this time to calculate the dimensions of a rectangle with given perimeter and area. Using the viewscreen Sandra tried to demonstrate to the class how the corresponding system of equations could be solved:

$$b + h = s$$

$$b * h = p.$$

First Sandra entered a re-written version of the first equation in itself:

$$\text{solve}(b + h = s \mid b = s - h, b)$$

The machine replied: true.

Rob commented: "She did not use the p, she did not use the second equation."

Sandra changed the command into:

$$\text{solve}(b + h = 20 \mid b = s - h, h)$$

Before the generalization the value of the parameter s had been 20, and apparently she felt the need to return to the concrete case. The result, $s = 20$, is logical but Sandra does not notice that. Then she realises that Rob was right, but she entered $b + h = p$ instead of $b * h = p$:

$$\text{solve}(b + h = p \mid b = s - h, h)$$

The machine replied $s = p$, which is also reasonable. At the end Sandra entered

$$\text{solve}(b + h = s \mid b = p/h, h)$$

and that gave the right answer.

Sandra's behaviour gives the impression that the parameters were an extra, complicating factor in the problem solving process. In earlier situations she had shown that she was able to apply this solution scheme correctly in concrete cases without parameters. It is not clear whether she really perceived a formula such as $b = s - h$ as an object that can be substituted.

Conclusion

The conclusion of the above exemplary episodes is that the didactical scenario to use parameters for generalization was confirmed by some of the students such as John and Rob. Computer algebra was helpful to clarify the problem solving strategy. For others, the use of parameters complicates the situation to a greater extent. In order to be able to solve parametric equations, it seems to be important that students are able to perceive formulas as objects. If a formula invites to a calculation process in the eyes of the student, s/he will find a general solution containing parameters hardly satisfying. Some students were able to distinguish the roles of the different letters, whereas others seemed to be confused by the amount of variables.

The conjecture that performing procedures in the computer algebra environment would enhance the understanding of the global mathematical conceptions behind the procedures was confirmed only to a limited extent. Some students did not overcome the difficulties with the instrumentation scheme for solving systems of parametric equations. The equilibrium between paper-and-pencil work and machine work during the instrumentation process may be quite delicate, and we probably did not pay enough attention to the 'traditional' approach. Meanwhile, the teachers reported that they could benefit from the students' machine experience while treating the solution of quadratic equations after the experiment.

As a consequence for teaching, we would conjecture that the development of both the technical and the conceptual side of the instrumentation schemes deserve explicit attention. This can be done by means of student interaction, classroom discussions and demonstrations, so that the instrumentation process gets a more social character.

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TAKING THE "FORM" RATHER THAN THE "SUBSTANCE": INITIAL MATHEMATICS TEACHER EDUCATION AND BEGINNING TEACHING

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This research paper explores the relationship between initial mathematics teacher education and beginning teaching. It describes a longitudinal study which tracked a cohort of seven student teachers through a one-year, full-time, preservice university-based teacher education course and then as secondary mathematics teachers into their first year of teaching in the Western Cape region of South Africa. Qualitative data in the form of interviews, students' reflective journals and other written materials as well as lecture and classroom observation data were collected over a two-year period. The study found that in constituting their teaching repertoires, beginning secondary mathematics teachers appeared to take the "form" rather than the "substance"[1] *of their mathematics method course. They used a number of discrete tasks introduced there, as well as a professional argot, a way of discussing the teaching and learning of mathematics. There appeared to be a disjuncture between the practices privileged on the preservice course, and those used by beginning teachers in mathematics classrooms. A socio-cultural explanation for this is offered which turns centrally on the extent to which student teachers are given access to the recognition and realization rules of "best" practice.*

Introduction

It is widely recognized that teachers do not always implement in their classrooms, or at least do not seem to implement, the practices they acquire on teacher education courses. If they draw on these courses at all, they appear to take on the "form" of these rather than their "substance". This disjuncture has been variously explained: Lortie (1975) attributes it to educational biography and the 'apprenticeship of observation'; Zeichner and Tabachnik (1981) to school setting; Lacey (1977) to differential engagement by students with their teacher education courses, and Cooney (1985) and Thompson (1992) to a failure to change student teachers' and teachers' belief systems. Less commonly, the seeming disjuncture is related to the structuring of the teacher education courses, be they preservice or inservice (although see for example, Borko et al, 1992 and Eisenhart et al).

This issue is an important one, since the entire rationale for teacher education rests upon its ability to prepare students as classroom teachers. In the South African context most particularly, where the educational system is in so many respects dysfunctional, the manner in which teacher education might be used to change teachers' classroom practice has particular saliency. The professional development of teachers, through both preservice and inservice teacher education programs, is regarded as a central pivot for educational transformation.

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[1] I am grateful to Karin Brodie and Lynn Slonimsky for the formulation of the problem in this way. See Brodie et al. (2000).

Inservice and preservice providers attempt to make available to teachers a privileged repertoire, a particular encapsulation of "best practice". How this is configured clearly varies according to the particular teacher education program on offer, but in all cases the question arises: how are the principles underpinning this privileged repertoire made available to students (or in the case of inservice education, qualified teachers), and to what extent do teachers draw upon it in developing their own teaching repertoires?

The study

These questions provided the focus of a two-year longitudinal study (see Ensor, 1999a, 1999b) that tracked a cohort of seven students through the mathematics method component of a preservice Higher Diploma in Education (HDE) program and then as beginning secondary mathematics teachers into schools. The aim of the study was to provide a theoretical account of the recontextualizing of pedagogic practices by beginning teachers from this mathematics method course. Recontextualizing is a notion drawn from Bernstein (1977, 1990, 1996) and Dowling (1996, 1998), and points to the transformation of discourses as they are disembedded from one social context and inserted into others.

The study was stimulated by a broad theoretical interest in the articulation between sites of practice (such as signaled in the work, for example, of Rogoff & Lave, 1984; Carraher et al, 1985; Lave, 1988; Noss & Hoyles, 1996; Walkerdine, 1988, and others). It was concerned with how privileged forms of knowledge and practice about teaching were made available to student teachers and the extent to which these were used by beginning teachers in classrooms. Drawing on the work of Basil Bernstein (1990, 1996) and Paul Dowling (1998), a model was developed to provide an analysis, from a socio-cultural perspective, of a mathematics preservice teacher education course, of secondary mathematics classroom teaching and the recontextualizing of practices between them.

A variety of data was collected for the purposes of the study. In the first year, I took field notes of sessions of the mathematics method course and interviewed the two teacher educators responsible for it. I also collected a range of materials written by student teachers, such as reflective journals, examinations, and a curriculum project, and I conducted interviews with students. In the second year of the study, I interviewed each of the group of seven teachers four times over the year, and in conjunction with the third set of interviews, video recorded a number of lessons with each teacher on a specific day. I also interviewed the head of the mathematics department in each school, or, where there was no head, a senior mathematics teacher.

From teacher education to classroom teaching

On the mathematics method course, students were exposed *inter alia* to a range of exemplary mathematics tasks, research in mathematics education as well as an approach to mathematics teaching that favored visualization and intuition as a gateway into formal, conceptual mathematics. "Relational" understanding was privileged over "instrumental" understanding. Students were given access to a range of resources: teaching resources such as geoboards and geostrips; the use of history in the teaching of mathematics; ideas for organizing classrooms, as well as exposure to issues such as racism and sexism. This privileged approach to teaching was made available through a range of exemplary mathematical tasks, and through explicit discussion. All of the teaching took place in the university context, however, and student teachers did not watch their teacher educators teach in classrooms, nor, whilst on teaching practice, did they gain the opportunity to put their own practices up for evaluation by mathematics specialists. In terms of the way in which the teacher education course as a whole was structured, students were supervised on teaching practice by lecturers who were not necessarily mathematics education specialists.

Over the period of the method course, students were given access to a *professional argot*, a way of talking about mathematics teaching which privileged, for example, "visualization", "verbalization", "relational over instructional understanding" and learners "discovering things for themselves". A professional argot comprises terms and modes of argument used by members of a profession when either engaged in, or discussing it. Those aspects of the argot that are foregrounded or backgrounded at any point in time, and the level of specificity of the language used, depend on the evoking context. Different features are likely to be foregrounded when discussing with a colleague, for example, than with a layperson. A professional teaching argot provides a student teacher with access to a vocabulary and modes of argument to describe "best practice". Of significance in the particular preservice course of my study was that this argot was elaborated independently of reference to actual classrooms, and thus embodied potential ambiguity about what practices would, for example, constitute "relational" and "instrumental" understanding.

Use of such an argot can be illustrated by the following comments by a student teacher, Thabo Monyoko. In his teaching practice journal and in an interview at the end of the HDE year, Thabo spoke positively about the mathematics method course, which he described as having effected a "complete revolution" in his thinking about teaching. He said he had been exposed to a new approach in terms of which "pupils come to discover some of the things on their own [...]. They actually see how some of the things they do in mathematics is practical and some of the things they discover on their own." He no longer "monopolized classroom activities", "standing in front like a priest", simply giving the formula "raw from the book" so that "people have got to ram it into their heads [..] In the past I would simply give the formula from the book and give them an exercise and they apply the formula, that's all". On his second teaching practice he said he tried "to implement some of, you know, the hands-on approaches [..] I remember I implemented some of these self-discovery approaches by pupils, I mean they were very fantastic, they were very interesting to the pupils and I think my lessons went pretty well, you know."

These ideas were re-iterated by Thabo when I interviewed him as a beginning teacher the following year. He indicated that the mathematics method course had "turned me around" in that he was now more "responsive to students' needs" and interacted with them more instead of "teaching from the front". For him, this meant walking around the class and "finding out what pupils were having difficulties with." This he related directly to lessons which I observed him teach. The following is an extract from the beginning of such a lesson on sequences with Grade 12 students.

> T: Let's say that the sixth term of a geometric sequence is 3125 and the fourth term is 125. Now find the eighth term. [as he speaks, he reads from a textbook and writes on the board:]
> 6th = 3125
> 4th = 125 Find 8th
> [He repeats again, given that the 6th term is 3125 and the 4th term is 125, find the 8th term]
> Now because you don't know the value of a, now remember in a geometric sequence the general term is
> $T_k = ar^{k-1}$ [writes this on the board and speaks as he writes]
> Now in order to find any term of a geometric sequence you must first find the value of a and the value of r. When you have a and r you can then find out the term. Now given that the sixth term is 3125 we can write
> $T_6 = ar^5 = 3125$
> Now let's call this our equation 1 [he writes 1 and circles it after the equation given above]
> Now the fourth term is 125. Therefore
> $T_4 = ar^3$ which is equal to [he turns his head to the class and cocks his head in expectation of a response. Someone says 125] 125 . OK we'll call this equation 2
> (Thabo: extract from transcript of recorded lesson 2)

Thabo continued in this relatively unbroken expository style to solve two simultaneous equations on the board. At one point he turned to ask a student to check a calculation. When he completed the solution to the problem, he turned to the class, and for the first time and ten minutes into the lesson, he asked students by name if they had any problems: "Mr. Nzo, Nyamende, Zola?" He then proceeded to solve another problem on the board. Again, students were required to listen and take notes. Occasionally he posed a question, normally to ask students to calculate for him. After his explanation, he chose a question from the textbook for students to try on their own and walked around the class, discussing with the students in Xhosa and English.

Two interesting issues emerge from this brief discussion of Thabo's practice.

Firstly, there appears to be a variation between Thabo's preferred teaching style and that privileged on the method course, as well as a variation between what he said about his practice and how he actually worked in the classroom. In discussing the lesson afterwards, Thabo pointed to his use of questions, and his circulation around the class towards the end of the lesson, as evidence of the practices he had acquired from the mathematics method course. For him, there was no disjuncture between this course and his own practice, and between what he said about his practice and the way in which he actually taught. Yet, what Thabo said, both as a student and as a teacher, appears to be at variance in both these ways, a variation which was evident across the interviews and classroom practices of all seven beginning teachers. All seven teachers used discrete tasks (exemplary mathematical tasks and pedagogic resources made available to them on the course) and a professional argot of varying range. Teachers tended to deploy this argot, descriptions such as "verbalization", "visualization" and "self discovery", in ways consistent with their *own* practice. Thabo, for example, drew on the professional argot ("not teaching from the front", and facilitating "student interaction") to describe his teaching style, which was in many ways different to the approach developed on the course. So from the viewpoint of the teacher education course, it would seem that Thabo was saying one thing and doing another, taking its "form" rather than its "substance". Yet from Thabo's vantage point, this was not necessarily the case. He had acquired a professional argot and turned it to his own purposes. The ambiguity associated with the transmission of the privileged repertoire made this possible, an issue which I will return to below.

All seven beginning teachers recontextualized a small number of discrete tasks and a professional argot from their HDE method course. However, in interviews with them and in observation of their lessons, I found that they were not able to demonstrate access to the principles of selection, production or evaluation which underpinned this course. For example, they said they could not produce tasks like those introduced on the course, tasks which encapsulated its particular, privileged view of mathematics teaching, and they found it difficult to evaluate their practice in the ways that the teacher educators might do. Putting this differently, I would suggest that these teachers had gained partial access to recognition rules (they could describe aspects of "best practice" via the professional argot, in the ways the teacher educators did) but not realization rules (they were unable to produce tasks themselves which were consistent with the principles underpinning the method course).

Recognition and realization rules

Bernstein (1990) distinguishes between recognition and realization rules in the following way:

> Recognition rules create the means of distinguishing between and so *recognizing* the speciality that constitutes a context, and realization rules regulate the creation and production of specialized relationships internal to that context. (Bernstein (1990), p. 15

emphasis in original)

Bernstein elaborates these rules as part of his code theory. The present analysis is differently motivated methodologically but the distinction Bernstein makes can be related to the present project. Here we are interested in the extent to which student teachers are given access to both recognition and realization rules, for how "best practice" is to be recognized and how it is to be realized in practice. In the sense in which I am using these terms here, access to recognition and realization rules enables appropriate use of a professional argot (that is, its use as the teacher educators might use it) as well as its realization in practice. Access to recognition rules alone provides students with the ability to differentiate "best" from "poor" practice and to describe "best practice" discursively.

As illustrated above, Thabo demonstrated only partial access to recognition rules in that he used the professional argot largely inconsistently with the forms of "best practice" privileged on the mathematics method course. He was also was unable to demonstrate access to realization rules. I have suggested that these issues of access are related to the structuring of teacher education as a form of knowledge.

Teacher education: knowledge forms and pedagogic modes

Teacher education can be regarded as a hybrid of explicit, discursive practices (exhibiting what Dowling, 1998, terms *high discursive saturation*) and implicit, tacit practices (exhibiting low discursive saturation). Highly discursive practices are relatively context independent and can be realized to a substantial degree in language, while tacit practices are more context-dependent, and less easily grasped linguistically. Mathematics teacher education can be thought of as making available to student teachers a form of "best classroom practice" which comprises both explicit and tacit elements. The principles of selection, production and evaluation which underpin "best practice" can therefore be made available explicitly, through language, but not exclusively so. To become a teacher, one needs to watch teachers teach, teach oneself and open one's efforts up for evaluation[2]. Just as the crucial aspect for the transmission of discursive practices is that the generative principles of the privileged repertoire are made explicit in language, the crucial aspect for the transmission of tacit practices is that they are made available and acquired in the site of practice, the school classroom, through demonstration and correction. While it is productive to speak to student teachers or teachers *about* "best practice", we also need to show them what this means, in actual classrooms, and allow them to put their own practice up for evaluation. This provides the basis for acquiring recognition and realization rules.

In the case of my study, students were exposed to "best practice", described in part above, exclusively in a university setting. As I have said, because of the way the HDE as a whole was structured, students did not observe mathematics teacher educators teach in classrooms, nor teach in classrooms themselves and have their performance evaluated by mathematics teacher educators. So while students were able to gain some access to discursive aspects in the university setting, the more tacit, implicit aspects, those which required elaboration and exemplification in classrooms, were not made available. The professional argot was thus invested with considerable ambiguity, and teachers were able to use it to describe practices that were in many ways quite different to those privileged on the preservice method course.

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[2] Cases have been used in interesting ways in teacher education (see for example Merseth, 1996). I would suggest that while the discussion of cases outside of classrooms can effectively provide access to recognition rules, access to realization rules requires this engagement (see Pi-Jen Lin, 2000).

Conclusion

In this paper I have briefly described a longitudinal study which explored the relationship between initial teacher education and classroom teaching. All seven beginning teachers in the study took from the course a professional argot and a range of discrete tasks (or, as some would suggest, its "form") rather than working independently with its underlying principles (its "substance"). This variation is commonly explained in research literature on the basis of educational biography and school setting. I have suggested elsewhere (see Ensor, 1999a) that school settings produced constraints on the teachers of my study in terms of how they developed their teaching repertoires, how they taught, and the extent to which they drew on their preservice course in doing this. School settings, however, did not seem to constrain teachers in a simple or obvious way, and did not appear to be decisive in shaping the recontextualizing of pedagogic practices for the teachers of my sample. A further factor which antecedent literature suggested might be important in shaping the development of teaching repertoires, that of educational biography, was also not decisive. What seemed overwhelmingly to affect recontextualizing was access to recognition and realization rules. Access to recognition and realization rules expands the range of tasks and approaches that teachers can draw on and provides the possibility for the production of new tasks and pedagogic choices. In the case of my study, lack of access restricted the recontextualizing potential to tasks actually encountered by students on the initial teacher education course and a professional argot which teachers deployed selectively, and not always appropriately, to describe their own practice.

This paper has been concerned with why teachers appear to practise differently in their classrooms from the ways privileged on the preservice and inservice courses they attend, and from the ways in which they speak about them. I have attempted to offer a possible reason for this in the particular structuring of teacher education discourse and its modes of pedagogy. Insofar as teacher education occurs exclusively in a site removed from that of classroom practice, it is difficult to make available fully the principles, the recognition and realization rules, that generate any particular view of "best practice". Instead, the latter will stand as a collection of resources for potential, selective recruitment by teachers in forming their individual teaching repertoires. For this reason they may appear to appropriate the "form" rather than the "substance" of best practice.

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A Didactics-Driven Intelligent Tutoring System

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Abstract

Most Intelligent Tutoring Systems (ITS) have remained research tools and have not succeeded in bridging the gap from the laboratory to the school. The Guided Autonomous Learning System (GALS) is a new ITS model that is grounded in a substantiated model of learning, that would provide diagnosis and adapted remediation of errors, while at the same time paying attention to environmental factors and User-Machine Interface. GALS has been applied to a primary school mathematics program that is currently in use in Israeli classrooms. The paper presents the principles underlying the development of the GALS model, and describes the results of its experimental implementation in Grade 1 classes.

Introduction

When the concept of Intelligent Tutoring Systems (ITS) was first proposed, great hopes were raised that efficient models would be developed and implemented in schools (Murray, 1999). In fact, however, only a few such systems have transcended the confines of research models and have actually been implemented in classroom settings (Urban-Lurain, 1995). According to Rosenberg, two main reasons explain the lack of use of ITS in actual education:

1) "The systems are not grounded in a substantiated model of learning".
2) "Testing is incomplete, inconclusive, and in some cases totally lacking" (Rosenberg, 1987).

Although it is clear that a good ITS must be able to handle student errors, Rosenberg points to the models by Burton and Anderson (Burton, 1982 and Anderson, et. al 1985) and asserts that "an exhaustive enumeration of error types is not a model" (Rosenberg, 1987, p. 8). He also argues that "lists of error types are only part of the raw data for a model of tutoring. Unanalysed listings do not illuminate the tutoring process."

The following paper presents a model of a new ITS, that is indeed grounded in a model of learning, and includes diagnosis and remediation tools. This didactically-based ITS has been implemented in the study of mathematics in primary schools in Israel, and the results of its classroom use will be described. At the time of this writing, the GALS ITS is being used by more than 2,000 Grade 1 students.

The Guided Autonomous Learning System (GALS) ITS Model

As early as 1973, Brown noted the positive synergism that could result from combining the power of the new computers and advances in computer languages with research in didactics (Brown, 1973). For this reason, the new GALS ITS uses object-oriented programming and multimedia resources and we incorporated findings from didactic research in its design.

Anderson asserts that "cognitivism does not imply outright rejection of decomposition and decontextualization" (Anderson, 2000). The GALS model applies two basic principles stressed by Anderson:

1. "Assessing learning and improving learning methods requires careful task analysis at the level of component skills, intimately combined with study of the interaction of these skills in the context of broader tasks and environments" (p.3).
2. "Assessing learning and improving learning methods requires research and instruction in contexts that are consistent with the scopes of the skills currently under investigation." (p. 4).

The GALS mathematics curriculum for first grade is comprised of more than 200 activities, each of which deals with one central concept. These activities are arranged in ascending order of difficulty, from the most simple to the most complex. The GALS model was first applied to mathematics rather than to other disciplines because the structure of mathematics enables the development of a logical program structure (Suppes, 1967).

Open models, such as simulations, play a very important role in the learning process; however, it is very difficult to deal with diagnosis and remediation in an open environment. For this reason, the GALS model, in which every activity has a well-defined goal, includes the following types of activities:

1) "Open-ended" activities, i.e. activities that can be approached in many ways and have more than one correct answer;
2) "Semi-open" activities, i.e. activities that have a goal, but which allow students to work in the way that they want;
3) "Closed" activities, i.e. activities that have a goal and only one way to reach it.

The inclusion of open-ended, semi-open, and closed activities in the GALS model enabled the integration of powerful diagnosis and remediation tools.

Each activity in the GALS model begins with an explanation that teaches the students a basic concept for the first time or reminds them of a concept that they have learned previously. The computer then demonstrates how the interface for the specific activity works (e.g. "just click," "draw," "select," etc.).

Following the demonstration, the student is asked to solve the identical exercise that the computer solved. Contrary to what one might expect, we did not find that students in such situations simply imitate without understanding; rather, it was found students who were not truly ripe for an exercise were not necessarily able to repeat the demo. An explanation for this can be found in the assertion in Vergnaud's book on Vygotsky: "We can only imitate what lays in the zone of our own intellectual capacities" (Vergnaud, 2000, p. 28).

After the student solves the first exercise, s/he is presented with 2-7 similar exercises. These exercises are generated by the computer pseudo-randomly, such that two different students working on the same activity simultaneously receive different exercises, and students who do the same activity twice encounter different exercises. If the student solves an exercise correctly, s/he receives a funny animation, gets a green light, is awarded points, and is automatically presented with the next exercise.

Behaviorist models, which consider students to be a "black box," resulted in major failures (Hativa, 1988). For this reason, any new ITS models should integrate an analysis of frequent errors and misconceptions, and should be able to deal with the reasons that underlie those mistakes. In addition, new ITS models should provide assistance and remediation built on mental models.

The strength of the GALS model is revealed when a student makes a mistake. For each concept presented in the program, the developers reviewed common mistakes and determined prevalent misconceptions. They then built state-machines that diagnose each foreseeable mistake. Findings of research regarding errors in specific domains were incorporated when available (e.g. Brown, 1978; Van Lehn 1982; Gray, 1994), and when there was no existing body of findings, research was conducted in order to map common errors.

As mentioned previously, it is not sufficient to know what the frequent errors are; rather, it is necessary to have a plan of action for instances in which errors of this nature are encountered. For example, the mistake most commonly found in response to the exercise "3 + __ = 8" is the answer "11", i.e. the student adds 3 + 8 instead of determining the missing number (Conne, 1988). What should be done when a student makes such a mistake?

Cox has noted the need for an appropriate teaching method for providing remediation for specific errors: "Research on what teaching methods are appropriate for remediating specific errors are almost inexistent" (Cox, 1975).

Two basic approaches to correcting specific errors can be described:

a. The Piagetian "accommodation-assimilation" model: In simple words, if a learner makes a mistake in a given problem, he must abandon the algorithm applied and construct a new one that better fits the situation. This process is called "assimilation." If the new algorithm works, the learner records the match between the algorithm and the situation, and correctly applies the algorithm upon encountering similar situations in the future. This is known as "assimilation" (Siegler, 1991). If we apply this model to our case, the appropriate remediation would be to inform the student that he made a mistake (e.g. by saying "you made a mistake," "you're almost there," or "try again") and to then allow him to build a new algorithm and to assimilate it.

b. The "private tutor" model: According to Anderson, a private tutor who is faced with a student who makes a basic mistake will commonly respond by simply teaching the concept again.

The basic problem with incorporating the Piagetian model is that many students are not bothered by the fact that their algorithm does not work when applied to a specific problem; moreover, even if they are bothered, they often are not capable of constructing a new algorithm independently. The "private tutor" model of remediation is also problematic, in that students do not build their knowledge base by themselves if they are simply taught concepts again. Consequently, the GALS model developed an alternative method of responding to errors, which falls between the two models described.

Using Vygostky's principle of a "zone of proximal development" (Vygotsky, 1986), the student's mistake can bee seen as an indication that s/he is having difficulty "climbing the step" that leads up to the new concept. In such situations, students should be given a "stool" that is appropriate to their specific problem, so that they can "climb the step" by themselves. In our design of the GALS model, these "stools" took the form of "hints" that were carefully constructed based upon an analysis of the causes of common mistakes and misconceptions, and that give a basic adapted remediation.

Thus, for example, in our case of the mistaken response to the equation 3 +__= 8, most students who had difficulty did not understand the order of the equation. They saw the 3, the plus sign, the 8, and the equal sign, and they simply solved the equation 3 + 8 =, perhaps wondering if the teacher who wrote the exercise was feeling quite right. In response to such a mistake, one simple but efficient hint could be: "3 plus 11 does not equal 8" (which points out that student did not arrive at the correct answer). Similarly, the hint could be phrased "3 plus what equals 8?" (which explains the order of the equation). After receiving the hint, the student could then be given a second chance to solve the exercise.

If the student fails again after receiving a hint, GALS implements a method close to the "private tutor" model and presents the student with the solution, but only after explaining to the student what is wrong with the answer. (It is important to note in this context that since the exercises are generated in real-time, the hints and corrections are also generated in real-time by powerful engines.)

This method of remediation proved to be very efficient. For most students, the hint (and if necessary the full explanation) was sufficient to enable them to build a correct algorithm. However, occasionally it was found that students were unable to build a correct algorithm despite the fact that they received help and were shown how to solve this type of problem. As described below, the GALS model's "Learning Manager" deals with such cases.

The Learning Manager

Lepper and Gurtner's presentation of the advantages of computers as personal tutors included the fact that computer programs can be individualized for learners with different skills and capabilities (Lepper and Gurtner, 1989). In the GALS model, at the beginning of the school year, each student begins the program with the same first activity (albeit with different generated exercises). If she succeeds in the activity, she gets the next one, and so on, following the order of the decomposition of the program in basic units, which each deal with one specific concept.

One might think that the students would all proceed through the program in a common, linear path; in fact, however, no two children ever followed the same course through the program. This is because if a student encountered difficulties in an activity, the "learning manager" would send him to a revision activity that would have him complete all the prerequisites for the activity in which he failed. This revision activity could either be an exercise that the student had completed previously or a special remedial activity that would only be encountered by students experiencing difficulty. Conversely, students whose performance was found to be very good are sent by the "learning manager" to special activities reserved for students with high levels of achievement.

Although in most cases, the learning manager and the expert systems for each activity enable students to bridge the knowledge gap and move ahead, occasionally a given student still does not master a specific concept. In such cases, the teacher receives a special report that lists the students who experienced pronounced difficulty, as well as the specific difficulties of each student. This enables the teacher to provide appropriate remediation that is focused on the specific problems of specific students, and even to directly access the computer activity that revealed the student's difficulty.

User-Machine Interface

According to Lepper and Gurtner's outline of the advantages of computers as personal tutors, the computer always remains patient, nonjudgmental, and supportive. Moreover, because the computer is always fair and impartial, it may minimize the pernicious effects of teacher prejudices or favoritism (Lepper and Gurtner, 1989). For this reason, effort was made to build a user-friendly interface, to create a program that is both patient and funny, to determine objective criteria of success, and to be sure that the interface includes a dynamic adaptation to each learner.

The Experiment and Implementation in Schools

In 1998, the Israeli Ministry of Education appointed a commission to supervise an experiment that has been conducted by an independent professional team. During the first year of this experiment, the GALS-based program was utilized in Grade 1. In each of the following years, participating students moved on to the next grade of the program, while new students began the program at each grade. At the present time, more than 800 students are participating in the experiment. In addition, at the start of the 2000-2001 school year, the program became commercially available to Israeli schools, resulting in its current use by over 2,000 first grade students.

Beyond the computer program with hundreds of multi-media challenges, the computerized GALS-based math program in use in Israeli schools today is accompanied by a set of tools that includes student workbooks, ready-to-use exams, enrichment worksheets, manipulatives, and a teacher guide with suggestions for hundreds of discovery activities that can be done by groups or individuals. Students using the program spend an average of one class hour a week on the computer. The learning manager tracks the progress of each student, and teachers receive reports about the achievement of the class as a whole as well as the about the performance of individual students.

Results

The main results found by the independent professional team have been the following:

1. A comparative summative test of achievement of first grade students showed significantly better achievement ($p<0.001$) in the research group (10 classes, 217 students) than in the control group (10 classes, 211 students).
2. The integration of computer technology in the learning process was proven to contribute greatly to student achievement.
3. Most students mentioned the integration of the computer as the main factor of satisfaction of the program.
4. Both teachers and students reported a high level of satisfaction with all the parts of the program.
5. In the GALS program, students receive explanations each time they do not succeed in an activity. In this manner, beyond its role as a mechanism for practice and as a diagnostic tool, the program serves primarily as a teaching tool that enables the students to correct their mistakes and to understand the reason of their failure.
6. In many cases, it was found that the computerized activities presented students with concepts that had not yet been covered in class; nonetheless, the explanations provided by the program enabled the students to succeed in these activities. In addition, it was found that students who were exposed to concepts for the first time on the computer succeeded in the parallel textbook activities without needing additional explanations.
7. Students who encountered problems in a specific activity usually received efficient help and corrections, and were able to succeed and to move on to the next activity.

8. Most students succeeded in all the activities either after their first attempt or upon returning to the activities after being given additional remediation activities.
9. It was found that almost no random answers were given by the students.
10. Teachers reported that the software provided weak students with a learning experience that allowed them to remain "tuned in" during the lessons. This was not found to be the case with such students in previous years (Milgrom, 1999, Milgrom, 2000).

It should be noted that following the initial experiment, the Israeli Ministry of Education recommended pursuing a three-year trial and committed itself to continuing support for the program's development.

Conclusions

These results seem to indicate that an Intelligent Tutoring System grounded in a substantiated model of learning can efficiently be used in classrooms settings.

More data about the effectiveness of the GALS model are expected to be available at the summative evaluation stage of the ongoing experiment, and could be validated in the course of a new experiment that has begun this year with first grade classes in France.

The Guided Autonomous System (GAL) model could be applied to the study of mathematics in higher Grades, as well as to other subject matters.

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INTERPRETING SYMBOLIC STATEMENTS AS TEXTS: AN EXPLORATORY STUDY

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This paper is devoted to the problem of the interpretation of mathematical texts[1]*. Some ideas on mathematical language are shortly discussed with the help of some constructs from functional linguistics. Some evidence regarding the interpretation processes of a symbolic text by groups of 10-graders, including both written answers and the transcriptions of spoken interactions is presented and discussed. The outcomes of this study show that students often try to interpret mathematical statements according to everyday-life schemes. This suggests that in school practice mathematical expressions should be dealt with as texts (rather than as abbreviations or local conventions) and that metalinguistic awareness should become one of the the goals of both linguistic and mathematical education.*

Recently various theoretical frameworks have been diffused that enhance the role of languages in the learning of mathematics. This holds specially for the neo-Vygotskian standpoint, which gives great value to communication as a way to promote learning. More recently, in investigations more focused on cognitive aspects, Sfard (2000a, 2000b) interprets thinking as communication and assigns to languages a more complex role than the traditional one: they are not regarded just as carriers of (pre-exixting) meanings, but as builders of the meanings. In a context where communication becomes central, it cannot be regarded but as "an activity in which one is trying to make his or her interlocutor act or feel in a certain way", i.e. an activity pertaining to the realm of pragmatics too. A thorough investigation of the languages of mathematics from the standpoint of pragmatics is far from being developed. Some example of application of pragmatic constructs to the learning of mathematics, such as Grice's[2] Cooperation Principle, may be found in Ferrari (2000). Morgan (1996, 1998) and Burton & Morgan (2000) have carried out investigations on mathematical language from the viewpoint of Halliday's[3] functional linguistics. They focus on some interpersonal aspects of mathematical language (such as the use of impersonal forms in academic mathematics textbooks) but take into account some aspect which are interesting from a cognitive viewpoint as well, such as cohesion[4]. In particular Morgan (1998) provides a general description of mathematical language as a set of registers as they are used in mathematical practice (rather than as they are usually accepted by mathematicians) which seems adequate to the needs of research and practice. The use of expressions like 'mathematical language' through this paper assumes Morgan's definitions and discussions. The new functions of mathematical language in education require researchers and practitioners to consider it as a complex system, taking into account all its components (verbal, symbolic, visual, ...) that very often are combined. Ordinary languages and mathematical one are different as regards not only the symbolic or the visual component, but the verbal one too. In the new perspective the verbal component cannot but play a crucial role. Still, the symbolic component has played a major role in the development of mathematical thought and may play an important role in mathematics education. A thorough discussion of the functions of the symbolic component is far beyond the aims of this report. Symbolic notation systems are not important just because they are possibly more precise or less ambiguous than ordinary languages, but because of the computational opportunities they provide. Moreover, they allow people to get rid of some of the meanings embodied in everyday-life words, when it is necessary to build new meanings.

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[1] Through the paper the word 'text' is used according to linguistics to mean any spoken or written instantiation of language, independently from its length or complexity.

[2] Grice (1975)

[3] Halliday (1974, 1985)

An important feature of educational pratice in mathematics is the need of using the same linguistic forms with different functions: to build and organize mathematical knowledge and to communicate with other people, their experiences and cultures. This plurality of functions may generate conflicts. For example, in contemporary mathematics, it is perfectly acceptable to name 'rectangle' a square shape, whereas this use is inadequate in some contexts as far as it violates pragmatic conventions, such as Grice's Cooperation Principle. In a similar way, in some cases, logical connectives (and, or, if...then, not, ...), no matter whether in verbal or symbolic form, are to be used according to their truth-functional definition (for example, when defining set-theoretical operations), whereas in communication practices within the class they are required to play functions that go far beyond truth-functionality, such as the organization of the texts and of the links between the sentences that occur. If the understanding of texts in mathematics becomes a goal for education, then students have to deal with texts with different format, organization and functions almost at the same time. Different kinds of text may propose different interpretative problems. For example, some of the questions raised by Sfard (2000b) as concerns discursive focus[5] are appropriate. A good share of the misinterpretations of texts by college students, for example, are related to failure in the grasping of the focus of the text as a whole rather than in the interpretation of single words or expressions. This may depend from the fact that often in mathematical texts focus is not marked in the same ways as other texts. A closely linked question is cohesion, i.e. the functional links among the various components of a text. Cohesion in mathematical language (in all its components) is usually less explicitly marked than in ordinary language, which may be an obstacle to students' interpretations.

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[4] According to Halliday (1985, p.309) 'cohesion' refers to non-structural resources designed to establish relations within the text that are not only semantic but also functional.

[5] In linguistics various constructs (such as theme/rheme, topic/comment, given/new, ...) have been proposed to deal with these issues. For detailed discussions see for example Halliday (1985) or Leckie-Tarry (1995).

This study is devoted to the ways some groups of 10-graders interpret a text made up of three symbolic expressions. We are interested in the ways students put together the occurring expressions and the interpretation schemes they adopt when the expected interpretation is not adequate from the standpoint of communication.

INTERPRETATION OF A SYMBOLIC TEXT

The following problem has been given, in December, 2000, to four classes of 10 graders.

PROBLEM[6]

The positive integers x, y are given. We know that all of the three following properties hold at the same time

(a) $x^2 < y^2$ (b) $3x > y$ (c) $x^3 > 10^6$

Based on the given data, for each of the following statements find whether it is true or false:

(i) $x \leq y$ (ii) $2 < y$.

Explain your answers.

A PRIORI ANALYSIS

This problem includes various critical points. First of all, there is the interpretation of '≤' in a true statement which violates pragmatic principles. A potential obstacle is the interpretation of inclusive 'or' in a statement which is of the form 'A or B' with A clearly true and B clearly false, which violates elementary pragmatic principles such as Grice's Maxim of Quantity, for 'A' alone would be less expensive and more effective from the viewpoint of communication. Also the coordination of (a) and (c) may result troublesome for a number of students. At this regard it must be remarked that cohesion among (a), (b) and (c) is explicitly pointed out (at the meta-textual level) in the preceding verbal text ('all of the following ...', 'at the same time', 'based on the given information'), but this may be not enough, since neither specific algorithms nor standard linguistic markers are available. The problem has been designed in order to prevent students from applying some standard algorithm with little control, forcing them to use methods based on the interpretation of the given statements.

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[6] The original texts of the problem, of students' answers and of the transcriptions of their spoken interactions are in a language other than English. Some of their linguistic features are lost in the traslation process. For example, we are afraid that the English translations of the texts produced in the interactions are not in the same register as the original ones. Nevertheless, in this paper we investigate aspects which are not too much affected by the translation. Anyway, the original versions of the materials are at disposal of anyone interested.

METHODOLOGY

The problem has been given to 4 classes of 10-graders (76 students altogether). Students have worked about 30' individually (producing written answers) and other 30' in small groups (2-3 students of different skill levels). Copies of their written individual answers were available to students during the interaction; the work of all the groups of one class has been recorded. We present some quantitative data on the whole sample and investigate the behavior of one group of 2 students more closely. Of course, the data of the first kind are gathered from texts actually written out by the students whereas the others are transcriptions of spoken interactions.

INDIVIDUAL ANSWERS

Question (i)

57 students claim that (i) is false, 16 that it is true and 3 give no answer. Negative answers mostly refer to the fact that "it cannot happen that $x=y$". Some other students are puzzled by the occurrence of '$\leq$' in (i), whereas in the data occur '$<$' or '$>$' only.

Question (ii)

44 students claim that (ii) is true, 11 do not answer, 8 claim that it is false and 13 claim they have not data enough to give a definite answer. Altogether 21 students seemingly do not recognize the links between (a) and (c). Moreover the number of non-answers is larger than in question (i). Most of the answers to (ii) have been given with no explanations. Among the explanations given we mention: "I cannot know if $2<y$, it depends on the values of x, y" or "I have no data on y" or "$2<y$ is false because if $x=1$, then y could be 2".

Let us see some more examples.

Valentina answers to (i)*: "False because x is never equal to y", whereas to (ii) answers: "*$x>100$ *(c),* $x^2<y^2$ *(a)* $\Rightarrow$ $y>2$*, since x, y are positive"*

Ivano: *"(i) is false because* $\sqrt{x^2 < y^2} = x < y$ *, (ii) is false too because y could be 2; if (ii) were with '=' it would be true"*

Andrea: *"(i) is false, because in (a) there is < whereas in (i) there is =. I do not know whether (ii) is true or false because I have data on x and not on y"*

Sergio: *"(i) is false.* $y = \sqrt{x^2 + k} = x + \sqrt{k}$ *but this do not imply* $2<y$*"*

Deborah: *"(i) cannot be, because in the hypothesis there is* $x^2<y^2$*; if it were* $x^2 \leq y^2$ *it would be true. (ii) is true"*

Enrica: *"(i) is false because y cannot be equal to x because if I square both they would be still equal; (ii) is true bacause else x would be less or equal to 1 and the third piece of information would be false"*

It is noteworthy that among the 'improper' answers to (i) a good share are explained quite clearly (showing some command of mathematical notations), whereas it is difficult to find well explained answers to (ii). Moreover, no student refers to (b) in his or her answer to (i) and (ii).

INTERACTIONS

Le us examine the transcriptions of the interactions of the group made up by Valentina (a girl with excellent grades in all subject matters including mathematics) and another girl named Ines (rated at average level).

Valentina: *"The first is false because x cannot be equal to y"*

Ines: *"If the square is less, the number too is less"*

V: *"Hmm, here there is 'less or equal'"*

I: *"It is the same!"*

V: *"It is not equal!"*

I: *"But it works all the same!"*

V: *"Why am I to write 'equal' if it is less?'*

I: [a bit vexed] *"Oh, it is like the elevator: there is 'Maximum weight three hundred kg' but you take it even when you are alone.* [laughing] *You do not weigh three hundred kg!"* [Valentina is small and slim]

V: *"Of course not. Maybe you are right. The second is true."*

I: *"We have no information on y"*

V: *"The cube of x is ten to the sixth. So x is equal to one hundred. If x is at least one hundred, y must be one hundred one, at least."*

I: *"y could be one and the statement would be false"*

V: *"If y were one, x would be zero, the cube of zero is zero"*

I: *"It could be:* [points at the occurrences of *x* and *y* in (a), (b) (c)] *x is zero, y is one, x is one hundred one"*

V: *"x is always the same. Okay, we know that y is more than x, and x is more than one hundred one, then y is more than one hundred one"*

DISCUSSION

A palpable outcome is students' uneasiness in recognizing that $x \leq y$. Most of them adopt the argument that x cannot be equal to y, so pointing at the communicative inadequacy of the formula rather than at its claimed falsity. Valentina's answer is quite clear also because she shows a good command of mathematical notations and steadily applies even 'ab absurdo' arguments. It seem reasonable to conjecture that these students (rightly) feel the inadequacy of the statement $x \leq y$ which violates not only everyday-life pragmatic rules but even implicit rules of school practice: usually to answer a question it is not accepted just a true statement, but the statement which is the most adequate to the question and the related context is required. Other answers point out a further aspect: the difference between the relation '<' occurring in (a) and '≤' occurring in (i). In this case it is questioned the adequacy not just of (i) but rather of the whole text.

As concerns (ii), the answers of students who fail in linking the question to both (a) and (c) can be classified into two groups. Who answers 'false' most likely focuses on (a) only and remarks that y could be 2. (b) is neglected by almost all students, maybe because in it occurs '>' in place of '<' or '≤'. Most likely (c) is not taken into account by some students as it does not involve y explictly. These answers seem depend on the lack of linguistic markers of cohesion between (a), (b), (c). The lack of cohesion induces some students to assign to each statement its own topic. The topic of (c) alone cannot be other than x, which is the only variable occurring. Some students try to apply algorithms to put data together.

The different features of mathematical language (in all its components) from the functional (not only grammatical or lexical) viewpoint may explain a number of students' difficulties. Most of the students refuse $x \leq y$ as inadequate compared to the data available. Even the troubles with question (ii) may be explained in a similar way. Ordinary language provides a number of ways to mark the links between the statements in a text (intonation in spoken texts, vocabulary, pronouns, connectives, ... in all the texts) whereas mathematical one (in its symbolic and often verbal component) cohesion is usually marked in other ways, such as the spatial disposition of the formulas or the availability of specific algorithms or the repetition of some symbol or letter. This happens for example in the solution of linear systems: standard methods automatically take into account all the equations involved, that are identified mainly by their spatial disposition and by the occurrence of brackets or braces.

The spoken interaction between Valentina and Ines points out some interesting processes. Valentina, who usually takes good grades in mathematics, gets stuck because of the occurrence of '≤'. Maybe Ines, who generally takes lower grades in mathematics than Valentina, is not completely aware of the question raised by her friend, but her indifference for the distinction between '<' and '≤' and her efforts to represent the data verbally play a positive role. Ines' attitude is clearly agonistic, as she seems to be moved mainly by the wish of prevailing against Valentina. As regards question (ii) Valentina, in order to explain her answer, provides a sequence of examples that use statements that are not consequences of the data ("the cube of x is ten to the sixth", "so x is equal to one hundred") afterwards rectified by others ("if x is at least one hundred"). Ines clearly does not grasp cohesion among the data and interprets the two occurrences of x as different numbers. Both the parts of the interaction enhance some features of verbal language. Ines and Valentina are both inaccurate in their interpretations. Ines tries to use 'less' to interpret both '<' and '≤', which is inaccurate, but succeeds in drawing Valentina's attention on some aspects of the meanings involved that are relevant to the answer. Moreover she uses an example (the elevator) which is not closely related to the problem, but where the pragmatic function of the warning 'Maximum weight three hundred kg' is made straightforward by the situation and one's everyday experience. In other words, the example of the elevator is pragmatically rather than semantically related to the problem situation. Also Valentina, as remarked above, is inaccurate in her examples. Her efforts to give x and y values compatible with the data seem useful steps toward the solution. In both cases, verbal language (in a spoken register) allows her to make inaccurate statements and rectify them afterwards without too much danger. If Valentina had written down her examples in symbolic form ("$x^3=10^6$", "$x=100$") and had applied to them standard algebraic transformations, she could have lost the control of the function of her productions. In other words verbal language (in both spoken and written registers) not only provides much more opportunities to mark some of the functional features of the texts (topic, cohesion, ...) but is also more flexible than symbolic one, as it allows people to produce inaccurate statements and to rectify them afterwards, or to mark them as conjectures, or examples, or other. Very often I find college students who write down formulas that are not consequences of the assumptions but only examples. Unfortunately, they often forget the functions of their writings, and apply to them algebraic transformations, and derive false conclusions. Their behaviors, that are sometimes labelled as 'incorrect applications of rules' could more effectively be regarded as examples of failure in the control of the functions of the texts produced.

TEACHING IMPLICATIONS

A possible interpretation of this data is: experiments of this sort have no relevant teaching implications as the problems assigned are tricky and unfair. This opinion is compatible with traditional teaching practices that mainly enhance the learning of standard procedures in standard formats. If we give a central role to communication the role of languages becomes more relevant. In particular it seem reasonable that students should interpret simple texts in mathematical language including those containing symbols, even if they are not in standard format (as happens in everyday-life communication). The outcomes of this experiment point out the need that students command the transitions between different languages or registers, with the related functional properties and conventions. This suggest that mathematical expressions should be studied as texts rather than just as local conventions or abbreviations. This implies a better coordination between the teaching of languages and the teaching of mathematics and a stronger focus on aspects like metalinguistic awareness, i.e. awareness of form and functions of a text, in addition to its meaning, as suggested by MacGregor & Price (1999). Of course, further research is needed to refine these ideas and to design the teaching methods more suitable to attain the goals suggested above, but we believe that anyway mathematical language should be considered in the context of actual interactions (rather than as a separate code) with all its components.

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ALGEBRAIC SYNTAX AND WORD PROBLEMS SOLUTION: FIRST STEPS

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In Filloy and Rojano (1989) we introduced the use of concrete models for the teaching of solving linear equations and studied the abstraction processes that take place when such models are put in work by pupils of 12-13 years of age. In this paper we discuss how concrete modeling in algebra influences the way children in the arithmetic/algebra transition can learn to solve word problems. We discuss the stage in which the algebraic syntax developed to solve equations has just been acquired and then used to translate the word problems into the algebraic code.

Previous research has been undertaken to probe cognitive processes that take place in solving word problems in the transition from arithmetic to algebraic thinking. Bednarz and Janvier (1996) have substantially contributed to this aspect of research in problem solving in algebra. Puig and Cerdan (1990) have formulated criteria to determine when a word problem can be considered as algebraic. From a different perspective, A. Bell (1996) has approached this matter by showing through examples how generic problems can provide algebraic experiences that develop manipulative algebraic abilities. Sutherland and Rojano (1993) have studied how a technological environment can help students to represent and solve word problems without having to take on board with the algebra symbolic code, from the very beginning. The present paper addresses the theme of *extending the use of basic algebraic syntax to the context of the solution of arithmetic/algebraic word problems*. Pupils of 12-13 years of age were interviewed just after they had learned how to solve linear equations with one unknown appearing on both sides. In a first stage of the interview, a teaching approach using "concrete models" for solving equations was used (items in sequence I) and in a second stage, children were given two sequences of word problems. Sequence A included problems of the type: "find a number that…"; and Sequence P, included problems posed in a variety of contexts and in which a progressive symbolization process is required to make up the equation that solves the problem. The purpose of facing children with problems of sequence P was to study their potential in transferring basic algebraic syntax, recently learned, to other contexts, different from those used for the teaching of solving equations. Items of sequences I, A and P of the interview constitute a teaching strategy that aims to provide with senses the learning of algebraic syntax (Filloy, 1991). With this strategy, it is possible to complete the teaching cycle:

I.-Meaningful introduction to algebraic syntax (through "concrete models" for solving linear equations);

II.-Immediate use of the elements of manipulative algebra, just acquired, in the resolution of word problems (progress towards a semantic use of algebraic syntax);

III.-Meaningful progress towards a more complex level of algebraic syntax.

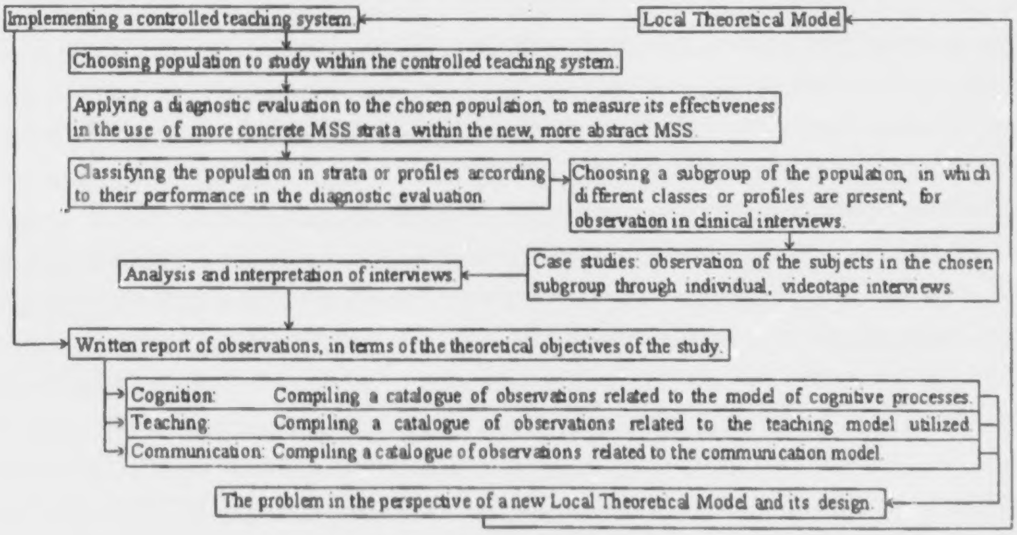
In this paper we report results from a case study with 12-13 year olds, who were introduced to the algebraic realm, using the teaching cycle described above. Among the twelve participants in the study, Mariana, a girl with a history of being successful in mathematics, got to complete the teaching cycle during a 1:45 hr session of clinical interview. In the following sections, we discuss some relevant issues from Mariana's interview, with regards the extension of syntactic skills through problem solving activity.

Theoretical and Methodological Framework

In Filloy (1990) we introduced the methodological framework of **local theoretical models** in which the object of study is brought into focus through four inter-related components:

(1) **Teaching models** together with (2) **models of cognitive processes**, both related to (3) **models of formal competence** that simulate competent performance of the ideal user of a Mathematical Sign System (MSS) and (4) **communication models** to describe rules of communicative competence, production of texts, texts decoding, and contextual clarification.

The following scheme describes the rationale of the case study:


Implementing a controlled teaching system.
Local Theoretical Model
Choosing population to study within the controlled teaching system.
Applying a diagnostic evaluation to the chosen population, to measure its effectiveness in the use of more concrete MSS strata within the new, more abstract MSS.
Classifying the population in strata or profiles according to their performance in the diagnostic evaluation.
Choosing a subgroup of the population, in which different classes or profiles are present, for observation in clinical interviews.
Case studies: observation of the subjects in the chosen subgroup through individual, videotape interviews.
Analysis and interpretation of interviews.
Written report of observations, in terms of the theoretical objectives of the study.
Cognition: Compiling a catalogue of observations related to the model of cognitive processes.
Teaching: Compiling a catalogue of observations related to the teaching model utilized.
Communication: Compiling a catalogue of observations related to the communication model.
The problem in the perspective of a new Local Theoretical Model and its design.

Results from the diagnostic test located Mariana in the category of students with high proficiency in a) solving "arithmetic" linear equations; b) solving arithmetic word problems; and c) numeric skills. Before the interview, Mariana had not had been taught any algebra.

THE INTERVIEW ITEMS (Sequences I, A and P).

We will write **I Mr.n** for the **nth** item of Sequence I Series in Mariana's interview.

Sequence I

I Mr.1	$x+2=2x$	I Mr.9	$5x+8=8x$	I Mr.17	$15x+1590=71x-202$
I Mr.2	$x+5=2x$	I Mr.10	$7x+468=19x$	I Mr.18	$11x+687=45x-27$
I Mr.3	$2x+4=4x$	I Mr.11	$113x+332=321x$	I Mr.19	$10x-18=4x$
I Mr.4	$2x+3=5x$	I Mr.12	$7x+23x+6$	I Mr.20	$8x-7=4x+13$
I Mr.5	$6x+15=9x$	I Mr.13	$13x+20=x+164$	I Mr.21	$7x-1114=3x+1001$
I Mr.6	$4x+12=x$	I Mr.14	$8x+12=4x+52$	I Mr.22	$7x-1114=3x-1001$
I Mr.7	$4x+12=6x$	I Mr.15	$28x+348=52x+12$	I Mr.23	$7x+1114=3x-1001$
I Mr.8	$5x+8=3x$	I Mr.16	$5x-3=2x+6$	I Mr.24	$113x-70=22x+1022$

Sequence A (a sample of items)

Item

A Mr.1 If you add five to a number, then subtract thirteen and the result is forty-five, what is the number?

A Mr.3 If you add two to a number and the result is double that number, what is the number?

A Mr. 5 If to forty-eight you add three times a number and the result is nineteen times that number, what is the number?

A Mr.6 Five times a number plus three is equal to double that number plus twelve. What is the number?

Sequence P (a sample of items)

Item

P Mr.1 There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are here in the yard?

P Mr.2 There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are there in the yard?

P Mr.4 Mariana is thirteen; Eugenio is forty. When will Eugenio be twice Mariana's age?

P Mr.5 Mariana is thirteen; Roberto is twenty-eight. When will Roberto be twice Mariana's age?

The Interview-Some issues.

MARIANA´S PERFORMANCE BEFORE THE INSTRUCTION PHASE

Mr. approaches the first items of sequence **I** with the trial and error method. She previously reads the equation: "I need to find a number that multiplied by 2, plus 4, equals ...". Like all the children in the same category as Mariana, she, very quickly, stops trying to use this method. Thus, in item I Mr. 4 $2x+3=5x$, which could be easily solved by trial and error, she demands being taught a (school) method. At this stage the instruction phase begins on how to operate with unknown quantities by using a "concrete" (geometric) model (see Filloy/Rojano, 1989, and Filloy/Sutherland, 1996, for a detailed description of this model).

MARIANA'S PERFORMANCE AFTER THE INSTRUCTION PHASE

Mariana's work with the concrete model to operate with unknown quantities is characterized by an immediate abstraction of the actions performed in the model to the level of algebraic syntax. This could be clearly observed in item I Mr.6. A factor that may have influenced this abstraction process is the parallel register that Mariana carried out using the algebraic code, while she was dealing with areas comparison in the model. In this way, the abstraction process towards the algebraic language was reduced to an immediate translation of her actions in the model to this language. This immediate abstraction of the model's actions to algebraic language is not uniform throughout Mr.'s performance. The presence of more complex modes of equation, or equations with a non-trivial numerical structure ($C<A$, in $Ax+B=Cx$), hindered the way operations were abstracted. Items I Mr. 8 and I Mr. 12 showed this phenomenon: If certain syntactic elements are locally generated (let's say, using a certain mode of equation) they are not transferred automatically to new modes. It is necessary to return to work with the semantics of the model to reconstruct the actions carried out and recover them at the level of algebraic syntax.

PROGRESS TOWARDS SEMANTICS (Solving word problems and the role of algebraic syntax)

Finally, in sequence P, Mariana was asked to solve the word problems that were presented in pairs. The second problem, of each pair, is obtained by changing only the data of the first one. The pairs in the sequence P are P Mr. 1 and P Mr. 2; P Mr. 4 and P Mr. 5. The interviewer intervened to help Mariana to build up the equation in the first of each pair, while in the second, Mariana showed to be able to produce the equation and to solve it, using the recently-acquired syntactic skills to solve equations with more than one occurrence of the unknown.

PMr1 and PMr2 (fragments of Mariana's interview)

There are chickens and rabbits in a yard. There are sixteen heads and fifty-two feet. How many chickens and rabbits are here in the yard?

The interviewer (**I**) suggests Mariana (**M**) to use letters to represent the number of rabbits and chickens. She writes down x for the number of rabbits and $16-x$ for the number of chickens.

I: How many rabbit's feet I can see?

M: eight

I: If there is one rabbit, how many feet?

M: four

I: If there are 2 rabbits?

M: eight

I: If I see this number of rabbits (x) how many feet?

M: I don't know

I: How could you name the number of feet, using this …? (he points out the 'x')

M: Interrupting the interviewer writes down: $4x$

I: How could you find out the number of chickens' feet? Do you have a name for the number of chickens?

M: The name of the chickens' number times two

I: And, how is it? Two times…

M: writes down: 2´ $(16-x)$

I: If 4x is the number of rabbits' feet and 2´ $(16-x)$ is the number of chickens' feet, How can I obtain the 52 which is the total number of feet?

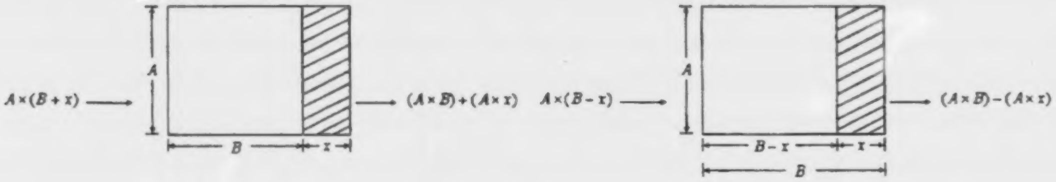
M: Adding, $4x$ to 2´ $(16-x)$

I: Write it down

M: writes: $4x$ + 2´ $(16-x) = 52$

I: This is an equation.

At this point, Mariana had not had instruction to remove the brackets in the left hand side expression. So, the interviewer intervenes with a piece of instruction to teach Mariana the distributive law in expressions that include 'x'. He uses a geometric representation, which in general terms corresponds to the following figure:



The interviewer asks Mariana to generalize the law to expressions such as 3´ (16-x); 4´ (16-x); etc. Finally, she writes down the equation: $4x + 32 - 2x = 52$

I: Can you solve it?

M: writes down the simplified equation: $2x + 32 = 52$

At this point, she used the syntactic elements she learned in the sequence I of the interview.

Mariana inverts the operations present in this equation and finds the number of rabbits.

M: Ten, is the number of rabbits

I: And, how many chickens?

M: points out the expression 16-x, and says "six".

When the item PMr.2 is presented to Mariana, she quickly recognizes that it is the same problem, but with different data and proceeds to solve it without any difficulty.

This strategy of using pairs of similar problems, is used in one part of sequence P. It could be observed that, once the greatest difficulty was overcome, that is the translation of the problem into an equation, the problem became a routine one for Mariana. This was because the syntactic elements that she had recently learned were applied immediately. This type of transfer allows the recently-acquired operational capacity to be considered as a potential tool to solve a larger family of problems, that is, a family that includes statements that lead to linear equations, which are here called 'non-arithmetic' (with more than one occurrence of the unknown).

FINAL REMARKS

The teaching strategy used to help Mariana to identify problems belonging to the same family consists in giving to the new type of expression, such as A´ (B+x) a treatment which is analogous to that given to the expressions of the type Ax + B = Cx. That is, translating these sorts of expressions into more 'concrete' terms in a context of areas and recovering the actions in the algebraic code. This strategy showed to be helpful to Mariana to extend the use of her syntactic skills to word problems, posed in a variety of contexts. In fact, Mariana's performance in the last items of sequence P, where she put into work new elements of algebraic syntax, provides evidence of a real progress towards the use of manipulative algebra in contexts different from those used in the teaching of such syntactic elements. Mariana's case illustrates how this type of return to the evolution of syntax can provide the elements of the manipulative algebra with *meanings* that enable them to be applied to the solution of entire families of word problems.

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GEOMETRY AT WORK
(or Pytagoras will never fail you!)

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Abstract: *This research report presents an empirical study of the geometrical knowledge 'hidden' in the working practices of Argentinian bricklayers. The aim of the research was to find clues to improve the mathematics education of adults within the vocational and professional training system. The results of the study concerning the construction of right angles are partially presented to illustrate some of the differences between the geometric knowledge situated in the school context and in the work place context. The paper finishes by raising some questions about the need and the possibility of trying to establish links between the different contexts for everyday mathematical practices and the school practices.*

Some of the reasons for the study

In the Argentinian province of Rio Negro, where the study was developed, the illiterate population reaches 34.04% (illiterate and functionally illiterate). Certainly, there are economic and social reasons that explain why a significant number of young people drop out school before reaching the end of compulsory school age. These reasons are reinforced by the academic failure of some students. Mathematics, as a school subject, has a big share of the responsibility for this failure, since it still operates as a selection instrument. This selection, moreover, takes place within a system where the meaning of the mathematical knowledge decreases for the student as she/he progresses.

On the other hand, only 3.1% of the illiterate adult population attend vocational schools, adult schools or literacy and numeracy programs. Moreover, the programs offered have little connection with the real needs of their potential students, or with their possibilities to gain knowledge, or with their abilities, or with procedures that would facilitate their effectiveness in the work place and their participation in the social life as citizens.

Within this context, the main goal of this study was to analyze the characteristics of the mathematical knowledge as used and valued by those that could benefit from attending vocational and adult programs[1] compared to the mathematical knowledge as it is promoted for them. In particular, we studied the practical problems that building workers face, as members of a community of practice, in their work-places to identify the geometrical knowledge, either hidden or explicit, that they use to solve them. We also analyzed how the workers relate to the knowledge learnt in school and to the knowledge in-practice, and how they understand and value them as knowledge. We chose to work with the community of bricklayers and building workers because of our interest in the teaching and learning of geometry and of spatial abilities and concepts.

Framework and research questions.

Our study is framed within the everyday mathematics cognition trend that sees everyday mathematics as a body of mathematical knowledge and practices accumulated through the development of everyday activities (Nunes, Schliemann and Carraher, 1993). Gerdes (1988, p.140) referring to the traditional forms and shapes given to objects affirms that this*'It constitutes not only biological and physical knowledge about the materials that are used, but also mathematical knowledge, knowledge about the properties and relations of circles, angles, rectangles, squares....'*. Our starting point is that the traditional procedures used by building workers also reflect a lot of 'hidden' or 'frozen' mathematics (Gerdes, op.cit.).

The authors of this paper would not make a strong argument in favour of the 'utilitarian mathematics culture' but are convinced that knowing more about this hidden mathematics in the adult context could contribute to the development of a realistic curriculum and would help to give meaning to mathematical tasks, to motivate adult students, and to them valuing their own situated knowledge. Nunes and Bryant (1996, p.108) referring to the mathematical knowledge that can be found embedded in many everyday activities, state that *'These points are not mere curiosities about mathematics: they have an impact on children's future chances because they can actually affect how well they do in school'*.

Our purpose is not to discuss whether or not the geometrical knowledge used at the work place to solve practical problems should be considered 'true' geometrical knowledge, but to find clues that contribute to create learning situations that establish links between the knowledge at work and the school knowledge.

With this purpose the research aimed at identifying and analyzing:

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[1] Due to length reasons, we do not report here the analysis of the formal educational system that refers to vocational, professional or adult schools.

a) work tasks, as developed in practice, that involved the use of geometrical knowledge and the planning and the verification of those tasks by the workers,

b) the use of work tools that materialize mathematical knowledge, and

c) the acknowledgment and valuing by the workers of their using of mathematical knowledge and its usefulness.

The problem that motivated our analysis was essentially to establish some of the differential traits between the two forms of geometrical knowledge, that embedded in work and that promoted at school.

Nunes (1992) points out that there are two distinct approaches to the study of cultural influences on mathematical knowledge, one based on the content of the knowledge and another based on the use of the knowledge. We wanted to know if the knowledge, in the sense of content of knowledge, present in both contexts of practice was the same and to identify and describe work tasks in which geometrical knowledge was embedded. However, our main focus of interest was on how this knowledge was used, what was its function in the thinking processes, in the way of approaching and solving the problems, not only from the point of view of mathematics, but also from the perspective of the relationship that the person establishes with the knowledge.

Research procedures

When deciding how to approach the world of the building workers it was clear to us that our research methods would depend not only on the research aims but also on the research context. We really felt outsiders in that particular world: our knowledge of the work tasks was sparse or zero, our gender made us 'uncommon' among them, our social norms came from the academic world, our register of language was completely different from theirs… and we had *'plenty of time to waste, while they were there to work'* (as Juan confessed when we reached a friendly relationship).

To overcome this obstacle the talking of the first author with the engineers, architects and responsible persons in the building company was a great help to us. Through those conversations, we were not only granted permision to develop the study, but more importantly, we were 'introduced' to the field. The languages and the contexts being closer, the building company's personnel understood the aims of the research, and they provided us with descriptions of the roles and the tasks performed at the work place, and they also helped with the selection of workers to be interviewed.

Observing the actual working practices, and reading some previous interviews, were useful to identify certain activities where geometrical knowledge was potentially in use or embedded. However, observation was not the main procedure used. We could have gained information about how knowledge is transmitted from experts to learners in the community of practice, but we discarded this method: it was too expensive on time, too dependent on our interpretation, and it would give us little information on how decisions were taken.

Interviewing the workers was the research method chosen. The first author interviewed 12 workers selected according different criteria: the tasks developed and level of responsibility in the work place, their experience and their abilities as workers, and their previous school experience. The suggestions from the company people were very useful, since they helped to ensure that we would interview those bricklayers who would be ready to help and open enough to make the interviewing smooth.

Only bricklayers, building carpenters and foremen were interviewed but not building labourers, since they, being unskilled workers, rarely took decisions on their own regarding the planning or supervising of the work. All of them had more than four years practice, and regarding their level of formal education there was a range varying from illiterate workers to some who had passed succesfully through part of secondary school education.

The content of the interview was directly related to their actual work or required to evoke it. This means, for instance, there was no question asking *how do you draw a right angle?*, but *how do you proceed to make the frame of a door?*. The questions asked the reasons for a technique observed or the explanation of a particular procedure. Some examples of the topics that guided the interviews are:

a) *how to ensure the perpendicularity of the walls and the floor, how to ensure that the floor is flat and horizontal, how to know where the walls have to be built, how to build a doorframe, a windowframe, an arch...*

b) *how to use some of the working tools like the carpenter's square or the angle iron, the plumb-line, the spirit level ...*

c) *how conscious they were of having learnt at school (if they attended) any knowledge that was useful to their job, of using any mathematical or geometrical knowledge and how they valued that knowledge ...*

The interviews took place in the work place, some of them in the open air, some of them in the storeroom. The interviews were tape-recorded. The difficulties of convincing them to allow us to use the tape-recorder discouraged the use of a video camera, despite the fact that the gestural language was of great importance in our study. During the interviews, drawing tools like those normally used in the work place were at hand and spontaneously used. The interviewer took notes of the gestural language and drawn schemes to help with the later transcription. Keeping a record of the gestural language was of great importance, because many times it was the clue to understand the non-formal language. Data was interpreted through succesive and recursive analysis.

Some findings

Following we present some of the findings to illustrate the tasks identified as involving geometrical knowledge and to describe how this knowledge is used. The findings correspond to the procedures used in the construction of right angles and its verification. Six of the workers interviewed referred to and explained the construction of right angles in different tasks. The procedures reported were repeated in different interviews and involved:

1. The Pytagorean relationship

One procedure uses a particular Pytagorean triplet to construct and control the right angle when building window or door frames. Once having fixed the two sides for the right angle - 0.60m and 0.80m - then the length of the hypotenuse being 1m will ensure that the constructed angle is a right one.

> **Juan:** *I tell you a trick that will never fail you. You make it here 80 and there 60 and then you open or close the frame until you get one meter, then the sides are squared.*

Some of the workers show a functional use of this knowledge that goes beyond just a routine use

> **Carlos:** *You measure 80 and then 60 and you have to have one meter, and you have there the squared corner and then you increase when the length of your square increases.*
>
> **Interviewer:** *And how do you increase?*
>
> **Carlos:** *Instead of 60, 120 and you increase that way, and the same there, 80, 160 and to obtain the square, you have one meter, two meters...*

We want to note that while in school the 'direction of the implication' of the Pytagorean relationship used is the 'if' (if you get a right angle, then the relationship verify itself), the workers use the other direction of the implication, the 'only if', to construct and to control the right angles:

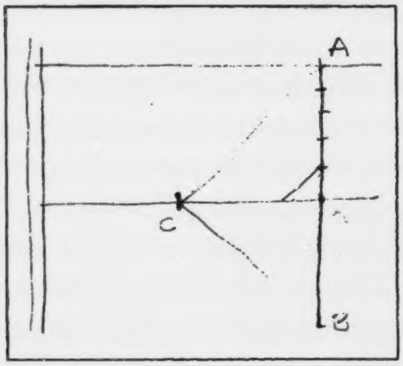
> **José:** *for instance, to square, we make on one side 60, on the other side we make 80, and we see, if we get one meter, then it is because it is squared ... but if we get a little more, this is out of square, it is out of shape.*

2. Properties of the diagonals of a rectangle

Another geometrical fact hidden in one of the procedures used to construct and control right angles concerns the properties of the diagonals of a rectangle or of those of a square. The first one is very similar to that found among Mozambican peasants by Gerdes (1988).

Santiago gave the following explanation, accompanied by a drawing, of how he proceeded when he needed to draw a right angle while 'copying' the basic plan of the house onto the actual ground.

> **Santiago:** (he wants a perpendicular to AO at O)...*like being squared here*[2] (pointing at O), *the axis* (the vertex O), ... *we start here* (O) *to all the sides, this is the main axis* (pointing at the segment AO), *we tighten the thread from here* (from A to B by O), *and another from here* (OC, a thread not yet fixed to the ground), *and then we have 5 meters* (as required by the plan) *here* (OA), *5 here* (OB), *and 5 here* (OC), *then to have this squared* (right angle at O) *you need* (to get) *the same* (length) *from here to here* (AC) *and from here* to here (BC).



Again the direction of the implication used by Santiago ('only if') is the contrary to the one more commonly used at school ('if').

3. Concretizing the idea of perpendicularity

Another procedure used to construct right angles when building door or window frames is based on the idea of perpendicularity concretized in the intersection of vertical and horizontal lines, concepts that correspond to the physical world and that are materialized in the working tools: the spirit level and the plumb-line.

> **Daniel:** *You hang the plumb-line at the two corners, then the frame is 'plumbed', and then you have to get it at 'level', at the corners, with the spirit level.*

4. Using a working tool that concretizes the geometrical concept

Finally, another procedure used was based on the use of the iron angle, a metal square, which materializes the geometrical concept of right angle.

> **Daniel:** *This is an iron angle, you put it at the corner of the door frame, and you square it, if your square is false, then you get it more open, and then with the iron angle you should adjust the frame, the frame is not yet fixed.*

Some results and further questions

The range and richness of content of knowledge we could identify throughout the different interviews is impressive, to the point that we found a procedure to construct angles of any measure smaller than 90° based on the concept of tangent. The content is related to the ideas of parallelism, angle, plane, straight line, proportionality, circle, triangle, square and rectangle. We also observed abilities related to measurement, visualization, interpretation of plans, and locating. Concerning how this knowledge is used there are some constant features that repeatedly appeared. We summarize them below staying with the example of right angles.

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[2] The letters in the drawing are ours to help the reader following the explanation.

In the interviews, most of the workers referred to the use of more than one procedure and related its use to the requirement of the task or the particular situation. We may talk about two different kind of procedures: those based on geometrical facts, and those based on the use of physical tools that materialize geometrical concepts. For instance, only one of the workers referred to using the iron angle to construct right angles; the other two that referred to the instrument explained how to use it to verify the squareness.

One of the workers, Eduardo, referred to using three different procedures (1, 2 and 4) and made explicit the fact that he is able to adapt the procedure to the particular requirement of the task. Eduardo is the one who has both a bigger responsibility in the work place and a 'better' academic history. The richness of the procedures used could depend on both the task to be solved and the worker's knowledge of relevant geometric facts, with this knowledge having been acquired 'in situ' or at school.

Referring to different uses of the knowledge we want to note that the different use of 'implications' we have illustrated in the previous section appears repeatedly and relates our findings to the idea of inversion in the works of Lave (1988), Carraher (1986) and Schliemann and Carraher (1990).

The interviews also showed us that the relationship established with the knowledge in use was different from that promoted in school. In particular, the workers made explicit that they could use more than one method in every situation, one to control the construction and the other to verify it. For instance, two of the workers use the Pytagorean relationship and the properties of the rectangle to control the construction process, and to verify the result one of them used the iron angle. It is important to note that this idea of further verification is one of the characteristics that makes a distinction with what is taught at school and is linked to the fact that in work places the idea of 'error' is linked to 'loss of effectiveness'.

We observed a wide range of possible procedures to solve particular tasks, and that the 'frozen' knowledge can be made operative, adapting it to the requirements of the context, for instance adapting the procedure to the size of the object by using proportionality facts. Also different personal experiences may engender different degrees of flexibility for the procedures used by the participants.

Some of the procedures used can be thought as being general, as being applicable to several circumstances, some of them being based on geometrical properties or facts, flexible in their application, and some of them being concretized in tools. Among the first we could include the use of the Pytagorean triplets, and among the second the use of the iron angle. Some of the procedures are specific, being like algorithms, as set of rules for getting a specific output from a specific input, like the use of the plumb-line and the spirit level to make a door frame. When geometrical knowledge is flexibly used we can think of the procedure as being a problem solving strategy, not a simple routine technique.

As a conclusion, we want to share with the reader further issues that we have raised ourselves. If one wants to obtain new insights for the teaching of mathematics, more detailed descriptions are needed about the informal and formal procedures for solving geometrical problems and, in particular, how the particular situation presents conditions that elicit different types of strategies. However, at which point can the informal content of the lessons become an effective foundation for more formal mathematical reasoning, and what are the limits of engaging students in contextualized reasoning and learning?

The situation in which the problems are embedded may have a strong impact on how they are solved, the impact seeming to result from the meaning that problems have for the ones engaged in problem solving. School mathematics promote classroom exercices that elicit algorithmic solutions, making the students lose track of the meaning of the problem within the situation and within the structure of the knowledge.

The divorce between understanding and the use of algorithms is a by-product of an educational system that leads children to focus not on meaning but on routines. The algorithmic approach promoted in our classes makes students use knowledge as a tool, not as an object in itself to be thought of. Therefore we have two more questions: Can everyday mathematics help with the transition from tools of knowledge to objects of knowledge? Can actual mathematics teaching benefit from teachers and text-book writers, among others, being acquainted with non-formal mathematical knowledge and everyday procedures?

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AUSTRALIAN AND US PRESERVICE TEACHERS' PERCEPTIONS OF THE GENDER STEREOTYPING OF MATHEMATICS[1]

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Historically mathematics was considered a pursuit more suited to males than females. Using a new instrument, contemporary high school students' beliefs about the gender stereotyping of mathematics have been measured and an apparent change in beliefs reported. The same instrument was administered to preservice teachers in Australia and the USA. These countries share common social and cultural characteristics and both were active in addressing identified female disadvantage in mathematics education outcomes. The preservice teachers were asked to respond to the survey items as they believed high school students would answer. The Australian and US preservice teachers' responses were compared. The results are reported and discussed in this paper.

Introduction

Over the past 25 years or so, researchers, practitioners, and policy makers have been active in attempting to redress gender differences favouring males in mathematics learning outcomes (for extensive reviews, see Leder, Forgasz, & Solar, 1996; Forgasz, Leder, & Vale, 2000). Areas in which females had been identified as disadvantaged included: enrolments in the most advanced mathematics subjects and in courses requiring these subjects as pre-requisites, and the attainment of well-above average scores. In the past, mathematics was strongly believed to be a male domain. This view was not only held by students and their parents but also by teachers (e.g., Leder, 1986). Researchers postulated that this belief contributed to females' decisions not to pursue studies in non-compulsory and/or challenging mathematics courses to the same extent as males.

More recently considerable attention has been placed on boys' educational issues. Views of boys' disadvantage, even in the traditionally male preserves of mathematics and science, are receiving increasing media publicity and coverage (e.g., Colebatch, 2000; Gough, 2000). The impact of gender on performance and participation in mathematics continues to be of concern to the community.

Background to the study

Mathematics as a male domain. It is widely accepted that "affective issues play a central role in mathematics learning and instruction" (McLeod, 1992, p.575). The Fennema-Sherman [F-S] *Mathematics Attitudes Scales* [MAS] are frequently used to measure students' attitudes towards mathematics (Walberg & Haertel, 1992). The MAS consist of "nine, domain specific, Likert-type scales measuring important attitudes related to mathematics learning" (Fennema & Sherman, 1976, p.1). The Likert format makes the scales easy to administer and score. One of the subscales of the MAS is the *Mathematics as a male domain* [MD] scale. Based on the assumption that "the less a female stereotyped mathematics as a male domain, the more apt she would be to study and learn mathematics" (Fennema & Sherman, 1976, p.7), the MD was designed so that high scores reflected less stereotyped beliefs and low scores more strongly held views stereotyping mathematics as a male domain. Consistent with the prevailing Western societal views of the 1970s when the MD scale was developed, it is not surprising that no allowance was made for beliefs that mathematics might be considered a *female* domain. Forgasz, Leder and Gardner (1999) provided research evidence to demonstrate that this view was no longer tenable and argued that many of the items on the MD scale were anachronistic and others no longer valid. The scale was, they claimed, much in need of revision.

Two new instruments. Two new instruments – *Mathematics as a gendered domain* and *Who and mathematics* - have been developed and trialed. The aim of both versions is to measure the extent to which mathematics is stereotyped as a gendered domain; that is, the extent to which it is believed that mathematics may be more suited to males, to females, or be regarded as a gender-neutral domain. Details of the process for the development of the items on the scales, and the establishment of the validity and reliability of the items are described elsewhere (see Forgasz & Leder, 2000; Leder & Forgasz, 2000).

The results of the administration of the two new instruments to 861 Australian grade 7-10 students during 1999 have been reported (Forgasz, 2000; Forgasz & Leder, 2000; Leder & Forgasz, 2000). The findings appeared to challenge notions of mathematics as a *masculine* endeavour.

In this paper, data from only one of the two instruments – the *Who and mathematics* scale – are presented. Specifically, two groups of preservice teachers – one in Australia and one in the USA[2] – were asked to complete the instrument as they expected students to do. Comparisons could thus be made between the two preservice teacher groups, and those of Australian students.

The *Who and mathematics* scale

An innovative response format was adopted for the Who and mathematics version of the instrument. Thirty statements were presented and for each statement, respondents had to select one of the following responses:

BD – boys definitely more likely than girls
BP – boys probably more likely than girls
ND – no difference between boys and girls
GP – girls probably more likely than boys
GD – girls definitely more likely than boys

Scoring. In order to interpret responses to items on the *Who and mathematics* instrument, the categories are scored as follows: BD = 1, BP = 2, ND = 3, GP = 4 and GD = 5. Responses are entered into a database and analysed using SPSS_{WIN}. Mean scores are calculated for each item. One-sample t-tests are conducted on the item means to test for statistically significant differences (at the p<.01 level) from the middle score (ND) value of 3.

Interpretation of results. For items with means not significantly different from 3, respondents, on average, believe that there is no difference between girls and boys with respect to the wording associated with the item.

For items with mean scores statistically significantly different from 3:

- mean scores <3 mean that, on average, respondents believe that boys are more likely than girls to match the wording of items, and
- mean scores >3 mean that, on average, respondents believe that girls are more likely than boys to do so.

The items, predicted responses. The 30 items, in the order they appear on the *Who and mathematics* instrument are shown in Table 1. The predicted gendered response directions for the items, based on previous research findings on perceptions of mathematics as a *male domain*, are also shown.

Previous findings from the new 'Who and mathematics' instrument. The previously reported response directions of the Australian grade 7-10 students (see Forgasz, 2000; Leder & Forgasz, 2000), based on mean scores for each item on the *Who and mathematics* scale, have also been included in Table 1. The data reveal that for only eight out of the 30 items students' responses were in the directions predicted by previous research in the field (Items: 2, 3, 10, 16, 21, 24, 28, 30).

The study

Aim. In 2000, the *Who and mathematics* instrument was administered to preservice teachers in Australia and the USA. The aim was to explore whether Australian and US preservice teachers held common or different views of contemporary high school students' beliefs about the gendering of mathematics. It should be noted that data from US high school students have also been gathered. As yet, comparisons between the US students' and US preservice teachers' views have not been reported.

Sample and methods

The sample sizes were: 394 Australian and 96 US preservice teachers. The 30 items of the *Who and mathematics* instrument (Table 1) were administered to the Australian and US pre-service teachers. Because of the wording of items, the instructions to the preservice teachers were slightly different from those given to the high school students. Students were asked for their reactions to each statement. The preservice teachers were asked to answer as they believed high school students would respond. Thus comparisons between the findings of high school students and preservice teachers reflect consistencies and differences in students' beliefs and pre-service teachers' beliefs about student beliefs.

The data gathered from the preservice teachers from both countries were entered into a database and analysed statistically using SPSS_{WIN}.

Analyses, results and discussion

Independent groups t-tests, by country, were conducted for each of the 30 items. Mean scores for all items and the p-levels for items with statistically significant different means are shown in Table 2.

Table 1. *Who and mathematics*: The 30 items, predicted directions of item responses (Pred), and response directions of 861 Australian grade 7-10 students (Aus students).

	ITEM	Pred	Aus students
1	Mathematics is their favourite subject	M	*F*
2	Think it is important to understand the work in mathematics	F	*F*
3	Are asked more questions by the mathematics teacher	M	*M*
4	Give up when they find a mathematics problem is too difficult	F	*M*
5	Have to work hard in mathematics to do well	*F*	*M*
6	Enjoy mathematics	M	*F*
7	Care about doing well in mathematics	M/F	*F*
8	Think they did not work hard enough if do not do well in mathematics	*M*	*F*
9	Parents would be disappointed if they do not do well in mathematics	M	*F*
10	Need mathematics to maximise future employment opportunities	M	*M*
11	Like challenging mathematics problems	M	*nd*
12	Are encouraged to do well by the mathematics teacher	M	*nd*
13	Mathematics teacher thinks they will do well	M	*F*
14	Think mathematics will be important in their adult life	M	*F*
15	Expect to do well in mathematics	M	*F*
16	Distract other students from their mathematics work	M	*M*
17	Get the wrong answers in mathematics	F	*M*
18	Find mathematics easy	M	*F*
19	Parents think it is important for them to study mathematics	M	*nd*
20	Need more help in mathematics	F	*M*
21	Tease boys if they are good at mathematics	M	*M*
22	Worry if they do not do well in mathematics	M/F	*F*
23	Are not good at mathematics	F	*M*
24	Like using computers to work on mathematics problems	M	*M*
25	Mathematics teachers spend more time with them	M	*nd*
26	Consider mathematics to be boring	F	*M*
27	Find mathematics difficult	F	*M*
28	Get on with their work in class	F	*F*
29	Think mathematics is interesting	M	*F*
30	Tease girls if they are good at mathematics	M	*M*

Table 2. Mean scores by country and significance levels of independent groups t-tests by country

Item No.	Australia	USA	p-level	Item No.	Australia	USA	p-level
1	2.68	2.38	<.001	16	2.18	2.31	nd
2	*3.07*[1]	*3.02*	nd	17	*3.02*	3.14	<.05
3	2.67	2.28	<.001	18	2.76	2.47	<.001
4	*2.98*	3.45	<.001	19	2.76	*2.86*	nd
5	3.14	3.33	<.01	20	3.12	3.41	<.001
6	2.76	2.54	<.01	21	2.69	2.79	nd
7	3.18	*3.15*	nd	22	3.23	3.41	<.05
8	3.33	3.43	nd	23	3.1	3.29	<.01
9	2.71	*2.83*	nd	24	2.45	2.48	nd
10	2.74	2.61	nd	25	2.87	*2.78*	nd
11	2.68	2.42	<.01	26	3.11	3.35	<.01
12	*2.92*	2.72	<.05	27	3.23	3.43	<.01
13	2.87	2.44	<.001	28	3.59	3.26	<.001
14	2.72	2.58	nd	29	2.84	2.55	<.001
15	2.73	2.6	nd	30	2.64	2.58	nd

[1] For each country, a one-sample t-test was conducted on the mean score for each of the 30 items to test for a statistically significant difference from 3 – the middle of the range of potential mean scores. Mean scores not significantly different from 3 (at the p < .01 level) are shown in italics in the Table 2.

The mean scores for the Australian and US preservice teachers are also represented graphically in Figure 1. The line down the middle of the graph is at the value 3 – the mid-point of the range of possible mean scores. Bars to the left of the centre line represent mean scores <3; bars to the right, mean scores were >3.

As is evident from Figure 1 (and Table 2), for all items except Item 4, the *direction* of the beliefs of the pre-service teachers from the two countries were the same. The means for Item 4 indicate that the Australians believed that high school students would respond that there would be no difference between girls' and boys' likelihood to "give up when they find a mathematics problem is too difficult"; the Americans, however, indicated that they believed high school students would consider girls were more likely than boys to do so (mean score >3).

When the directions of the preservice teachers' responses are compared to the predictions from the research (see Table 1), it is clear that the pre-service teachers in both countries believe that high school students hold views consistent with previous research findings. In other words, the preservice teachers believe that high school students still have traditionally stereotyped views of mathematics as a male domain.

Table 2 reveals that there were 17 items (1, 3-6, 11-13, 17, 18, 20, 22, 23, and 26-29) for which there were statistically significant differences in mean scores by country.

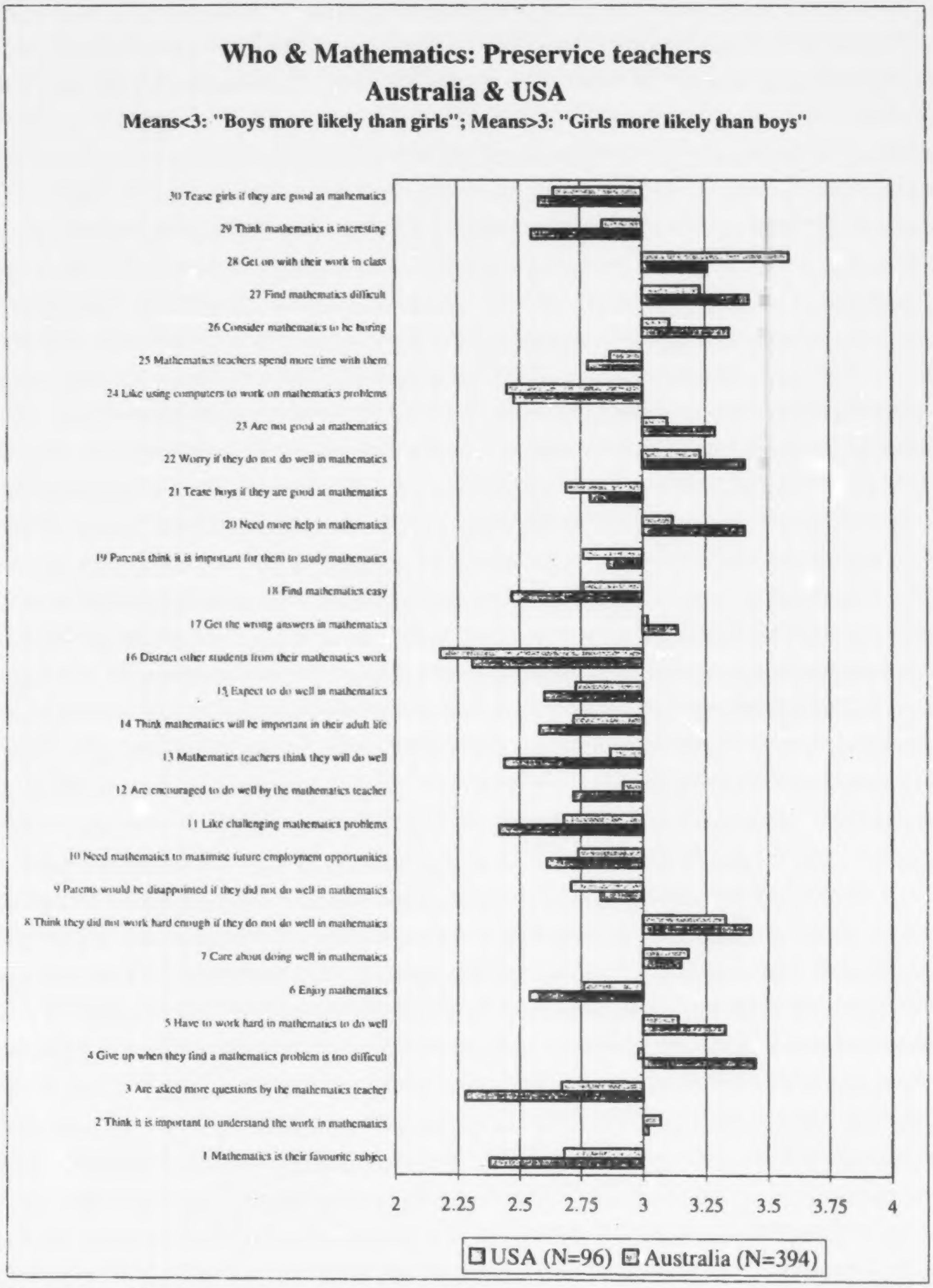

Figure 1. Mean scores: Australian and US Pre-service teachers

From the Australian data, it is clear that the preservice teachers' views (Table 2 and Figure 1) did not match with the students' views (Table 1). The gendered directions of their responses differed on many of the items. It is tempting to speculate that the same will be found when the US student and preservice teacher data are compared.

Final words

The responses of the pre-service teachers to the *Who and mathematics* instrument were consistent with the predicted gendered directions of responses based on previous research in the field. The preservice teachers in both countries held similar gender-stereotyped expectations that high school students' beliefs would continue to reflect views that mathematics was a male domain.

In reality, it is highly likely that the pre-service teachers' views of high school students' beliefs actually reflected their own personal experiences and beliefs. The Australian preservice teachers would have been in grade 7 at least six years earlier; others, earlier still. When the Australian high school students' views are considered, it seems that the preservice teachers (about 10 years older than the participating grade 7 students), are 'out of touch' with contemporary high school students' views on the gender-stereotyping of mathematics. It is interesting to speculate what will happen when the preservice teachers enter the profession to find that many of their expectations differ from those of their students.

The data gathered from the *Who and mathematics* instrument appear to raise more questions than answers. Alternative data gathering methods would be needed to try to understand what has brought about the apparent change in student views of the gender stereotyping of mathematics and why preservice teachers' understandings of the high school students' views differ so much from the teenagers' beliefs.

Interestingly, in both countries the majority of respondents was female – Australia: 83% and USA: 89% - a telling reflection of the gender profile of the future teaching profession in both countries. To undertake a valid gender analysis of the data (particularly from the USA) and make comparisons, larger samples would have been required to compensate for the gender imbalance in participant numbers.

In the study reported here, other socio-cultural factors were omitted – for example, developed/developing nation status, ethnicity/race, culture/religion, and socio-economic background/class. Researchers using the *Who and mathematics* instrument and including these factors may uncover differences in views that would assist in more effectively identifying gender-based inequities demanding action. It remains important, however, to continue monitoring broadly-based gender issues in mathematics.

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Endnotes

[1] Gilah Leder and I are co-researchers on this project.

[2] With thanks to Peter Kloosterman and his colleagues for administering the instrument in the USA.

STUDENTS' CHOICE OF TOOLS IN SOLVING EQUATIONS IN A TECHNOLOGICAL LEARNING ENVIRONMENT

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This paper presents several findings on 13-14 year old students' ability to choose, employ and integrate various tools in their solution of four equations. We analyzed the ability and preferences of six pairs of students to use paper and pencil and four computer tools in their solutions. The interviewed students were able to produce various solution methods, make transitions between them, and find a complete solution of each equation. With regard to their choice of a preferred tool, the students displayed in their actual work a preference for manual, algebraic algorithms. In their comments, however, they frequently expressed a preference for a technological tool, and could provide an explanation for their opinion.

Introduction

Many researchers in mathematics education recommend the use of computerized tools in algebra (Dörfler, 1993; Balacheff & Kaput, 1996; Yerushalmy & Schwartz, 1993). These claims are backed by research on students' work on algebraic tasks with specific computerized tools – such as spreadsheets, graph plotters and algebraic symbol manipulators. On the other side, research on students' ability to choose, employ and integrate various tools in the solution of an algebraic task is scant. Our limited knowledge on this issue is also related to the fact that regular classroom tasks in algebra frequently recommend explicitly the representations and tools that should be employed.

The goal of this paper is to present several findings on (a) students' solutions of algebraic equations, in an environment that provides a variety of tools, (b) students' ability to choose, employ and integrate various representations and tools in their solution process, and (c) students' view of their solution tools.

The study was conducted within the learning environment of the *CompuMath* project (Hershkowitz et al., in press). The project developed a junior high school mathematics curriculum, integrating an interactive computerized learning environment. The *CompuMath* students work one to two (out of five) weekly lessons, in a computer lab. The topics of the algebra course include algebraic generalizations in the first year (with spreadsheets as a technological tool), solutions of equations in the second, and functions in the third year (with a graph plotter and a symbol manipulator as technological tools). Learning is based on a cyclic sequence of investigating open problem situations in small heterogeneous groups, followed by consolidations of mathematical concepts and processes and reflective actions. The curriculum supports a broad understanding of the equation concept, both as an algebraic or numerical equality, and as a particular point in a continuous variation.

Procedure

Six pairs of 13-14 year-old higher, and average ability students were interviewed. The students belonged to a selective, but not mathematically oriented urban school. They were in the first month of their last year of a three-year algebra course. The findings reported here are based on written protocols taken during the interviews and on students' written work from these interviews. Each interview lasted about 90 minutes.

The students were presented a sequence of two single equations and two systems of equations (one linear and one quadratic in each case). The equations are presented in the leftmost column of Table 1. Before presenting the first equation, the interviewer explained to the students, that they will receive a sequence of four algebraic equations and will be required to solve them in *as many different ways* as they can. The students were also shown four possible tools, to accomplish the task: paper and pencil, and three computerized tools -- graph plotter, algebraic symbol manipulator and spreadsheet. During their work, the students received no instructions or hints with regard to their choice of tools or solution methods. At the conclusion of each of the four tasks, each student was asked separately (but in the presence of his/her peer) about his preferences towards the employed tools.

Solution methods

During the interview, the students used a variety of solution methods (an average of 3.8 methods per pair, per equation) and employed all four available tools. All pairs found the solution of each equation, even when they were unable to provide a paper and pencil solution. Table 1 presents the students' solutions of the four equations. Due to limitation of space, some secondary variations in solution processes were not included.

Remarks: (a) During their course work, the students did not encounter the algebraic algorithm for solving quadratic equations or systems. (b) During their course work, spreadsheets were used to describe and investigate variations, but were not employed as a tool for solving algebraically presented equations. (c) A spreadsheet solution of equations with two variables is very complex and inconvenient. (d) The students had no previous experience with parallel work on several computer tools and with the need to choose a tool according to their considerations.

The first remark explains the students' relatively low level of success in the use of paper and pencil to solve quadratic equations, whereas (b) and (c) may explain the relatively low frequency of their use of spreadsheets, as compared to other tools. Another fact that should be mentioned is that (d) did not seem to have a limiting

Table 1. Solution methods employed by the interviewed students (number of pairs using each method*).

Tool / Equation	Paper and Pencil	Graph Plotter	Alg. Symbol Manipulator	Spreadsheet
1.2 (x – 0.5)= 8.4	• Expanding the expression and solving for x. (6) • Dividing first by 1.2 and then solving for x. (2)	• First, an unsuccessful attempt to graph the equation by entering it as a whole, and then a) give up. (1) or b) graph each side separately and trace the intersection point (eventually changing step size or scaling). (2) or c) graph the left side, trace and monitor the y-coordinate. (1) • Direct performance of stage b) or c). (2)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation, as a formula, copying it downwards and looking for the appearance of the right-side value (8.4), or the True value. (4) • Unsuccessful attempts (confusing the role of the independent and dependent variable, omitting the sequence of independent variable). (2)
$x^2 - 5x + 6 = 0$	• Dividing by x (correctly) and unable to continue.(2) • Dividing by x (incorrectly) and receiving an incorrect solution. (3) • No attempt (awareness of an unknown method). (1)	• Graphing each side separately and tracing the intersection point (possibly changing step size or scaling). (2) • Graphing the left side, tracing and monitoring the y-coordinate. (3) • Mentioning graphing, but no attempt. (1)	• Entering the equation and pressing the "Solve" key. (6)	• Entering a sequence of numbers in Column A (e.g., from 1 to 10 in steps of 0.1), possibly changing the step size. Then, entering in Column B the left side or the whole equation as a formula, copying it downwards and looking for the appearance of the right-side value, or the True value. (3) • No attempt. (3)

* Occasionally, the same pair produced more than one written (paper and pencil) solution for the same equation.

Table 1 (continued). Solution methods employed by the interviewed students (numbers of pairs using each method).

Tool / Equation	Paper and Pencil	Graph Plotter	Alg. Symbol Manipulator	Spreadsheet
$x + y = -4$ $x - y = 8$	• Adding the two equations and solving. (3) • Subtracting the two equations and solving. (2) • Using one equation to express one of the variables and substituting the obtained expression in the second equation. (5)	• Graphing each equation separately and tracing the intersection point (possibly changing step size or scaling). (6)	• Entering the two equations as a system and pressing the "Solve" key. (6)	• Entering sequences of numbers in Columns A and B. Then entering the sequence of A+B in column C. Realizing the dependence between the x values (in A) and the y values (in B) and giving up. (3) • Attempting to search in the "Functions" menu. (1) • No attempt. (2)
$y = x + 1$ $y = x^2 + x$	• Substituting for y (as expressed in the first equation) in the second equation and then a) complete solution (2) or b) partial solution (2) or c) incorrect solution (1) • Substituting for x (as expressed in the first equation) in the second equation and unable to obtain a solution. (2) • Considering any pair (x, y) satisfying the first equation as a solution. (1)	• Graphing each equation separately and tracing the intersection point (possibly changing step size or scaling). (6)	• Entering the two equations as a system and pressing the "Solve" key. (6)	• No attempt. (6)

on students' facility in their transition from one tool to another and on their ability to make connections between their work on different tools. Thus, for example there were at least ten cases of going back from a technological tool to previously performed paper work, in order to find out "what went wrong". Instances of students making connections between outcomes obtained on different technological tools were also observed. For example, pairs E and F reported that they used the results obtained on the algebraic manipulator, in order to verify their previous work on the graph plotter.

Choice of solution tools.

We will report here several findings on students' preference for solution tools. The analysis was based on several sources: (a) observations of their actual work, (b) students' comments made after the solution of each equation and (c) spontaneous comments made by students during their work.

All six pairs started each task with paper and pencil. The fact that the solution algorithm for solving a quadratic equation was unknown did not deter them from employing an algebraic approach as a first attempt. For example, pair A attempted to solve Equation 2 as follows:

A1: *Let's do –6* [Writes $x^2 - 5x = -6$] *Can we divide by x?*

A2: *May be [square] root of x?*

A1: *Let's do +5x and then the root* [Writes $x^2 = 5x - 6 \ / \ \sqrt{x}$]

A2: *We were not taught to take this apart [separate x].* [Writes $x = -6 + \sqrt{5x}$]

A1: *We'll put it together, take it apart, put it together, take it apart, until we get [the solution].*

Table 2 presents the interviewed students' preference of tools, as expressed explicitly at the end of solving each of the four equations.

Table 2. Students' preference of tools for each of the four equations (N = 12).

Tool Equation	Paper	Graph plotter	Symbol manip.	Undecided
$1.2 (x - 0.5) = 8.4$	10		2	
$x^2 - 5x + 6 = 0$	2	5	3	2
$x + y = -4$; $x - y = 8$	2	2	7	1
$y = x + 1$; $y = x^2 + x$	2	5	4	1

Although in their actual work, all six pairs chose to start each of the four solutions on paper, the students were less committed to this tool in their comments made at the end of each task. Table 2 indicates that in the case of the linear equation, most students preferred a solution on paper. In the other three tasks, however, most students chose one of the two computerized tools as their first preference.

The preference for paper and pencil was backed up by several reasons. We cite selectively because of space limitations.

- Level of involvement in the solution process.

  My knowledge is better on paper. You don't make efforts on the computer. That's good for the lazy ones. I like to use the paper, but sometimes it gets complicated ... I like challenges. (E2, Eq. 1)

  If I use Derive, it does not mean that I did it. The computer did it. (F2, Eq. 2)

- Availability.

  Chances are that I may be without a computer – on the matriculation exam there are no computers. (B1, Eq. 1)

- Need for understanding and for transparency.

  Derive did not show anything ...just gave the answer and did not show anything. (B2, Eq. 2)

  The paper seems to me safer, although Derive solves everything. [Working with Derive] on complicated equations, it may get complicated and you don't understand what did you get ... Derive is more convenient, but if [the equation] is complicated, I don't understand how did I get the x. (D2, Eq. 1)

- Teacher/Interviewer expectations.

  Derive is the best and is quick, but in exercises the teacher wants us to show the way. (B2, Eq. 4)

  The computer is better, easier, but this is not a way. On a test they won't let us use it. If we were allowed, of course. (F2, Eq. 3)

- Personal attitude.

  I don't want to deal with graphs. (C1,C2 Eq. 2)

  I like the paper. It's comfortable. (E1, Eq. 1)

Most students could recognize, at least at a declarative level, the advantages of computer tools. In view of the objective difficulties raised by solving an equation in a spreadsheet (*Excel*) environment, we will relate here to students' attitude towards the graph plotter (*Mathemati-X*) and the symbol manipulator (*Derive*). Students mentioned (mainly when their knowledge did not allow them to solve the given equation manually) the computer's ability to solve any equation.

The computer can find every possible result and I can't (C1, Eq. 4)

Students mentioned as the main advantage of the graph plotter its transparency, whereas the symbol manipulator's main strength, according to them, lies in its speed and operational easiness.

The solution [on paper] is a little difficult. I prefer the graph. It gives the intersection point. By substituting numbers [i.e., trial and error] we could have thought that there is only one solution x = 2. We could have stopped at 2. Here [on Mathemati-X] we see the graph going up and down and we can find both solutions. (A1, Eq. 2)

A preference for the graph plotter was frequently attributed to the fact, that the manipulator did not satisfy the need to understand the solution process.

Between Derive and Mathemati-X, I prefer Mathemati-X, because it's more concrete. When we see the graph, we understand, we see it concretely, clearly. (D1,D2, Eq. 2)

Derive was the best and the quickest. (B2, Eq. 4)

In some cases, the legitimacy of a symbol manipulator as a solution tool was questioned altogether.

Derive is not really a way [for solving equations]. (F1, Eq. 1)

[Derive] is easier – but it is not a way. (F1, Eq. 3)

Conclusion

The findings clearly indicate that the interviewed students were able to employ a variety of solution methods. They made connections between various meanings of the equation concept and of the solution process employed with each tool.

Some researches showed that even when students were able to solve standard problems in both symbolic and graphical representations, their actual understanding of connections between representations was often superficial and vague (Yerushalmy & Schwartz, 1993; Knuth, 2000). This difficulty was not observed in our case. The interviewed students were able to present the concept of equation in various representations, move between tools and between representations and connect the outcomes. They were able to produce various solution methods, make transitions between them, and find a complete solution, even when a standard algebraic, paper and pencil method could not be produced.

With regard to their preferences of the four available solution tools (paper and pencil, graph plotter, algebraic symbol manipulator and spreadsheet), the picture is less clear. For affective, cognitive and external reasons, students displayed in their actual work a preference for manual, algebraic algorithms. In their comments, however, they expressed a less univocal stand. They frequently chose a technological tool as their first preference, and explained the reason for their expressed preference. The main criteria that influenced the students' choice of tools were a potential to display the solution process, a potential to allow a higher extent of student involvement in the solution process and compliance with accepted norms of work.

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CONSTRUCT VALIDITY OF A DEVELOPMENTAL ASSESSMENT ON PROBABILITIES: A RASCH MEASUREMENT MODEL ANALYSIS

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ABSTRACT

The aim of the study reported in this paper was to obtain construct related evidence of a test on pupils' understanding on probabilities. The theoretical background underlying the design of the test is presented. The test was administered to year 4, year 5 and year 6 Cypriot pupils (n=623). The Extended Logistic Model of Rasch was used and the data were analysed by using the computer programme QUEST. A scale was created for the test and analysed for reliability, fit to the model, meaning and validity. It was also analysed separately for each of five groups (boys, girls, year 4, year 5 and year 6 pupils) to test the invariance of the scale. Analysis of the data revealed that the instrument has satisfactory psychometric properties. Five levels of probabilistic thinking were also identified. Despite the fact that there is a linear sequential hierarchy among the five levels, a big difference between the second and the third level was found. The findings are discussed with reference to intended uses of the assessment. Suggestions for further research are also drawn.

I) INTRODUCTION

In recently published literature on mathematics education there is a movement to introduce elements of probability into the elementary school curriculum. Thus, nowadays probability is one of the major areas of mathematics in primary curricula (NCTM, 2000; DfEE, 1999; Ministry of Education, 1994). The main purposes of the studies which have been developed by educational psychologists and mathematics educators about probability were the investigation of the conceptions of pupils about probability and the understanding of the concept of probability (Jones et al, 1999; Konold et al, 1993), the investigation of the misconceptions of children (Garfield & Ahlegren, 1998; O'Connell, 1999; Ayres & Way, 2000) and the proposition of valid framework which would enable young children's probabilistic thinking to be described and predicted across levels (Jones et al, 1997; Amir & Williams, 1999). It has been shown that early primary pupils are able to develop conception about probability prior to classroom instruction and the experiences of the six years old pupils are useful for the teaching of probability (Ayres & Way, 1999; Ojeda, 1999). Children bring informal knowledge acquired in daily life from their culture which might interfere with their learning of probability. Ayres and Way (1999) reveal that children without any formal probability schooling can make decision based on likelihood. However, Amir and William (1994) argue that it is possible for pupils of twelve years old to encounter difficulties in predicting the probability of tossing a coin, believing that it depends on how you toss it.

Although there has been considerable research into students' probabilistic thinking, there has been almost no research on the development and evaluation of instructional programmes in probability. Instructional programmes are expected to be flexible and guided by formative assessment of pupils' understanding of the subject, according to the cognitive stages of children. Jones et al. (1999) supported that pupils' thinking about probability could be divided into four stages. At the first stage pupils predict the probability of an outcome on the basis of subjective judgement. At the second stage they predict the probability of an outcome after a combination of quantitative judgements and subjective judgements. At the third stage they compare the probabilities of different types of outcomes on the basis of consistent quantitative judgements and they distinguish "fair" and "unfair" probability generators on the basis of valid numerical reasoning. At the fourth stage they solve different types of problems on probabilities. This kind of theories empowers the construction of appropriate instructional programmes with purposes and school activities which are connected with students' thinking about probability. In the absence of a framework for systematically describing and predicting young children's thinking on probability the instruction of probability in primary education is possible to be inappropriate. According to Shaughnessy (1992) there is a need to develop appropriate tasks to assess students' conceptions of probability, their understanding of probability and their ability to solve problems on probabilities.

In this context, the main purpose of the study was the development of an assessment tool for measuring primary pupils' ability on the understanding of probability and the examination of the construct validity of the test. A test's construct validity is defined by the degree to which a set of items measures the theoretical construct it was designed to measure (Allen & Yen, 1979). Construct validity is an ongoing process whereby a test is evaluated in the light of a specific construct. It is therefore important to collect data to verify that the measured attribute behaves in concordance with the underlying theory (Cronbach, 1990). Eventually, the purpose of the study was not only to construct a valid tool of assessment on probabilities for pupils of primary school but also to identify levels of probabilistic thinking which could be helpful for diagnostic teaching of probabilities at primary education.

II) THE DEVELOPMENT OF THE TEST: SPECIFICATION OF THE CONSTRUCT DOMAIN

The construction of the test was guided by existing research and theory in the following two areas: a) current philosophy on Mathematics Education and b) research and theory on developmental assessment. Moreover, a key requirement in designing the test was its alignment with the mathematics curriculum that was operative in the area where the study is conducted. It was therefore taken into account that probabilities consisted a major part of the Cyprus primary mathematics curriculum. A content analysis of the national textbooks in Mathematics was conducted which helped us to identify seven main aims of teaching probabilities at primary schools and the emphasis which is given to each of them. Moreover, a documentary analysis of the National Standards of the USA (NCTM, 2000) and of the English National Curriculum (DfEE, 1999) was conducted. It was found that these seven aims are also implied in the national standards of USA (NCTM, 2000) and the English national curriculum (DfEE, 1999). Thus, the test tasks were constructed according to these seven aims (see Table 1). As far as the influence of research and theory on developmental assessment, the applications of developmental assessments for measuring proficiency in cognitive abilities and content areas (Brown et al, 1992) were taken into account. Two essential concepts derived from these works were as follows: a) the developmental ordering of tasks on a continuum of difficulty and b) the provision of controlled, interactive support to examinees during the testing process. It was therefore decided to include in the test assessment tasks related with each aim on pupils' skills in probabilities which will cover a range of item difficulties. Moreover, in the instructions given to teachers who were asked to administer the test information were provided on the kind of support that pupils could have.

Table 1: Specification table of the Probability Test (PT)

Aims	**Tasks of the test at different levels**				
Pupils should be able to:	**1**	**2**	**3**	**4**	**5**
Describe events as impossible, likely or certain.	1, 2				
Compare events according to their degree of likelihood.	5, 8, 9	6, 10			
Estimate the probability of an event by using the formula of Laplace.		3, 4, 11			
Predict the change of the probability of an outcome of type A if the conditions of the experiment are changed.			12, 13		
Compute probabilities for simple compound events, using such methods as organised lists or tree diagrams.			7, 16,17		
Use Laplace formula of estimating the probability of an outcome of type A to compute the total number of possible outcomes or the number of outcomes of type A.				14, 15	
Examine the fairness of a game.					18, 19

III) METHODS

Once the final version of the test was developed, a table which indicated the relations between the tasks of the test and the aims of the teaching of probabilities was created (Table 1). The specifications and the tasks were content validated by two experienced primary teachers, the authors of the national textbooks, two postgraduate students of Mathematics Education, and two members of the Educational staff of the

Department of Education at the University of Cyprus. The "judges" of the content and the face validity of the test were asked to mark-up and to make comments on the items. In the light of their comments minor amendments were made. The final version of the written test was administered to 623 Cypriot primary pupils of year 4 (220), year 5 (218) and year 6 (185). Additionally, 318 of the subjects were girls and 305 were boys.

The Extended Logistic Model of Rasch (Rasch, 1980) was used and the data were analysed by using the computer programme Quest (Adams & Khoo, 1996) to create a scale satisfying the seven measurement criteria set out by Wright and Masters (1981) which have to be met in order to claim that the items form a valid and reliable scale. The scale is based on the log odds (called logits) of pupils' abilities to answer correctly the 19 items of the Probability Test (PT). The items are ordered along the scale at interval measurement level from easiest to hardest. The Rasch measure produces scale-free measures of pupils' abilities and sample free-item difficulties (Wright & Masters, 1981). This implies that the differences between pairs of measures of pupils' abilities and item difficulties are expected to be sample independent.

IV) FINDINGS

The data were analysed initially with the whole sample (n=623) and all the 19 items together. There were two items (16 and 17) which did not fit the model. Thus, the analysis was repeated with the whole sample and the 17 remaining items. Then, the analysis was repeated with each of the five groups of the sample. This was done to investigate whether the test is used consistently by boys, girls, year 4, year 5 and year 6 pupils and is part of the measurement criteria set out by Wright and Masters (1981).

Table 2: Statistics relating to the test for the whole sample and the five groups

Statistics	Whole (n=623)	Boys (n=305)	Girls (n=318)	Year 4 (n=220)	Year 5 (n=218)	Year 6 (n=185)
Mean (items)	0.00	0.00	0.00	0.00	0.00	0.00
(persons)	-1.18	-1.21	-1.16	-2.55	-1.17	-0.22
Standard deviation (items)	1.67	1.84	1.61	1.84	1.32	1.58
(persons)	1.39	1.35	1.36	0.96	1.22	1.17
Separability* (items)	0.99	0.99	0.99	0.96	0.99	0.99
(persons)	0.89	0.86	0.89	0.86	0.91	0.90
Mean Infit mean square (items)	0.99	1.00	0.99	1.00	0.99	1.00
(persons)	0.99	0.99	0.99	1.00	0.99	1.00
Mean Outfit mean square (items)	1.02	1.00	1.03	1.03	1.02	1.01
(persons)	1.02	1.01	1.04	1.02	1.03	1.01
Infit t (items)	-0.04	-0.05	-0.05	0.03	-0.04	-0.06
(persons)	-0.01	-0.01	-0.01	-0.03	-0.01	-0.04
Outfit t (items)	-0.07	0.06	0.11	0.10	0.11	0.09
(persons)	0.04	0.11	0.08	0.15	0.02	0.04

* Separability (reliability) represents the proportion of observed variance considered to be true. A value of 1 represents high separability and a value of 0 represents low separability.

Table 2 provides a summary of the scale statistics for the whole sample and the five groups. The following observations arise from this table. First, we can observe that for the whole sample and for each group the indices of cases and item separation are higher than 0,85 indicating that the separability of the scale is satisfactory. Second, the infit mean squares and the outfit mean squares are approximately 1 (0,99 up to 1,03) and the values of the infit t-scores and the outfit t-scores are approximately zero (-0,07 up to 0,11). And since the mean squares are within 30% of the expected values, calculated according to the model, it can be claimed that there is a good fit to the model. It is also important to note that all the items have difficulties which could be considered invariant across the 5 groups, within the measurement error (0,15). Thus, an important aspect of creating a scale (sample-free item difficulties) has been achieved. Third, the mean scores of pupils' performance indicate that an increase of performance by age can be identified. However, even the mean of year 6 pupils is relatively low (-0,22) and thereby the mean of the whole sample is very low (-1,18). Fourth, the standard deviations of the abilities of each year group but year 4 are relatively high. This implies that there is a big variation among the responses of year 5 and year 6 pupils whereas the performance of year 4 pupils was generally very low.

Figure 1 illustrates the scale for the remaining 17 items of the test with item difficulties and the whole group of pupils' measures calibrated on the same scale. The following observations arise from Figure 1. First, the items are well targeted against pupils' abilities in probabilities. More specifically, pupils scores range from -4,18 to 3,04 logits and the item difficulties range from -3,34 to 2,68. It can be, however, claimed that the targeting of the items at pupils' abilities in probabilities could be improved by adding some very easy items (Thresholds ≈ -4,00). Second, the most important weakness of the test is the absence of moderately easy items to moderately hard (i.e. from -0,76 to 0,91 logits). Thus, although the psychometric properties of the test seem to be satisfactory, the Probability Test could be improved by adding items which are neither easy nor hard. Third, five levels of probabilistic thinking can be identified. These levels are very similar to the levels mentioned at the specification table of the test. More specifically, pupils who are at the first level (i.e. below –2,00 logits) are able to describe events as impossible, likely or certain. They are also able to find which of two events is more likely to happen and their decision is based on the fact that they have realised that the probability of an event A depends on the number of outcomes of type A. However, pupils who are at the second level (-2,00 up to –0,75) are able to use the formula of Laplace for computing a probability and to compare events according to their degree of likelihood. After the second level, there is a relatively big area where none item is included. This could be attributed to weaknesses of the test either in including neither easy nor difficult tasks or in the fact that another level of probabilistic thinking should be included in the design of the specification table. However, this finding may reveal a gap between the second and third level and that pupils have to make an important progress in order to move from the second to the third stage. At the third level (0,90 up to 1,30), pupils are able to compute probabilities for simple compound events (e.g. throwing two dices), using such methods as organised lists or tree-diagrams. They are also able to predict the changes of the probability of an outcome of type A when the conditions of the relevant experiment are changed. Pupils who are at the fourth level (1,30 up to 2,00) they can not only find the probability of an outcome of a specific type but are also able to use Laplace formula in order to compute the total number of possible outcomes or the number of outcomes of a specific type. Finally, pupils at the fifth level (above 2,00) are able to examine the rules of a game and find out whether the game is fair.

Figure 1: Scale for the Probability Test (N=623, L=17)

```
--------------------------------------------------------------------
High Achievement in Probabilities        Difficult items
 3.0 Thresholds                    X  |
                                      |
                                      |
                                      |      19
                                  XX  |
                                      |      18
 2.0                                  |
                                  XX  |      15
                                      |
                                 XXX  |      14
                                      |
                                      |
 1.0                            XXXX  |      13
                                  XX  |       7        12
                                      |
                              XXXXXX  |
                                      |
                              XXXXXX  |
  .0                                  |
                                      |
                       XXXXXXXXXXXXX  |
                                      |
                                      |       6
                 XXXXXXXXXXXXXXXXXXX  |
-1.0                                  |       3        10
                                      |
                    XXXXXXXXXXXXXXXX  |      11
                                      |
                                      |       4
                    XXXXXXXXXXXXXXXX  |
-2.0                                  |       8         9
                                      |
                                      |
                     XXXXXXXXXXXXXXX  |       5
                                      |
                                      |       1
-3.0                                  |
                         XXXXXXXXXXX  |
                                      |       2
                                      |
-4.0                          XXXXXX       |
  Weak achievement in Probabilities        Easy items
-----------------------------------------------------------------
```

Note: Each X represents 5 pupils

V) DISCUSSION

The Extended Logistic Model of Rasch was useful in creating a good interval level measure of the Probability Test identifying primary pupils' abilities in probabilities and for investigating its validity and reliability. The Rasch model was also helpful in analysing the conceptual design of the test. The findings of this study reveal that the Rasch analysis supports the conceptual design of the instrument. The underlying trait, that is primary pupils' abilities in probabilities, seems to be an overarching concept comprised of five different levels of probabilistic thinking. Thus, the Probability Test and its Rasch scale may help teachers decide how to identify and meet pupils' learning needs in relation to the five levels of probabilistic thinking and how to use their teaching time and their resources. An important implication of the identification of learning needs is that decisions about the next learning steps follow from it and pupils could be helped to improve their abilities and move from a lower level of thinking to a higher level. However, teachers should be aware of the fact that although the five levels follow a linear sequential hierarchy, there are pupils who are at the same level but their abilities may differ. Moreover, there is no clear distinction between the levels but between the second and the third level. It is important to note that acceptable fit was also obtained using structural equation modeling procedures for the theoretical five-factor first-order structure. A two second-order factors structure was also supported revealing that factors 1 to 2 (i.e. levels 1 and 2) were explained better by a second-order factor variable which was substantively different from that which was explaining factors 3 to 5. However, further research regarding the levels of probabilistic thinking is needed in order to examine whether a new level covering aims of teaching probability which are not mentioned in the Cyprus curriculum should be included in order to cover the area between the second and the third level of probabilistic thinking. Finally, the analysis leads to suggestions for improving the targeting of items against pupils' measures through the addition of one very easy item and some neither easy nor very difficult items (-0,5 up to 1,0 logits). Thus, further validation studies of a new version of the Probability Test may be needed in order to obtain a better targeting against primary pupils' abilities in probabilities.

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RESEARCH REPORTS

DIFFICULTIES WITH NEGATIVE SOLUTIONS IN KINEMATICS PROBLEMS

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***ABSTRACT**: The article reports a Case Study (Heidi) described in a wider research work on Problem Solving in Kinematics, involving 28 high school students. The study provides evidence that Heidi's proficient academic performance leads her to the correct answers and verbal description of the physical phenomena involved. However, when explaining the answer in algebraic languaje. Heidi cannot interpret it because of her lack of acquaintance with negative numbers as final results, although she has dealt with them as part of the intermediate procedures.*

***RESUMEN**: Este artículo reporta un Estudio de Caso (Heidi) descrito en una investigación más amplia sobre Resolución de Problemas de Cinemática, realizado con 28 estudiantes a Nivel Secundaria. Los resultados de este Estudio muestran que el alto desempeño académico de Heidi le permite llegar a las respuestas y dar descripciones verbales de los fenómenos físicos involucrados en los problemas. Sin embargo, al expresar las respuestas en lenguaje algebraico, no logra interpretarlas debido a que Heidi desconoce a los negativos como resultados finales a pesar de trabajar con ellos en procesos intermedios de resolución.*

INTRODUCTION

A research project was recently carried out whose central problem was the study of negative numbers in their interaction with the languages and methods used to solve equations and problems (Gallardo, 1994). The general methodology of the project dealt with the interaction of these categories on two levels, the historical and the didactic. The historical-critical analysis carried out in this work allows us to conclude that the presence of subtractive terms and the laws of the signs appear in remote times, as do the elements necessary for the operativity of signed numbers. It can be said that a crucial step in the recognition of these numbers is the acceptance of negative solutions. The empirical analysis is based in the preceding historical study of negative numbers in the resolution of algebraic equations. In this research, we identified different stages of conceptualization of negative numbers which appear both in the historical and didactical spheres.

THE STUDY

Once research on negative numbers in the arithmetical-algebraic domain has been concluded, a second stage is initiated where the incidence of these numbers in high school physics is analyzed. The methodology used in this particular stage is the same methodology applied in the project as a whole, that is, using the historical-critical method in order to search for elements of analysis that may explain students' difficulties in interpreting negative magnitudes in elementary kinematics problems. Galileo and Newton's physics were revisited. In Galileo's text, motion laws are described and interpreted through dialogues that lead to the birth of the New Science

(Crew, 1914). Newton sets up kinematics' axioms which, as their name indicates, are expressed mathematically (Whiteside, 1972). In relation to our purpose, we must mention that neither Galileo nor Newton present any occurrence of negative numbers as solutions.[1]

This article reports a first experience in the teaching of kinematics. The traditional teaching-learning process involving the motion laws is analyzed. The problems dealt with herein are those applied in the high school context and were picked out of the Lima, Perú High School Physics Program (Encalada, 1999). Three out of 20 considered "Typical Problems" (Laglois et al, 1995) were selected. An exploratory questionnaire including these three problems was developed and applied to 28 students from four different groups of a same teacher. The researcher picked out the brighter students for the clinical videotaped interview purpose. Such selection was based on the fact that a good performance in physics and mathematics is necessary condition to do well at solving physics problems (Lang Da Silveira et al, 1992).

The case of one of the students, Heidi, is featured in the paper. A Formula Table containing conventions regarding the plus and minus signs[2] is handed out to the students together with the problems (Fig. 1).

<table>
<tr><th>Uniformly accelerated motion</th><th>Free fall</th><th>Vertical thrust upward</th></tr>
<tr><td>The positive sign (+) is used when the mobile is moving.
The negative sign (–) is used when the mobile slow down or stops. $V_f = V_0 \pm at$;
$V_f^2 = V_0^2 \pm 2ae$; $e = V_o^t \pm \frac{at^2}{2}$; $t = \frac{V_f - V_0}{a}$</td><td colspan="2">The positive sign (+) is used when the mobile drops.
The negative sign (–) is used when the mobile moves upward. $V_f = V_0 \pm gt$;
$V_f^2 = V_0^2 \pm 2gh$ $h = V_0 t \pm \frac{gt^2}{2}$; $t = \frac{V_f - V_0}{g}$</td></tr>
</table>

Fig. 1: FORMULA TABLE handed out to the students.

As shown hereafter, the use of the Formula Table leads to positive solutions on both problems.[3] In fact, the general formulas of the Uniformly Accelerated line Motion are the following: $V_f = V_0 + at$; $V_f^2 = V_0^2 + 2ae$ where letters may have any value, including negative solutions.

SOLUTION PROCEDURES OF TWO KINEMATICS PROBLEMS.

> ***PROBLEM ENUNCIATION 1:*** *The brakes of a car travelling at 200m/s are applied and the car comes to a complete stop after 80 meters. Estimate the car's acceleration and the time it takes to stop.*

---

[1] It is important to mention that historical analysis of this second stage of the project in still under way.

[2] All of the letters take on positive values or zero only.

[3] Two problems with negative solutions were selected. The third one has a positive solution and is not presented in this article.

***Analysis of the first question*: Estimate the acceleration**

- The student writes problem's data and unknown.

  V =200 m/s; V_f =0; e=80 m; a=? ; t=?

- She explains that "V=200m/s is start off velocity" [Note that initial velocity is designated by V and not by Vo]

- She recognizes that final velocity is zero when she asserts that "velocity decreases and the car comes to a stop".

- She checks her Formula Table (See Fig.1) and writes: $V_f^2 = V_0 - 2ae$. (Mistaken because initial velocity is not squared).

- She inserts her data and obtains: $(0)^2$=200-2a(80). She continues the algebraic procedure and states the following: "This two $\overparen{(0)}^2 = 200 - 2a(80)$, comes here $\overparen{0=(200)}^2 - 2a(80)$". The error in formula $V_f^2 = V_0 - 2ae$, is compensated when placing the exponent 2 on the numerical term that corresponds to Vo. She adds: "zero $\overparen{\ \ 0} = (200)^2 - 2a(80)$, you don't put anything else. =40000 – 160a, because it's zero". For a moment, the first member of the expression disappears. This indicates that she does not have full understanding of the equation concept. The process continues correctly: 160a=40000; a= 40000/160; a=250m/s (Incorrect unit). Deficiencies in the acceleration concept can be perceived in the following dialogue: Interviewer asks: "what does acceleration mean?"

Heidi answers: "Acceleration appears to be the relationship between the distance and... and the distance. Well, that is to say, the distance it covers during the time it's moving[4]. That is veloc...acceleration. Distance over square time". She adds exponent 2 to her previous answer: $a = \frac{250m}{s^2}$

When obtaining a positive solution, she does not retrieve the interpretation she herself verbalized before she looked at her Formula Table when she stated: "velocity is decreasing and the car stops".

If the student had used the general formula: $V_f^2 = V_i^2 + 2ae$, she would have arrived at a negative solution $a = \frac{250m}{s^2}$.

The minus sign could have helped her provide a physical interpretation of the problem instead of repeating mechanically that acceleration is "the distance divided by the square time", as occurred when she used the "positive expression" that appeared on the Formula Table. The previous fact identifies a difficulty in the teaching of kinematics at high school level.

---

[4] She's not clear about the acceleration. She mistakes it for velocity. The correct relationship established by the acceleration is between velocity and time.

***Analysis of the second question*: Estimate the time it takes to stop.**

- Once the acceleration is known, the student realices she must find out the time. She then writes: $V_f = V_o - at$. She explains this by indicating "because I have all those data and only the time is missing".

She inserts her data on the previous formula, obtaining: 0 =200 - 250 t. She then writes: 200–250t and states " Here 200-250t is zero, so you don't put it". Again, the equation disappears in front of zero. She now adds the zero and the equal sign in order to do an operation. She obtains 0+250t=200; 250t=200; t=0.8s (correct answer).

- The researcher poses her an additional question to analyze the velocity magnitude.

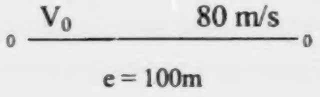
***ADDITIONAL QUESTION TO PROBLEM 1:** A car starts from rest. Its velocity is 80 m/s after running 100 meters. Estimate the acceleration.*

- She asserts that "the car is at absolute rest so it doesn't move and the velocity is zero". She writes the problem's data and unknown:

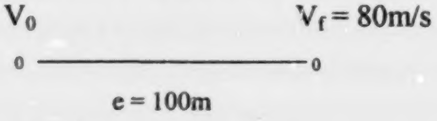
$$V_o=0; \quad e = 100; \quad V = 80 \text{ m/s}; \quad a = ?; \quad V_f = 0 \text{ (error)}.$$

Because of the error on V_f [final velocity] and because Heidi writes down three velocities, the researcher asks her "what does V=80m/s mean?" She replies: "It starts from rest, so $V_0=0$ and V = 80 m/s is the velocity it goes at; it is just the velocity. It cannot be the final velocity because…" (silence).

She is mistaking uniform straight line motion, where velocity V is constant, with uniformly accelerated motion where the velocity changes at each time unit at the so called acceleration rate. The researcher uses a scheme to elucidate the difficulty to the student.



She adds the velocity's notation V_f
The diagram now appears like:



This way, the student recognizes that V is the final velocity V_f.

- Heidi indicates she will use the formula: $V_f^2=V_0^2-2ae$, but with the positive sign, namely, $V_f^2=V_0^2+2ae$, "because the mobile is moving".

- She inserts her data$(80)^2=(0)^2+2a(100)$. She goes on with the procedure:
6400 = 0+200a; 6400 = 200a ; a = 6400/200 a = 32 m/s^2 (Correct result. She spontaneously adds the measurement unit).

SOLUTION PROCEDURE FOR PROBLEM 2

PROBLEM 2 ENUNCIATION: *A balloon*[5] *rises at a constant velocity of 5m/s. When it is 30 m. away from the ground, a stone is let down from it. At what velocity and after how long will the stone reach the ground?*

- The student begins with the second question (time). She indicates: "This is the height reached by the stone" and writes h =30 m (She does not consider the balloon). She writes the problem's unknowns V=?, t=?; she mentions that "when a body is at height, there is gravity". She writes g = 9.8 (She omits the measurement unit). Note that she recognizes an implicit datum. Gravity does not appear in the problem's enunciation.

- Heidi now goes to the Formula Table and writes $h = at^2/2$ (She does not consider gravity as g in this expression). The researcher asks her about this: "This formula $h = at^2/2$ Why?". The student answer:

"They ask for the velocity, they ask for the time, but I have none. There are two unknowns here ($h = at^2/2$) and I must find one".

She then points out at the data on the paper (V=?, t=?, g= 9.8) and recovers the gravity's acceleration. She substitutes acceleration (a) for (g) on the formula, obtaining $h= gt^2 / 2$

She inserts the height as well (h=30m) and carries out the inverse operations correctly: $30=\frac{9.8(t)^2}{2}$; $60=9.8t^2$, $\frac{60}{9.8}=t^2;2.47=t$ (She omits the measurement unit)[6]

- She now goes to the first question of the enunciation: what is the stone's velocity when reaching the ground? She writes: $V_f^2=V_o^2+2gh$, and explains "This is (V_f) what they are asking me for ". She places the value of V_o on the expression and obtains

$V_f^2=(0)+ 2gh$, and explains "because initial velocity is zero" (error).

- She inserts her other data in the formula: $V_f^2 = (0)^2 +2(9.8)(2.47)$. She mistakenly justifies initial velocity zero stating that: "it starts from rest and then it goes up". She also makes an error when substituting time for height (she writes 2.47 where it should be 30).

Observe a contradictory situation on the following dialogue:

E: "What is the stone's velocity when you let go of it?"
H: "Well, the same. If you go up at that velocity and come down at the same velocity, it is going to be the same. When going up, the balloon and the stone go at the same velocity V_o=5m/s, but when reaching a 30 m. height, the stone gets there at V_f=0 because it is going to come back down".

---

[5] Hidrostatic balloon containing the stone.

[6] Only the positive square root is considered in teaching. So time is always not-negative. This solution is incorrect because the stone's initial velocity is different from zero, namely V_o = 5m/s.

Note that taking the balloon into consideration helps her conceive $V_o \neq 0$. When she omits the balloon and only takes the stone into consideration, she obtains $V_o = 0$. After she provides the answer $V_o = 5m/s$, the researcher has her check her previous formula to compare both results.

E: "If V_o=5m/s, so why did you put zero in here $\widehat{V_f^2= (0)}^2 +2(9.8)(2.47)$?"

The student remains silent. She now goes on to the first question of the problem. She does not notice that she made a mistake when placing time instead of height and goes on with the procedure: $V_f^2 = 0+48.4$; $V_f = 6.95$ (Omits the measurement unit). Her final result is wrong, it should be 25 m/s (stone's final velocity when reaching the ground)[7]

- The researcher poses one more question to elucidate student's confusion in relation to the initial velocity.

ADDITIONAL QUESTION TO PROBLEM 2.

> *A stone falls from a 100 m. height. Estimate the time it takes for it to reach the ground and the velocity when reaching it.*

- Heidi writes the problem's data and unknown: $h = 100$ m; $t = ?$; $V = ?$; $g = 9.8$ (Omits the measurement unit).

- She then writes the following formula: $h = gt^2/2$

- She goes on with the correct insertion of her data and the inverse operations:

$100=9.8t^2/2$; $200=9.8t^2$; $t^2=200/9.8$; $t^2 =20.4$; $t= 4.5$ (Omits the measurement unit)

- She writes $V_f^2 = V_o^2 + 2gh$. She asserts that "velocity is zero because I consider the stone only when it is let down" (she refers to the initial velocity). She inserts her data and obtains: $V_f^2 = (0)^2 + 2(9.8)(100)$; $V_f^2 = 1950$

$V_f = 44.2$ (Omits the measurement unit). This solution is correct.

- When the researcher asks her to compare problem 2 with the additional question, Heidi asserts "they are the same because you use the same formulas to solve them; but they follow a different procedure, because the balloon and the velocity 5m/s, which I never used, have a part in problem 2. On the other case (she refers to the additional question to problem 2), the velocity is null and the balloon does not appear". It is plain to see student's misconception that at free fall initial velocity is always zero. She forgets that an object may be hurled with an initial velocity different from zero.

STUDY CONCLUSIONS.

The article shows the difficulties that a highly qualified student faced in order to interpret the solution of kinematics problems. Such difficulties derived from the previous teaching on the subject. Such teaching encourages the use of the Algebraic Formula Table leading always to positive solutions. It is worth noting that the convention of the plus and minus signs in formulas implicitly contains a Cartesian reference system that is not mentioned to the students.

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[7] See Appendix where correct solving procedure is shown.

Teachers resort to the use of signs instead of referring to signed magnitudes which, at a higher level, would be considered as vectorial magnitudes. Also, given the fact that difficulties to these problems lie in the relativity of time and space where physical phenomena take place, defining the participating objects and their motion reference point or the time interval where they take place is necessary. Again, elucidating the reference system in use is very important.

Another problem in teaching is the omission of the dimensional analysis. The measurement units associated to magnitudes should be kept in mind all throughout the solving procedure. This consideration would help verifying if the answer is correct and might bring to light possible errors on the designated formulas and on the operations carried out with the corresponding magnitudes. The most generalized tendency among students is to fit in the measurement units once the final result has been obtained.

As far as the student's performance is concerned, our conclusions are the following: She shows better understanding of the physical phenomena after reading the problem and before consulting the Formula Table. She is not able to put in algebraic language what she expresses in natural language, since the former is mediated by a convention of signs that appears unintelligible to the her, and which cover up the coordinates system in use. So in problem 2, she mistakes the velocity rate as null because she was not aware of the reference system; had she been aware, she would have placed the origin at 30 m. from the ground, and would have considered the balloon containing the stone, and would have noted that both move at initial velocity of 5m/s (See Appendix).

It is worth mentioning that during the Problem 1 solution procedure and Problem 1 Additional Question, arises what we have called an error compensation, that is, Heidi mistakenly omits exponent two in the algebraic expression but then puts it back in when carrying out an erroneous transposition of the members of the equation. So the second error makes up for the first error. On the other hand, we know the student has not consolidated the equation concept because she does not write the complete equation, and integrates it only when carrying out the operations. As far as the physical context is concerned, Heidi seems confused at the types of motion she has been exposed to, since she is not aware of the different velocities involved in each of them. This, in turn, manifests her not having full understanding of the acceleration concept.

APPENDIX: CORRECT ANSWER TO PROBLEM 2

ENUNCIATION OF PROBLEM 2: *A balloon rises at constant velocity 5m/s. When it is 30 m. away from the ground, a stone is let down from it. At what velocity and after how long will the stone reach the ground?*

DATA: $g = -9.81\ m/s^2 \cong -10$; $V_o = 5m/s$; $h = -30m$; $t = ?$; $V_f = ?$

Note that the problem involves the use of vectorial magnitudes, since it introduces a coordinates system. The origin is considered at 30 m above the ground, that is, at the point where the stone is let down.

SOLUTION:

- ***Time it takes the stone to reach the ground (Second question).***

$h = V_0 t + \frac{gt^2}{2}$. Replacing the data we have: $-30 = 5t - \frac{10t^2}{2}$

Solving the quadratic equation we arrive at answers: $t = 3$ y $t = -2$

Result $t = -2$ is discarded because that would mean the stone should have been let down two seconds before. If it were so, it would have not spanned the mentioned distance but more meters. So, the time it takes the balloon to reach the ground is 3 seconds.

- ***Velocity at which stone reaches the ground (First question).***

$V_f = V_o + gt$. Inserting the data, we have: $V_f = 5 + (-10)(3) = -25$

Thus, the stone will reach the ground at a velocity of 25m/s.

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REVEALING AND PROMOTING THE STUDENTS' POTENTIAL: A CASE STUDY CONCERNING INEQUALITIES

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ABSTRACT: This paper presents a study concerning novices' cognitive apprenticeship in the field of inequalities. A specific educational context was designed with the purpose of revealing and enhancing the students' potential in dealing with inequalities according to a functional approach. A preliminary analysis of students' solutions is provided.

1.Introduction

In many mathematics domains, mathematics education research must face widespread, strong difficulties in teaching and learning specific subjects. Difficulties met by teachers and students frequently bring to postpone those subjects and/or reduce their teaching to procedural aspects. In some cases epistemological, didactical and cognitive analyses can help planning teaching experiments which allow researchers to better understand the reasons for these difficulties and, possibly, reveal students' potential in dealing with those subjects. Innovative educational choices are an expected reasonable outcome (cf Arzarello and Bartolini, 1998). In particular, didactical analyses may point out conditions under which difficulties take place and peculiar didactical choices related to them; while epistemological analyses may enlinghten the nature of involved mathematical concepts. Cognitive analyses seem to be necessary in order to detect crucial mental processes underlying specific mathematical performances, with the aim of enhancing them through classroom work.

In this report we present the guidelines and some preliminary results of a research program conceived according to the above perspective and concerning the approach to inequalities in 8th - grade. Our working hypothesis is that a functional approach to inequalities (i. e. an approach based on the comparison of functions), when suitably managed by the teacher, can reveal (from the research point of view) and allow to exploit (from the curriculum design point of view) a students' potential which goes far beyond the mathematics content involved (inequalities). Our preliminary results support this hypothesis and enlighten some conditions concerning the educational setting which students' success seem to depend on.

2. Inequalities: a challenge for teaching and research

In most countries, inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner that avoids, in particular, the difficulties inherent in the concept of function. For instance, in Italy and some other countries students are taught to deal with second order inequalitities depending on a parameter (e.g. $x^2+Kx+1>0$) in a very rigid, prescriptive way: they must solve the equation $x^2+Kx+1=0$ (distinguishing between the three cases: no real solution, two coincident real solutions, two distinct real solutions); then they must build up a schema where "concordances" and "variations" of signs, in relationship with the values of the parameter K, provide the student with the solutions for the given inequality. We observe that this approach implies a "trivialisation" of the subject, resulting in a sequence of routine procedures, which are not easy to understand, interpret and control.

As a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (cf Boero, 2000; Malara, 2000), at the entrance of the university mathematics courses in Italy most students fail in solving easy inequalities like $x^2-1/x>0$. In this task, proposed to a sample of 58 students entering the Faculties of Science of Genoa and Pisa Universities, less than 60% engaged in solving the inequality (the others answered *"I am not able", "I did not study it"*); most of them performed the following transformations: from $x^2-1/x>0$ to $x^2>1/x$ to $x^3>1$. Few students took care of the case $x=0$; less than 10% of the whole sample made a distinction between the case $x>0$ and the case $x<0$. In general, graphic heuristics were not exploited and algebraic transformations were performed without taking care of the constraints deriving from the fact that the > sign does not behave like the = sign (see Tsamir et al., 1998, for a deep analysis of such behaviour). Similar phenomena were described in some studies concerning the French situation (see Assude, 2000; Sackur and Maurel, 2000).

This brief presentation brings to the following conclusion: the prevailing manner of teaching inequalities in school neither is efficient (as concerns the results, in terms of capacity of dealing with a large set of rather simple inequalities) nor results in acquisition and/or reflection about the mathematics concepts involved. We may ask ourselves what are the reasons of this situation.

One reason could be the fact that equations (and inequalities) are considered (in most of European countries, including Italy) as a typical content of school Algebra; this subject matter is distinguished from Analytic Geometry and does not include functions. This might explain why inequalities (and equations) are not dealt with in those countries from a functional point of view. But even in countries where functions (and Analytic Geometry) belong to school Algebra (cfr. NCTM Standards, 1989) the procedural, algebraic approach prevails in many curricula and even in innovative proposals (cf Mc Laurin, 1985; Dobbs and Petersen, 1991).

Another possible hypothesis is that inequalities are a very complex and demanding subject; dealing with few and well codified cases in an algorithmic way appears as a consequence of these intrinsic difficulties. In order to support this interpretation we may observe that (from an epistemological point of view) the functional aspect plays a crucial role, both for equations and inequalities. Indeed, let us analyse mathematicians' work when they solve equations with approximation methods, deal with the concept of limit or treat applied mathematical problems involving asymptotic stability: the functional aspect of inequalities plays a crucial role. This fact is often neglected in traditional teaching: as suggested at the beginning of this Section, we may recognize that the traditional teaching of inequalities avoids the "function" concept and reduces the difficulties ineherent in the "variable" concept and the complexity of the solution process by treating inequalities as a "special" case of equations.

Under the same perspective we can make the hypothesis that an alternative approach to inequalities based on the concept of function could provide an opportunity to promote the learning process of the difficult concepts involved and the development of the inherent skills. It could also ensure an high level of control of the solution processes of equations and inequalities (Sackur and Maurel, 2000).

Finally, we must remark that in spite of the importance of inequalities in mathematics and of the difficulties met by teachers and students in dealing with them, few studies in mathematics education concern the school approach to inequalities (see References). It is like if the subalternity of inequalities to equations in the teaching of mathematics, was reflected in a scarce relevance for inequalities in mathematics education research; in particular, as concerns the functional approach to inequalities.

3.Method

Keeping the previous analysis into account we have planned a teaching experiment in two VIII-grade classes with two rather limited aims: investigating the feasibility of an early functional approach to inequalities; and revealing students' potential and difficulties in dealing with this subject as a special case of comparison of functions. We choose to guide VIII-grade students in a cooperative, gradual enrichment of tools and skills inherent in the functional treatment of inequalities. Then we have analysed how (in relatively complex tasks) they were able to use their knowledge and increase their experience in an autonomous way.

3.1. The educational context

36 VIII-grade students (divided into two classes) were involved; as usual in Italy, they had started to work with the same mathematics and science teacher in grade VI. The didactic contract established in grades VI, VII and at the beginning of grade VIII was coherent with the methodological choice of a cooperative, participated, guided enrichment of tools and skills in the planned activity. A rather common routine of classroom work consisted in individual production of written solutions for a given task (if necessary, supported by the teacher with 1-1 interventions), then the teacher guided classroom comparison and discussion of students' products; possibly, the adoption of other students' solutions in similar tasks followed. Another aspect of the didactic contract included the exhaustive written wording of doubts, discoveries, heuristics, etc.

3.2. Specific content and educational choices

As concerns the **content,** the concepts of function and variable have been approached through activities involving tables, graphs and formulas. At the beginning the geometrical context (area, perimeter, etc.) was prevailing, then it has been progressively left aside. At the beginning the function was presented as a machine trasforming x-values into y-values (*machine view* in Slavit, 1997), then classroom activities focused on the variation of y as depending on the variation of x (*covariance view*). By this way a dynamic idea of function gradually prevailed on the static consideration of a set of corresponding pairs (*correspondence view*). As a consequence, a peculiar aspect of the concept of variable was put into evidence (a variable as a "running variable", i.e. a movement on a set of numbers represented on a straight line) (cf Ursini, 1997). Finally, the approach to inequalities was realised by comparing functions.

The didactical contract demanded to compare functions as global, dynamic entities. Students knew that they had to compare functions by making hypotheses based on the analysis of their formulas. The point-by-point construction of graphs was discouraged. As a consequence, the ordinary table of x, y values was sometimes exploited as a tool to analyse how y changed when x changed (column-vertical analysis) and not as a tool to read the line-horizontal point-by-point correspondence between x-values and y-values. Finally, we can remark that the algebraic and the graphical settings were strictly related (formulas were read in terms of shapes in the (x,y) plane, while graphs evoked formulas).

As concerns the **educational choices** related to classroom management of the functional approach to inequalities, the following points were considered crucial:

- classroom discussions about "what do we loose and what do we earn" when a function is represented through formulas or graphs or tables or common language;
- different ways of describing given functions have been encouraged. For instance the discovery that the formula $y=2x^2$ corresponds (within the table of the x and y values, read vertically according columns) to an *"irregular"* increase of y which is different from the *"regular increase"* in the case of $y=2x$, and that the irregular increase results in a curved line when the graph is drawn (later on the teacher will call it *"parabola"*) allows to link different ways of representing functions (see Duval, 1984: coordination of different linguistic registers). They will become personal tools exploited and transformed (see Ex.1 in the next Section) to compare functions. Even the metaphors used by students to describe the role of different pieces of the same formula have been encouraged and discussed.

3.3. Individual task

We will consider the following task:

"Compare the following formulas from the algebraic and graphic points of view. Make hypotheses about their graphs and motivate them carefully, finally draw a sketch of their graphs. A) first function. B) second function"

Slightly different fuctions A and B were chosen according to the students' levels. In particular in this paper we will analyse the solutions of three students engaged in solving the problem with the following functions (of mean level of difficulty):

A) $y=x^2-4x+4$; *B)* $y=-x^2+4$

In every case the above task was significantly more difficult and complex than the previous ones: they concerned the comparison of functions like $y=x^2$, $y=-x^2+4$

We expected that (according to the didactical contract and previous experiences) most students could explore both functions in a dynamic way, trying to answer the following questions:

- *What does happen with y when* $x>0$ *or* $x<0$*? Does it increase? Does it decrease?*
- *Are there meeting points between the two graphs? Where? When?*
- *Does the first graph overcome the second one? Where?*

All individual solutions were collected (some protocols contain questions and comments written by the teacher during the 1-1 interactions). We collected also some recorded interviews performed after the end of the activity and concerning the strategies produced by students and their general ideas about inequalities.

4. Preliminary results

4.1. Some quantitative data

- 5 students out of 36 did not succeed in tackling the problem (they did not understand it, or were stuck)
- 7 students (out of 31 who were able to tackle the problem) arrived to incorrect conclusions (sometimes due to trivial mistakes in calculations).
- 3 students (out of 31) build up graphs point by point (against the didactical contract), while the others compare functions in a dynamic and global way with different strategies.

4.2. Some qualitative data.

From a qualitative point of view collected data are interesting for the following reasons:

a) 28 students (out of 31) show good skills in coordinating different linguistic registers (formula, graph, verbal language, etc.)

b) a plurality of strategies of comparison between the two functions, frequently consisting in a personal blend of strategies and moves compared and discussed in previous classroom activities. In particular we can find:

b1. Strategies based on the relationships between formula and shape of the graph (6 out of 28 who keep the didactical contract).

Formulas, or some of its parts, are associated with a peculiar shape of graph (Ex. 1, Davide).

In the case of Davide x^2 evokes the shape "parabola" and the other elements of the given formula are interpreted in terms of transformations of the prototypical graph: for instance the presence of a negative sign before X^2 suggests the "reversed U shape" (for a similar behaviour cf an example provided by R. Hershcowitz: the student detects the graph of $y=-x^3$ by referring it to the well known graph of $y=x^3$). The dynamical aspect of the solution consists in the interpretation of the different parts of the given formula as transformations of the prototypical formula.

Davide seems to ask himself questions like "What does it mean $+4$ in the graph? And what about the sign - on the left of x^2 *? What does it mean $-4x$ in the graph?*".The most interesting fact is that the graph seems to be "an object", a shape which can be moved as a whole.

We can ask ouselves how the relationship between the shape "parabola" and the formula " $y=x^2$ " was established by Davide. Considering the background of his class and his personal history we may make the hypothesis that such relationship was probably acquired through the activities of drawing graphs and that the dynamical dependence of y-values on x-values that was observed and discussed in the classroom (with the help of the table) could have "condensed" in a shape. This hypothesis implies that for Davide the shapes of the graphs are not mere shapes, because they bear the peculiar modality of increase of *y* when *x* increases. A careful analysis of his protocol seems to confirm our hypothesis.

Ex.1, Davide: compare the two functions $y= x^2- 4x+4$ and $y= - x^2+4$

In the graph x^2 and $- x^2$ are two parabolas: the former is over 0 and looks like U, the latter looks like a reversed U. In both functions there is +4, thus we can say that both parabolas stand over the x-axis of +4

(Fig. 1: he sketches the graphs of $y= x^2+4$ *and* $y= - x^2+4$*)*

In function A there is an operation (-4x) which moves and transforms this parabola.

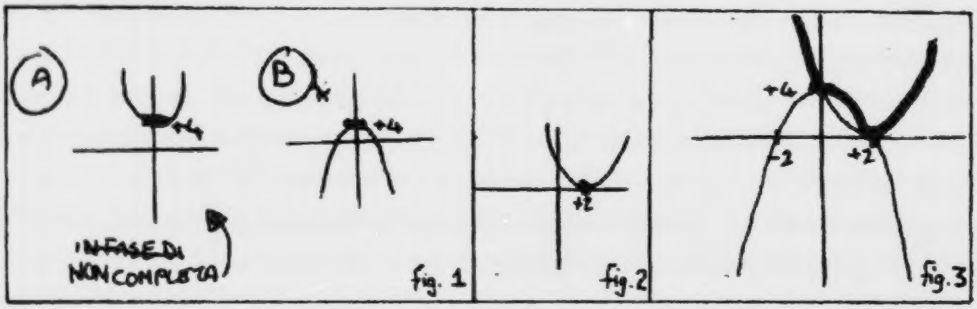
A) when x=2, y=0, thus we can say that , when y=0, the parabola stands in x=2.

(Fig. 2: a parabola with the vertex in the point (2, 0)

This parabola does not go under zero when x is negative.

This parabola starts slowly, because -4x decreases its speed ,but successively goes up quickly.

(Fig. 3: superposition of parabolas A and B) [...].



b2. Strategies based on the relationship between formula and increase or decrease of y when x increases (14 out of 28).

In this case students analyse the behaviour of the formula according to *x>0* or *x<0* (Ex. 2, Lorenzo). We can imagine that Lorenzo asks himself questions that are very different from those considered by Davide: "*What does it happen if x<0? And if x>0? Does it increase? Does it decrease?*"). In this case the only dynamical aspect concerns the analysis of how *y* changes in relationship with *x*. Also in this case the approach is global: it is the whole function that decreases or increases when *x* changes. For Lorenzo the graph represents a synthesis and the visual validation of his previous analysis.

Ex.2, Lorenzo: compare the two functions $y= x^2- 4x+4$ and $y= - x^2+4$

- In both formulas you have always found a +4. In the former x^2 you always have a positive result, while in the latter you never find it, because there is a minus before.
- If x<0: The former increases more because there is x^2 , always positive, and the sign changes also in -4x, which turns out +4x.

On the contrary, the latter dicreases only because I take out an x^2 and at a certain point the +4 cannot keep the $- x^2$ above the zero. There is no meeting point.

- If x>0: The former decreases for a certain period (from 0 to +2)because the +4 and the x^2 don't keep the - 4x, while after that *[the function]* increases.

The latter for a certain period stays above zero (from 0 to +2) because the +4 supports the $- x^2$ while after that it decrease below zero.

Meeting points x=0, y=4 and x=2, y=0

[He draws the superposed graphs with a comment]
The latter is bigger than the former from 0 to +2

b3. Strategies based on the search for "remarkable points" (8 out of 28). (Ex.3, Laura)

These students through the discovery of remarkable points (*$x=0$ $y=\ldots$, $y=0$ $x=\ldots$*) perform an analytic-inductive analysis of the functions under comparison and get an idea of the shape of the graph. In particular Laura finds some points of the first function, then declares:"*It does not touch the origin and y is never negative!!*". This behaviour can be considered (in the case of the two functions) as intermediate between the preceding two. Indeed neither the use of the graph is prevailing.(like with Davide), nor the analysis of how y changes in relationship with x (like with Lorenzo), but these two aspects are intermingled.

Ex.3, Laura: compare the two functions $y= x^2- 4x+4$ and $y= - x^2+4$
Both formulas have +4 in the end, and thus we can neglect them (translate of +4 with respect to the y axis). They both are parabolas.The first one (A) is the greatest.
Now I proceed by means of "oriented" computation.
[......]
$x=0$ $y=0$; $x=-4$ $y=+16-(-16)=+16+16=32$; $x=+4$ $y=0$; $x=2$ $y=-4$
between +4 and 0 the result is <0!! Now I add +4 !!
A) $x=0$ $y=+4$; $x=+4$ $y=+4$; $x=+2$ $y=0$
It does not touch the origin and y is never negative!!
B) $x<0$ y=negative, except for $x= -2$, where $y=0$
$x>0$ y=negative, except for $x=+2$, where $y=0$
It does not touch the origin and it is positive (only) between $x=-2$ and $x=+2$.!
Curiosity: what is y like, when x is greater than –2 and smaller than +2?
$x=-1$ $y=+3$ $x=+1$ $y=+3$
The matching point of the two graphs is +4 in the y axis and the second point is in $x=+2$ $y=0$
(drawing: superposition of the parabolas A and B)
As I observed above, A is the greatest, but there is a little overcoming of B between $x=0$ and $x=+2$

c) Plurality of meanings of pieces of formulas

In the task of comparing the two functions $y= x^2- 4x+4$ and $y= - x^2+4$ a crucial difficulty concerns the role of "*-4x*" in the first fuction. We can observe how, according to the peculiar strategy of each student, "*-4x*" takes diferent meanings. For instance, in the case of Davide (Ex.1, underlined parts): "*-4x*" is responsible for "*moving and transforming the parabola*"; later on "*-4x*" takes another meaning: it "*diminishes the speed of increase*". For Lorenzo (Ex. 2, underlined part) "*-4x*" is "*responsible for the decrease of the function between 0 and 2*", in relationship with the other parts of the formula. Other students declare that "*if -4x did not exist the two parabolas would be equal: the former oriented upwards and the second oriented downwards*". This diversity of meanings attributed to the same piece of the formula enriches the interpretation and representation of the whole formula. The same phenomenon occurs for other pieces of the same formula: for instance "*+4*": for some students "*+4*" means that "*the parabola does not pass through the origin*" of cartesian axes, for other students it means that "*it moves the parabola upwards*".

Conclusion

As a consequence of epistemological analyses (which reveal the importance of the functional aspects of inequalities in mathematicians' work), and didactical analyses (which show how the current approach to inequalities based on purely algebraic procedures produces very limited learning results), and keeping into account cognitive studies about functions, a teaching experiment aimed at exploring the feasibility of an early functional approach to inequalities with VIII-grade students was planned and analysed. Collected data seem to support our didactical and educational choices; in particular, inequalities appear an interesting issue to study the students' construction of different aspects of the concepts of function and variable, and a promising learning context for these concepts. This means that the functional approach to inequalities, originally intended to better understand and possibly solve some teaching and learning problems concerning inequalities, offers a possibility for studying and improving teaching and learning of functions.

A relevant aspect of our study concerns the educational choices related to the aim of a cooperative enrichment of tools and ways of reasoning useful to deal with inequalities. Here we may say that the approach to inequalities based on a global, dynamic approach to functions seem to fit very well with this aim: graphic representations, gestures, metaphors concerning functions were very easy to share in the classroom (once the teacher decided to encourage their use by students).

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MATHEMATICS TEACHER BELIEF SYSTEMS: EXPLORING THE SOCIAL FOUNDATIONS

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Abstract

There is a considerable literature on teachers' beliefs and conceptions and their effect on the teaching of mathematics, but much of this literature either is located within a psychological paradigm, or where a more eclectic perspective is adopted, fails to locate the sources of beliefs in the social world. In this paper I look at some of the present models and approaches to teacher belief systems and argue that these can only give us an incomplete picture leaving as they do, the source of belief systems in the social world unexamined or unproblematised. I offer some sociological concepts that can help us understand better how belief systems are constructed upon teachers' ideological foundations.

The social role of mathematics education

Naturally there is diversity and variety in all mathematics teaching – a diversity that finds its rationale in mathematics teachers' belief systems. While what teachers do is based upon what they believe – what they do has both unintended as well as intentional consequences. It can hardly be contested that we live in an uneven and unjust society where access to education and to justice depend on the capital one can appropriate and accumulate. There is so much evidence in the literature to support this contention that it is hardly now contentious. Injustice is a process that goes on all around us, even when - and arguably especially when - we do not look for it or recognise it. Mathematics plays a significant role in organising the segregation of our society, as Sue Willis cogently argues:

> Mathematics is not used as a selection device simply because it is useful, but rather the reverse.
>
> (Willis 1989, p 35)

In other words, mathematics education plays its part in keeping the powerless in their place and the strong in positions of power. It doesn't only do this through the cultural capital a qualification in mathematics endows on an individual. It does this through the authoritarian and divisive character of mathematics teaching. Mathematics thus performs a social function, and by engaging in mathematics teaching, teachers are consequently involved in a social function. Hence in order to understand better the nature and functioning of mathematics teaching we need to look for foundations, predilections and structuring frameworks that would support a social model for understanding the discipline. Such an approach requires us to locate ourselves within a dynamic and dialectical analysis of the relationships between human agency – the will of the individual – and social structure – the wider enabling and constraining forces operating on us. Indeed, it requires us to look for social forces not only as acting *on* us, but also as acting *in* us. Karl Mannheim takes this argument a little further.

> Strictly speaking it is incorrect to say that the single individual thinks. Rather it is more correct to insist that he participates in thinking further what other men have thought before him. He finds himself in an inhabited situation with patterns of thought which are appropriate to this situation.
>
> (Mannheim 1936 (2nd Edition 1952), p 3)

There is much evidence to suggest that school mathematics has a firmly established cultural tradition. This "*school mathematics tradition*" (Cobb, Wood, Yackel, *et al.* 1992) can be classified as teacher centred, where classroom routines incorporate the introduction of a new technique, presentation of examples and setting of exercises. In this routine, the teacher does most of the talking, directing and instructing pupils. Mathematics is presented as little more than replication of procedures demonstrated by the teacher (Brown, Cooney and Jones 1990) with a focus on memorisation and drill making the subject dull and uninteresting (Ball 1990, p 12). This characterisation of mathematics teaching is not only widespread (Romberg and Carpenter 1986), but is also historically persistent as a dominant model over the past 100 years (Cuban 1984). In addition, this pattern has been described as "*the most consistent and persistent phenomena known in the social and behavioural sciences*" (Sirotnik 1983, pps 16 – 17). There is considerable evidence that curriculum innovations become largely incorporated into teachers' existing teaching approaches and styles. Furthermore there is evidence that teachers portray their own teaching as more open than it might be described objectively (Edwards and Mercer 1987). This is a serious situation, which requires us to look for creative explanatory models of human behaviour.

Jeff Gregg has reported a study into the reasons for the persistence of the school mathematics tradition (Gregg 1995a, b). Synthesising research over the past 20 years, he suggests, "*there are certain beliefs about mathematics and its teaching as well as certain classroom practices that are taken-as-shared by many in our society*" (Gregg 1995b, p 443). He claims the hegemonic nature of these beliefs may be responsible for the widespread failure of the history of reform in mathematics education. Jeff Gregg describes a process whereby not only are teachers socialised into a culture of teaching, but "*teachers, students and administrators actively participate in the production and reproduction of these processes*" (Gregg 1995b, p 461). He suggests that by separating teaching from learning, and adopting a view of ability as capacity, teachers are able to act without questioning the taken-as-shared beliefs and practices of the dominant school mathematics tradition (Gregg 1995b, p 462). Whilst Jeff Gregg's study and analysis is a useful insight into the acculturation of mathematics teachers, what he lacks is an explanatory framework for understanding the nature and roots of the phenomenon he describes. To provide this, we need to look more deeply into the organisation and source of teacher belief structures.

Mathematics teacher belief structures – some theoretical limitations

Studies of teacher beliefs often focus attention on beliefs as if they existed in a social and political vacuum drawing fundamentally upon a psychological paradigm, which seems unable to account adequately for the difficulties of teacher change. However, as the Centre for Contemporary Cultural Studies has suggested, beliefs do not exist in a social vacuum:

> If we are interested in the ways in which consciousness is formed, we cannot stop at the level of lived beliefs. Beliefs, conceptions and feelings are not only carried in the minds of human subjects; they are also written down, communicated, 'put into circulation', inscribed in physical objects, reproduced in institutions and rituals and embodied in all kinds of codes.
>
> ((CCCS) 1981, p 27 - 28)

Hence, beliefs have a wider and deeper dimension, rooted in cultural norms and forms that are themselves rooted in social structure. These have a huge influence over consciousness and ideology, setting many agendas and putting boundaries around what is considered as possible, describable or even legitimate.

Studies of teachers' belief and knowledge structures have increased considerably during the 80s and 90s. Kenneth Zeichner, Robert Tabachnick and Kathleen Densmore (Zeichner, Tabachnick and Densmore 1987) suggest that we need to consider adopting approaches to teacher development that recognise the complexity of the nature of knowledge (Zeichner, Tabachnick and Densmore 1987, p 24). They identify a lack of consensus in the literature on teacher socialisation and challenge the view that student teachers change and modify their views on teaching through experience and teacher education. Rather what happens is an elaboration of previously existing perspectives and a selective focus on experiences that validated their own perspectives. Again providing evidence for the inherent stability of belief systems.

Thomas Cooney and his associates have worked for some time on the knowledge and beliefs of preservice secondary mathematics teachers. They recognise that beliefs about mathematics and how to teach it are influenced by experiences with schooling long before prospective teachers enter professional training and that these beliefs seldom change (Brown, Cooney and Jones 1990). Such a worrying state of affairs requires us to try to understand therefore not just what it is that teachers believe, but how these beliefs are structured and organised (Cooney, Shealy and Arvold 1998).

Alba Thompson worked for a number of years on mathematics teacher beliefs. She claimed that teachers' patterns of behaviour characteristics are a result of consciously held beliefs acting as a '*driving force*'. In addition, practice can be the result of unconscious beliefs and intuitions (Thompson 1984). What is unclear is the nature of these '*driving forces*', where they emanate and how they become operationalised. Alba Thompson suggested that more research was needed on the stability of teacher beliefs.

This phenomenon of teachers modifying new ideas and practices by adapting them to fit existing practices is now well understood (Thompson 1992, p 140). Underlying this problem is the issue of stability of the structures of commitment that we hold as individuals acting within a social world. This is an example of the difficulty in changing deeply help ideological dispositions and underpinnings. We may change words, we may change the context to (pseudo) real life situations, and we can even change the architecture of the school buildings, but little changes in the relations of power and domination in the mathematics classroom. "*Unfortunately the literature on teacher change, though rich with tips, does not offer explanations for this phenomenon*" (Thompson 1992, p 140).

Alba Thompson reviewed much of the research on mathematics teacher beliefs (Thompson 1992) yet it becomes clear in her review that much of the driving force in this research comes from the belief that it is the *teacher's view of mathematics* that is responsible for classroom practice. Such a view is typically represented by two comments:

> One's conception of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it. The issue then is not, what is the best way to teach it, but what is mathematics really about?
>
> (Hersh 1986, p 13)

> All mathematical pedagogy even if scarcely coherent rests on a philosophy of mathematics.
>
> (Thom 1973, p 204)

These positions need to be questioned and deconstructed. Without further clarification, one reading is that one's conception of mathematics is the deciding factor in structuring one's teaching. Rene Thom seems to go further in using the word '*rests*' – a spatial metaphor that has a sense of dependency embedded in it. He question this begs is – on what does ones philosophy of mathematics itself rest.

Teachers' social perspectives – the missing dimension?

Much research undertaken on various aspects of teacher beliefs tends "*inadequately to explore teachers' social beliefs*" (Liston and Zeichner 1991, p 61). Teachers' *social* knowledge

> tends to be inadequately addressed in most accounts of teacher knowledge, is rarely examined in teacher education curricula and is awkwardly handled in the prominent models for cultivating reflective thinking and action in teachers.
>
> (Liston and Zeichner 1991, p 61)

As social beings, mathematics teachers do not come to the classroom devoid of social and political motives and intentions. Yet nor can we merely append '*social knowledge*' to a growing list of categories of professional knowledge alongside 'knowledge about children', 'pedagogical content knowledge' etc. because of the fundamentally constitutive nature of social beliefs.

In developing a theoretical framework, we need to be able to conceptualise this dialectical relationship between the individual and the social. Pierre Bourdieu offers a way through this in his appreciation of the interplay between objective social structure and subjective personal dispositions, which forms the central methodological and conceptual organisation of his work and informs his empirical studies (Bourdieu 1972, 1990b). It is his assertion that objective structures are actualised and reproduced through subjective dispositions (Bourdieu 1972, p 3). This does not mean that subjective dispositions have a primacy over more objective social structures. Rather that the development of individual dispositions is influenced and constrained by objective structures, the nature of hierarchy, the form of hegemonic positions and so on, which in their turn reinforce the objective structures. What distinguishes Pierre Bourdieu's approach is the way in which social structural properties and social and economic conditions are always embedded in everyday lives and events of individuals (Harker, Mahar and Wilkes 1990, p 8). Of course implicit in here is a readiness to accept that:

> There exist in the social world itself, and not merely in symbolic systems, language, myth etc. objective structures which are independent of the consciousness and desires of agents and are capable of guiding or constraining their practices or their representations.
>
> (Bourdieu 1990a, p 14)

Within this framework, there are two main conceptual tools that can be incorporated into research on teachers' beliefs that will give us access to some previously un-illuminated routes to the roots of the systemic logic of teachers' belief and values systems – the *habitus* and ideology.

Habitus

For Pierre Bourdieu, this symbiosis can be examined and understood through the elaboration of the *habitus*. I explore this in more detail, theoretically and empirically, elsewhere (Gates 2000), but briefly, the *habitus* is the cognitive embodiment of social structure. Our *habitus* forms the generative principles that organise our social practices leading to social action and provide us with systems of dispositions that force us (or allow us) to act characteristically in different situations. The *habitus* thus resides within patterns of interactions and needs to be explored using techniques that dig deep enough into the logic underpinning the observable practices - a logic that is interwoven with the social origins of one's predispositions.

The mathematics teacher's *habitus* will be at the root of the ways in which teachers conceptualise themselves in relation to others; how they enact and embody dominant social ideas and well as how they transform and adapt them. The *habitus* is at the bottom of how we react, judge and evaluate.

Working with the *habitus* though is not enough. To help us understand the social foundations of mathematics teachers we have to look also at how the *habituses* of groups of teachers gel into ore organised forms of thinking and cooperation. Teachers become pulled together or 'interpellated' into social groupings and formations through the sedimentation of the individual *habitus* and predispositions into more socially organised ideological frameworks.

Ideology

Fundamentally, what distinguishes ideology from general sets of ideas is that ideology is about the relationship between ideas and society and the relationships between individuals. What typifies ideological ideas is their relation to the conflictual nature of economic and social relationships. Ideology thus relates to matters of power and social structure as well as relating ideas and activity to the wider socio-cultural context Hence, looking for ideological underpinnings require us to look at language forms used, to explore the social imagery adopted, to elaborate on how individual teachers categorise and organise their ideas especially in relation to others. These in particular will need to connect with ideas on the nature and form of society and how it operates. In addition, these will need to be tied to issues of practicality, which embody relations of domination. Relating this discussion to teaching, ideological underpinnings appear as ideas and assumptions about human nature, about learning and educational difference, the role of education, the role of the teacher and ideas about priorities for teacher professional development. Ideology can thus be represented as relatively stable, deep structures of ideas. Our ideological makeup, establishes us as being the same as and different from the individuals and groups with whom we associate or work – but a positioning process founded not upon some philosophy of mathematics, but upon one's social frameworks.

Conclusions

I am arguing here that while strictly psychological models of teacher beliefs can give us considerable insight into the structure of teachers' knowledge, they have some limitations when we come to want to look at some of the wider and possible unintended consequences of the education system. This can be informed by adopting models of belief systems as well as research techniques that look at how an individual constructs a system of beliefs both structurally and temporally. As researchers it requires us to go beyond the data of observable practices and into the realms of those patterns and generative principles of which the teachers themselves may not even be aware. Underlying such an approach is the ideological predisposition that sees thinking as a social act, and systems of beliefs as representing dominant and objective social structures as well as helping those structures to operate and reproduce.

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PUPILS PERCEPTION OF THE LINKS BETWEEN DATA AND THEIR ARITHMETIC AVERAGE[1]

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The results presented in this paper are part of a larger study on the strategies used by pupils to solve arithmetic average problems. The arithmetic average being a widely used concept, it is important to study pupils' knowledge and their difficulties in order to improve their learning. Two questions arose from previous studies: 1)up to what point do pupils perceive the links between data and their average? and 2)does the introduction of a null datum(=0) complicate the situation? A first task asked for an estimation of the effect of the modification of a datum on the average and a second required finding the datum which would leave the average unchanged. The results tend to demonstrate that pupils have a fairly good perception of the effect of data modifications on the average with better results if the datum is zero.

Les résultats présentés sont extraits d'une étude sur les stratégies utilisées par les élèves pour résoudre des problèmes de moyenne. Vu l'utilisation répandue de ce concept, il est important de sonder les connaissances et les difficultés des élèves pour en améliorer l'apprentissage. Deux questions se sont posées à la suite d'études précédentes: Jusqu'à quel point les élèves perçoivent-ils les liens entre des données et leur moyenne? et La présence d'une donnée nulle(=0) rend-elle la situation plus difficile? Nous avons construit deux tâches dans lesquelles on demandait d'estimer l'effet du changement d'une donnée sur la moyenne et de donner la valeur qui laisserait la moyenne inchangée. Les élèves semblent avoir une assez bonne perception de l'effet de la modification d'une donnée et ceci encore mieux dans le cas d'une donnée nulle.

The arithmetic average is certainly a widely used concept and not only in the context of statistics. In fact, we can trace attempts to make many observations of the same phenomenon as far as the Babylonians (500-300 BC). It is not before the work of Tycho Brahe (16th century) that the use of the average becomes clearly distinct. However, as it is largely documented, today's students have trouble understanding this concept. In the past, many studies have stressed students' difficulties with problems involving averages, particularly when weighted averages are concerned (Pollatsek, Lima, Well, 1981; Gattuso, Mary, 1996). Clearly, this is not a simple computational algorithm, and it is not well understood.

Young children already have conceptions (or misconceptions) of representativeness, and a study from Mokros and Russell (1995) reveals that they perceive the average in five different ways: as a mode, an algorithm, a reasonable value, a midpoint or as a point of balance. However, Cai (1995), questioning sixth-grade pupils, found that although 90 % of them knew the computational algorithm for the average, less than half had reached a conceptual understanding of the concept. Other results show that 8th grade students are able to find a weighted average without previous specific instruction, and that instruction may even interfere, since older students have a poorer performance (Gattuso, Mary, 1998). This conclusion agrees with earlier findings of Pollatsek (1981), where only fourteen out of 37 college students answered correctly to a weighted average problem. Leon and Zawojewski (1990) also established that most students can understand the mean as a computational construct, but have more difficulty seeing it as a representative value. Young children seem to have a fairly good perception of the average, but this does not seem to develop into a deep understanding of the concept and difficulties show up if the problem asks for more than a simple computation, and there is no real improvement with age.

In addition, zero presents a particular problem, as in other mathematical situations, such as division by zero or 0 as an exponent. Looking into the properties of the average, a study revealed that college students with basic statistical education had a tendency to apply the four axioms which constitute an additive group to the computation of means (Mevarech,1983). Particularly, in the case where one datum is 0, some pupils consider it as the neuter element and assert it does not change the mean. Strauss and Bichler (1988) observed difficulties among 8 to 14 year-olds while looking at the development of children's concepts of the arithmetic average. Not only do children have problems understanding that the sum of the deviations to the mean is zero, but they also fail to understand that the average is representative of the values averaged and in to take into account a zero value in the computation of the average. Again, this fact seems to be independent of the age group.

RESEARCH ISSUES

In this paper we focus firstly on the understanding of the links between the data and the mean. More precisely, the issue of "representativeness" was translated in a simpler question: "Do pupils perceive that changing even one of the data affects the mean?" The second aim is to examine, in parallel, the difficulties encountered by the introduction of a datum equal to zero: "Are similar situations dealing with a datum equal to zero more difficult?"

To improve the understanding of such a concept, it is important to know how children cope with it in different situations, how much they know about it and what are the difficulties. The final aim is to investigate if pupils perceive the connections existing between the data and the average and to analyse their strategies and reasoning for eventual use in the construction of teaching interventions. Ultimately, we hope that the results of this study will help setting up future teaching experiments.

METHOD

Context of the study

Although simple arithmetic average problems are part of the elementary mathematics curriculum, it is not before the 9th grade that Québec students encounter the concept of weighed average. The results presented in this paper are part of a larger study on the strategies used to solve problems of weighted averages by children through their high-school years. A total of 638 high school students from grades 8, 9 and 10 (ages 13 to 15) participated in the study. Each student answered 5, 6 or 7 questions depending on whether they were in 8th, 9th or 10th grade respectively. A total of 24 different tasks were designed, each of which was administered to part of each year group. This paper discusses the results of 6 of these tasks.

Description of the tasks

The tasks were meant to investigate the pupils' understanding of the link between the data and the average and to see if the same situation using a datum equal to zero would pose greater difficulties. The tasks were framed into three different situations consisting of modifying the data set by: 1) adding one datum, 2) replacing one datum by another and 3) removing one datum. In each situation, two options were considered, using a non null datum or a datum equal to zero (see appendix for examples). Two questions were asked: a) Does the average increase, remain unchanged or decrease? b) If you think the average is modified, what value would leave it unchanged? Even though the first question calls for a qualitative answer and the second one does not really require computations, the numbers were chosen so as to make the calculations easy if ever the pupil found the need to do some.

We posed two hypotheses. The first one was that the question that proposed the replacement of one datum by another would be the most difficult because it required more steps that the one presenting the addition or removal of a datum. Between the latter two, adding a data should be easier. Secondly, because previous studies had shown that students do not seem to take into account zeros, it was assumed that problems involving a zero would produce poorer results.

RESULTS

Performance

Looking at Table 1, we see that for cases involving a datum different from 0, the results seem to confirm the fact that it is more difficult for the students to estimate the modification of the average if there is replacement of data; in this case, one datum smaller than the previous average was replaced by one datum greater than the original average. For problems involving addition or removal of one datum, results are higher, showing very little difference between the two.

Cases involving a datum equal to zero produce a different outcome. Our hypothesis is not supported: the case involving the replacement of one datum is no longer the most difficult one. The case where one datum is removed has the lowest results. Furthermore it seems that problems involving a datum equal to 0 are easier than previous cases involving a data $\neq 0$. Here, a datum smaller than the original average was replaced was by a datum equal to zero (also smaller than the average).

Table 1: Question a) correct answers

Data	Adding one datum	Replacing one datum	Removing one datum	Total
$\neq 0$	68/98	52/107	74/109	194/314
	69,40%	**49,50%**	**67,89%**	**61,78%**
$= 0$	91/101	75/104	68/105	234/310
	90,10%	**72,12%**	**64,76%**	**75,48%**

Looking at the results of the second question asking what should the modified datum be so the average would not change, the overall pattern is similar although the results are clearly lower. Again, the situations involving a zero seem to be easier except in the case of the removal of one datum. Let us emphasise the fact that no calculations were absolutely necessary for answering these questions. We also examined the results as a function of age group. Although there is no obvious pattern, we can say that the 8th grade students perform often better or at least as well as their older counterparts.

Table 2: Question b) correct answers

Data	Adding one datum	Replacing one datum	Removing one datum	Total
$\neq 0$	61/97	46/106	53/109	160/312
	62,89%	**43,39%**	**48,62%**	**51,28%**
$= 0$	77/101	61/103	39/105	177/309
	76,24%	**59,22%**	**37,14%**	**57,28%**

Strategies for question a)

We analysed the strategies (or reasoning) used by the students to try to understand these results. In general, in cases where the modified datum is $\neq 0$ (Table 3), we can say that the strategy used most successfully and most often by the students is to compare the modified data to the original mean. The student asserts: "a value greater than the mean is taken out, so it will decrease...". However another reasoning also focusing on the data leads to wrong answers when a datum is replaced because it says, for example: "We take out something and put back in compensates…" without any regards to the values of the data and with no reference to the mean value. They add the effects of the modified data instead of subtracting them.

Problems in which there is a zero (Table 4) are easier because the reasoning, in the case of adding or removing a datum, involves only the number of data, the total remaining the same. In the case of the replacement of one datum by another, what makes it easier is the fact that students mostly reason on the difference in the total, which in this case is the modified value: a datum $\neq 0$ is replaced by a zero, so the total decreases while the number of data remains unchanged. Here also, only one factor varies.

Table 3: Strategies: Data ≠0 Question a)

Data ≠0 a)	Adding one datum ≠0	Replacing one datum by ≠0	Removing one datum ≠0
Correct strategy mostly used **Don**	"A datum smaller than the mean is added" 45,92%	"A datum smaller than the mean is removed and a datum greater than the mean is added" 33,64%	"A datum greater than the mean is removed" 47,71%
Incorrect strategy mostly used	(Average + datum)/2 **Wwe** (without weight) 16,33%	False Compensation of effects **Fcom** 17,76% **Wwe** (as much as in Adding) 16,82%	Nothing particular

Another noticeable result is the fact that many errors are due to reasoning based on a misconception or, more accurately, a conception inadequately used. More than half of these answers are of the type: "0 is removed, it doesn't change anything…", treating 0 as the neuter element. Another erroneous reasoning is based on the conception that each datum is equal to the average. Even though it is mentioned in the problem that the datum removed is equal to zero, some pupils still think that each datum being equal to the average "Nothing changes, they all have the mean...".

Table 4: Strategies: Data =0 Question a)

Data =0 a)	Adding one datum = 0	Replacing one datum by 0	Removing one datum=0
Correct strategy mostly used	"There are more persons for the same total" **Eff** 53,4%	"The total decreases" **Tot** 38,46%	"There are less persons for the same total" **Eff** 47,0%
Incorrect strategy mostly used	Nothing particular	No explanation or answer **Nex** 12,50%	Misconception 24,8%

Strategies for question b)

The strategies change with the questions when students are asked to give the datum that would leave the mean unchanged (Table 5). In **T/n** -if you add a datum it has to be equal to the mean so the ratio (Total/nb of data does not change)– and **Mean,** –take out a value equal to the mean– we may say that the fact that "if all data should be equal, then each datum should be equal to the mean" is correctly used although it is not explicitly said. It is however difficult to distinguish between a correct answer and one misusing a conception saying that all data are equal to the mean. A great number of "no answer" or "no explanation" is also noticeable in the case of the removed and replaced data ≠ 0 that did not appear so much in the other problems[2].

Table 5: Strategies: Data ≠0 Question b)

Data≠ 0 b)	Adding one datum≠0	Replacing one datum by ≠0	Removing one datum≠0
Correct strategy mostly used	"If there is a person more, we add the mean to the total then it's unchanged **T/n** 24,74%	"The same as what is removed" **Com** 38,68%	"It should be the mean so it does not change" **Mean** 30,28%
Incorrect strategy mostly used	Various calculations without weight **Wwe** 19,59%	Various calculations without weight **Wwe** 7,5% And **Nex** 27,36%	No explanation or answer **Nex** 35,78%

The case of the datum equal to zero gives rise to the usual misconceptions: "Adding (or removing) 0 does not change anything..." and when the datum is replaced: "it should be the average value so everyone has the same". Many students think that every datum is equal to the mean unless otherwise specified. It is in fact possible, and also it is often said "that the mean is the value each datum would have if every data should be equal" the second part of the sentence (**if**...) seams to have disappeared.

Table 6: Strategies: Data =0 Question b)

Data = 0 b)	Adding one datum=0	Replacing one datum by 0	Removing one datum=0
Correct strategy mostly used	"It should be the mean so it does not change" **Mean** 36,63% and **T/n** 21,78%	"The same as what is removed" **Com** 50,96%	"It should be the mean because everyone has it" "Misconception" 15,24%
Incorrect strategy mostly used	Misconception 14,5%	Misconception 11,54%	Misconception 9,52%

DISCUSSION

The results call for a different point of view on the situations involved. Our hypotheses were based on previous research and on the complexity of the mathematical operation involved, but children naturally use comparison confronting the modified data with the mean or looking at what is changed, the number of data or the total. These comparisons seem easier if the datum is zero because only one factor varies. _

If we look at each modification separately, we see that adding a datum, affects the mean in the same direction. If the added datum is greater than the average, the average increases and so on. The opposite is true when removing a datum. The situation were a data is replaced is not as difficult as was thought a priori, since if you look at the modification of the total while the number of data remains the same, the changes also vary in the same direction. These facts should not be presented directly but children should experience these different situations and be encouraged to estimate the results. A datum equal to zero should be treated as another datum (often smaller that the average in simpler contexts).

However, many errors are due to an incorrect application of some conception of the mean, particularly, one giving equal value to each datum. Various analogies are used in teaching but they present some limits. One example, often used in teaching the mean, is "equalising a pile" or distributing a quantity equally. The fact that all the data are not equal at the beginning should be emphasised, and probably working inversely, asking to construct a list of data given a certain mean (with or without additional constraints, like no data equal to the mean) could help the pupil to see that the data are not necessarily all equal. Every time an analogy is used, we should insist on the limits of it's validity.

Certainly, these results are incomplete and are not meant to be generalised but a majority of students do perceive the link between data and their mean, and their 'natural' strategies should be looked into and analysed for further applications in constructing teaching situations. As a matter of fact, these types of situation are scarcely used in the classroom, students should be encouraged to rely more on estimation before indulging in automatic computation.

[1] Unless otherwise specified, in this text the words "average" and "mean" are used synonymously and refer to the "arithmetic mean" or "arithmetic average".

[2] This problem was not the last one in the questionnaire.

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APPENDIX

Replacing one data = 0

A group of eight friends empty their pockets. They have an average of 11$ each. Peter decides to take back the 4$ he put in and goes to work. At the same time, Jean-Philippe joins the group but he does not have a penny. What happens with the average?

a. 1) It decreases 2) It does not change 3) It increases

b. If you think that the average has changed, how much should Jean-Philippe have so that the average remains the same?

Removing one data ≠ 0

The average age of the first seven first persons attending Geneviève's party is 21 year old. When Jean-Philippe who, is 27 year old, leaves. What happens to the average age?

a. 1) It decreases 2) It does not change 3) It increases

b. If you think that the average has changed, what age should Jean-Philippe have so that the average remains the same?

RESEARCH ON ATTITUDES IN MATHEMATICS EDUCATION: A DISCURSIVE PERSPECTIVE

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Abstract: Attitude to mathematics is generally assumed to be a stable and reliable construct. Closer examination, however, reveals, that the specific interrelated conditions in which attitudes are both expressed and studied have a striking impact on the way people respond to research instruments. In this paper, an exemplary analysis of the discursive practices and the resources which are used in the organisation of an interview is carried out. This is leading to a discussion of the problems that emerge when the categories of observers are rashly superimposed on the utterances of the observed.

Introduction

In mathematics education as a theoretical field language plays a crucial role. It is not only the vehicle with which findings are reported and the recent questions of the field are discussed. It also is the most important resource for describing the practice of mathematics teaching and learning. Moreover, this practice mainly is accessible via the language that is used within it (e.g., Adler, 1995; Dowling, 1996; Krummheuer, 2000; Pimm, 1987; cf. Mason and Waywood, 1996). Apart from that, language even serves as the predominating means to examine mental orientations like attitudes where the mental is taken to subsume cognitive, affective and enactive aspects of the psyche. In this contribution, the role of language in the research on attitudes and similar concepts is discussed.

It appears to be well accepted that the mathematics teacher's expectations and attributions on students' learning as well as her beliefs, attitudes and orientations have a substantial influence on classroom practice (c.f., McLeod, 1992; Ruffell, Mason and Allen, 1998; Thompson, 1992). On the side of the students, their attitudes are considered to be very significant factors underlying their school achievement (c.f., Leder, 1992; Ponte et al., 1992). These judgements are based on the assumption that attitudes and the like are relatively stable and reliable constructs so that they can be used for describing a person's mind. According to Ajzen's (1988) definition, attitude is "a disposition to respond favourably or unfavourably to an object, person, institution or event" (p. 4). If, in contrast, an individual's attitude was something situational bound, the concept would lose some of its effects. On the other hand, if too stable, the concept would go down in value for descriptions of change processes in mathematics teacher education or student's learning.

After having undertook a set of empirical studies of attitude, Ruffell, Mason and Allen (1998) doubt the stability and reliability of what is considered an attitude. In their research, they experience attitudes as highly influenced by the social and emotional conditions in which they are both perceived and observed. In the end, they challenge the very construct of attitude as a fruitful taxonomy for research or for the practice of teaching. In essence, it can be considered as questionable whether people actually possess attitudes and the like, or whether these constructs are just categories of observers who wish to account, by language, for what they claim to see. This looks very much as if a discussion of this issue under a discursive perspective could be promising.

Theoretical position

According to Austin (1962) and Searle (1969), language is a human practice. People use it to get things done. The fundamental tenet of Austin's and Searle's speech act theory is that all utterances state things and do things. What constitutes a speech act can be analytically divided into three parts (c.f. Ricœur, 1978): The act of the speech itself (propositional act), that what we do in the speech (illocutional act), and that what we do by means of the speech (perlocutional act). The words 'shut the door' can be used as a request, or with the force of an order. Alongside the specific meaning of the words (their proposotional character), the specific way in which the words are spoken make for the illocutional act of the speech. Finally, this very way in which the utterance 'shut the door' is spoken may have the stimulating effect of making the hearer anxious or annoyed (the perlocutional act). It may produce a certain reaction on the side of the hearer.

Austin and Searle offer a highly social perspective on language: Ways of talking are considered as social acts. They draw the attention to conventions in the achievement and performance of actions through talk. In their view, language use is embedded in a social environment that has a strong impact on the way in which a speech act is perceived. On the surface, this issue is intuitively taken into consideration when an interview about a person's attitudes to mathematics is organised. The interviewer may arrange the interview conformably to the social conventions for interview situations, she may care for an undisturbed room and a relaxed atmosphere where the interviewee feels safe and willing to express her opinion about mathematics.

Less obvious, and rarely considered in empirical studies on attitudes, is the fact that the interview talk itself is a social construction in which the speakers do things with words. The interviewer tries to stimulate the interviewee to put into words what she thinks or feels about mathematics. This is done in a more or less sophisticated way but it has to be methodically controlled. What the interviewer's questions and utterances intend to do is part of the interview strategy. On the other side, the interviewee is not just the victim of the interviewer's questions. With every of the interviewee's utterances something is stated or done. Or, as Ricœur (1978) put it, the utterances in themselves state things and do things. Thus, by giving a specific response to the interviewer's stimuli the interviewee is constructing a particular version of what the interviewer later on will call an attitude: the interviewee's talk carries an action. The interviewee organises her utterances along social conventions of how language is used in interview situations but not without taking into account her own goals and aims within the interview.

Traditionally, the conventions of how language is used in the way social life is put together are studied within the field of conversation analysis (e.g., Sacks, 1992; Sacks, Schegloff and Jefferson, 1974). Conversation analysis, an approach based on the discipline of ethnomethodology (e.g., Garfinkel, 1967), reflects upon every utterance as locally meaningful and conditional for the course of the talk. By studying talk as an object in its own right, conversation analysts found out that the way talk is formally organised is not a creation of individual persons but shared across collectivities. For instance, turn taking within everyday conversation is predominantly organised by a structural feature known as adjacency pairing: a question requires an answer, a greeting a return greeting. Results like that allow access to the modes of operation of how words do what they do.

Taken together, these features suggest language should be of enormous interest in the study of attitudes. Such a discourse oriented and sociopsychological perspective has been elaborated under the label of 'discursive psychology' (e.g., Edwards, 1997; Potter and Wetherell, 1987; Smith, Harré and van Langenhove, 1995; van Dijk, 1997). In order to draw a clear dividing line between cognitive and discursive psychology, it can be stated that cognitive approaches generally try to configurate unambiguous results by systematising the reactions of the participants in a specific setting. In contrast, discursive psychology reflects upon descriptions, explanations and justifications given in the course of a talk or a written report. The analytical task of a discursive psychology is to take apart, to split up such descriptions and justifications. It is studied in which ways consciousness is constituted through discourse (Lerman, 2000): when and how peoples express explanations, how they position them strategically compared with alternative justifications, and in which sequence descriptions, justifications and explanations are produced.

> We reject a product-and-process psychology of mental development, where mind is viewed as an objective development outcome. In its place is a discursive-constructive notion of mind as a range of participants' categories and ways of talking, deployed in descriptions and accounts of human conduct. (Edwards, 1997, p. 48)

The discursive psychological perspective is sensitive to the relation between an object and its description. It challenges both the existence of attitudes in the mind of interviewees and the alleged objectivity of interviewers' classifications. To put it in slightly drastic terms: The attitude categories used by the observer to classify the interviewees' utterances may tell us more about the observer than about the interviewees. Discursive psychology, however, determines the discursive practices of the people under study as well as the resources (e.g., systems of categories, narrative characters, interpretative repertoires) which are used in the organisation of the discourse. Discourse is considered as cognition-in-action.

Under the perspective of discursive psychology, some traditional sociopsychological concepts have been revisited (e.g., Edwards and Potter, 1992; Potter and Wetherell, 1987), but not within the field of mathematics education. In the following, an example from my own research is analysed in order to demonstrate how the discursive approach has an impact on the practice of research on attitudes in mathematics teaching and learning. It is not intended to give a comprehensive discourse analysis but to focus on some details that already show the significance of a discursive psychology for mathematics education research.

Example from research

The passage I want to discuss originates from an audio-taped interview with a student teacher. The interviewer is a peer and the interview is done at the interviewee's home. As a matter of course, the passage is derivative in at least three aspects: firstly, it is part of the transcript of the interview and as such already focussed on, secondly, the focus is on the words and not on their pronunciation, lastly, it is a translation from the German original. Points in brackets symbolise seconds of silence and do not represent parts that have been omitted, italics refer to paralingual recordings.

Interviewee: all right then with mathematics I firstly connect (..) with mathematics I firstly connect numbers (...) then any formulas and (.) yes mathematics is simply driving me to desperation *(laughs)*

Straightaway, this counts as a negative statement to mathematics. In terms of attitudes, the interviewee explicated the construct mathematics and gave her opinion on it. Thus, the interviewee should be considered, or classified, as a person with a desperate attitude to mathematics, in particular to numbers and formulas. On a fictitious Likert-scale to an item 'I like mathematics' this could be translated into a marker at the total-disagreement end of the scale, both by the participant of the study and by the observer. But when we look at the course of the interview the situation gets more complex. After the laughter of the interviewee, the interviewer remained silent for a moment. According to conversation analysis, such ignorance of a turn giving marker may result in an explanation (or, sometimes, a repair) of what has been uttered previously. Apparently, the silence, here, invited the interviewee to extend her response:

Interviewee: all right then with mathematics I firstly connect (..) with mathematics I firstly connect numbers (...) then any formulas and (.) yes mathematics is simply driving me to desperation *(laughs)* (...) well from first to fifth grade mathematics was really great fun that is it was not my favourite subject or so (.)

> but I liked it like all other subjects and it was absolutely okay (.) then we had a new absolutely stupid incompetent teacher she was absolutely unqualified (.) she has then no mathematics at all (..) been able to teach us and from this moment onwards mathematics was nothing but driving me to desperation (..) didn't like

Now, what previously has been said about the interviewee's attitude to mathematics should be withdrawn and modified. For grades 1 to 5 mathematics had not been driving her to desperation: *Mathematics was really great fun*. Therefore, we cannot certify her anymore a totally desperate attitude to mathematics. Moreover, the passage reveals that less the numbers and formulas account for the partly desperate attitude, but the *absolutely stupid* new teacher. Apparently, the personality of the teacher dominated the mathematical content as long as the generation of a positive or negative attitude to mathematics was concerned.

In the end of the passage, the interviewee uttered *didn't like*. On one hand, we can interpret this as a judgement about the teacher, but this is not so surprising when we take into account that the teacher was previously attributed to be *absolutely stupid, incompetent* and *absolutely unqualified.* Rather, the moderate expression would be astonishing. On the other hand, *didn't like* can be read as in relationship with mathematics. Thus, it could have served as a mitigation of the beforehand expressed desperate attitude to it. According to this interpretation, on the fictitious Likert-scale the interviewee would have made a second marker because she felt bound to develop her first opinion. As a matter of fact, this is more easily done within interview situations than on questionnaires. As a result, the context by which utterances are framed is influential in two ways: Firstly, the organisational and methodical setting in which the research is conducted influences the manner of the utterance. Secondly, the surrounding textual association of an utterance is leading to a more profound or, at least, modified understanding of it. This issue is well acknowledged in traditional cognitive psychological research when it operates with interviews or observation. In the case of data collection by means of questionnaires the textual context in which single items occur, that is their sequence, is often neglected and the responses to the items are treated as isolated judgements.

Further, in the first part of the passage the interviewee characterised mathematics as numbers and formulas and immediately concluded that mathematics was driving her to desperation. So, on a first view, the numbers and formulas take the responsibility for the negative attitude. But this is inconsistent with the fact that numbers in particular are of paramount importance in primary mathematics, and that was a time when mathematics was really great fun. The clue, here, are the first words in the passage: *with mathematics I firstly connect*. This introduction, especially the word 'firstly', points to the possibility that the following statement should rather be regarded as spontaneous and immediate than as a reflected definition. Consequently, in the second part of the passage where the interviewee starts to reflect on her school experience, this first characterisation of mathematics is of less importance. Thus, the passage under study results to be stratified: After the short common sense introduction, an explanation on a higher level of reflection is given. What firstly appeared to be an inconsistency within the interviewee's attitude now proves to be an organisational aspect of the text.

Another striking aspect of the passage is the choice of words by which the interviewee tried to convincingly describe her attitude to mathematics. At several points, she used what Pomerantz (1986) has called an "extreme case formulation". If, for instance, somebody argues *'nobody in his right mind ever needs calculus in his life'* this can express an argumentative support of his own lack of understanding of calculus. It is the purpose of extreme case formulations to verify a judgement at its extreme limits. In the first passage, mathematics was *simply* driving her to desperation, formulas were *any* formulas. Then, mathematics was *really great* fun, it was *absolutely* okay. The new teacher was *absolutely* stupid, *absolutely* unqualified and has not been able to teach mathematics *at all*. The interviewee's repeated use of 'absolutely' can be interpreted as putting all the blame on the teacher in order to release herself from any participation in the generation of the desperate attitude. The interviewee's extreme case formulations can indicate that she is used to explain her relation to mathematics as outwardly determined.

Implication for future research

These observations clarify two points: Firstly, some information on the parts of a discourse that surround an expression can be sufficient for challenging what beforehand has been putatively proved as a reasonable interpretation of the utterance. Secondly, it has been demonstrated that interviewees' responses, as any discourse, are organised according to certain aims. In the case that has been analysed, the interviewee's construction and formulation of the response make it easier for her to impute accountability for her desperate attitude to the teacher.

These points are of the greatest importance for the attribution of attitudes to individual persons, and their classification. The interpretation of a single isolated utterance can lead to oversimplified, distorted or wrong evidence - not to think of starting from a simple marker on a scale. Apparently, attitudes, if ever regarded as a meaningful construct, are far more complex than all what can be deduced from a one-dimensional judgement. The transcript analysed above offers some insight in what the interviewee considered spontaneously as mathematics and how she judged these issues. But it is highly problematic to take this specific judgement for the interviewee's attitude to mathematics.

This critique signifies that a simplistic separation of the construct 'attitude' from its position related to a single dimension of judgement is critical. Whenever attitudes to mathematics are imposed, this implies the existence of a homogeneous and consistent understanding of what mathematics is. But this is, obviously within the passage, a fiction. Before giving a judgement, the interviewee explained what it is that she is going to judge. Again, this explanation is not a neutral description of mathematics. Instead, the concept 'mathematics' is actively reconstructed by the interviewee's utterance, it is defined in a specific situation. This argument also invalidates a simplistic separation between the object and its description.

If we want to find out how and what people think and feel about mathematics, we should resist the temptation to rashly superimpose our system of categories on people's talk or people's responses. A more sensible approach starts with the reconstruction of the systems of categories, the narrative characters and strategic devices of the people under study in order to understand the organisation of the discourse first. Such scrutiny may reveal that what people think and feel about mathematics is not stable and consistent but strongly situational and more or less determined by context. Admittedly, the discursive approach tears us away from, and beyond, attitudes.

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DEEP STRUCTURES OF ALGEBRA WORD PROBLEMS: IS IT APPROACH (IN)DEPENDENT?

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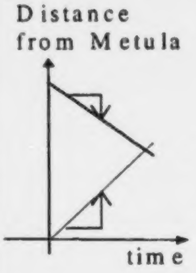
Research of word problems in arithmetic as well as in algebra has long stated that meaningful categorization that should lead to problem solving with understanding takes into account the deep structure of the problem. But is the "deep structure" approach independent? In this article we discuss the possible impacts of the domain of functions on the ability and the difficulty to solve motion problems in algebra. We studied students who participate in the VisualMath curriculum (a function approach to algebra in grades seven to nine). What might be a pedagogical obstacle was encountered in work of students learning by this functions-based approach. It suggests that construction of equations as a comparison of two functions is harder when the equation cannot explicitly describe the situation model unless another variable or a parameter is introduced. Thus, the emphasis on algebraic symbols being meaningful modeling language rather than solely the objects of manipulations may set a new cognitive sequence that removes previously known obstacles and introduces new ones. The finding that the categorization of word problems seems to be approach-dependent represents a more general view about emergent research of curricular change.

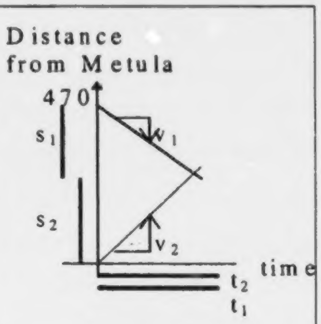
DEEP STRUCTURES OF WORD PROBLEMS

A word problem is examined at two levels of abstraction: the *quantitative structure*, which describes arithmetic operations and relations among symbolic or numerical entities, and the *situational structure*, which describes relations among physical properties of the entities within a story problem (Bednarz and Janvier,1996; Hall et al., 1989; Shalin and Bee, 1987; Yerushalmy and al, 1999). Each one of these two structures determines the deep structure of a problem and therefore might be responsible for the problem's difficulty. Several studies have shown that performance on solving word problems is a result of an interaction between an individual and a problem, so it needs to be understood in light of both the knowledge and skills the individual brings to the solution process and the nature of demands imposed by the problem (Sebrecht, 1996; Bednarz and Janvier, 1996; Nesher and Hershcovitz, 1994).

The traditional approach to algebra centered on symbolic manipulations, solving for unknowns and structures of algebraic expressions. This strand usually navigated the solving of algebra word problems to concentrate on the quantitative structure, by assigning symbols to unknown quantities and arranging them in an algebraic relation to answer the questions posed (Nathan and Koedinger, 2000). On the other hand, studies demonstrated that pre-algebra students often think of a problem purely in situational terms (Baranes, Perry and Stigler, 1989). Yet another finding is that students representing story problems in formal problem model terms tend to disregard the meaning associated with the equation of the values and thus may provide solutions which are physically or situational implausible (Silver, 1988). Observations of highly competent solvers show their skills in using the situation model and the quantitative model within a problem (Hall et al., 1989) in an integrated fashion. The coordinated use of situation model with one's formal problem model appears to be fundamental to problem solving with understanding, in a variety of domains (Koedinger and Anderson, 1990).

Problem 1
Metula and Eilat are 470 km apart. A truck and a cab started traveling at the same time towards each other. The cab traveled from Metula to Sdom at an average speed of 80 km per hour. The truck traveled from Eilat to Sdom average speed of 56 km per hour. Both drivers reached Sdom at the same time.
a. How long after starting their journey did the two vehicles reach Sdom?
b. How many kilometers did each of them travel until they arrived to Sdom?





v_1=56 km per hour
v_2=80 km per hour
s_1+s_2=470 km
$t_1=t_2$ hours

Figure1a: An example of a typical rate problem to which we refer in this paper

Figure1b: Qualitative graph of the situational structure of problem1

Figure1c: Qualitative graph of the situational structure and the quantitative structure within the same model

The concept of function provides a set of terms for mathematical modeling that can turn routine symbolic work into model construction to describe the problem's situation in numerical, graphical, and symbolic terms (Heid, 1995; Yerushalmy, 1997; Nemirovsky 1996). More specifically, situations involving a single variable can be described in a two-dimensional Cartesian system by use of a triad of quantities, two quantities representing the independent and dependent variables respectively, and the third representing the rate of change of the second quantity as a function of the first. Constant-rate problems are frequently present in algebra texts. Such problems (as in figure 1a) are usually solved by means of DeRT tables as an organizing structure to the construction of an equation. When experiencing a new function-based algebra curriculum based on intensive use of functions' graphing tools (Yerushalmy and Gilead, 1999), we learned that functions can turn solving such common algebra rate problems into a meaningful activity that emphasizes both the modeling and manipulation skills. The situation can be modeled graphically by two linear functions of time. Semi-quantitative sketches of functions (figure 1b) or graphs that accurately describe the given quantities form a visual presentation of the situational structure and may help students to express correctly in algebraic terms the quantitative relations. A set of given quantities, unknowns, and some constraints, formed by arithmetic operations between pairs of unknowns, form the quantitative structure of the problem. Thus the graph (as in figure1c) models the situational structure of a problem and the quantitative structure (Chazan, 1993).

Studying our students' problem-solving attempts, we observed unexpected achievements and unexpected obstacles. Often we were surprised to see how successful was the solving of problems considered complex by the traditional approach, and at other times how very similar problems were far harder. A first attempt to investigate further what seemed to be a phenomenon was a systematic analysis of the domain of constant rate problems according to the range of possible interactions between quantitative structures and situational structures (Gilead, 1998; Yerushalmy et al., 1999; Yerushalmy and Gilead, 1999). We suspected that differences between constant rate problems would not be fully explained by differences in the quantitative structures or by the differences in situational structures, but by the interaction between the two. We also wondered whether this complexity is an epistemological or a didactic obstacle and whether it is an outcome of the function approach. An empirical study was designed to test our conjecture on the effect of the situational structure, the quantitative structure, and their mutual correspondence on students' performance in solving constant rate problems. The work presented in this paper is part of this research.

METHOD

Following the analytic categorization we prepared for this study 21 different rate problems, each representing one of the types of problem as defined in Yerushalmy et al. (1999). A questionnaire containing four problems, randomly selected for each participant, was administered in 17 different classes of ninth graders who had already learned to solve rate problems by a function approach within the VisualMath[1] curriculum.

These same problems were administered in another 17 classes of ninth graders who had learned to solve rate problems by the equation-unknown strand (which below we call the non-functional approach). About 100 students of the two populations solved each problem. In this paper we refer only to four combinations of two situational structures and two quantitative structures (Table 1).

The pair of problems 3 and 4 have the same quantitative structure of the type: v_1=given, v_2=given $t_1=t_2$, s_1-s_2=given, while the pair of problems 1 and 2 share Another quantitative structure of the type: v_1=given, v_2=given $t_1=t_2$, s_1+s_2=given.

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[1] Visual Mathematics is an intensive technology function approach algebra curriculum developed by the Center of Educational Technology, Israel.

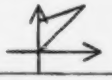
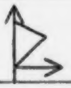
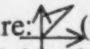
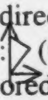
Situational structure Quantitative structure		
v_1=given ,v_2=given, $t_1=t_2$ s_1-s_2=given	Problem 3	Problem 4
v_1=given ,v_2=given, $t_1=t_2$ s_1+s_2=given	Problem 2	Problem 1

Table 1: The four combinations of the two situational structures with the two quantitative structures and the number of the problem that presents each combination

The pair of problems 2 and 3 share the same situational structure:  (two vehicles driving in the same direction at the same time). Problems 1 and 4 both share another situational structure:  (two vehicles driving towards each at the same time). Students' performance was scored as correct or incorrect. A correct model was either formed symbolically (a correct equation), graphically (reading the solution from accurate graphs), or numerically (reading the solution from a table of values).

FINDINGS

Following the traditional observations of algebra word problems, we started by analyzing what effect the quantitative structure had on students' performance in each population. For the function approach students, no significant difference was found between the two quantitative structures of problems 3 and 4, and problems 1and 2 (Table 2).

Quantitative structure	**Function approach** Percentage of students who built a correct model	**Non-functional approach** Percentage of students who built a correct model
v_1=given,v_2=given, $t_1=t_2$ s_1-s_2=given	64% n=187	44% n=208
v_1=given,v_2=given, $t_1=t_2$ s_1+s_2=given	65% n=206	63% n=208

Table 2: Percentage of correct solutions in the two quantitative structures

This was not the case for the non-functional approach population (correctness × quantitative structure significant: $p\{\chi^2(13.948;1)\}=0.001<0.01$). Among these students the problems with the quantitative structure v_1=given, v_2=given, $t_1=t_2$, s_1+s_2=given seemed easier (with 63% success) than the problems with the quantitative structure v_1=given, v_2=given, $t_1=t_2$, s_1-s_2=given (with only 44% success). The latter structure is reported as problematic in other studies (Mayer, 1982; Clement, 1982) because of the relational proposition involved in it (one traveled 45 km more than the other). The higher success (64% against 44%) with this problematic structure within the function approach population might be explained by their use of graphs of functions to describe the situation. This visual representation of the processes might have helped to state correctly the algebraic relations among quantities and form a right algebraic model (Hall et al., 1989).

A similar analysis was conducted for the situational structures of the pairs of problems 2,3 and 1,4 (Table 3). No significant differences were found in the two populations.

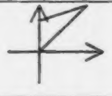
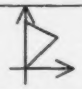
Situational structure	**Function approach** Percentage of students who built a correct model	**Non-functional approach** Percentage of students who built a correct model
	63% n=205	54% n=208
	64% n=188	53% n=208

Table 3: Percentage of correct solutions in the two situational structures

However, Table 4, which presents the correct solutions to the four possible combinations of the two situational structures and the two quantitative structures, reveals that some combinations were easier than others for the function approach population. Problems 1 and 3 seemed to be of the same level of complexity (89%) and significantly easier than problems 2 (39% success) and 4 (33% success).

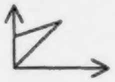
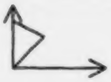
Situational structure Quantitative structure		
v_1=given ,v_2=given, $t_1=t_2$, s_1-s_2=given	89% n=103	33% n=84
v_1=given,v_2=given, $t_1=t_2$, s_1+s_2=given	39% n=102	89% n=104

Table 4: Percentage of correct solutions to the four problems by the function approach students

We assume that the combination of the specific correspondence of the quantitative structure and the situational structure of problems 1and 3 were responsible for the results. For each function in this case, a point and a slope are given, which make it possible both accurately to graph and explicitly to describe the linear function symbolically. We termed such combinations between the situational structure and the quantitative structure *canonical* (Yerushalmy and Gilead, 1999). The situational model of problems 2 and 4 can only be sketched, not drawn by accurate graphs. Thus the symbolic model (the equation) cannot be fully described by the given quantities when phrased according to the situational model. A symbolic representation would have to use a parameter or another independent variable in addition to the time-independent variable, and the solution cannot be determined by straightforward comparison of two functions G(x)=F(x) as it can be determined in the canonical problems. This quality of the four rate problems is detailed in Figure 2.

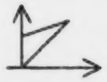
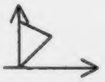
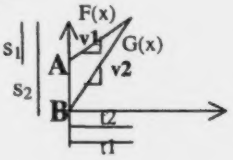
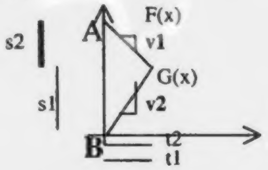
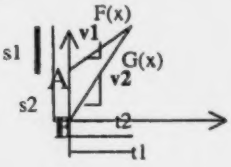
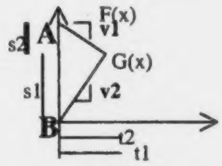
Situational structure Quantitative structure		
v_1=given v_2=given $t_1=t_2$ s_1-s_2=given	**Problem 3**  **Canonical** F(x): determined by **point A** and **slope v_1** G(x):determined by **point B** and **slope v2**	**Problem 4**  **Non-canonical** F(x): **point A** is not determined G(x):determined by **point B** and **slope v2**
v_1=given v_2=given $t_1=t_2$ s_1+s_2=given	**Problem 2**  **Non-canonical** F(x): **point A** is not determined G(x):determined by **point B** and **slope v2**	**Problem1**  **Canonical** F(x): determined by **point A** and **slope v_1** G(x): determined by **point B** and **slope v2**

Figure 2: The contribution of each quantitative structure to information about the two functions in each visual situational structure

Analyzing the performance of the function approach students according to the canonical and non-canonical terms, we found (table 5) significant differences $(p\{\chi^2(117.222;1)\}=0.001<0.01)$.

Type of combination	**Function approach** Percentage of students who built a correct model	**Non-functional approach** Percentage of students who built a correct model
Canonical	89 % n=207	50% n=224
Non canonical	36% n=186	57% n=192

Table 5: Percentage of correct solutions to canonical and non-canonical problems.

Differences among the non-functional approach students (table 5) were not found significant (p $\{\chi^2(2.209;1)\}=0.137>0.05$).

Performance on canonical problems was significantly (p $\{\chi^2(73.473;1)\}=0.001<0.01$) better (89%) within the function approach population than within the population of the traditional solution approach (50%). However, performance on non-canonical problems was significantly (p $\{\chi^2(17.687;1)\}=0.001<0.01$) better (57%) within the population of the traditional solution approach than within the function approach population (36%).

Thus, the multiple representations that is one of the foundations of the function's approach students were taught to use, was especially helpful in problems in which the equation could be explicitly derived from the situational model. It was less helpful, and we conjecture that it was even an obstacle, where the symbolic model (equation and solution) could not be directly derived from the situational model. An informal review, of different "function approach to algebra " texts that we conducted, revealed that the majority of constant rate word problems are of the canonical type, while in the traditional approach texts canonical and non-canonical types are interwoven. The findings have implications for the use of graphic tools (e.g; Graphic calculators) that most sequences of function approaches to algebra suggest, as an exploratory support for solving word problems. These tools might prove useful and helpful only for the canonical problems.

SUMMARY

The paper discussed the possible impact of the domain of functions on the ability and the difficulty to solve rate problems in algebra. Some of the problems (the canonical ones) turned out to be significantly easier by the function solution approach than by the traditional solution approach. The non-canonical problems proved more difficult within the function solution approach. This finding that the categorization of word problems seems to be curriculum-dependent may represents a more general challenge for emergent research on curricular change. New curricula may help to eliminate known pedagogical obstacles, but may also generate new unexpected ones. And new curricula may have to revise the previously used set of tools for analyzing students' knowledge.

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MICHAEL'S COMPUTER GAME: A CASE OF *OPEN* MODELLING

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We report on work from the Playground project, in which young children (6 to 8) are designing their own computer games using a specially built iconic language to represent rules that determine the behaviour of objects in the emerging game. We report on one child's development of a maze game to illustrate how conventional modelling theory might be extended to encompass situations in which children legitimately construct and reconstruct the problem being modelled and in which children are simultaneously learning about the potential and utilities of the modelling tools themselves. We conclude that a framework for such modelling needs to acknowledge the complexity of activity and to incorporate new perspectives on validation and on modifying the goals.

Introduction

The notion of modelling that interests us describes the activity of creating and testing a model that represents critical elements, including typically the mathematical structure of a problem or system. Much of the research on modelling activity involves the use of software; indeed Schecker (1993) proposes that model building software be recognized as a category of tools in its own right, and defines this new category as:

> "… context-free software tools that support the user in representing a part of the 'touch-and-show' reality in the form of an abstract, quantifiable system of parameters and their relationships (the *model*), which predicts the behavior of the real system." (p. 162)

Such software has been used by students to explore ready built models (for one of many examples, see White, 1984) and to express the learners' own ideas for models that represent a particular situation or problem. (this distinction is discussed by, amongst others, Bliss & Ogborn, 1989.)

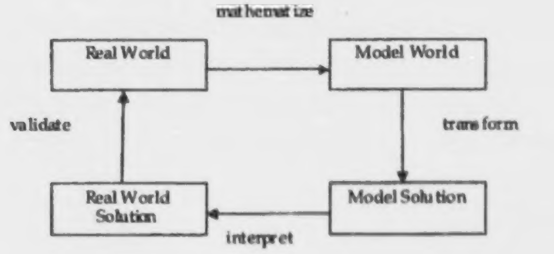

Fig. 1: The modelling cycle

Our interest in modelling stems from our work in the Playground Project[i]: we are studying young children (6 to 8 years old), as they use specially designed software to make their own computer games. Our focus is on the way young children express their ideas for games through the articulation of rules. We wish to modify or extend modelling theory to encompass situations like ours where the learner possesses considerable control over the nature of the problem: the child is involved in problem posing, an activity that falls outside the classical modelling cycle as expressed in Figure 1.

We also hope to adapt the modelling cycle by acknowledging the central role of tools in shaping the activity. Our starting point owes much to the situated cognitionists (Lave, 1988) in that we regard tools and learners to be in a dialectical relationship; that is to say, as the learners become more familiar with the tools, they become aware of new opportunities and utilities of those tools. Through using the tools, the learners re-construct their understanding of them. This shapes the way that the learners think about their solution to the problem, and, where problem posing is legitimate, the problem itself.

In summary, we aim to illustrate the evolutionary nature of a form of modelling that encompasses conventional modelling activity but adds to it two important processes: (i) an openness that makes it legitimate to change the problem being modelled, and (ii) an interleaving of learning about the tools themselves.

Our approach

We report the work of one child, Michael, who is 7 years old. He has been building a maze game using *Pathways*, one of the pieces of software being developed in the Playgrounds project. *Pathways* is driven entirely through the use of icons with the intention that it is accessible to very young children. The software allows children to choose from many different background screens and to place a number of objects on the screen. The objects can be given various shapes and they can be made to move automatically or to be dependent on the joystick or the mouse. A significant feature of the software is that objects can be given rules that determine their behaviour when the game is played. An underpinning hypothesis is that young children can create complex behaviour by teaching objects a relatively small number of simple rules expressed in an iconic language. An object can hold several rules. For example, the pathways in Figure 7 apply to the walls of a maze and they express the rules: 'When I touch the tiger, I reveal a rabbit' and 'When I touch anything, bounce off me' (i.e. any other touching object should bounce away from the object that holds the rule). Children move from defining rules to playing the game by clicking a switch that turns the game on but they can return at any time to making rules by switching the game off again.

The game evolutions are captured in clinical interviews, in which we work as participant observers alongside pairs of children (usually pairs, though in the episode reported in this paper Michael is working on his own). The children are between 6 and 8 years of age. The interviews normally last between 1 and 1.5 hours. Game evolutions usually extend across 4 or 5 such interviews.

The data for the research are transcriptions of the recordings on videotape of the children's activity, captured from their computer screen and from discussions between the children and the researchers. Also, there are many versions of the emerging game, saved at regular or significant moments.

Fig. 2: Michael's game at the beginning of this episode

Evolution of a Maze Game

This episode began towards the end of the second session with Michael, who was building a type of maze game in which the player controls the tiger with a joystick and tries to 'catch' the rabbit (see Figure 2) at which point the rabbit disappears as if eaten by the tiger. He has built a timer (in the top right corner of the screen), which counts the time that has elapsed since the game was switched on. The game contains several pieces of wall (the blue strips) but so far Michael has not given the walls any rules. This episode describes how Michael's game evolved to include rules for the walls. (Note: In the transcribed sections below, M refers to Michael and R to the researcher(s) who were present.)

At the beginning of this episode, Michael played the game and found that the tiger passed straight through the walls. Michael wanted to turn the walls into barriers.

1. M: Can we make it not be able to go through?
2. R: There is something we can try on that but let's worry about that a bit later.
3. M: Would it be the bounce button?
4. R: Yes. How did you know about the bounce button?
5. M: When I was searching.

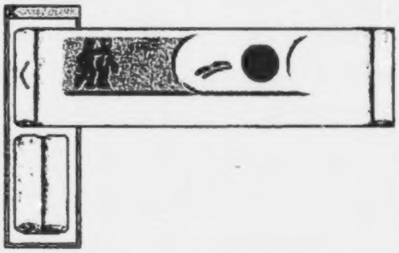
Fig. 3: Michael's rule for bouncing

Michael had remembered some earlier exploration in which he had come across an icon for bouncing. When this icon is used in a rule, anything else that touches the object with the rule bounces off that object. Michael opened up the pathways for one piece of wall and began to enter each element of the wall's rule: 'When I am touching the tiger, bounce off me' (Figure 3).

Michael checked this rule by playing the game but found that the tiger passed straight through the wall. The researchers explained that there was a conflict in Michael's rules: Michael had already written a rule for the tiger which meant that it was controlled by the joystick and so the tiger did not know whether to follow the joystick or to bounce off the wall. In fact, the tiger would momentarily bounce off the wall but then immediately pick up the position of the joystick again. Michael was not content.

6. M: I wish it would just not go through … it wouldn't really be a proper maze because the idea is that you can't go through stuff like walls and stuff. That's the whole point of mazes.

With some encouragement from the researcher, Michael began to consider alternative solutions that would resolve the bouncing conundrum. He considered penalising the player somehow when the tiger collided with a wall.

7. M: If you touched the wall, I was thinking …Oh, I've actually just had an idea. It changes its speed. It speed goes up so it's harder to control. So basically it goes, "Oh no I've touched the wall". It gets faster and faster and faster.

The researcher explained that the speed of the tiger was controlled manually by the joystick, and so any programmed change to the speed of the tiger would in any case be overridden by the player.

8. M: I've just thought of the worst one ever. The game stops.

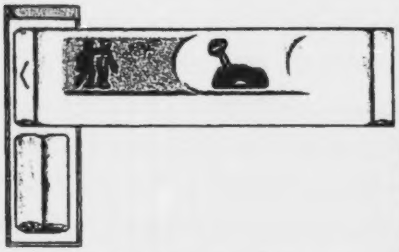

Fig. 4: Michael's new rules for each circular obstacle

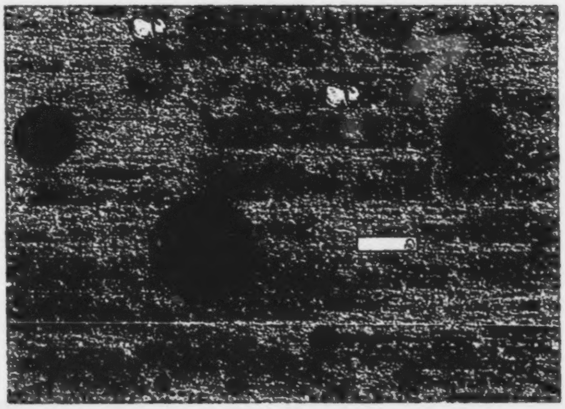

Fig. 5: Michael's game with circular obstacles

Michael entered the rule: 'When I touch the tiger, I stop the game' into a blank pathway for one piece of wall (Figure 4). He then copied this rule for each of the other pieces of wall.

Michael kept this rule for the walls for most of the remainder of his time working on the maze game. During this time he concentrated on other aspects of his game. By the time we pick up the story once more, Michael has changed the walls into circular obstacles (Figure 5). He has also added two extra rabbits, each of which moves automatically and bounces off the walls.

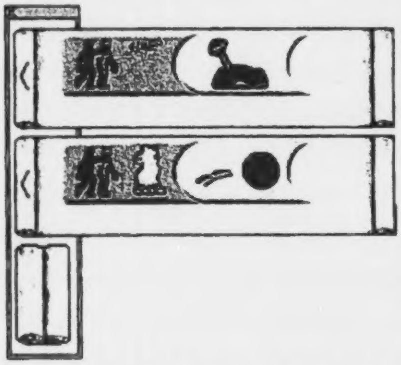

Fig. 6: Michael's stopping rule for the walls

Michael had created this effect by applying the earlier idea of bouncing off the walls to the rabbits. Thus, each circular obstacle contained the pathways in Figure 6.

The second rule can be read: 'When I touch anything, bounce off me'. Michael had already learnt that the tiger was controlled by the joystick and could not (visibly) be made to bounce off the walls, so in effect only the rabbits would now bounce off the obstacles.

However, Michael was still worried about his first rule, which made the game stop too often. He wanted to penalise the player some other way when the tiger collided with an obstacle.

9. M: Can I make … oh yeah … I could change that rule (pointing to a circle) until …make it, could I make it one rabbit appears every time you touch it?

The researcher suggested that Michael could start with some hidden rabbits and make them appear when the tiger touched an obstacle.

10. M: …also when you … like you can destroy one, you can get one but then you go in … and you catch one, it appears again but, would that happen?

The researchers asked for clarification.

11. M: Well, tiger gets the rabbit but then he accidentally goes into a ball and the rabbit appears again, sort of.
12. R: Oh I see, one that's been eaten appears.
13. M: Yeah.

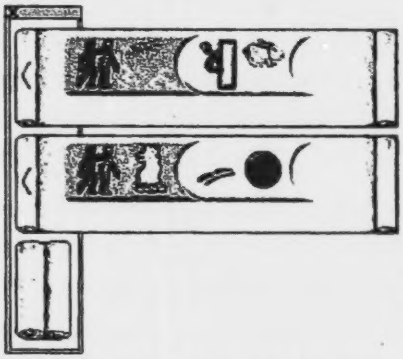

Fig. 7: Michael's rule for making the rabbits re-appear

Michael decided that since he has six obstacles and three rabbits, he could allocate two obstacles to each rabbit, so that whenever the tiger touched an obstacle it would make a particular rabbit appear and, if that rabbit was not already hidden, nothing would apparently happen.

Michael erased the 'I stop the game' icon from the pathway of one of the obstacles. With some help, he found an icon for 'I show' and inserted it into the rule (see Figure 7) and he made sure that the correct rabbit was included in the rule. Michael repeated this process for each obstacle.

Discussion

We have noticed three very different phases of activity in the pupils' work. These phases have no clear order and they are often not distinct from one another.

Phases of activity

1. Game-oriented activity (G): There are some points during the sessions when the main focus of the students' work is on the game itself. When the pupils are in G their thinking is focused on the final outcome of their efforts, i.e. the game they are designing. They are concerned with issues such as the context for the game, the background screen, what happens, who controls which objects, how many players are involved and how any victory is accomplished. Lines 6, 7 and 8 above are comments from Michael at times when his thinking was clearly game-oriented.
2. Tool-oriented activity (T): There are other stages during the pupils' activity when they need to learn about the tools that they are using. Whenever computers are used, some time needs to be devoted to understanding the software and how it operates. This is quite normal and there are bound to be stages when the pupils have to find out how to change the background picture on the screen or which particular icons in which order might be used for one of their rules. In line 5 Michael refers to a tool-oriented stage when he was "searching", by which he means he was exploring the software.
3. Rule-oriented activity (R): The pupils create particular rules for particular objects in their game to make things actually happen when the game is played. Michael has

given a rule to the walls in his game so that when the tiger touches any of them it will bounce, rather than carry on in the same direction (Figure 2). Later he changed the rules for the walls (which had become circles) so that when the tiger touched one of them a new rabbit was made to appear (Figure 6). When the thinking is clearly focused on a rule to be included in the game, we say it is rule-oriented activity.

Complexity

Michael's activity highlights two issues that have implications for how we think about the modelling cycle.

1. Order of the phases: It must already be clear that there is no prescribed order for the three phases (G, R and T) described above. For instance, the students are engaged in game-oriented activity when they use the icon in the toolbar at the foot of the screen to switch their game on. This might happen at any stage, either for the sheer fun of it or to validate their changes. Thus the place of G in the complete picture cannot be determined in advance. It is also evident that T and G might occur in either order: on occasions students learn about a particular tool before they attempt to use it in the game (line 5); at other times they learn about a new tool because its need has become evident in the game (activity leading up to the rule in Figure 7). The preferred approach will vary with the student and with the particular situation.
2. Links between the phases: The complexity of the phases is also attributable to the close links that exist between them. Activity in G often suggests new possibilities for R (see the text leading up to and including line 1). In lines 6, 7 and 8 Michael is in G but he is moving towards R because he is starting to think about particular rules. During the course of this short extract he uses the phrase "If you touched the wall" which is very similar to the one used by the software's mouth when it is applied to certain rules. Students need to learn to think about their game's rules formally in the language of the software so they can change the game and Michael has started to do this very early. There is, however, no clear link in the data that has been collected between T and G and this leads to the hypothesis that R is critical within Pathways and an equivalent mode needs to be found for other situations that depend on the acquisition of tools.

Modelling

Our analysis of Michael's work highlights several features that are similar to those in the conventional modelling cycle. We see analogies between the real and model worlds of the conventional modelling cycle with Michael's game-oriented and rule oriented activity. We also recognise the processes of interpretation and validation. Furthermore, in describing the complexity of the interleaving of the different phases of activity, we are observing no more than has been reported by several other researchers who have commented on the lack of the linearity implied by the conventional modelling cycle (Lesh & Doerr, 2000). Because of the commonality between Michael's activity and modelling theory, we believe it is productive to consider the episode as a case of modelling. Nevertheless in the next two sections we wish to elaborate on some aspects of the data, which strike us as different from conventional modelling.

De-goaling

Michael resolved the conflict over bouncing by changing the game so that it simply stopped whenever the tiger touched a wall (line 8). Michael forged new insights into the utility of the tools involved in trying to make the tiger bounce and this caused him to redefine his objective. Later he adopted the bouncing idea for the rabbits (lines 9-13) and so changed the aim of his game once more. A feature of the above episode is that, during his tool-oriented activity, Michael often sought out new possibilities, and exploited new discoveries. In an earlier incident, Michael explicitly recalled his prior experimentation (lines 2-5) and subsequently used the bounce stone during a phase of rule-oriented activity in order to create the effect of walls as boundaries (Figures 3 and 4).

We refer to the process of changing the aim of the game as *de-goaling*. The purpose of the activity, as understood by Michael, was to create a computer game. This explicit purpose made legitimate, indeed encouraged, de-goaling, a process that is not normally associated with modelling.

Validating

Another feature of Michael's activity is the way in which he validated his ideas. The move from R to G was often triggered by the need to validate his recently expressed rules. In the rule-oriented activity that preceded line 6, Michael had formulated a bouncing rule (Figure 3). He validated this rule by playing the game. On this occasion, the validation was initially negative – the rule did not work – but subsequently very positive since he found new ways of resolving the dilemma as discussed above.

Similarly, the creative game-oriented activity in which he imagined disappearing and re-appearing rabbits (lines 9-13) is stimulated by the validation process (playing the game) that triggered a dissatisfaction with the game stopping repeatedly.

We are struck by the natural way that Michael moved from the model-world of rules to the 'real' world of his game because of the constant need to validate his progress and the legitimacy of changing the problem being posed, his maze.

Open Modelling

The feature of de-goaling and the nature of validation lead us to believe that we should regard Michael's activity as a variation of conventional modelling. We refer to the type of modelling illustrated by Michael's activity as *open* modelling. Open modelling describes activity where the learner is encouraged to reformulate creatively the problem itself as a result of learning about the utilities of and potentials for the modelling tools. In the literature on conventional modelling, concerns have been expressed about the lack of evidence of validation strategies when children are expressing their own models (Doerr, 1996). Matos (1995) reports on 10th grade students using spreadsheets to model the length of paper in a roll of given diameter. Matos found that the spreadsheet's use was transformed during the activity: the software started as a tool for expressing models but became the reality within which the students worked. As a consequence, Matos' students hardly ever referred back to the problem after the initial stage. Validation rarely occurred in the conventional modelling cycle. In contrast, Michael engaged naturally and almost continually in validation processes. The creative challenge of de-goaling provides a drive to move between real and model world activity. There is some evidence that, because of the legitimacy of changing the problem being modelled (i.e. de-goaling), validation occurs more naturally and takes on a more positive and creative role in open modelling than it does in conventional modelling activity reported by other researchers.

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[i] The Playground Project, co-ordinated by the Institute of Education, University of London, is an EU ESPRIT project (No. 29329): http://www.ioe.ac.uk/playground (For a discussion of children's understanding of rules, see Hoyles, Noss, Adamson, & Lowe, submitted.)

STUDENTS' CONCEPTIONS OF CUBIC FUNCTIONS

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This paper explores some aspects of students' conceptions of cubic functions of the form $f(x) = a(x-h)^3 + k$ *. This was done by analyzing students' responses to a problem dealing with family of functions. On the basis of these answers, a series of hypothesis was formulated. These hypotheses were corroborated with a second group of students who solved the problem and were asked to correct and comment a pre–arranged solution to it. It was found that an important proportion of students develops a consolidated and invalid conception of the cubic function with special characteristics concerning its domain.*

Introduction

In this study we wanted to explore some aspects of students' conceptions of cubic functions of the form $f(x) = a(x-h)^3 + k$. We did so by considering the performance of a group of university students solving a problem dealing with graphics of cubic functions. Our purpose was to describe some characteristics of the students' graphics and to explore, on the basis of those graphics, some aspects of the students' conceptions of the cubic function. The study was done in two phases. During the first phase, we constructed a set of categories that enabled us to characterize the students' graphics and we formulated some conjectures concerning the possible conceptions that were behind that behavior. During the second phase, working with a different group of students, but the same problem, we confirmed the previously found characterization for the graphics and were able to test the proposed conjectures. We formulated as well a partial description of the students' conceptions of this type of cubic function.

In what follows we discuss some conceptual aspects concerning the understanding of the notion of function, in general, and of the cubic function, in particular. We then describe in detail the instruments we used to collect, codify and analyze the students' performance. Finally, we present the results found and draw some conclusions.

Understanding of the notion of function

The understanding of the notion of function has drawn some attention recently (Harel & Dubinsky, 1992; Tall, 1991; Romberg et al., 1993; Leindhardt et al., 1990). In particular, Sierpinska (1992) has produced a list of acts of understanding and epistemological obstacles related to the notion of function. For example, an act of understanding concerning the representation of functions is the discrimination between different means of representing functions and the functions themselves. An epistemological obstacle concerning the graph of a function is that the graph of a func-

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tion is a geometrical model of the functional relationship. It does not have to be exact and can contain pairs (x,y) such that the function is not defined for x.

The notion of *representative* (Schwarz & Dreyfus, 1995) has become important as technology is used more widely in the teaching of functions. This notion has to do with the fact that a given function can have multiple representatives within a given representation system. For example, $f(x) = (x-2)^2 + 3$ and $f(x) = x^2 - 4x + 7$ are two representatives of the same function. Likewise, a function can have an infinite number of representatives in the graphical representation system, depending on the range and scale used for the axis.

The process of understanding can be seen as proceeding by "states". Each state corresponds to a certain partial knowledge (a *conception*) that has worked with previous experience and that allows the student to feel comfortable while solving tasks. For a given type of problems a conception can correspond to either a valid or invalid version of the mathematical knowledge at stake. We will say that a conception (valid or invalid) is consolidated whenever the student having it "feels comfortable" putting it into play while solving problems. As it will be explained later, this sense of "comfort" can be observed through the coherence of the answers of the student to a series of related questions. On the other hand, a conception can be in an unconsolidated state. When this happens, partial knowledge has not been established and the answers of the student do not follow a coherent pattern.

Understanding of the cubic function

There is little research made on the understanding of the cubic function. Curran found that students exhibit links between their understanding of the graph of a cubic function and their understandings of the graphs of linear and quadratic functions (Curran, 1995).

We considered a specific form of the cubic function: $f(x) = a(x-h)^3 + k$. We did so, because the precalculus course under study followed a strategy of teaching translations and dilatations in the construction of graphs of functions. The general form of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is introduced later on.

We were interested in exploring some aspects of the students' conceptions of the cubic function. In particular, we wanted to see if it was possible that students develop consolidated but invalid conceptions concerning the domain and range of the function. This interest came from a first phase of the study in which students were asked to solve a problem concerning family of cubic functions (the problem is shown later). We found that many students draw graphs similar to those shown in table 1.

The proportion of answers of these types made us think that they could be a consequence of an invalid and consolidated conception of the cubic function, instead of simply being a consequence of drawing mistakes or specific circumstances related to the specific problem at hand. This conception could be expressed as follows:

> *The domain and range of the cubic function is a proper subset of the real numbers and can be seen as an interval around the inflection point of the function.*

We felt that if a student has this kind of conception, then he/she could agree with situ-

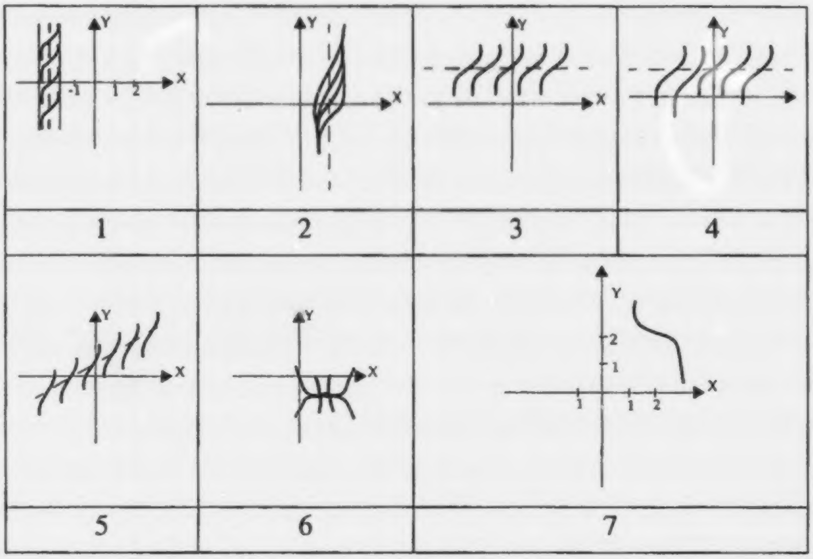


Table 1: Types of graphs produced by students

ations as the following:

- ▲ If the inflection point of the function is on the y-axis or close enough to it, then the graph crosses that axis (Graph types 3, 4, 5, 6).
- ▲ If the inflection point of the function is not close to the y-axis, but it is not too far away, then the graph of the function has the y-axis as one of its asymptotes (Graph type 2).
- ▲ If the inflection point is far enough from the y-axis, then the graph of the function has other asymptotes (Graph type 1).
- ▲ If the inflection point of the function is far enough from the x-axis, then it does not cross that axis (Graph type 3).

In order to test these hypotheses we decided to work with a different group of students and the same problem. This was the second phase of the study.

Context and data collection

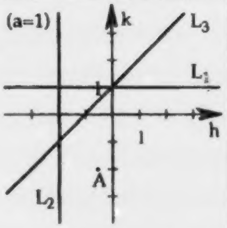
The study was done with first–semester students of a Precalculus course in a private university in Bogotá, Colombia. This course was taught on the basis of a curriculum innovation that involved graphic calculators (Gómez et al., 1996). The course concerns an introduction to the study of functions with special emphasis in the relationship between the symbolic and graphical representation systems and problem solving. One fourth of the course deals with linear functions, followed by the study of cuadratic, cubic, polynomial, rational and radical functions. Special attention is given to the graphical role of the parameters in the different possible symbolic representations of a function.

Some results are already known concerning this curriculum innovation. Mesa and Gómez (1996) found no differences in some aspects of understanding between the students who took the traditional course and those who took the curriculum innovation. Gómez and Rico (1995) found that the students of this group participated more actively in social interaction and in the construction of the mathematical discourse, changes that can partially be attributed to a different behavior of the teacher. Even though she changed her behavior, Valero and Gómez (1996) found that the teacher could not change completely her beliefs system. Carulla and Gómez (1996) found that the teachers and researchers who participated in the curriculum innovation underwent significant changes on their visions about mathematics, its learning and teaching. Gómez (In press) found that the effects of technology use on achievement depends on the way it is integrated into the curriculum.

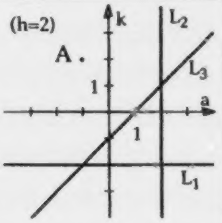
The problem used was one of the problems scheduled to be done at the time cubic functions were taught in the course. We wanted to explore whether students of this new group produced graphs similar to those found in the first group; and whether those graphs were a consequence of a consolidated but invalid conception of the cubic function. In order to perform this exploration, we collected and analyzed three different types of information: 1) the answers of the students to the problem; 2) the way students corrected and commented a solution to the problem produced by us that contained most of the mistakes corresponding to the consolidated and invalid conception; 3) the comments made by the students to a series of statements related to the above solution and to the hypotheses presented earlier.

The last two instruments were designed in order to make sure that the mistakes found in the graphs were not a consequence of drawing problems and to induce the students to put into play their conceptions under different circumstances concerning the same problem. The problem proposed to the students was the following.

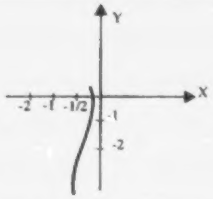
1) Assume that a = 1. The figure shows the plane **h–k** plane, where the point (h, k) represents the cubic function $y = (x - h)^3 + k$. Draw in different cartesian planes **x–y** the functions or family of functions corresponding to a) A; b) L_1; c) L_2; and d) L_3.



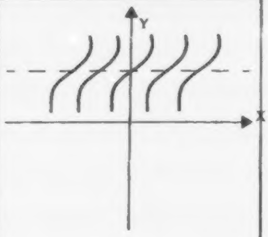
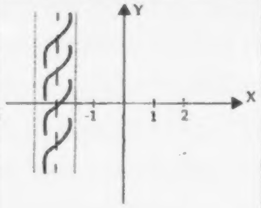
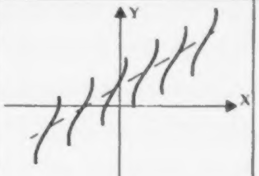
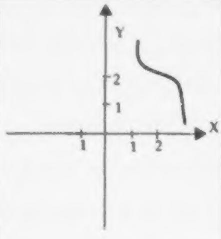
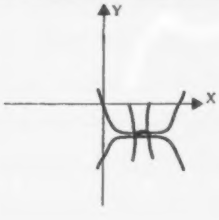
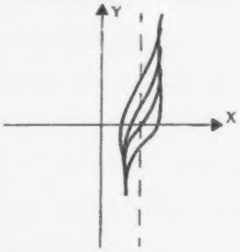
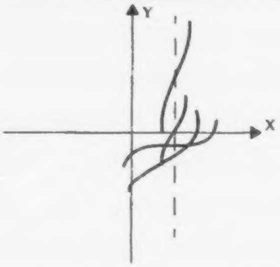
2) Assume that h = -2. The figure shows the **a–k** plane, where a point (a, k) represents the cubic function $y = a(x - 2)^3 + k$. Draw in different planes **x–y** the functions or family of functions corresponding to e) A; f) L_1; g) L_2 and h) L_3.



The following is an example of the solution to the problem that the students were asked to correct and comment and the statement they were expected to comment.

1) a. The point A corresponds to a graph whose inflection point is $(-1/2, -2)$. Its symbolic representation is $y = (x + 1/2)^3 - 2$.		The graph does not cross the y-axis.

The following are the rest of the graphs and statements that were proposed to the students for correcting and commenting on.

			
1) b.The graphs do not cross the x-axis.	1) c. The graphs seem to have two asymptotes.	1) d. Some graphs cross the y-axis and others do not.	2) e. The graph has restricted domain and range.
			
2) f. Some graphs do not cross the y-axis. The domain is the same for all the functions.	2) g. For big values of y the graphs join each other.	2) h. Those graphs with big enough value of k do not cross the x-axis.	

Analysis

For the solution of the problem the students were divided in two groups: 13 students were asked to solve problem 1 and 9 students, problem 2. All students were asked to correct the solution proposed and to comment on the statements proposed. We calculated the following percentages:

▲ The percentage of students that, having produced a graph, drew a graph of one of the types expected, as described in table 1.

- For each answer corrected, the percentage of students that marked it as correct.
- For each statement commented, the percentage of students that accepted it as valid.

Furthermore, an analysis of each student's answers was made on the basis of his comments to the statements. The purpose of this analysis was to explore the coherence of the comments of each student to the series of statements, and be able to conclude about their conceptions. We considered only those students who commented at least three statements. We considered that a series of answers were coherent if at most one comment was contradictory with the other comments (from the point of view of their validity).

Results

Table 2 shows the percentage of students that, having produced a graph, draw a graph of one of the types expected, as described in table 1.

Graph type	1	2	3	4	5	6	7
Percentage	79	100	8	75	78	88	46
Total number of answers	19	1	12	12	9	8	13

Table 2: Graph types percentages

All students marked as correct all the answers presented in the solution proposed to them. Table 3 shows, for each statement commented, the percentage of students that accepted it as valid.

Statement	1.a	1.b	1.c	1.d	2.e	2.f	2.g	2.h
Percentage	32	54	57	38	56	75	11	75
Total number of answers	19	13	14	8	9	4	19	4

Table 3: Comments percentage

Table 4 shows the percentage of students that had a coherent (valid and invalid) series of comments to the statements, together with the percentage of students who proposed an incoherent series of comments. There were 9 students with less than 3 answers. For the other 10 students, the percentages were as follows.

	Coherent invalid	Coherent valid	Incoherent
Percentage	40	50	10

Table 4: Coherence percentages

Discussion

The results show that students of the second group continue drawing graphs of the types found with the first group of students. When asked to correct the pre–arranged solution, all students marked as correct all the invalid answers proposed to them. However, when asked to comment on the statements proposed, the reactions of the students differed. Many answers were found in which students did not agree with the statements. Nevertheless, this was not a consequence of a "random" reaction from the students to the statements. The analysis of each students' comments to the series of statements show that the students can be categorized into three groups: those with a coherent and invalid series of answers, those with coherent and valid answers, and those with incoherent answers. The relevant point here is that the group with coherent but invalid answers represents an important proportion of the total group. This leads us to think that many students can develop a consolidated and invalid conception of the cubic function of the form $f(x) = a(x-h)^3 + k$. However, our hypothesis concerning the range of the cubic function is not clear, given the results to questions 2g and 2h.

Conclusions

There might be many reasons why students develop this type of conception of the cubic function. From the mathematical point of view this seems to be a natural situation. Since cartesian planes are traditionally drawn with the same scale in both axis and the cubic function grows rapidly, the part of the domain of the function that can be "seen" in the graph is usually a subset of the real numbers. Furthermore, it seems that most textbooks and teachers tend to consider cubic functions for which $a \geq 1$.

Teaching and textbooks do not help either. We found that the textbook used in the course did not present graphs in which it could be seen that the domain were the real numbers. Furthermore, when checking some teaching materials drafts from one of the teachers, we found graphs similar to those drawn by the students.

It is very likely that this problem exists with cuadratic functions as well. As a matter of fact, when performing informal interviews with the students, one of them justified the "asymptotic" behavior of the cubic function as being the same as the one for the cuadratic function. In this sense, one can say that there is a link between the understanding of the cuadratic and the cubic function. However, even though we do not have data to justify, we think that this link is broken when cuadratic and cubic functions are compared to linear functions, a fact that would not corroborate Curran's results in this respect.

One may argue that the problem proposed to the students was about family of functions and not about the domain and range of the functions. Therefore, students could have been more concerned about answering the questions and correcting the answers proposed with respect to what they considered relevant in the problem and this might be the reason why all of them marked as correct all the answers proposed to them. However, this was not the case with the statements they had to comment on. Those statements referred to the graphs themselves and made no direct connections to the text of the problem.

Finally, technology might have played a role as well. The "window" problem identified by Schwarz and Dreyfus with their "representative" concept might induce students to construct their invalid conception. This is a somewhat paradoxical situation given that graphic calculators and computer software allow students to easily change the scale and range of the axis. However, this could be evidence of the fact that students do not take advantage of those features.

The learning difficulties found in this study would not be very important if they could be interpreted as a consequence of drawing problems related to the specific context of the task at hand. However, as we have found elsewhere (Carulla & Gómez, 1997), when technology is involved, students tend to construct their understanding based mainly on the graphical representation of the concepts. This might be the reason why we found these consolidated and invalid conceptions.

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Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics

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In this paper we propose a theory of cognitive construction in mathematics that gives a unified explanation of the power and difficulty of cognitive development in a wide range of contexts. It is based on an analysis of how operations on embodied objects may be seen in two distinct ways: as embodied configurations given by the operations, and as refined symbolism that dually represents processes to do mathematics and concepts to think about it. An example is the embodied configuration of five fingers, the process of counting five and the concept of the number five. Another is the embodied notion of a locally straight curve, the process of differentiation and the concept of derivative. Our approach relates ideas in the embodied theory of Lakoff, van Hiele's theory of developing sophistication in geometry, and the process-object theories of Dubinsky and Sfard. It not only offers the benefit of comparing strengths and weaknesses of a variety of differing theoretical positions, it also reveals subtle similarities between widely occurring difficulties in mathematical growth.

Introduction

The theory presented here builds on work that has developed steadily over the last two decades (Gray, 1991; Tall & Thomas, 1991; Gray & Tall, 1994; Tall, 1985, 1995). But it is not a simple restatement of earlier theories. A simple switch of viewpoint is seen to reveal powerful insight into very different ways in which individuals construct mathematical concepts. To gain insight into this viewpoint, we consider the situation in which *embodied objects* are perceived by and acted upon by individuals. (The precise nature of embodied objects will be discussed in more detail shortly, but essentially they begin with human perceptions using the fundamental senses, and become more mentally based through reflection and discussion over time.) Our viewpoint then compares the developing embodied meanings of the objects and their configurations with other mental constructions relating processes and concepts through the use of symbolism. Our purpose is to compare the meaning embodied in the objects and their configurations on the one hand with process-object abstraction on the other. We seek a theory with the power of both explanation and prediction of the varied nature of cognitive development throughout mathematics. We require a viewpoint that is theoretically sound yet has a simple and practical meaning relevant to the spectrum of practitioners from teachers of young children to university mathematicians.

One of our hypotheses is that the theorised encapsulation (or reification) of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon (which we henceforth refer to as *base objects*). We observe that the embodied configurations are more primitively meaningful than the encapsulated mental objects and yet lack the flexibility and power of the distilled essence of the symbolism that links dually to both mathematical concept and mathematical process. The consequence is that the embodied approach can give fundamental meaning to mathematical ideas but that such embodied representations prove to be complex to handle when they are applied to increasingly sophisticated problems. Progress to more subtle levels of mathematical thinking requires eventual access to the powerful and compact use of symbolism. We observe that a practical, real world understanding of simple mathematics can very well benefit from a focus on the operations on the base objects, and such a perspective is satisfactory, even insightful in everyday situations. However, an *exclusive* focus at this level can act as an epistemological obstacle barring the way to the more sophisticated theory that is required for subtle technical and conceptual thinking. As was observed in Gray, Pitta, Pinto & Tall (1999), those students who consider only their perceptions of embodied objects remain at a more primitive level, whilst those who succeed move on to more sophisticated levels, with an easy movement forward to focus on the symbolism or back to consider the configurations of the base objects. Some of those with a developing hierarchy maintain the full range (being 'harmonic' in the sense of Krutetskii (1976)), others become successful by focussing on the higher levels (increasingly 'analytic' according to Krutetskii), losing contact with the real world and becoming 'formal' thinkers (Pinto & Tall, 1999).

Embodied objects

We take the notion of 'embodied object' to begin with the mental conception of a physical object in the world as perceived through the senses. Examples include a Jersey cow, a hamburger, a paper bag, geometric objects such as triangles, arrays of objects such as the dots on a domino, the drawing of a graph of a function, a Venn diagram, and so on. In addition to our direct perceptions through our physical senses, we also think about what we perceive, compare our sense of one embodied object with another and share these ideas with others. In this way our perceptions take on an increasingly subtle meaning. On the one hand our mental conception may be in the form of a "skeleton" or a prototype, having general properties that provide a basis for communication until the addition of specific properties lead to the particular. (For instance, the word 'dog' may bring to mind a domestic animal with fur that barks; but as we consider new information, such as a dog whose fur is cropped into artistic shapes, we might then home in on a poodle, or to a specific poodle belonging to a friend.) On the other—and this is highly relevant to the development of mathematics—our perceptions may become abstractions which no longer refer consciously to the specific objects in the real world. An example of the latter is the idea of a 'straight line' which is initially seen as a line drawn with a tool such as a ruler that makes it 'look straight'. By talking about the idea we move on to consider a mental concept that is 'perfectly straight', 'having no width', 'arbitrarily extensible in either direction'. None of these properties is true of an actual line in the real world, but it is *based* on real-world perception and can only be constructed mentally by building on the human acts of perception and reflection. In this way we see an increasing sophistication in the notion of 'embodied object' that begins with sensory perception and is refined in mental thought through the use of language to give increasingly refined precision and hierarchies of meaning. This gives an increasingly sophisticated conception of embodied objects in a general manner which has been specifically described by van Hiele (1985) with reference to geometric objects.

We use the term 'embodied object' in a manner similar to the theory of Lakoff and his colleagues who speak of 'embodied cognitive science' (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000). However, we note that Lakoff's theory does not explicitly use the notion of 'embodied object'—the term does not appear in the index of either Lakoff & Johnson (1999) or Lakoff & Nunez (2000).

Our approach makes a closer analysis of the nature of mathematical concepts and sees a significant distinction between embodied objects (such as a triangle or a graph) on the one hand and the symbols of arithmetic and algebra on the other. The latter symbols act as pivots between processes and concepts in the notion of procept (Gray & Tall, 1994), providing a link between the conscious focus on imagery (including symbols) for thinking and the unconscious interiorized operations for carrying out mathematical processes. In particular, we empathise with Dörfler (1993) who claims that, although he can imagine five *objects*, nowhere in his mind can he imagine a mental object for the number ''five'. From the perspective we are adopting in this paper, we agree that the imagery for the number 'five' is not an embodied object, although a mental image of 'five fingers' clearly is. This emphasises that thinking involving embodied objects is likely to differ significantly from the kind of thinking involved in the successful development of arithmetic and algebra. However, it does not mean that we cannot call a number an 'object' to manipulate, simply that it is not an *embodied* object. In fact, our linguistic use of number as a noun—'five is a number'—gives it a semblance of being an entity, even though this is no more an embodied object than the gerund 'running' in 'running is good for you'. We refer to a number as 'it', we operate on 'it' and with 'it' in arithmetic and—far more important—the symbol for the number allows us to switch flexibly between mental concept and mental process.

Encapsulation of procedure - process - procept

Gray & Tall (1994) adopted the distinction between procedure and process of Davis (1983, p. 257) whereby the term *procedure* is a step-by-step algorithm in which the individual needs to complete each step before taking the next. A *process* occurs when one or more procedures (having the same overall effect) are seen as a whole, without needing to refer to the individual steps, or even the different procedures. For example, "count-all", "count-on", "count-on-from-largest", "known fact", are all different procedures for the process of adding two numbers. When the symbols act freely as cues to switch between mental concepts to think about and processes to carry out operations, they are called *procepts*, These can be composed and decomposed at will to derive new facts. For instance, 8+6 may be calculated by decomposing 6 into 2+4, composing 8 and 2 as 10, and 10 and 4 as 14, or as decomposing 8 into 4+4, then recomposing 4+6 as 10, and then the other 4 plus the 10 makes 14. More particularly, it is now relatively easy to see the implications of the distinctions we make between the process of addition and the concept of sum. The former suggests *doing* the arithmetic whereas the latter emphasises a proceptual structure that consists of a theory of related procepts, including the base objects on which the processes act, the symbols as process and object, and the concept image related to the use and meaning of the procepts. Thus, in the example above, the procepts are symbolised whole numbers with the related process of addition; the base objects are initially physical objects, but then become figural objects and later become redundant as they are subsumed in a counting process which is itself compressed into the concept of sum.

It is clear from this discussion that the spectrum of procedure-process-procept is not a classification into disjoint classes; we explicitly mentioned the *ambiguity* of the symbolism as process or concept in the title of our paper (Gray & Tall, 1994). It is a categorization into a spectrum of improving sophistication in which the categories blend one into another, even regressing on occasion to a more primitive case. One of us remembers adding up marks in mathematics examinations and getting to 'know' most of the required facts, yet regressing to add 89 and 2 with a quick count-on as '89, 90, 91'. What matters with the increasing sophistication is that a 'process' usually (but not always) may be performed by a specific finite procedure (a counter-example lies in the general process of convergence to a limit). A 'procept' relates to a thinkable concept and a process carried out by its corresponding procedures.

What is clear, however, is the steady development of entities operated on, from physical objects including fingers, to imagined fingers or configurations of counters, to mentally operations with the number symbols themselves. The increasingly sophisticated arithmetical knowledge developed by children (see Steffe *et al.*, 1983) is exemplified by an increasing detachment from immediate experience, the development of different aspects of counting and a change in the form of unit counted. Within four of the counting types, the perceptual, figural, motor and verbal we may see the gradual shift in the nature of the base object from a perceptual unit to a mental embodied object. Cobb (1987), has suggested that it is "only at the level of abstract counting that number words or numeral signify conceptual entities that appear to exist independently of the child's actual or represented sensory motor activity" (p.168). We suggest that it is at this stage that the transition from process to concept can occur that forms the basis for understanding the numeration system. Though the system is straightforward for those who understand it, it remains a source of difficulty for many, particularly when shifted beyond whole numbers and extended to decimals. We suggest that it is the formation and reliance upon embodied configurations in the whole number context that is the basis for this difficulty. The recognition of proceptual structures provides the flexibility.

Sophistication and a spectrum of performance

Figure 1 (expanded from figure 1 of Gray, Pitta, Pinto & Tall, 1999) shows the possible outcomes of different levels of sophistication from pre-procedure through to procept. It shows that a problem requiring only a routine procedural solution will distinguish between failure and success only in terms of the change from pre-procedure to procedure. The availability of different routes at the process level introduces the possibility of alternative methods allowing checking for possible errors in execution, even to an underlying unconscious feeling that something is wrong when an error is made (Crowley & Tall, 1999). The procept level moves to a higher plane where the symbols act dually as process and concept, allowing the individual to think about relationships between the symbols in a manner which transcends process alone.

For example, we may recognise that the procedure ‘add 3 to a number and double the result’ and the procedure ‘double a number and add 6’ both give the same process. Symbols can be effective for these two procedures in terms of the expressions $(3+n)\times 2$ and $2\times n+6$, or the more standard notations, $2(3+n)$ and $2n+6$. These all represent the same input-output *process* operating on the (value of) the number n. A student still at the procedure level might find these various expressions and their procedural meanings a considerable barrier to understanding expressions as processes.

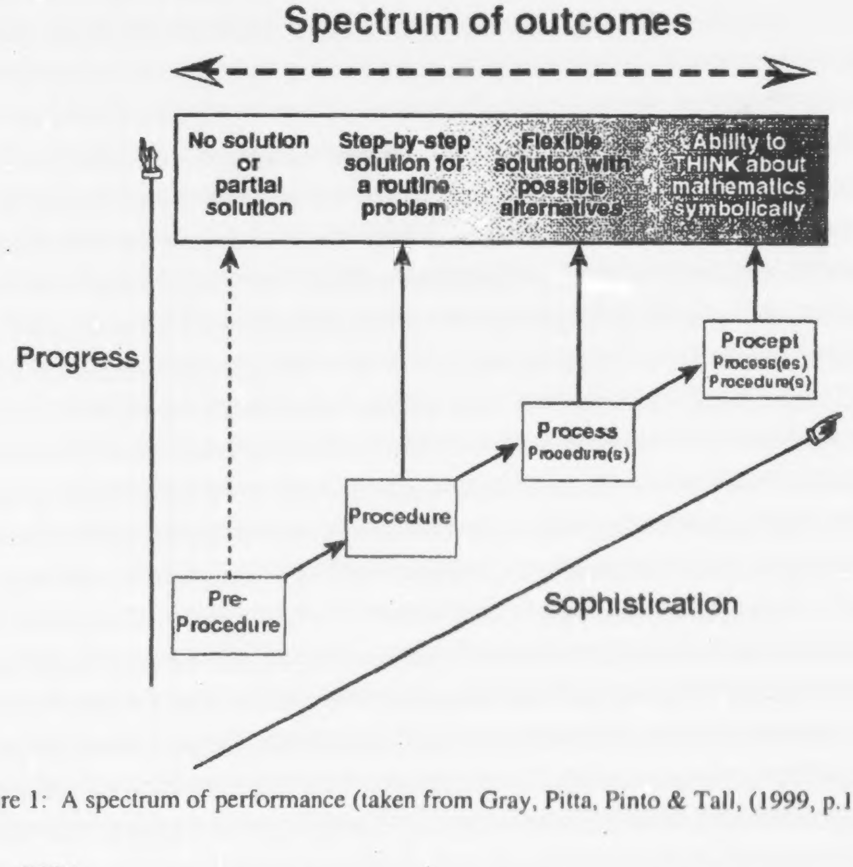
Spectrum of outcomes
No solution or partial solution
Step-by-step solution for a routine problem
Flexible solution with possible alternatives
Ability to THINK about mathematics symbolically
Progress
Procept
Process(es)
Procedure(s)
Process
Procedure(s)
Procedure
Pre-Procedure
Sophistication

Figure 1: A spectrum of performance (taken from Gray, Pitta, Pinto & Tall, (1999, p.121)).

Refocusing – a possible explanation

How does this process of refocusing from operations on physical objects, to operations on mental entities, to operations *with* mental entities occur. A probable solution is given by Edelman & Tononi (2000, p.57) who report many studies that show that initial problem solving causes activity in a wide range of brain centres, measurable using brain scans, but, as the solution processes become more routine, the required brain areas become fewer as alternatives are no longer required. Edelman and Tononi hypothesise that conscious thought requires a combination of both high correlation between different areas of the brain (which they term 'integration') at the same time as there is a range of possible choices to make (termed 'differentiation'). Routine mathematics becomes unconscious because it requires little differentiation in parts of its activity, only coming to the surface when a particular decision must be consciously made. Thus it becomes possible—but not inevitable—that the focus on the basic objects being manipulated becomes less necessary, and the links, first to inner perceptions, then to increasingly unconscious processes, gives a natural sequence of development for the human brain.

The relationship between embodied objects and encapsulation of processes

It is at this stage that our theoretical positions begin to diverge from both embodied object theory and process-object theory. The former could be a viewpoint in which all mathematical concepts are embodied objects. Such a view fails because the concept of number is not an embodied object, although the concept of 'five things' is. In saying this we misinterpret Lakoff and his colleagues who say that all *thought* is embodied rather than *all we think about* is embodied. However, we consider it important to lay the ghost of the idea that all mathematical concepts are conceived as embodied objects. For several years now (for example, Tall, 1995; Gray *et al* 1999; Tall *et al* 2000), we have been homing in on three (or perhaps four) distinct types of concept in mathematics. One is the *embodied object,* as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. Another is the *symbolic procept* which acts seamlessly to switch from a 'mental concept to manipulate' to an often unconscious 'process to carry out' using an appropriate cognitive algorithm. The third is an *axiomatic concept* in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory. (Here the fourth type of concept might occur by distinguishing between those concepts evolving from embodied objects and those from encapsulated processes (Tall, *et al*, 2000).) Expanding the theory based on 'perception, action and reflection', we see the different kinds of mental entities arising as in figure 2 (overleaf).

We begin by considering the classical situation where the individual performs *operations* on *embodied objects*. We have already considered the case of number concepts where the base objects are physical objects and the encapsulated concepts are number concepts represented by number symbols. Here we note an interesting fact. *Because the counting process operates on physical objects, the seemingly abstract*

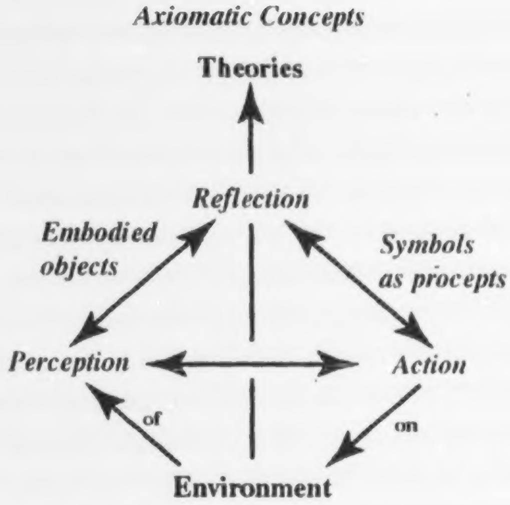

Figure 2: different kinds of mental entities arising through perception, action and reflection

concept of number, theoretically formed by a process of encapsulation, already has a primitive existence in the physical configurations of the base objects. It is by elaborating this simple idea that we come to a distinct view of the role of base objects in the formation of the higher order encapsulated concept. Essentially we see this role of the base objects as a stepping stone to the higher order concept, whilst at the same time having specific meanings for some individuals that act as epistemological obstacles preventing the hierarchical development that is essential for progress to more sophisticated mathematics.

Consider, as a second example, the idea of 'rate of change' and the subtle mathematical process of differentiation and its related concept of derivative. Here we see the picture of a graph as an embodied object that represents the function concept visually. It can be drawn and seen either with a pencil or with a wave of the hand in the air. This embodied action conveys the sense of the changing gradient of the graph as it changes slope. It proves to be natural for many students to develop an insight into the changing gradient by simply 'looking along the curve' and plotting the visual numerical value of the (signed) gradient as a graph. This can be done visually and enactively *without any numerical calculation or symbolic manipulation*. The more formal ideas can come (shortly) after the fundamental embodied activity has been constructed with support from the bodily movement of the individual.

This brings us to our major difference with theories of process-object encapsulation, particularly formulated in the sequence action-process-object-schema (Czarnocha et al, 1999; Sfard, 1991). Our observations of human activity reveal that the 'encapsulated object' is not simply produced by 'encapsulation' or 'reification' of process into object, but is greatly enhanced by using the configuration of the base objects involved as a precursor of the sophisticated mental abstraction.

This is not to say that the higher sophistications of calculus and analysis always remain consciously linked to fundamental embodied objects. They don't. The connections may remain but become unconscious, so that the brain can move on to focus on essential details selected as the basis of axiomatic development. This starts with formal definitions (based on useful, generative, properties) and continues by a *process* of formal deduction of theorems. Each established theorem then becomes available as a *concept* for use in the proof of later theorems. Different students learn formal mathematics in different ways. Many do not develop beyond their existing embodied perception. Some build on their concept imagery, modifying and extending their conceptual hierarchy to grow naturally into the formal ideas. Others have grown in sophistication and no longer evoke conscious links to their embodied sense of the world, extracting meaning from the formal definitions by formal deduction (Pinto, 1998).

In this way we see abstractions rooted in embodied objects of the biological brain providing a basis not only for geometric thought in a developing van Hiele sense, but also a foundation for symbolic process-object encapsulation and on to axiomatic thought.

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ANALYTIC-SYNTHETIC ACTIVITIES IN THE LEARNING OF MATHEMATICS

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We consider the well-known in psychological literature, but not yet sufficiently investigated in application to mathematics education mechanisms of thinking – the analytic-synthetic activities. Examples of using such means of mental activity as «synthesis» and «the analysis through synthesis» in solving geometric and algebraic problems are given. From these examples, one can deduce that various problems require various aspects of analytic-synthetic activities, and the most complicated means of such activity – "analysis through synthesis" can be developed only as a result of serious and specially designed teaching. The important task of mathematics education is to develop an effective system of teaching pupils to use all means of analytic-synthetic activity.

In this theoretical essay we will consider the well-known in psychological literature, but not yet sufficiently investigated in application to mathematics education mechanisms of thinking – the analytic-synthetic activities. The thoretical framework of our paper is Soviet activity approach in psychology and, particularly, S. L. Rubinshtein's conception of means of mental activity, or *mental operations* (S. L. Rubinshtein, 1989, p. 377). S. L. Rubinshtein (1958, p. 28) wrote : "The process of thinking is first of all the analysing and the synthesising of what is resulted in the analysis, then processes of abstraction and generalisation which are derivatives of analysis and synthesis. Regularities of these processes and their interrelations with each other are essentially basic interior regularities of thinking".

Analysis (in the ancient Greek *decomposition, partition, dissection*) is a procedure of mental, and frequently as well real dissection of a subject (phenomenon, process), of a property of a subject (subjects) on its components; extraction from a subject of aspects of its study; drawing out of a subjects their sides, properties, relations between them.

The analysis frequently is represented as multistage process. Something, reached as a result of the initial analysis, becomes then subject of the deeper analysis in the next stage. This passage from one level of the analysis to other, deeper level, is determined by the requirements and character of new tasks arising during the process of cognition.

Synthesis (in the ancient Greek *junction, compiling, integration*) is a mental combination of parts and sides, extracted by the analysis, in some new mental unity, in which the typical features of the analysed subject are fixed. The synthesis is connected with the simplification of what is analysed, with the detection in a subject of essential internal connections constructing the mental unity, with obtaining new piece of knowledge.

The synthetic activity of generalisation and meaning-making of the results of the analysis together with the analogy and other means of mental activity is a powerful tool of the discovering new truths, of obtaining principally new scientific results; of constructing revolutionary ideas creating new landmarks and directions in the development of sciences.

The synthesis reproduces the analysed subject, but is connected with the improvement, enrichment, deepening of that knowledge about a subject as a whole, which we had before the analysis. Synthesis already uses methods of that scientific theory, within the framework of which the analysis is produced, and, hence, those idealisations and abstraction, on the basis of which that theory is constructed. As a result of the analysis and synthesis the subject is reproduced in its essential and necessary relations and with that degree of the precision and adequacy, which is determined, on the one hand, by the contemporary state of the science and experimental technique and, on the other hand, by the character of the problem of research.

Thus, the synthesis is a procedure, inverse with respect to the analysis, but with which the analysis is frequently combined and intertwined in practical or cognitive activities. The analysis and synthesis are studied by such sciences as psychology, epistemology, logic, pedagogy and didactics of various disciplines.

Historically, the analysis was considered as a path (method of thinking) from the whole to the parts of the whole, and synthesis as a path from the parts to the whole. Therefore, analysis and synthesis are practically inseparable from each other. They accompany each other, complete each other, constituting the united analytic-synthetic method. The analysis assumes synthesis, and the synthesis is impossible without the analysis.

The analysis and synthesis as the methods of scientific knowledge play the important role in mathematical research. Similarly, extremely great is their role in mathematics teaching, in which they appear in the different forms: as methods of the solution of problems, of the proof of the theorems, of the study of properties of mathematical concepts etc.

Some of synthetic methods of solving problems involving geometrical constructions (methods of geometric loci, of similarity) were known still by ancient Greek geometers, and the distinction between analytical and synthetic methods was introduced to mathematics by Euclid. He considered the analysis and synthesis as two kinds of the "syllogistic" method of proof. In the thirteenth book of the "Elements" Euclid wrote: "In synthesis we begin with what is already proved, and come to the inference or to the knowledge of what is necessary to prove" (Euclid, 1956).

One can also find the definitions of the analysis and synthesis in the works of Francois Viete, who remarked that "there is a way of investigating the truth in mathematics, and the invention of that way is assigned to Plato; Theo has named it "analysis" and has defined as follows: "we consider the required as known and pass from a corollary to a corollary until we are convinced in the truth of the required; the synthesis consists in the following: proceeding from known, we, from a corollary to a corollary, come to the discovery of the required" (Yushkevich, 1970).

In the didactics of mathematics the terms "analysis" and "synthesis" traditionally meant two oppositely directed courses of reasoning, usually used in problem solving and in proving of theorems; the analysis is the reasoning directed from what is required to find or to prove to what is given or established earlier; the synthesis is the reasoning in the opposite direction.

Kolyagin et al. (1975, p. 52) wrote that "nowadays one understands the analysis as the means of thinking, leading from the corollary to the reason generating that corollary, and the synthesis is understood as the means of thinking, leading from the reason to the corollary generated by this reason".

Giving much attention to the means of thinking "analysis" and "synthesis", we certainly understand, that the mental activities are not reduced to these means only. The importance of such means as abstraction, concretisation, generalisation, analogy etc. for thinking in general and for mathematical thinking in particular is well known and widely admitted.

The analysis and synthesis can be combined with each other.

S. L. Rubinshtein distinguished the important form of the analysis – one which is carried out through synthesis. The essence of such analysis is the following: "the object of thinking is being repeatedly included in new connections and thus it arises in new appearances, with new qualities fixed in new concepts; thus, new contents are repeatedly taken out of the object, it turns repeatedly to new sides; new properties of the object come to light"(Rubinshtein, 1958, p. 98-99).

Thus, the important means of thinking arises: "the analysis through synthesis". Its role in psychology is connected with the detection of new qualities, sides and properties of objects. Therefore, this means is connected to the creative processes. S. L. Rubinshtein (1976) named this means «a quintessence of thinking». In our view, the possession of this means of mental activity is the highest level of person's development in general and her/his mathematical development in particular.

As a result of the using of means of mental activities «synthesis», «analysis» and «the analysis through synthesis» the special kind of intellectual activity – analytic-synthetic activity is born. This new kind of activity, in its turn, generates various analytic-synthetic methods of reasoning and problem solving in teaching/learning mathematics.

We will illustrate the essence of means «synthesis» and «the analysis through synthesis» by examples of problem solving.

The first example is a problem which can be solved by «pure» synthesis.

Problem 1. The perimeter of an isosceles triangle is equal to 1 meter, and the base of the triangle is equal to 0,4 m. What is the length of the lateral side (fig. 1)?

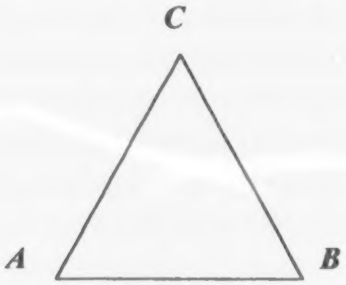

Fig. 1

Solution:

We will denote properties by the letter P with appropriate numbers. We have:

P1: the triangle ABC is isosceles (given).

P2: the perimeter of a triangle ABC is equal to $AB + AC + BC = 1$ m (given).

P3: $AB = 0{,}4$ m (given).

P4: $AC = BC$ (from P1).

P5: $AC = BC = ?$ (it is required to find).

Compare the properties.

P6: $0{,}4$ m $+ 2\,AC = 1$ m.

P7: $AC = 0{,}3$ m (P6).

Thus, in this problem we, deducing corollaries from the condition, come to the answer. This is the most simple example of the use of *synthesis*.

In the next problem the synthesis is accompanied by the analysis.

Problem 2. FOX: OX = 5. Find numeric values of the letters in this equality.

Solution:

We begin with synthesis, i.e. with deriving corollaries from the given equality.

P1: FOX: OX = 5 (given).

P2: FOX = 5 OX (P1).

The further reasoning is based on the possibility of the representation of a number as a sum of digit summands.

P3: 100 F + OX = 5 OX (P2).

Here we have distinguished OX, i.e. have used the analysis in addition to the synthesis. We remark also, that OX is a two-digit number. Further:

P4: 100 F = 4 OX (P3).

P5: 25 F = OX (P4).

P6: 25 F is a two-digit number (P5).

Now the analysis will be used once more. It is necessary to answer a question: for what numbers F, the number 25 F is two-digit? It is easy to find, that there are only three possibilities: F = 1, F = 2, F = 3. Hence the answer is: FOX can be deciphered in three ways: 125, 250, 375.

In this example synthesis was accompanied by the simple, but very essential for finding a solution analysis. Therefore, it is possible to name the mental activity used in this solution «synthesis through analysis».

Finally, we will consider two examples (geometric and algebraic) of application of the most advanced and complicated means of mental activity – «analysis through synthesis».

Problem 3. Prove that the triangle ABC, in which bisector coincides with a median, is isosceles (fig. 2).

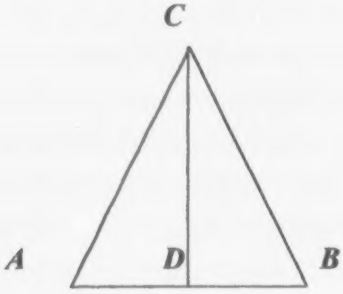

Fig. 2

Solution:

P1: ∠ *ACD* = ∠ *BCD* (i.e. *CD* is a bisector).

P2: *BD* = *AD* (i. e. *AD* is a median).

P3: The triangle *ABC* is isosceles (required to prove).

From P1 and P2 nothing can be received, and other data is not present. What should one do in such case? There is a necessity in the analysis, and the analysis here is rather difficult.

It is necessary to prove, for example, the equality of sides *AC* and *BC* of the triangle *ABC*. What is necessary to know (to prove) for this purpose? The answer to this problem is rather difficult for the 7-th grade pupils of the Russian schools, where this problem is offered.

The more mathematically gifted pupils can use the sign of an isosceles triangle: the triangle is isosceles, if its base angles are equal. The idea is to construct, taking advantage of the equality of angles *ACD* and *BCD*, an isosceles triangle with the same angles and with a lateral side equal simultaneously to *AC* and *BC*. Thus, one can think about the inclusion of angles *ACD* and *BCD* and sides *AC* and *BC* into new connections, so that the equality of sides *AC* and *BC* becomes revealed. For this purpose it is possible to draw a segment *DE* on the prolongation of the segment *CD*, so that *DE*=*CD*, and, considering equal triangles *EAD* and *CBD*, one can show that the triangle *CAE* is isosceles and, hence, *AC* = *AE* = *BC* (fig. 3).

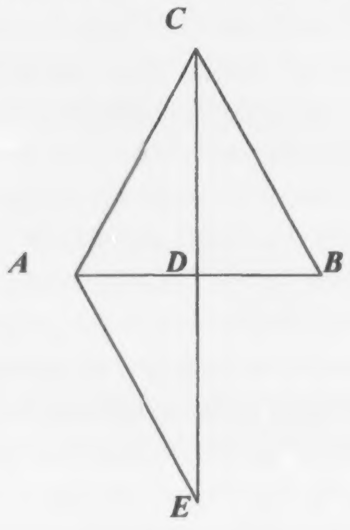

Fig. 3

The solution of this problem is a vivid example of the application of the *analysis through synthesis*. The analysis leads us here to the necessity of the rather complicated additional construction.

In the following problem the analysis through synthesis is applied to solving a non-standard algebraic problem .

Problem 4. Find natural solutions of the equation:

P1: $1 + x + x^2 + x^3 = 2^y$.

Solution:

We begin with synthesis, trying to transform this equation.

P2: $1 + x + x^2 (1 + x) = 2^y$ (P1).

P3: $(1 + x^2)(1 + x) = 2^y$ (P2).

What more can be deduced from this representation of the equation? One can read it as follows: the product of two natural numbers (x is a natural number, hence, $1 + x^2$ and $1 + x$ are also natural) is equal to the non-negative integer power of 2. In which case is it possible? Obviously, it is possible only when factors are also integer non-negative powers of 2. Thus, we have

P4: $1 + x^2$ and $1 + x$ are integer non-negative powers of 2, or, more concretely:

P5: $1 + x = 2^m$, m is a non-negative integer number.

P6: $1 + x^2 = 2^n$, n is a non-negative integer number.

P7: $x = 2^m - 1$ (P6).

P8: $(2^m - 1)^2 = 2^n$ (P6, P7).

P9: $2^{2m} - 2x2^m + 2 = 2^n$ (P8).

Note that if $n > 0$ then both parts of the equality can be simultaneously divided by 2. Consider separately the a case $n = 0$. In this case $x = 0$, but thus equality contradicts to the condition of naturality of number x. Therefore, $n > 0$. We have

P10: $2^{2m-1} - 2^m + 1 = 2^{n-1}$ (P9).

P11: $2^m (2^{m-1} - 1) + 1 = 2^{n-1}$ (P10).

All the transformations we accomplished are rather simple, however, it might seem unclear, for which purpose we obtained the equality P11. Here the analysis through synthesis has been accomplished, which has lead to the equality in which for all but one values of parameter n there appears an even number in the right part and odd one in the left part.

Now we will consider, for which values of m and n this equality is possible. Obviously, n does not exceed 1 (if $n > 1$, we have an even number on the right and odd one on the left), and m is greater than 0 (if $m = 0$, the number on the left would not be integer). As we have established above, $n > 0$, therefore, $n = 1$, whence $2^m (2^{m-1} - 1) + 1 = 1$. Hence, $2^{m-1} - 1 = 0$, whence $2^{m-1} = 1$ and, hence, $m = 1$. Finally we obtain $m = 1, y = 2$.

In this non-standard and rather difficult example there are three subtle moments (ideas): 1) one should see the conclusion P4; 2) denotations P5 and P6; 3) when transforming the left part of equality P8, it was necessary to foresee the purpose, that on the right and on the left there should appear expressions with values of different parity for almost all values of parameters.

In these examples, one can see that various problems require various aspects of analytic-synthetic activities, and the most complicated means of such activity – "analysis through synthesis" can be elaborated only as a result of serious and specially designed teaching. The important task of the theory of mathematics education is to develop an effective system of teaching to all means of analytic-synthetic activity, including the most difficult one – analysis through synthesis.

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RELEARNING MATHEMATICS - THE CASE OF DYNAMIC GEOMETRICAL PHENOMENA AND THEIR UNEXPECTED CARTESIAN REPRESENTATIONS

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Abstract This study describes the processes underwent by two mathematically sophisticated students during the investigation of geometrical phenomena with Dynamic Geometry. The investigation took place in an environment which included a computerized setting and ad hoc designed investigations to provide opportunities to deal with unexpected outcomes. In such an environment and in order to make sense of what they did and found, the students reviewed and enhanced their knowledge and thought processes, consolidating connections, establishing new ones, and developing new ways of work.

Introduction

In the last decades, there were considerable advances in the design of computerized environments to promote different and possibly more powerful ways of learning mathematics. Papert (1980) conceived the idea of "microworlds" in which students had the possibility to talk mathematics and to construct their ideas by designing. Pea (1986) defined "cognitive technologies" as a tool which may help transcend the limitations of the mind, supporting fundamental changes in the thinking processes. DiSessa (2000) described new "literacies" as rearranging "the entire intellectual terrain" (p. 19) making advanced knowledge more accessible.

In line with these and many others, our starting point rests on the broad assumption that computerized technologies enable students to approach, learn and understand mathematics in ways that were not possible before. There are many research studies reporting about learning in such settings (e.g. Kaput, 1992; Schwartz et al., 1993). Our focus is on "re-learning" (in the sense of Arcavi & Nachmias, 1989): we explored how students (and teachers) revisit and enrich their known mathematics in ad-hoc designed learning environments. In this paper, we briefly describe such an environment, and we present a case of two mathematically sophisticated students at the end of secondary school working with it. The mathematical topic of the exploration was familiar to the students, and they had all the needed pre-requisite knowledge to succeed, and they did. However, as in previous studies with students learning mathematical topics for the first time (Hadas & Hershkowitz, 1998, 1999, Hadas et al., in press), in this study the re-learning process of these students was led by their need to produce explanations of surprising findings which contradicted their expectations. We present data from their work and analyze the ways in which their prior knowledge and school habits (uses of procedures and formulae, repertoire of functions, expectations, courage to integrate and discuss intuitive ideas, etc) were engaged and changed during their work.

The environment

In our study, by learning environment we mean: (a) a Dynamic Geometry (DG) microworld with the possibility of juxtaposition and use of multiple representations (geometrical drawings, measurements, Cartesian graphs) and (b) specially designed geometrical investigations (involving unexpected outcomes) presented through a coherent sequence of open ended tasks (leading students to make and check conjectures and explain results not in line with their conjectures).

The following is one such sequence of tasks which includes two parts. The first deals with the area variation of an isosceles triangle (with equal sides of length 5 units) as a function of (i) the variable base (task 1), (ii) the variable angle between the two equal sides (task 2), and (iii) the variable altitude to its base (task 3). In the second part, the investigation turns to the area of a scalene triangle (sides of length 4 and 5), again, as a function of the variable third side (task 4) and as a function of the variable altitude to it (task 5).

Firstly, students are requested to examine the variation visually by dragging, measuring, and hypothesizing about the main graphical characteristics of the expected Cartesian graph of the relationship (domain, range, increase, decrease, maximum, symmetry, graph shape etc.) and to relate these to the geometrical features at stake (for a full description and analysis of the tasks, see Arcavi & Hadas, 2000).

After the students propose hypothesis/predictions, they are asked to make use of the software (Geometry Inventor, 1994) in order to draw the graph in real time, as they drag and change the triangle. Then they are requested to compare the graph on the screen with their predictions, and to explain the findings. Here, we concentrate on data from tasks 1 & 5, which in our experience are almost certain to produce surprises, contradictions and a subsequent motivation to search for an explanation.

The students

MI and US, 12 graders at the time of this study (age 17), were friends who studied together in the same advanced mathematics class, at the highest level of their last year in a prestigious (public) secondary school. They had had at least two years of calculus, geometry and algebra. They worked through the sequence of tasks described above in the presence of an interviewer who asked them questions to clarify what they said and requested for explicit verbalizations of their thought processes. At the end, they talked very openly about their views of the experiment in relation to their school experiences. The interview was videotaped and almost fully transcribed.

Framework for the analysis

Our analysis of the relearning focuses on three interrelated components:

- The role of prior knowledge - How pieces of the students' entrance knowledge determine what students see (and thus possibly hinder progress), and at the same time, how their knowledge is refined and enhanced by their experiences in this environment.

- The role of the views of the subject matter and on "doing" mathematics.

- The role of real time experimentation - How the objects of their observations and the features of the software support the change in their knowledge and in turn how the growth of their knowledge support new uses of the software?

Data and Analysis

At the beginning, MI and US correctly worked out the values of the domain of the variable base (0 -10). When they turned to the range, without any prompt, they said:

US: *between 0 and..., a right angled triangle will be the largest.*
MI: *When it is 90, its five...* [inaudible, maybe "squared"] *divide by 2.*
US: *its 25/2.*

Later, US explained this conclusion:

US: *I simply saw it is 12.5, the only variable* [between the two equal sides] *is* α *and when the angle is 90° the sine is 1, and if it is not 90° then the sine value is less than 1 and that decreases the area.*

The calculation they used ("*5 squared divide by two*") corresponds to the formula side-times-side-times-the sine of the enclosed angle which was the most common for them from their trigonometry course. Then they were asked to predict the graph of the area as a function of the triangle's base.

US: *It is growing like ...* [traces with his hands a parabolic shape with a minimum], *... you have to use a theorem...*
MI: *Why not linear?*
I: *A nice question.*
MI: *If it is the base times the altitude over 2 it must be linear... but the altitude decreases and the base increases ... Anyway, that means that is not linear, but it is not parabolic either, I think. ... for a parabola there must be here a quadratic function, and I don't see it.*
US: *He is right, shall I explain?*
I: *Yes.*
US: *I am not sure, just a second... Suppose I define the length of BC as x, so how do I find the area? I divide it by two and draw the altitude, I can do x times the altitude...*

US and MI brought many knowledge resources to this problem: they identified the maximum area correctly and efficiently. Their repertoire of functions was beyond the prototypical ("*It is not linear, but it is not parabolic either*"). The connections between the graph and the formula were properly and spontaneously invoked ("*for a parabola there must be here a quadratic function, and I don't see it*"). They could use another area formula ("*Suppose I define the length of BC as x...*").

We propose that in this part of the session, their actions and discussions were clearly determined by their looking of how to apply what they already knew (which they did and quite successfully). They switched from one formula to another according to the context, and spontaneously discussed why the function is neither linear nor quadratic. However, they made almost no use of the environment and the findings it may suggest as a source of information or insight. One obvious reason for that, is their lack of previous experiences with a dynamic software. But, we claim that another, more subtle reason for not relying on the environment was their implicit views on doing mathematics. Their actions and discussions revealed a "formalistic approach", (e.g. finding quickly what is easy to find -the maximum- via a known calculation, attempts to invoke 'ready-to-go' procedures – "*you have to use a theorem...*" - and proceeding to a "secure" building of a symbolic model, as the only basis to envision the characteristics of the graph rather than from the features of the situation observed on the screen.

Their initial reliance on displaying and using their knowledge and their initial reluctance to use the dynamic environment, was confirmed by MI and US themselves, when they were debriefed after the interview. They mentioned that at school they "simply follow the formulae", sometimes even "without understanding".

In the following exchanges, they gradually began to rely more on the information provided by their experimentation with the software in order to produce qualitative arguments.

US: *Symmetry? I don't think that there will be symmetry, here there will be higher values* [points to the zone in the graph up to the maximum area value*] and here* ["after" the maximum*] it will come down slowly.*
I: *Why do you say so?*
US: *I imagine the rate of change*
I: *You imagine it from seeing it?*
US: *From sort of physical vision, not formula or anything.*
I: *MI, do you agree?*
MI: *It looks like it first grows a lot and then it decreases a lot.*
I: *So you agree with non-symmetry here?*

For the first time, in order to predict the shape of the graph, they abandoned the formulae. Instead, they relied on their visual perception of the area variation in order to predict how the rate of change should be reflected on how the graph should look like. They seemed to agree on the qualitative difference between rates of change on both sides of the maximum. However, MI made a surprising comment which reflected the interference of his symbolic knowledge.

MI: *No, I think it will be symmetry because again if we consider the formula side times side times the sine of the angle between them, then, kind of, if sinus from 0° to 180° is symmetrical. From looking at it as it seems...*
I: *So it looks like that but it is not like that. Is that what you say?*
MI: *Yes. Also usually functions like these are symmetrical. I think on the basis of what...*

Two strong reasons made MI disregard what he had just "saw" and to reject their agreed qualitative conclusion about the asymmetry of the graph. The first is the knowledge of the formula for the area which focus on the angle as the independent variable (in which case the graph is indeed symmetrical) and not the base. From what they said later, we learn that their school experiences did not include calculus problems in which students model the same dependent variable as a function of different independent variables. Thus, in this case, MI could not disentangle what he imagined as symmetrical (a function of the angle) from the situation he is investigating (a function of the base). The second is related to MI's views of their past practices of mathematics, rather than to a dominant piece of prior knowledge: the area functions they have met were symmetrical. Prior knowledge and past school practices "join forces" here in order to reject a qualitative correct observation.

After drawing the graph with the software (see figure 1), the interviewer asked them whether they got what they have expected, and they both laughed:

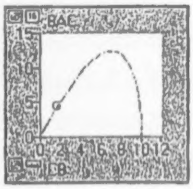

Figure 1

MI: *No, absolutely not.*
I: *Does it make sense?*
US: *I said the opposite... I thought it was non-symmetrical from an absolutely non-mathematical view. I imagined the side varying and I imagined the increase and saw that at first the area increases with a fast rate and when it decreases, it decreases with kind of a slower rate, therefore I thought it is not symmetrical..*
MI: *It does not, it does not make sense... Why it is not symmetrical?*

US focuses on why the asymmetry he mentioned before turned out to be the opposite of what they obtained. Whereas, MI was very puzzled by the asymmetry itself. Later on, when discussed the algebraic representation, he returned to his dilemma:

MI: *I know that in order to get a symmetrical graph, the maximum value ought to be when the angle between the equal sides is 60°, and it is at 90°, but from the point of view of the formula it still does not make sense to me.*

When they obtained the graph for task 2 (Figure 2a below), MI realized that his expectation of symmetry was right, but it referred to the area as a function of the angle and not as a function of the base. MI realized that a dependent variable can be a function of two different independent variables, and thus settled his dilemma. Figures 2b and 2c show the graphs they obtained for tasks 3 and 4. Their expectations were correct and made them feel comfortable with both the environment and their work.

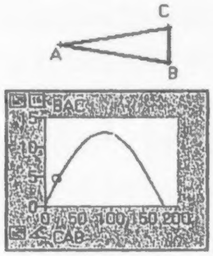

Figure 2a

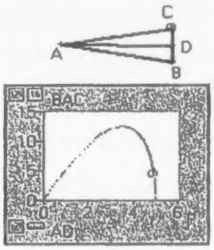

Figure 2b

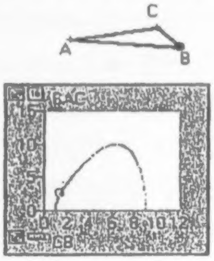

Figure 2c

In task 5 (investigating the area of a scalene triangle of sides 4,5 and x, as a function of the altitude to the variable side), MI and US found another point of confrontation with their previous knowledge and habits of learning presenting another opportunity for the creation of new connections. Firstly, they predicted that the graph would be similar to the three graphs they already saw in the previous tasks.

MI: *why should it be different than before?*
US: *I don't think that it makes any difference if the triangle is isosceles or not.*

But, at this point and in contrast to their work on the first task, they felt the need to experiment before making a final prediction. After discussing the domain of variability of the altitude and the range of the area, US suggested to use the DG tool to measure the altitude AD while changing the triangle. As US dragged the vertices on the screen and drew his predicted graph on the worksheet MI suddenly took the mouse from US, dragged the figure on the screen and looked unsatisfied.

I: *What disturbs you, MI?*

MI: *It is like we have a double domain. You have a certain length* [of the altitude AD] *here and also...*

US: *Something is wrong, it is returning back, the altitude values are growing from 0 till 4 and then, if you continue, it is going back.*

I: *I want to understand, does it change your prediction about the graphs' shape?*

US: *It is not the graphs' shape, I am trying to solve MI's question. It is a function. It is a graph that satisfy some sort of function so it can't be that there are two images for the same pre-image, two different values of the area fitting the same value of the altitude. It simple can't be.*

In order to strengthen the argument, MI measured:

MI: *For 3.12 you have the maximum* [10 square units] *and also this area* [2.18]. [He produced by dragging the two situations on the screen one after the other, as seen in figure 3.]

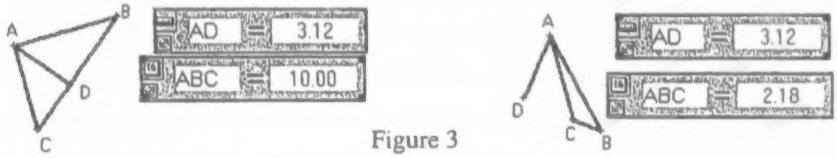

Figure 3

We note that at this point, MI and US used freely the DG environment to challenge their own initial prediction ("*why should it be different from what happens with an isosceles triangle*"). However, what they found seemed unnatural and different from their school experiences with "nice" graphs, let alone those who do not represent functions. (Reluctance from relationships which are not functions is described in detail in Even, 1993.) Thus when the interviewer asked them again if they want to correct their predicted graph, they requested to draw the graph with the software, at once, instead of risking a prediction. Although they already knew that "it will return back" (see figure 4) they were surprised and laughed:

MI: *How come that we didn't guess it.*

US: *It is very surprising.*

MI's comment can be interpreted in at least two different ways. It may mean that given their analysis, they could have guessed the shape of the graph. However, judging from the somehow ironic tone of his voice, he could have meant that this graph is something impossible to anticipate.

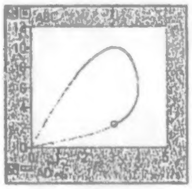

Figure 4

Then they tried to reflect on their prediction and explain the result.

US: *It is as MI said, for the same altitude there are two area values.*

MI: *And this fits what we see here.*

US: *In one case the altitude is inside the triangle and in the other outside.*

But they were still concerned.

MI: *It is impossible to write it as a function.*

US: *I never saw a graph that does not satisfy the definition of function...*

MI: *After what we have said ... it is not so surprising.*

The interviewer suggested using the algebraic representation. As they were used to do so, they had no problems in writing two symbolic expressions for the two situations $[0.5x[\sqrt{(25-x^2)} + \sqrt{(16-x^2)}]$ and $0.5x[\sqrt{(25-x^2)} - [\sqrt{(16-x^2)}]$. When they proceeded to make sense of how these expressions depict the situation, their comments seemed to add another layer of meaning in connection to the geometrical phenomenon.

> MI: *Looking at the graph, the functions meet when the altitude length is 0 or 4. 4 is not the maximum of the function, it is the maximum of the altitude. At 4 it will be the same in the two formulas, the value of* $\sqrt{(16-x^2)}$ *is 0.*

Discussion

We take the data above as an illustration of what it may mean to re-learn a known mathematical topic within a new dynamic environment which creates the need to settle results contradicting expectations. MI and US have not seen these tasks before. However, from their school experience, they were used to explore several types of functions and solve max-min problems by modeling and using calculus tools (e.g. derivatives). Thus the mathematical topic of their exploration was very familiar to them, and the following are some of the characteristics of their re-learning process:

- The unfulfilled expectation of symmetry (in task 1) and the non-function (in task 5) created engaging dilemmas. In task 1, MI's correct envisioning of the variation as a function of the angle, affected what he saw when the variation was a function of the basis. In task 5, their notion (and repertoire) of functions did not allow a non-function to depict a situation. In the first case, their knowledge helped them realize that the same dependent variable can vary according to different independent variables (something they were not used to but were able to understand and experience). In the second case, their knowledge and ability to produce a symbolic model and make sense of it, coupled with their initial qualitative analysis led them to accept the surprising graph as the right representation. In both cases, they enlarged their repertoire of functions.

  Therefore, we can say that the students previous knowledge which seemed to hinder their learning was also the basis of their re-learning. Their knowledge was precisely the basis upon which meaningful dilemmas were created. They were able to integrate the DG tools in order to: (a) produce qualitative arguments, (b) make connections between different representations and (c) resolve their dilemmas and apparent "contradictions".

- The students approach to the first task indicated habits of work and views of mathematics clearly established by their school practices. As their work progressed, they started to use the tool to allow themselves to operate differently from what they were used to. Later, in task 5, they incorporated the dragging and graphing options to support more qualitative reasoning. They began to think about the predicted graph not only in terms of what kind of formula it is tied to, but also on how the graph has to reflect the features of the situation using the dragging and measuring tools offered by the software.

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CHILDREN'S GRAPHICAL CONCEPTIONS: ASSESSMENT OF LEARNING FOR TEACHING

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We report a study of 14-15 year old children's graphical conceptions and misconceptions using a diagnostic instrument developed from the research literature to suit the UK National Curriculum. Rasch measurement methodology was used to develop the instrument with a pilot sample and the final instrument and resulting scale is here evaluated and reported based on the full sample of 425 children. The result is that a hierarchy of responses is confirmed, each level of which is described as a characteristic performance including key misconceptions. We compare this with previous work on graphs and functions, explore a small group of teachers' knowledge and discuss the applications of the work in schools in the present stage of the programme.

Introduction and background

This study builds on previous work on misconceptions in children's graphical thinking, and especially in their interpretations of graphs (Clement, 1985; Even, 1998; Janvier, 1981; Kerslake, 1993; Sharma, 1993). Unfortunately, as Leinhardt et al (1990) said, "of the many articles we reviewed almost 75% had an obligatory section at the end called something like 'Implications for teaching' but few dealt directly with research on the study of teaching these topics" (p. 45). We would add that the 'teaching implications' drawn from research on the psychology of learning mathematics are in any case in general problematic: for many reasons these implications rarely impact on practice. Williams and Ryan (2000) argued that research knowledge about students' misconceptions and learning generally needs to be located within the curriculum and associated with relevant teaching strategies if it is to be made useful for teachers. This involves a significant transformation and development of such knowledge into pedagogical content knowledge (Even, 1998) which requires its own study. In particular teachers need to know at which stages of their development pupils are likely to exhibit the researched misconceptions and errors and where in the curriculum they are relevant. Ryan and Williams (2000) produced such data for errors scattered across the curriculum. The present study develops this work by focussing in depth on the area of 'graphical understanding and interpretation' relevant to years 9 and 10 of the mathematics curriculum, and by connecting the research with teachers' knowledge of these misconceptions. In particular this study:

- developed an instrument from the research literature to assess children's learning and misconceptions on a scale related to their curriculum, which we claim is a prerequisite for transforming this knowledge into professional practice, and

- explored the development of this into an instrument for assessing teachers' pedagogical content knowledge.

The development of the assessment instrument involved the tuning of, or the development of, diagnostic items from the research literature on graphicacy to fit the school curriculum. This developed from an analysis of the key work in the field of children's thinking, identifying items which related appropriately to:

- slope-height confusion: the height is a distracting feature when interpreting the slope (Clement, 1985);
- linearity-smooth prototypes: pupils tend to sketch linear graphs and expect some form or reasonableness, such as 'smooth', 'symmetrical', 'continuous' (Leinhardt et al, 1990);
- the 'y=x' prototype: pupils' tendency to construct the y=x graph;
- the 'Origin' prototype: graphs are drawn through the origin;
- graph as 'picture': many pupils, unable to treat the graph as an abstract representation of relationships, appear to interpret it as a literal picture of the underlying situation (Clement, 1985);
- co-ordinates: pupils' tendency to reverse the x and the y co-ordinates and their inability to adjust their knowledge in unfamiliar situations (Kerslake, 1993);
- scale: pupils prototypically read a scale to a unit of one or ten (Williams and Ryan, 2000).

Originally it was also intended to incorporate the misconception related to misinterpreting time-dependent and time-independent graphs (Hitt, 1998), but the items did not work effectively with our sample in the pilot and this was dropped from the study.

The scaling of the test provides a measure on which pupils, item-difficulty and errors can be located (following the methodology described in Ryan and Williams, 2000 and Ryan et al 1998). The pilot study, previously reported in Hadjidemetriou and Williams (2000), involved:

- interviewing 4 teachers about these items to confirm their curriculum relevance;
- testing 50 pupils, identifying errors and development of a possible hierarchy into a measurement 'scale' using Rasch methodology, and
- interviewing the children who made the expected errors to validate the error categories as significant misconceptions in the children's reasoning which are in accord with the literature.

This paper reports the confirmation of the instrument in the main study of 450 pupils, and the beginnings of our research into teachers' related pedagogical content knowledge (the instrument and report of the pilot will be found at http://www.man.ac.uk/cme/ch).

Method

The two study samples (pilot: n=50 and main: N=425) were all of year 10 pupils in the North West of the UK, whose teachers were interviewed (n=4 and N=12) to check that the test was regarded as fair and valid. The test results were subjected to a Rasch analysis in the usual way, including the coding and analysis of errors on the same scale as the items. (For a summary of this method see Williams and Ryan, 2000: the Rasch scaling is the modern stochastic development of the Guttman scaling model used in the CSMS studies in Hart, 1981). The result is a single difficulty estimate for each item and an ability estimate for each child consistent with the Rasch measurement assumptions, (only 4 mark points fell outside a model 'infit' tolerance of mean square 0.7 to 1.3 in the pilot, and one in the main study). Several items were modified between the pilot and main study, as described in detail in the pilot report in Hadjidemetriou and Williams (2000). The main study data was used also to scale the 'common' errors on the same scale (using the average ability parameter of the children who made the error). We are aware of the debate in mathematics education about the nature of such hierarchies: we accept that there may be serious dangers in fixing such constructs which may make improvements in curriculum and methods difficult. On the other hand teaching in the UK is dominated by a national curriculum which is so structured, and the engagement of teachers in practice requires us to adapt to this.

In addition to the test analyses, we drew on interviews with groups of children about the test items to gain some insight into the cause of the errors and their relation to significant misconceptions. In these interviews children were selected who made the interesting errors and discussions were organised in groups which allowed us to confirm our diagnoses of the children's thinking (along the lines described in Williams and Ryan, 2000). Furthermore we interviewed teachers and asked them to complete a questionnaire in which they were asked to suggest how difficult their children would find the items on the pupils' test, and to suggest likely errors and misconceptions the children would make. This data is used here to explore the validity of the research data on misconceptions and also the state of knowledge of this small group of teachers.

Results

The table below (Table 1) shows the resulting hierarchy of children's performance and thinking. A comparison with CSMS results reveals a comparability between the CSMS levels 1 and 2 and our own levels 1 and 2. The underlined statements in the figure are common to the hierarchy that Kerslake assembled (Kerslake, 1981; Sharma, 1993). However the emphasis is different in our test, because we included many more items which involved interpretation and sketching of graphs within contexts involving understanding of rates of change and associated misconceptions. Therefore our hierarchy at level 3 branches from the relatively common levels 1 and 2.

Interviews with children to validate the instrument

The main purpose of the pupil interviews was validation of the test, in particular our interpretation that the errors in the test are symptomatic of the misconception discussed in the literature. In this section we illustrate with interviews of a couple

Level	**Typical performance at each level (Performance descriptions underlined are descriptors of the parallel levels in Kerslake's heirarchy for comparison.)**	**Typical common errors (Scaling logits are in italics)**
5 *(Logit: 1.6 – inf)*	Sketching complex graphs to tell a story, including non-linear, two part and interpreting discontinuous graphs. Understanding calculation of gradient of a graph (y=4x)	Gradient = x/y instead of y/x *(logit 1.2)*
4 *(0.4 to 1.6 logits)*	Harder interpretation of 'constant rate' graphs. Overcoming the 'graph as picture' misconception by pointwise interpretation. Interpreting the meaning of (0,0) in context. Harder interpolation on y=x-squared. Sketching linear graphs to tell a story.	Linear prototype errors. (in drawing a graph where a curve is expected: *0.5, 0.92, 1.2)* Constant rate graphed as y=x prototype *(0.6)*
3 *(-0.5 to +0.4 logits)*	Parallel graphs have the same gradient, speeds interpreted as slopes: same speeds are drawn parallel on graphs. Understands varying slope of a curve (eg y=x-squared) and rate of change in an interval. Compares y-ordinates of two graphs in context. Distinguish slope from height. (Graph and its algebraic expression… not in our test)	Unit (or tens) prototype for scales *(0.2 and 0.4 logits)* y=x prototype (graph is linear and through the origin) *(logit 0, 0)* graph interpreted as picture *(-0.2)* slope-height confusion in context and out of context *(-0.2, -0.2, -0.3)*
2 *(-1.5 to –0.5 logits)*	Reading coordinates off a graph by interpolation and extrapolation. Recognise the slope as rate of change in interpretation of graphs of y on x: eg negative slope is decrease and steeper slope is greater rate of change than shallower. Use of scales in graph reading, interpretation of simple travel graphs.	Confusion of axes (*logit –0.5, also 0.1* a harder case where Sq-root 6 = 36) Misinterpretation of the origin (0,0) in context *(-.55)* Reversal of coordinates *(-0.7, -0.8)*
1 *(-Inf to – 1.5)*	Understanding of coordinates (interpret in context), and change or no change and 'steepness' of a graph. Use of unfamiliar coords.	
	Performance of children at higher levels includes those indicated for lower levels.	**The errors listed are most likely to be made by children at the level adjacent or below**

Table 1: Hierarchy of performance and errors

of children's slope-height confusion in relation to their errors in interpreting a graph of the growth of girls and boys in their teenage years. From the graph presented it can be concluded that the girls at age 14 are bigger (graph height) but the boys are growing faster (graph slope). The item (question 7 of the test) was developed from one of from Janvier's, and is designed to probe for slope-height confusion:

INT: So 'which group is growing faster at the age of 14'. You have 'girls' and why did you choose girls?
Nicole: Went up 14 and it's more than the boys.
INT: It is more…
Nicole: It is more than the boys at the age of 14.
INT: Right, and what do you understand when I am asking you 'which group is growing faster at the age of 14'?
Nicole: Which one is growing faster, which one is heavier.
INT: Which one is heavier. All right, is that what you understand Sara?
Sara: Yeah yeah
INT: Which group is growing faster? You went for the one which is growing, who is…
Sara: Heavier
INT: Heavier. So you put girls.
Sara: Yeah

Later when asked to interpret the curve of the graph for boys' growth, Sara effectively explained that their growth was slow up to the age of 12 and then grew fast, then stopped:

Sara: They grow quite quickly
INT: Quickly? Till which age do they grow quickly?
Sara: 'Till about 12
INT: 12. And then what happens after 12?
Sara: They are growing even quicker.
INT: Even quicker. And then?
Sara: The line just like … stops.

The interviewer subsequently confirmed that both girls Sarah and Nicole thought that the 'girls' were growing faster and would not change their minds. It seems clear that the classic 'slope height' confusion operated, in that the height of the graph serves as a powerful distractor in interpretation of the graph, leading to the error we sought to confirm.

Interviews and questionnaires with teachers

In the main study the test was given as a questionnaire to the teachers with instructions that they should record their perception of the difficulties of the items on a Likert scale, and suggest misconceptions students might have that would cause difficulty. We built a rating scale from these data and the item-perception-difficulty measures that resulted were correlated with the children's actual difficulty as estimated by the test analysis (rho = 0.395). In addition we sought to confirm the teachers' responses through informal interviews, where we also began to explore their teaching practices.

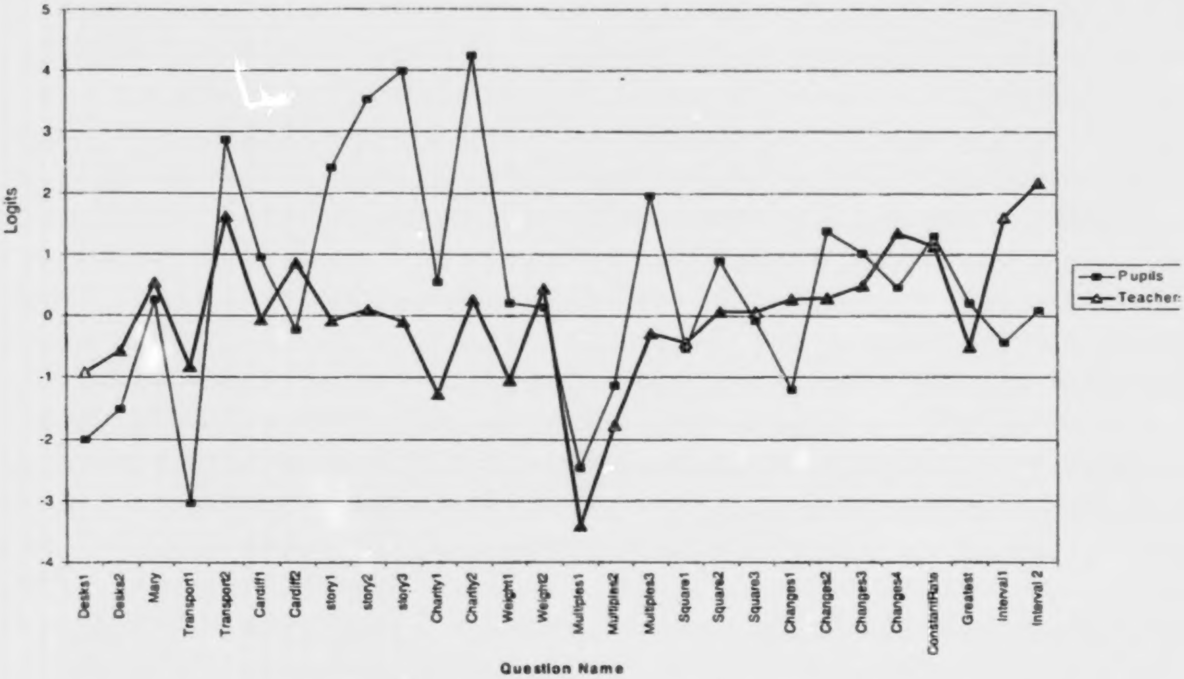

Figure: Teacher estimate and actual pupil difficulty

However, the teachers' estimates were significantly awry on a number of items (see figure, in which teachers' ratings of difficulty were scaled on a rating scale analysis, and plotted against 'actual' scaled values of the pupils difficulties).

The 'discrepant' items were examined for face validity and found perfectly acceptable as test items. However, the teachers' mis-estimation of their (relative) difficulty could be explained by one of two reasons:

(a) in at least three items the teachers underestimated the difficulty for the children because they apparently misunderstood the actual question themselves, i.e. they had the misconception the item was designed to elicit, or they had a limited understanding that did not receive full credit; or

(b) on two items the teachers' overestimated the difficulty because they did not realise the children could answer the question without a sophisticated understanding of gradient.

In the questionnaire and interviews, the teachers were encouraged to list the misconceptions that children might exhibit. Here we summarise the misconceptions mentioned by the 12 teachers we worked with:

Teacher Misconception	1	2	3	4	5	6	7	8	9	10	11	12
Slope height	✓			✓	✓	✓			✓			
Linearity											✓	
Y=X prototype					✓				✓			
Origin prototype					✓				✓			
Picture as graph	✓		✓		✓	✓			✓	✓		✓
Co-ordinates			✓		✓	✓		✓	✓			✓
Scale	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		

Table 2: Misconceptions mentioned by 12 teachers in interview or in the questionnaire

Conclusions and discussion

We have developed an instrument and a hierarchy describing children's graphical thinking and misconceptions which respects their curriculum and is regarded as valid by the, admittedly small, sample of teachers involved. The hierarchy summarised in the chart in this paper suggests how more and less sophisticated pupils behave with graphs and what their main misconceptions are. This is linked approximately at the lowest levels with the hierarchy for graphs described by Kerslake, but covers the literature on graphical interpretation.

The evidence causes us to doubt whether teachers are aware of the common misconceptions the instrument reveals in this field, and we believe that many teachers would benefit from using the instrument in their teaching. In our current work we are investigating this. In small numbers of observations of lessons, we have not yet seen teaching which takes account of the major misconceptions spontaneously.

We are aware of the criticism that the misconceptions children exhibit may arguably be strongly associated with the particular problems they are presented with and the tools they are given to handle them with. Indeed Roth's work (see http://www.educ.uvic.ca/faculty/mroth/Papers_Available) has shown that even expert scientists exhibit misconceptions when presented with tasks which demand interpretation in contexts outside their familiar experience. We agree also with Ainley (2000) that graphical work in general and interpretation skills in particular become transparent or fused when children embed them in their activity or social practice. This is entirely consistent with our other work (eg, Wake et al, 2000; Williams et al, in press) which suggests that experts interpret and use graphs effortlessly in their daily practice, when the graph as a semiotic tool is fused with its interpretant. Nevertheless we believe that learning to interpret graphs in new contexts is an important, if demanding skill, which requires its own practice. The difficulty of research on mathematics in use is that only rarely do we see mathematics actually being learnt rather than used.

We therefore persist in believing that work in academic settings which purports to represent unfamiliar situations is a valuable part of the curriculum, and that children's difficulties therein need charting and pedagogical attention. In the next stage of our work we will examine more closely teachers' use of such diagnostic instruments and how they might develop their practice in this respect.

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COMPOSING ANALOGICAL WORD PROBLEMS TO PROMOTE STRUCTURE ANALYSIS IN SOLVING ALGEBRA WORD PROBLEMS

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A short instructional unit was constructed to promote awareness of problem structure. The instructional unit included problem mapping, schema abstraction and analogical problem composition. The unit was tried with students and teachers. In this paper we report the results for 75 students, in 8th grade and 11th grade. It was hypothesized that following instruction students would improve in constructing analogical problems and would be less affected by context in sorting problems. The change in the sorting task was mostly in the expected direction but was not significant. Both age groups improved in problem construction but the change was significant only for the 8th graders.

Word problem solving was considered an Achilles' heel in mathematics education. In the last decade, however, it is perceived as an opportunity to acquire understanding of mathematical structures.

Traditionally, algebra problems were organized in topics such as transportation problems, or work problems (Mayer, 1981). These methods resulted in the categorization of problems by context in students' minds. Nowadays math educators promote the development of meaningful knowledge through the construction of connections between mathematical ideas (Hiebert and Carpenter, 1992). These links can be achieved, according to English (1997), through a process of analogical reasoning.

Analogical reasoning in problem solving occurs when previous problem solving experience (source) is transferred to a new problem solving situation (target). Such transfer requires, first of all, good understanding of the structure of the source problem (Gentner, 1983). As it turns out, this is not sufficient and more conditions have to be met. The transfer process includes three crucial and related actions that have to be performed: recognition of a potential problem to transfer from, abstraction of source problem structure, and mapping of corresponding elements and relations in the source and target problems.

Quite a lot of researchers detail conditions that promote transfer. Some of these works suggest, for example, that the abstraction of a general schema is facilitated by comparison and mapping of several examples in different contexts, and by the use of a combination of simple and complex examples (Gick and Holyoak, 1983; Reed and Bolstad, 1991; English and Halford, 1995).

In this work we try to promote the abstraction of problem schema by asking children to compare elements of pairs of problems, and requiring a generalization of problem components. We then ask them to compose analogical problems of their own. The task of composing a problem subject to given constraints is usually used for detecting students' conceptions (Rowell & Norwood, 1999). We use it here as a part of the instruction unit and view it as an important tool in encouraging focus on structure.

It should be noted that *structure*, much like *schema*, has different definitions. Reed (1987) defines similar-structure algebra problems as problems that have the same equation. Weaver and Kintsch (1992) suggest that equations play a lesser role and that the dominant factor is some deeper conceptual structure.

In this work we used problems that have the same equation and the same conceptual structure as examples of similar-structure problems. We use the terms *analogical* and *isomorphic* interchangeably to denote similar-structure problems. The construction of problem space and similarity tasks that test whether context or structure affect children's conceptions was based on ideas from Reed's (1987) work on a structure-mapping model.

METHOD

The study was conducted in the course of the school year. It took 6-7 class sessions and consisted of three parts: a pretest, an instructional unit, and a posttest. Both pretest and posttest included a problem sorting task and an analogical problem composition task.

The purpose of the sorting task was to identify the criterion children use in categorizing problems. Students were given 12 problems created by the product matrix of 4 context categories: transportation, work, pipes, and mixtures, and 3 different problem structures.
Using Reed's (1987) terms, three types of relations exist between these problems:
Isomorphic (or analogical) problems – same structure, different context
Similar problems – different structure, same context
Unrelated problems – different structure, different context

According to the research hypothesis children were expected to sort by context in the pretest. A shift towards sorting by structure was expected in the posttest as an outcome of applying the instructional unit.

A problem composition task, given several times along the study, checked children's ability to write an isomorphic problem to a given (or composed) source problem. The composition task had a significant role in the instruction. It was expected that children would perform better in this task in the posttest, and that this would be part of the explanation for the change in the sorting task.

The study sample consisted of 100 participants, studying or teaching mathematics, including: 75 students (61 students in 8th grade and 14 students in 11th grade); 12 student-teachers in mathematics; 13 mathematics teachers in middle school and high school. In this paper we report the results for the 75 middle school and high school students. We focus on the analogical problem composition task, and discuss the changes in students' performance.

The instructional unit was especially designed for the study and consisted of different types of activities:

1. Mapping between two given problems
2. Abstraction of problem components
3. Practice in writing analogical problems
4. Matching a general schema to a given problem

Examples of some of these tasks are detailed in the following section, as a part of the description of individual children's performance.

RESULTS

We will present quantitative results and proceed with examples that demonstrate the change in problem composition.

In the pretest sorting task students identified (put in the same category) more similar problems than isomorphic problems as being related. This trend was significant in the 8th grade group. Following instruction the 11th graders identified fewer connections between similar problems than they did before instruction, and both groups identified more connections between isomorphic problems than they did before instruction. These changes are in line with our hypothesis, but they are not significant.

Table 1 presents percentages of students who successfully composed an isomorphic problem in the first composition task (a simple problem given in the pretest), and in the last task (a complex problem given in the posttest).

Table 1: Isomorphic problem composition before and after instruction.

	Pretest	Posttest
8th grade	37%	71% *
11th grade	67%	80%

* significant difference

A comparison of problem composition in the pretest and posttest shows a significant change for 8th grader, where the difference in performance was 34%. In the 11th grade the difference in performance was 13% and non significant. However, as will be discussed later, the problem composition tasks included problems that differed in structure. As the student progressed, the structure for which he was asked to build an analogical problem became more complex.

It is interesting to note that the change in 8th grade occurred mainly for the girls. In the pretest 17% of the girls and 50% of the boys could construct an isomorphic problem. In the posttest 65% of the girls and 75% of the boys could compose an isomorphic problem.

The shift in ability to construct analogical problems can be seen in the work of an 8th grader and an 11th grader. It should be noted that with the students, for instructional purposes, we used (and defined by examples) the term *similar problems* instead of *isomorphic problems*.

Hilla's case (8th grade):

Table 2: Hilla's first problem composition task.

task	Source (given)	Composed by Hilla
Pretest: compose a similar problem using machines that produce parts.	*Two persons started walking one towards the other on foot and at the same time from 2 different cities, the distance between which is 96 km. One person was walking at the rate of 5 km/h, and the second at the rate of 3 km/h. How much time will it take before they meet each other?*	*Two machines in a factory that produces parts were operated at the same time. One machine works at the rate of 25 km/h and the other works faster than the first by 5 km/h. I have no idea how to continue this.*

During the instructional stage Hilla improved gradually and managed to compose a part of the analogical problems correctly. In the last task, which presented a complex source problem, she exhibited good structure analysis (as seen in Table 3).

Table 3: Hilla's last problem composition task.

	Source	Composed by Hilla
Posttest Task: write a similar problem with different context (not specified)	*When a student reads a favorite book at the rate of 50 words/min it takes him a certain amount of time. Once he read for 8 minutes at his usual pace and then increased it to 60 words/min and finished 2 minutes earlier than usual. How much time does it usually take him to read the book?*	*A cyclist usually rides at the rate of 10 km/h. After 12 minutes of riding at his usual pace he increased it to 14 km/h. How much time did he ride if we know that he arrived 4 minutes earlier?*

Hilla is also able to specify the similarity between the problems: *They both deal with the same thing that starts at a certain rate and changes to a new rate.*

Stav's case (11th grade):

Table 4: Stav's first problem composition task.

task	Source (given)	Composed by Stav
Pretest: compose a similar problem using machines that produce parts.	*Two persons started walking one towards the other on foot and at the same time from 2 different cities, the distance between which is 96 km. One person was walking at the rate of 5 km/h, and the second at the rate of 3 km/h. How much time will it take before they meet each other?*	*Two machines in a computer factory produce CD-s. One produces 4 records/hour and fills up a CD case in 16 hours It works 24 hours and then rests. The other machine produces 6 CD/h and it too works for 24 hours. How much time will it take the machines to fill up the same amount of cases?*

Stav's composed problem, presented in Table 4, is different in structure from the given problem. Stav does not really use the simultaneous work of the machines. There is no focus on a certain amount that is produced by the two machines working together.

The main change occurred on composing one of the problems in the instructional session task presented in Table 5.

Table 5: Stav's problem composition during instruction.

	Source (given)	Composed by Stav
Task: Compose a similar problem.	Back and Forth *A person leaves the city of Bally riding his bikes for 2.5 hours at the rate of 20 km/h he gets to the city of Gat. He does the way back on foot and walks for 10 hours from Gat back to Bally. What is his walking rate?*	*Yesterday a student was preparing his homework at the rate of 16 problems per hour for 3.5 hours. At what rate did he work today to finish the same amount of problems, if it took him 6.5 hours?*

Stav solved his own problem by writing and calculating: (3.5x16)/6.5=8.6 and writing the answer in words: The student solved 8.6 problems per hour.

While Stav managed to perform well on the Back and Forth (see Table 5) problem, another 11th grade student, Jenny, exhibited some interesting difficulties. Jenny wrote: *A student prepares his homework in the afternoon and checks his work again in the evening. If he prepares his work at a rate of 1 problem per minute it takes him 10 minutes. But when he checks his work at the rate of 2 problems per minute it takes him 5 minutes. How many problems were there?*

Jenny wrote: 5x2xX=10x1xX She realized that the same person is doing two similar actions, each at a different rate. She also used (correctly) the same total production: in both cases the same amount of problems is used. However, she did not have a good mapping strategy for creating an analogical problem, as a result she had trouble in choosing the unknown.

In the above examples students composed an isomorphic problem to a given problem. Another type of task involved mapping problems. Students were given examples of pairs of isomorphic problems, and asked to map their elements. Following several examples they were asked to compose a pair of isomorphic problems and map their elements. In this task Sagi, an 8th grader, wrote the problem pair presented in Table 6.

Table 6: Sagi's composed pair of isomorphic problems.

Sagi's 1st problem:	Sagi's 2nd problem:
The Truck: A truck travels on a 100 km track. It travels in the rate of 50 km/h. How much time will it take it to finish the track?	The Fisherman: A fisherman has to catch 100 fish. He catches them at the rate of 50 fish per day. How many days will it take him to catch 100 fish?

Sagi map of the corresponding problems' elements is detailed in Table 7.

Table 7: Sagi's problem mapping.

The Truck	The Fisherman
distance -	total number of fish
rate of driving -	rate of catching fish
number of hours -	number of days
In this problem we calculate: Distance/ rate = time -	In this problem we calculate: Amount/rate = time

Sagi's example demonstrates very good performance of a young child on a problem composition task, which, as will be further discussed, is not an easy task for students (or, in fact, also for teachers).

DISCUSSION

Using a short instructional unit this research tried to change students' focus in problem solving from context-centered to structure-centered. The instructional unit included several ideas suggested by different researchers, such as mapping between problem elements and comparing given examples. In addition to these, we tried another task, different in nature, but also more difficult, the task of writing an isomorphic problem. As it turned out, when problem structure was simple, such as a rate x time = production, a third of the 8th graders and two thirds of the 11th graders could compose it with some instructional guide. When problems became more complex, it apparently became more difficult to perceive a complete picture of problem structure and compose a problem of the same structure. This is not surprising in light of what is mentioned in the introduction, that understanding of the source problem structure is a necessary condition for analogical reasoning.

Although the tasks were difficult, some change in the direction of more attention to problem structure and less to context similarity occurred. The students' awareness of structure was also expressed in class discussions, and brought up in a follow up questionnaire.

The instructional unit together with the pretest and posttest took very few class sessions, and can be considered as an exploration of the power of the unit's task. In view of the effect of this short experience, we suggest to use these tasks and construct a more comprehensive program.

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SPONTANEOUS TENDENCY TO FOCUS ON NUMEROSITIES IN THE DEVELOPMENT OF CARDINALITY

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In this paper we report on a longitudinal study where we investigated whether there are differences in 3 – 4 year-old children's tendency to focus on numerosities, and whether these differences are related to the children's development of cardinality. It was found that the two groups of children at the age of three formed on the basis of their spontaneous tendency to focus on amounts of objects differed in their development of recognising and producing small amounts. Those children who spontaneously regarded the numerosities as relevant factors in the test situations developed faster in cardinality related skills than those children that focused on other features in the tasks.

Introduction

Recent research in cognitive, comparative and developmental psychology supports the position that infants have biologically primary quantitative abilities, which encompass their implicit understanding of numerosity, ordinality, counting and simple arithmetic (for a review, see Geary, 2000). However, in these studies on infant abilities there seems to regularly appear a small group of children who do not present these quantitative abilities at all.

In the research on the development of exceptional skills it has been established that the amount of deliberate practice as well as the age at which it is started is related to the level of performance (Ericsson & Lehmann, 1996). The child's social environment influences the ways in which he/she engages with the mathematical world around him/her. Also the differences in opportunities the child finds in his surroundings to practice pre-mathematical skills might explain the differences in the development of early mathematical skills of children. The world could appear to be full of numerosities and opportunities for practising early mathematical skills to some children, whilst others focus on other features in the environment and involve themselves much less with pre-mathematical ideas. According to Ginsburg et al.'s (1999) observational studies, 5 year-old children deal with mathematical activities as much as 45 percent of their time of free play in day care, of which 11% is enumeration.

In the development of cardinality a child moves from innate, preverbal counting system called subitizing to fluent usage of traditional counting in determining numbers of objects. Counting skills enable the child to recognise and produce bigger amounts than innate capability of subitizing does. "The cardinal number word refers to the whole set of entities and tells how many entities there are, that is, describes the manyness of the set" (Fuson, 1988, 5). The concept of cardinality grows out of children's experience with counting in a manner similar to that in which a symbolic concept of print grows out of children's experiences with the alphabet. Cardinality requires both explicit knowledge of the relation between numbers and quantity, and the attentional procedures for focusing on that which is counted (Bialystok & Codd, 1997.) According to Schaeffer, Eggleston, and Scott (1974), cardinality results from the integration of two prior processes: subitizing and counting. Among other primitive processes, subitizing provides the basis for acquisition of the number concept and orients a child to the numerosities of sets (s)he experiences (English & Halford, 1995, 60). Subitizing, or enumeration of small sets up to about four elements, is a rapid and accurate pre-attentive process, which requires no explicit teaching, and possibly little or no experience (Gallistel & Gelman, 1978; Sathian et al., 1999).

A major step in early mathematical development is learning to count. Counting enables the child to make quantitative determinations of amounts, rather than relying on perceptual or quantitative judgements. Cardinal situations and counting are, at first, separate and different situations for children, but are gradually connected when children understand that counting is not an isolated activity, but has a result (Fuson, 1988, 206.) The remarkably slow development of cardinality and counting skills (e.g. Fuson, 1988; Wynn, 1990) could be explained by major differences in preverbal and verbal counting systems (Wynn, 1992a, Wynn 1992b). One potential explanation for the differences in pre-schoolers mathematical skills (see, e.g Reusser, 2000) might be children's different amount of spontaneous dealing with mathematical activities.

In the present study, we followed up children from the age of 3 to the age of 4 years in terms of spontaneous focusing on numerosities and cardinality. This study investigates whether there are differences in children's focusing on numerosities, and whether the groups formed on the basis of differences in focusing on amounts differ in their development of cardinality. Our hypothesis is that the differences in spontaneous focusing on amounts might indicate the children's overall tendency to focus on numerosities in his/her surroundings, and this could cause considerable differences on the amounts of practice in enumeration and counting skills. So, those children who do not regard numerosities immediately as relevant factors in the tasks would not develop as quickly in recognising and producing small amounts.

Method

Participants

39 children (18 girls and 21 boys) from the city of Turku participated in this one year follow up study. The children's mean age at the start of the follow up was 2 years and 11 months (s=1,5 months). The children were tested for the second time when they were on average 3 years and 5 months (s=1,5 months) old, and for the third time when they were on average 3 years and 11 months (s=1,5 months) old. The children attended seven day-care centres located in the middle-class areas of Turku at the beginning of the follow-up period.

Procedure and tasks

The video-recorded tasks were presented individually in a familiar room of the child's day-care centre during the morning. The carrot task took approximately 5 minutes and the caterpillar/pig task about 10-15 minutes, depending on the child's skills.

Spontaneous focusing on numerosities was assessed at the beginning of the follow up by the carrot task and the spontaneous focusing section of the caterpillar task. *Cardinal tests* were presented at the age of 3, 3½ and 4 years. At the beginning and in the middle of the follow up period the cardinal section of the caterpillar task was presented, and in the end of the follow up a modified but parallel version of the caterpillar task, called the pig task, was performed.

Carrot task

The experimenter placed two similar cuddly toys in the form of "bunnies" and then a plate of 5cm long "carrots" on the table in front of the child. The bunnies and the carrots were identified together with the child. The experimenter asked the child, "Look at what I do. I do this, this. Now, you do what I did." While saying "this", the experimenter lifted two carrots, one at a time, into a row in front of the experimenter's bunny. The child imitated the experimenter as well as (s)he could and placed carrots in front of his/her bunny.

The amounts of carrots in the items were 2, 1, 3, 4, 5, ...10. If the child did not lift as many carrots to his/her bunny as the experimenter did, the failed quantity was repeated. After two failures with the same quantity, the experimenter returned once again to the previously successful quantities. In every case the first items with two, one and three carrots were presented to the child.

Caterpillar task

The materials in the caterpillar task were a black sack, a small (17cm x 14cm) box, ten sewed green fabric "caterpillars" (length 60cm), who had either 1,2,3... or 10 legs (length 6cm) every 2.5cm. Feet were 3.5cm in length. The one-legged caterpillar had its leg in the middle, the two-legged caterpillar had a leg in the middle and a leg 2.5cm from the other leg. This was the way in which all the caterpillars' legs were situated. There were also 24 red socks in the small box, which were elastic and suitable for the caterpillars. In the task, a child sat at a table. The experimenter sat to their left, and the sack that contained the caterpillars was on the experimenter's left. The box with the socks was placed on a tall stool on the opposite side of the table at the beginning of the task.

The caterpillar tasks did not to assume that the child would use any verbal number words. The child had to determine the number of objects (legs on the caterpillar), keep it in mind for few seconds and pick up the same amount of objects (socks) and bring them to the original objects (legs). The tasks enabled the child to spontaneously use either subitizing, estimating or counting to solve the items. The child could check the original number of legs on the way whilst picking up the socks, if s/he wanted to. The location of the caterpillar and the box of socks prevented the use of one-to-one correspondence. The child could not easily see the legs of caterpillars when standing next to the box of socks.

In the spontaneous focusing section of the caterpillar task the experimenter asked the child what clothes (s)he put on when (s)he goes out (e.g. hat, gloves, shoes and socks). Then the experimenter told the child about the sack, in which there lived a family of friendly caterpillars. The caterpillars were about to go out, but first they needed to put on socks. After introducing the first 6-legged caterpillar, which already had socks on, the experimenter showed, by pointing to the legs one at a time, that there was a sock on every foot of the caterpillar. The child was told that was the way that the other caterpillars should be dressed too. The 6-legged caterpillar "waited" in the sack and the 2-legged caterpillar was lifted from the sack. The child was asked to bring the caterpillar as many socks as the caterpillar needed from the box on opposite side of the table. All the socks that the caterpillar needed were to be brought in one go. After bringing the socks, the child was asked if (s)he had brought the correct number of socks. Then the experimenter put the socks on the caterpillar and asked if the caterpillar was ready to go. If the child did not bring exactly two socks for the caterpillar, the item was repeated to make sure that the child considered the amount of legs as a relevant factor in the task. The spontaneous focusing on amounts was assessed from these two trials in the first section of the caterpillar task.

The cardinal section of the caterpillar task was conducted immediately after the spontaneous focusing section by advising the child to bring exactly as many socks as the caterpillar needed if the child had not brought the right amount of socks for the caterpillar. This was to make sure that all the children understood that it was necessary to focus on the amount of legs and socks in this task.

The caterpillars were presented in the order 2, 1, 3, 4, 2, 5, 6, 1, 7, 8, 2, 9, and 10, where the numbers represent the number of legs of the caterpillars. The one- and two legged caterpillars were presented between the sequence of test caterpillars for two reasons: to motivate and relax the child with easier tasks, and to break the growing order of test caterpillars. If the child did not bring the right number of socks for the caterpillar, the failed amount was repeated, and if the child failed twice with the same amount, the previous successful amount was repeated. The amount at which the child got both attempts right determined the level of the child's performance.

Pig task

The materials in the pig task were a flat box, where the sows stayed, a small (17 x 14cm) box for pigs, 12 plastic sows (length 60cm), which had either 1,2,3…or 12 teats (length 3.5cm) every 2cms. The teats were located similarly to the legs of the caterpillars. There were also 24 piglets in a box on the opposite side of the table. The child and the experimenter sat as in the caterpillar tasks at a table.

In the pig task, the experimenter told the child about the sows that stayed in a box, and needed help. The child was asked, if s/he knew how sows feed little pigs, and the 6-teated sow, who nursed six pigs was introduced. The experimenter showed that this was the way in which sows feed their pigs: a piglet suckling at every teat. The children were instructed to help the other sows feed their piglets. The 6-teated sow went back to feeding her piglets in the box, and the 2-teated sow was lifted from the box. The child was asked to bring as many piglets as the sow could feed from the box on the opposite side of the table. The pig task continued in a manner similar to the caterpillar task. The maximum amount of teats the sow had was 12.

Analyses

It was carefully observed if the child immediately focused his/her attention on the numerosities of objects instead of other features in the first items of the tasks. The child was considered as spontaneously focusing on numerosities, if (s)he produced the same number of carrots as the experimenter had produced, or if (s)he immediately brought the same number of socks as there were legs on the caterpillar in the first items of the caterpillar task. The child was considered as not spontaneously focusing on amounts if (s)he did not focus on the amount of carrots, and either tried to imitate the way in which the experimenter lifted the carrots, or concentrated on feeding the bunny. Consequently in the caterpillar task the lack of spontaneous focusing on amounts appeared if the child did not pay attention to the number of legs and socks and brought either a handful of socks or just one sock for the first 2-legged caterpillars.

Cardinality was analysed from the cardinal section of the caterpillar task and the pig task. The carrot task could not be used because those children who did not focus on the number of carrots in the task obviously considered the whole task as an imitation of movements or feeding task instead of producing the same number of carrots for the bunny. After the child's first attempts in the caterpillar task, it was made clear to the child that the number of socks was a relevant factor in the task.

The level of a child's skills of recognising and producing numerosities was determined through successful trials in bringing socks for the caterpillar. The child had to succeed twice with the same number of legs to be considered as capable of producing the given number of socks. The task was finished after two failures at the same amount.

Results

Spontaneous focusing on numerosities

The children of this study had differences in spontaneous focusing on numerosities. There were 14 children who did not spontaneously focus on numerosities in either one (11 children) or both tasks (3 children), called the *non-spontaneous group*, and 25 children who focused immediately on numerosities in both tasks in the beginning of the follow up, called the *spontaneous group*. Four children did not focus on the amount of carrots in the carrot task, and 11 children did not focus on the numbers of legs and socks in the caterpillar task. Those children who did not focus on the amounts either imitated the movements of experimenter, concentrated on feeding the bunny, or brought only one sock or a handful of socks to the 2-legged caterpillar. In the caterpillar task, 10 children out of those 11 who did not focus on the amount of legs and socks in the first item started focusing on the number of legs when advised, and proving that they were capable of recognising and producing the amounts of one and two.

Cardinality

To examine the question of whether the non-spontaneous group of children differed from the spontaneous group in their cardinal skills during the follow up period we investigated their achievements in cardinal tests (see Figure 1.).

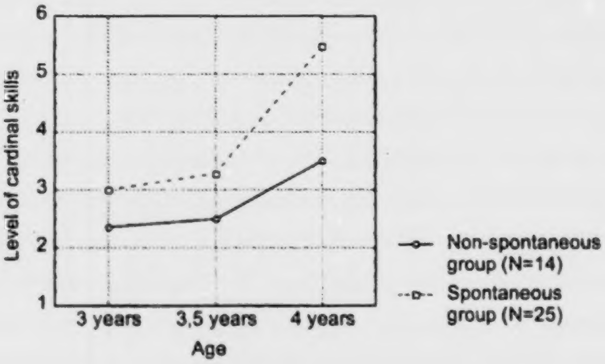

Figure 1. The results of the non-spontaneous and spontaneous groups in cardinal tests in the age of 3, 3½ and 4 years.

Separate 2 x 3 (group x age) ANOVAs with repeated measures were performed for their performance, together with Tukey's HSD tests, from 3 to 4 years of age. The main effects of group (F(1,37)=7,18; p=0.011) and age (F(2,74)=21,20; p<0.001) were statistically significant. The non-spontaneous group displayed weaker skills in cardinal tests and the children's skills developed during the follow up period. Symptomatically significant (F(2,74)=2,92; p=0.060) interaction of Group x Age in performance suggest that there were developmental differences in cardinal skills between the groups. The spontaneous group developed faster than non-spontaneous group in recognising and producing small amounts.

In spite of the slight trend of differences in the groups' performances in favour of the spontaneous group, Tukey HSD tests revealed that there were no statistically significant differences in the groups' results at the age of 3 and 3½ years. However, after 12 months, group differences began to appear. The cardinal skills of the non-spontaneous group were weaker (p<0.001) when the children were 4 years old. The results of group comparisons were confirmed by Kruskall-Wallis's non-parametric test.

Discussion

The children of this study had differences in their tendency to focus spontaneously on numerosities. The spontaneous group of children (25 out of 39 children) interpreted the carrot and the caterpillar task instantly as numerical and focused their attention to the amounts in the tasks. Those children (14 out of 39) who did not spontaneously focus on the numerosities in the tasks when they were three years old, developed more slowly in cardinality than those who immediately perceived the task situations in terms of various numbers of objects or events. The spontaneous group outperformed the non-spontaneous group at the age of four years unlike than at the age of 3 and 3½ years in their skills to recognise and produce small amounts.

According to the results, it seems that there are significant differences in the ways in which children pay attention to numerosities in situations and these differences in the amount and the quality of spontaneous activity might be of great importance for the children's later development of number concept. The fact that all the children except for one in the non-spontaneous group were able to bring the 2-legged caterpillar two socks when advised to focus on the amount of legs in the caterpillar task, supports the idea that spontaneous focusing on numerosities is a separate process from enumeration. It is possible that every child manages to deal with numbers within their subitizable range once they realise that number of objects in the situation is a relevant feature for his/her action. What causes these differences in the ways that children interpret their perception is a subject for a later study, where the follow-up of children will begin earlier than in this study. This should be investigated because this study raises the question of whether the differences in the spontaneous tendency to focus on numerosities produce differences in the development of cardinality, or vice versa. There is also a question about the nature of spontaneous tendency to focus on numerosities: is it inherently different in children, or as a result of social support? Could there be such differences in the linguistic skills or subitizing range of the children which would explain the different development of cardinal skills?

In Hannula's (2000) study it was reported that the spontaneous group of this study tried to use more counting to solve the caterpillar task at the age of three years (64% of the children) than the non-spontaneous group of children (21% of the children). Considering the long time (e.g. Fuson, 1988; Wynn, 1990) that counting skills take to develop accurately, it is plausible that the spontaneous group of children was further in their development of counting skills at the beginning the follow up, though the levels of children's performances did not differ. It is a common phenomenon in the development of skills that the level of performance does not rise immediately, but after a training phase, when a more advanced and demanding strategy is mastered. In this case the spontaneous group of children would have focused on numerosities more often, because the quantities of objects had become more meaningful to them. Even so, the differences in the skills of spontaneous and non-spontaneous groups in the beginning of follow up do not eliminate the possibility of spontaneous numerical tendency to have a significant role in the development of cardinality. The effects of differences in the amount of training quantitative skills produced by spontaneous activity could still derive to different developmental profiles in early mathematical skills. Therefore, our preliminary hypothesis for future studies is that spontaneous tendency to focus on numerosities serves as one of the building blocks for the child's early numerical development, possibly besides innate quantitative abilities.

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SELF-ESTEEM AND PERFORMANCE IN SCHOOL MATHEMATICS: A CONTRIBUTION TO THE DEBATE ABOUT THE RELATIONSHIP BETWEEN COGNITION AND AFFECT

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This research work was aimed to analyze the relationship between cognitive and affective aspects in the particular context of mathematical education. Self-esteem was chosen to represent affective pole involved in this process, while mathematical performance at school was focused in terms of performance in a specific mathematical evaluation test. Data collected suggest empirical evidence for a connection between the level of self-esteem, interaction patterns and mathematical performance inside the pairs, high-level self-esteem pairs with cooperative patterns of work performing clearly better than low self-esteem level subjects. These data seem to reinforce the idea that it is not possible to split children difficulties at school in a dichotomy of cognitive and affective aspects.

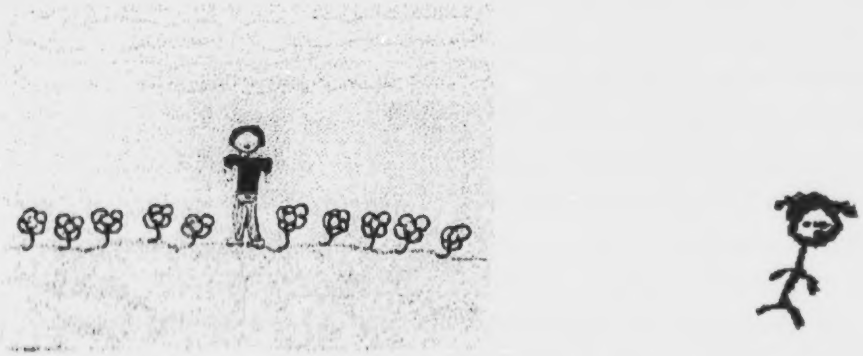
1. Introduction

The theoretical question about the place of affectivity in mathematical sense-making activities emerged strongly in previous research efforts of our research group (see Da Rocha Falcão and cols., 2000; Brito Lima & Da Rocha Falcão, 1997; Da Rocha Falcão, 1996), and is the main motivation of this research. These two poles of human psychological functioning cannot, of course, be scientifically addressed in such generic terms; because of this, the first step in operationally building this research enterprise was the translation of "affectivity" and "mathematical activities" is better-delimited and narrower representative variables. On the other hand, we share the perspective of V.A. De Bellis and G.A. Goldin about the affective system as including *changing states of feeling* (local affect) as well as *more stable, longer-term constructs* (global affect) (De Bellis and Goldin, 1999, pp.250; italics added). We are here particularly interested in the sphere of the so-called global aspect, concerning "(...) emotions about and within emotional states, emotions about and within cognitive states, and the *monitoring* and *regulation* of emotion (De Bellis and Goldin, op. cit., pp. 250; italics added).

Self-esteem was then chosen as representative of the global-affective sphere, since this complex process can be seen as a component of the general representation that someone has of himself/herself (Palacios & Hidalgo, 1995; Aberastury & Knobel, 1992). This representation has a more conceptual pole, the *self-concept*, and an affect-evaluative pole, the *self-esteem*. The data shown here try to offer empirical evidence of a connection between self-esteem and performance in school mathematics; this attempt, in fact, intend to go further, offering evidence for the debate about connections between affect and cognition in psychology. This debate is a hard theoretical task; since René Descartes, there is a strong tradition of splitting human nature in rational/spiritual aspects (*res cogitans*) and somatic/emotional/animal aspects (*res extensa*) (Descartes, 1973). This philosophical and epistemological background nourished theoretical systems in psychology stressing one of these poles (in detriment of the other one), without an integrative approach showing the functional interconnection between affectivity and cognition. Two important theoretical systems in psychology are good examples of this Cartesian heritage: Piagetian genetic epistemology and Freudian psychoanalysis. Cognition, from a Piagetian point of view, is related to a biological need of equilibration, where affective aspects are seen as "combustible" for logical structures (the "engine"): "(...) *affectivity is considered as the energetic pole of behavior*" (Piaget, 1980, pg. 135). Freud, for his turn, will stress unconscious pulsional (libidinal) aspects as central in the theoretical explanation of human behavior, viewing cognition (or epistemophilic motivation) as a derivative of libidinal impulse by sublimation or neurosis (Freud, 1973). There are certainly other important theoretical contributions addressing this specific aspect of an integrative view of human behavior (see, for example, Henry Wallon, Erick Erikson and Donald W. Winnicot's works on child development); none of them, nevertheless, seem to propose empirical data concerning specific aspects of cognition (i.e., specific knowledge domains or conceptual fields) taking into account affective aspects as constitutive, not merely adjuvant. The present research work tries to contribute in this direction, showing empirical evidence of the interaction between self-esteem and school achievement in mathematical problem-solving tasks.

2. Procedure

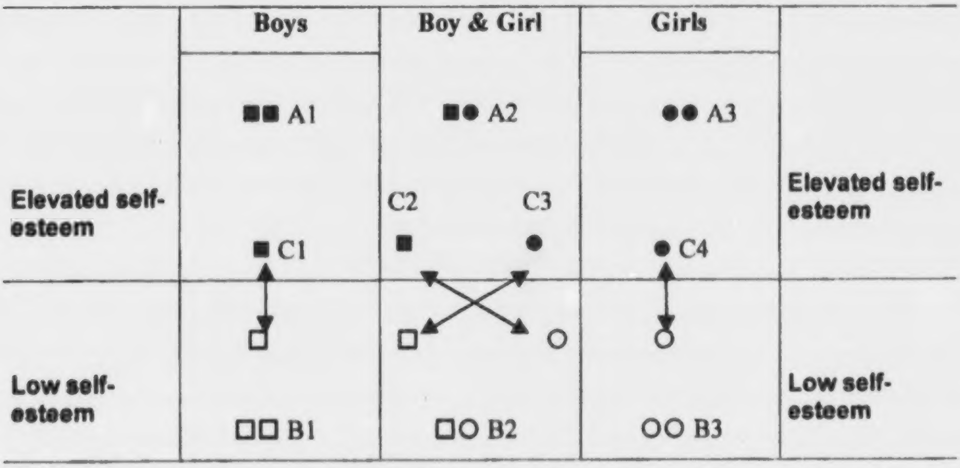
The sample of this study was initially constituted by 81 students of the 5th grade (elementary level) of a public school from Recife (Brazil), with ages varying from 12 to 14. This specific school level was chosen because it represents the first important moment of clear school difficulties in mathematics for Brazilian children (Ministério da Educação e do Desporto-Brazil, 1995). This initial sample was submitted to the HTP (House, Tree, Person) Test, used here to access levels of self-esteem among the students, according to administration criteria proposed by E. Hammer (Hammer, 1991). We could then arrive to a research sample of 20 students showing low or high levels of self-esteem (see Figure 1 below for examples of HTP data). These students were put together in pairs, controlling gender and self-esteem, as illustrated by Table 1 below. These ten pairs of students were submitted to a set of 20 evaluation questions in school mathematics, created specifically for northeastern Brazilian children (NAPE Evaluation Test – for more detailed information, see Neves e Souza, 1997). These questions addressed aspects considered important by Brazilian educational authorities for this school level, and are briefly described in Table 2. Both the achievement as well as the problem-solving strategies and difficulties in the NAPE Evaluation Test were analyzed, together with descriptive data like constitution of pairs concerning gender and self-esteem level (according to the procedure described in Table 1 below) and patterns of interaction and leadership showed by the subjects during problem-solving sessions. This set of data was analyzed with the help of multidimensional descriptive tools (cluster analysis of nominal data; for more detailed information about these statistical descriptive multidensional procedures, see Rouanet, Bernard & Le Roux, 1990); results are described in the next session.



Subject N: drawing occupying more than 1/3 of the paper, presence of details in the body, presence of the soil and additional elements (sun, clouds, flowers) (Obs: this drawing was originally produced with color pencils).

Subject R: drawing occupying less than 1/3 of the paper, few details in human body, absence of the soil (Obs: this drawing was originally produced with black pencil only).

Figure 1: Drawings of a person produced by two subjects considered, respectively, as being examples of high-level (subject N.) and low-level (subject R.) self-esteem, according to Hammer (op. cit.) classificatory propositions for HTP analysis.



	Boys	Boy & Girl	Girls	
Elevated self-esteem	■■ A1 ■ C1	■● A2 C2 ■ C3 ●	●● A3 ● C4	**Elevated self-esteem**
Low self-esteem	□ □□ B1	□ ○ □○ B2	○ ○○ B3	**Low self-esteem**

Table 1: Design for the constitution of 10 pairs of students, taking into account gender and self-esteem level (Boys= ■ and □; girls= ● and ○).

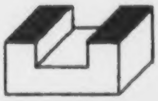
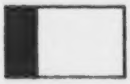
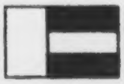
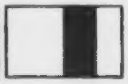
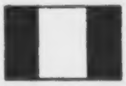
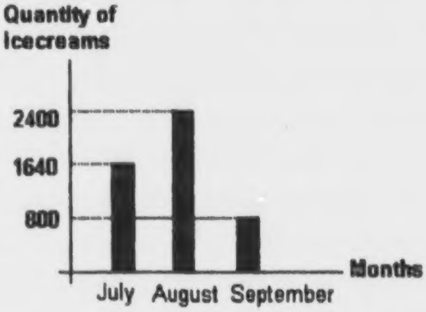
Mathematical aspect addressed [G_n =Group] and number of questions	**Example of a test item**
G_1= Problems of addictive structure **- 3 questions**	An airplane must fly over a distance of 962 Km in two stages: the first stage will cover a distance of 642 Km. How many kilometers this airplane will have to cover in the second stage?
G_2= Problems of multiplicative structure **- 3 questions**	A ship can carry a maximum of 200 kilograms per trip. What would be the minimum number of trips that this ship has to do in order to carry 8 people, each person weighting 60 kilograms?
G_3= Questions involving the use of operational algorithms **- 2 questions**	Make these operations: 3529 ÷ 15 3847 + 5 + 98
G_4= Questions involving the comprehension of the numerical decimal system of notation **- 2 questions**	The number ***eight thousand and two*** written in hindu-arabic algorisms is: __________
G_5= Questions of geometry **- 3 questions**	 Look carefully to this piece of a fitting game: how would be this piece of game seen from up to down? [*Options given are shown below:*]    
G_6= Questions involving fractions **- 3 questions**	The figure below represents a chocolate bar. Draw in black the part that corresponds, in this figure, to the following addition: 2/6 + 1/6 of the chocolate bar.
G_7= Questions concerning the comprehension of statistical descriptive graphics and mesures. - In the graphic on the right, how many icecreams would be sold if this amount was half the icecreams sold in September? **- 4 questions**	

Table 2: Profile of questions in the NAPE Evaluation Test.

3. Results

Two descriptive multidimensional clusters of pairs of subjects were obtained from data analyzed (see Figure 2 below). These clusters represent a hierarchical classification of the pairs of subjects according to two sets of information: classification ① was obtained from descriptive data concerning gender and self-esteem level in the composition of the pair, interaction and leadership patterns verified in the pair interactions during problem-solving activities, and need of help from the teacher during problem-solving. Interpretation of this first classification showed that two descriptive variables combined, *level of self-esteem* (LSE) and *pattern of interaction of the pair* (INT), could explain the partition obtained (the other variables having low contributions to the general variance of the data, i.e., contributions under the average value of explained contributions; this interpretation was confirmed by correspondence factor analysis). The first of these variables, LSE, admitted three analytical categories concerning the composition of the pairs in terms of self-esteem level of each subject: a) high-high; b) low-low; c) high-low.

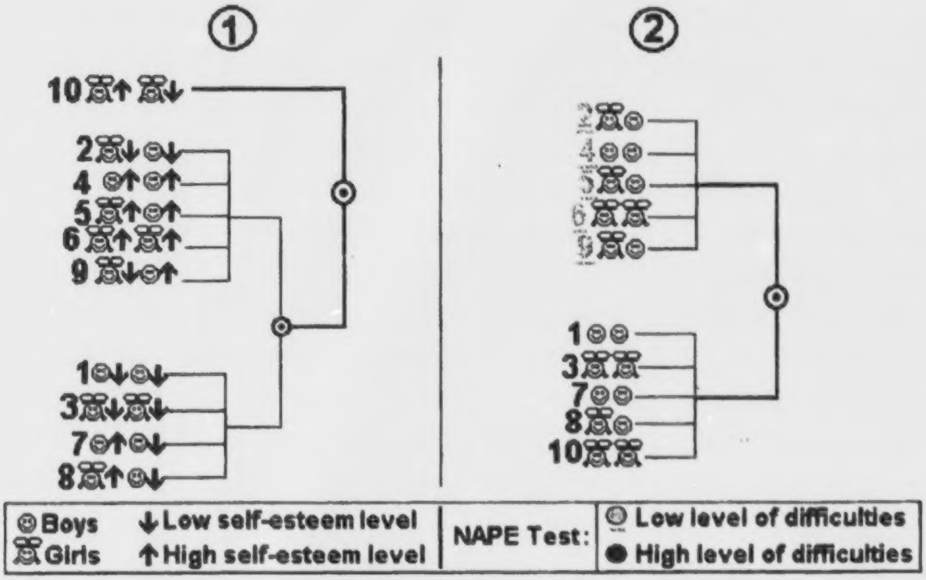


Figure 2: Hierarchical classificatory trees produced from data issued of pairs of subjects

The variable INT, for its turn, had equally three analytical categories concerning the patterns of interaction showed by the pair during problem-solving activities: a) Strong cooperation between subjects, with dialogues and step-by-step sense making in problem-solving procedures; b) One of the subjects is clearly the "guide", thinking aloud in order to share with the other subject his/her procedure; this other subject supports and encourages the first one; c) One of the subjects ignores the other, working alone and silently, the ignored subject accepting this scenario. Classificatory tree number ① shows two main clusters of pairs (and an isolated pair [number 10], which will be referred latter): the first one is formed by pairs 2, 4, 5, 6 and 9, and the second one being formed by pairs numbers 1, 3, 7 and 8. The first cluster is formed by three LSE high-high pairs, one low-low and one high-low; the second cluster is formed by two LSE low-low pairs and two high-low; the first pair was strongly characterized by INT pattern type (a), while the second cluster was characterized by INT pattern type (c). Briefly, we can interpret the partition obtained in terms of two groups characterized respectively by low self-esteem, combined with important difficulties in interaction between subjects of the pair (cluster 2: 1, 3, 7, 8), and high self-esteem, combined with good interaction pattern (cluster 1: 2, 4, 5, 6, 9). It is important to observe that this is not a *pure* partition (since there are pairs more and less representatives of the partition scheme), but the most suitable one, in terms of the contribution to the total explained variance of the analyzed data. In this context of analysis, the pair number 10 is explained as a member of cluster 2, since clinical analysis of protocols shows the same LSE and INT characteristics of this cluster. The exclusion of this pair from cluster 2 seems to be explained by very peculiar and strong characteristics detected in clinical analysis, specifically a clear INT pattern type (b) and a very strong dependence to the teacher during the problem-solving session.

Classification ②, for its turn, was obtained from data concerning the performance of pairs along the problems of NAPE Evaluation Test. The categorization of these data took into account the specific groups of mathematical aspects (in accordance with summary reproduced in table 2 above). The same multidimensional classificatory algorithm used for classification 1 analyzed these data, and the partition obtained showed a very interesting result: this partition is almost identical to the first one, *these two classifications being produced by different set of data.* The interpretation of classification 2 showed that the partition obtained is strongly explained by aspects that could be summarized in terms of difficulties in solving 6 of the 20 problems (the other 14 problems showed an irrelevant contribution to the partition obtained); cluster 1, formed by pairs numbers 2, 4, 5, 6 and 9, showed very few difficulties, while cluster 2, formed by pairs numbers 1, 3, 7, 8 and 10, showed important difficulties, summarized by table 3 below. A global interpretation of the main differences between clusters 1 and 2 of the second classification seems to indicate that the cluster 1 is clearly able to capitalize help from the teacher, and also from discussion inside the pairs; cluster 2 has important difficulties in dealing with formal representations like graphics, numeric systems and school algorithms. The partition obtained in classification 2 reproduces almost perfectly the previous partition obtained in classification 1, based upon self-esteem and patters of interaction inside the pairs; this result offers empirical support to a possible connection between the two sets of data.

Relevant questions of NAPE Evaluation test	**Pattern of problem-solving shown by cluster 1**	**Pattern of problem-solving shown by cluster 2**
Question 8: question involving fraction [G_8]	Has some difficulty in considering part-whole relations, but arrives to the solution after help offered by the teacher.	Absence of one specific pattern for this question.
Questions 9 and 10: Questions involving the comprehension of statistical graphics [G_7]	Can easily discuss and solve these questions.	Important difficulties in interpreting graphic representation even after teacher 's help.
Question 14: Question involving the comprehension of the numerical decimal system of notation [G_4]	Shows comprehension of the numeric-decimal system.	Has difficulties in dealing with intermediary zeros in transposing four-digit numbers from current language to algorism representation
Question 17: Problems of multiplicative structure [G_2]	Has some difficulty in making explicit the multiplicative structure of the problem, and in explaining how to solve it, but arrives to the solution.	Absence of one specific pattern for this question.
Question 19: Question involving the use of operational algorithms [G_3]	Shows comprehension of the numeric-decimal system, and of the algorithmic procedures related to this system.	Difficulties in intermediary operational steps of the use of algorithms (especially when dealing with division).

Table 3: Relevant questions of NAPE Evaluation Test (with respective patterns of problem-solving performance) which contributed for the partition obtained in classification ②.

4. Concluding remarks

This study tried to offer empirical evidences to the need of taking into account both affective and cognitive aspects in the research about mathematical learning. Efforts were made, first of all, in proposing clearly defined variables in order to arrive to a replicable study. As a result of this effort, the affective pole was operationally defined in terms of self-esteem, which cannot be considered as covering affectivity in its complexity, but seems to be an important aspect of *global affect* (De Bellis & Goldin, op. cit). This variable was analyzed together with other descriptive variables, and an important relation between self-esteem and interaction patterns was detected. On the other hand, mathematical activity was operationally defined in terms of performance in a specific evaluation test, what is certainly a narrow sample of mathematical performance, but is a perfectly public, explicit tool in the context of research.

The two very similar partition of subjects obtained as a function of descriptive variables like self-esteem and interaction patterns and performance in the NAPE Evaluation Test is a first empirical evidence of a possible connection between performance in school mathematics and affective aspects. This connection is still a very tentative interpretation, since we cannot assume a causal relation in any direction, neither we can discard the role of other intervening variables not addressed by this study. Interaction patterns, an aspect already explored in previous research (see, for example, Leikin and Zaslavsky, 1997), appears here in contact with self-esteem, and this empirical connection certainly deserves additional research.

Affect is undoubtedly a difficult aspect to take into account in the context of research and theorizing in psychology of mathematics education. Nevertheless, data collected here reinforce the need of taking mathematical learning into account as a human activity, impregnated of fears, self-evaluation, social roles and interaction possibilities.

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INTEGRATION, COMPENSATION AND MEMORY IN MENTAL ADDITION AND SUBTRACTION

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This paper reports on a study of Year 3 children's addition and subtraction mental computation abilities, and the complexity of interaction of cognitive and affective factors that support and diminish their ability to compute proficiently (accurately and flexibly). In particular, the study investigated the part played by number sense knowledge (e.g., numeration, number facts, estimation and effects of operations on number), metacognition, memory and affects (e.g., beliefs, attitudes). It found that proficient mental addition and subtraction was a consequence of the integration of all factors, but that accurate mental addition and subtraction could occur when some factors were impoverished if there was compensation.

Skilled mental computers are disposed to making sense of mathematics; they use a variety of strategies in different situations (depending on numbers and context) (Sowder, 1994). Thus, proficiency in mental computation involves both accuracy (achieving the correct answer) and flexibility (using a variety of strategies as efficiency requires). This paper looks at why some children are more proficient (more accurate and flexible) at mental addition and subtraction than others. In particular, a fundamental aim was to identify factors, and relationships among factors, that influence this proficiency.

Research on mental computation has proposed specific connections among mental computation and aspects of number sense, in particular, number facts and estimation (e.g., Heirdsfield, 1996). Other research relating to computation (in particular, children's natural strategies) has reported connections with numeration (e.g., place value) and effects of operation on number (e.g., Kamii, Lewis, & Jones, 1991).

Relationships have been posited between mental computation and affects (e.g., Van der Heijden, 1994), where affects cover beliefs (with respect to mathematics, self, teaching, and social context), attitudes (including self efficacy and attribution) and emotions (McLeod, 1992). Beliefs about the nature of mathematics can be manifested in a student's disposition – mastery orientation or performance orientation (Prawat, 1989). In relation to computation, mastery oriented students would aim for understanding and flexibility. Here, monitoring, checking, and planning might be evident. Whereas, performance oriented students would tend to aim to complete a task as quickly as possible, and not attend to understanding and reflection.

Proficient mental computers are flexible in their choice of strategies. Such effortful, reflective and self-regulatory behaviour should involve metacognition (e.g., Sowder, 1994). Metacognition can be considered to have three components: *metacognitive knowledge* (knowledge of own thinking), *metacognitive strategies* (planning, monitoring, regulating and evaluating), and *metacognitive beliefs* (perception of own abilities and perception of a particular domain) (Paris & Winograd, 1990). It is believed that metacognition, particularly metacognitive knowledge, is domain specific or even task specific (Lawson, 1984).

With regard to memory, Hope (1985) argued that superior short term memory was not necessary for proficient mental computation; rather, interest, practice, and knowledge were more important factors. Heirdsfield (1999) found that a superior short term memory was unnecessary for a student who was accurate and flexible with mental addition and subtraction. A well-connected and accessible knowledge base and efficient mental strategies were sufficient for the student. However, Heirdsfield also found that the *mental image of pen and paper algorithm* strategy always used by accurate/inflexible students tended to place heavy demand on short term memory.

Two aspects of memory seemed to be significant for mental computation: retrieval of facts and strategies, and concurrent calculation. Hunter (1978) suggested that the first aspect, demand for retrieval of facts and strategies, is met by long term memory. In his study of expert mental calculators, Hunter posited that these experts not only build up vast resources of numerical equivalents (e.g., number facts and other more complicated numerical equivalents), but also a vast store of ingenious strategies. In this way, complex calculations can be handled more easily by accessing long term memory for facts and strategies, thus eliminating the need for massive calculations and demands on temporary storage. However, his model did not account for the second aspect, concurrent processing of calculations. To encompass this, a model for working memory (Baddeley, 1992; Logie, 1995) consisting of a central executive, a phonological loop and a visuospatial scratchpad has been proposed. The *central executive* provides a processing and co-ordinating function, including information organisation, reasoning, retrieval from long term memory (access), and allocation of attention. The *phonological loop* (PL) is responsible for storage and manipulation of phonemic information, for instance, rehearsal of interim calculations. The *visuospatial scratchpad* (VSSP) deals with holding and manipulating visuospatial information. This may involve representation of numbers in the head, or positional information of algorithms.

In summary, research on mental computation and number has proposed connections among mental addition and subtraction, number sense (e.g., number facts, estimation, numeration, effects of operations on number), affective factors (including beliefs, attributions, self efficacy, and social context in classroom and home); and metacognitive processes. Further, it appeared that memory might have an effect on mental computation.

The study

The research consisted of two studies, a pilot study and a main study. Both studies were based on interviews developed to investigate mental computation (strategies and accuracy) and other aspects that were identified from the literature. The findings of the pilot study informed the main study. For the purposes of this paper, findings of both the pilot and main studies will be combined.

Subjects. The subjects were Year 3 students from two Brisbane independent schools that served high and middle socioeconomic areas. The students (13 in all) were selected (from a population of 3 classes, 60 students in all) after participating in a structured mental computation selection interview. As proficiency in mental computation was defined in terms of both flexibility and accuracy, both these factors were considered when selecting the students. As a result of their performance on the selection items, students were identified as accurate and flexible (4 students), accurate and inflexible (2 students), inaccurate and flexible (3 students), and inaccurate and inflexible (4 students).

Instruments. The students participated in a series of semi-structured clinical interviews that: (1) addressed mental computation strategies, number facts, computational estimation, numeration, and number and operations, (2) investigated metacognition and affect; and (3) administered memory tasks. The number sense, metacognition and affect tasks have been described elsewhere (Heirdsfield & Cooper, 1997). The memory tasks (Lezak, 1995) consisted of: (1) the Digit Span Test, a test of short term recall that requires verbal rehearsal and/or verbal recall; (2) a modified version of a short term retention test; and (3) a mazes test that addresses the central executive, for example, planning and decision-making.

The Digit Span Test specifically addressed the phonological loop (Gatherole & Pickering, 2000). Evidence from research investigating working memory in six- and seven-year old children found that existing visuospatial tests did not actually measure visuospatial memory (Gatherole & Pickering, 2000). Therefore, for the purposes of this study, no specific tests addressing visuospatial memory were administered. However, evidence of this component of memory was sought from observations of students' responses as they computed mentally. In a similar manner, evidence for the utilisation of the phonological loop and the central executive came from witnessing students' rehearsal of interim calculations, students' self-reports of "seeing things/numbers in the head", evidence of planning and choosing strategies, and other elicitations.

Interview procedures. The students were withdrawn from class to a quiet room in the school for the interviews. For most students, the series of interviews took four sessions of twenty minutes each. All interview sessions were videotaped.

Analysis. Students' responses on the interviews were analysed for: (1) accuracy and strategy choice for mental addition and subtraction (which, in turn, was used to determine flexibility, the use of a variety of strategies); (2) knowledge and strategies for numeration, number facts and computational estimation, and knowledge of the effects of operation on number; (3) metacognition and form and extent of affects; and (4) scores and strategies on memory tasks. For the purposes of identifying flexibility, mental computation strategies were classified using a scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) that divided strategies into the following categories: (1) *separated* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

Each student's ratings for number sense, metacognitive, affective and memory factors were summarised. These summaries were combined for each of the computation types: accurate and flexible, accurate and inflexible, inaccurate and flexible, and inaccurate and inflexible to produce a composite figure to represent that type. Factors were identified as commonly present (representing), varying and not present for each type. Analysis then moved from within types to across types to identify factors which by their presence or absence would show a relationship to accuracy and flexibility. Analysis focused on what was not present during failure as well as what was present during success.

Results

In this study, accurate/flexible mental computers employed a variety of efficient mental strategies to alleviate demands on working memory, while accurate/inflexible students resorted to one automatic strategy (*mental image of pen and paper algorithm*). Only one other student reported using automatic strategies. She was an accurate/flexible student, but her "automatic" strategies included a variety of efficient mental strategies. Inaccurate/flexible students also employed a variety of strategies (but low level strategies), while inaccurate/inflexible showed little in terms of strategies. This last group of students possessed poor knowledge, metacognition and memory.

Comparing accurate/flexible and accurate/inflexible students' responses, accuracy in mental computation was found to relate predominantly to fast and accurate number facts. Those students who scored poorly in the number facts test (slow and/or inaccurate) were inaccurate in mental computation. This would make sense, as fast and accurate recall of number facts from long term memory would result in less load on working memory, when more complex calculations are involved. Thus, fast and accurate number facts were found to be essential knowledge for accuracy in mental addition and subtraction.

In contrast, comparing accurate/flexible and inaccurate/flexible students' responses, flexibility in mental computation was found to relate to a mixture of factors, to number facts strategies and numeration, and, in part, to understanding the effects of operations on number, and metacognition. Fast and accurate number facts were not found to be related to flexibility (inaccurate/flexible students often had poor number facts). Students who were flexible in mental computation employed a variety of efficient number facts strategies (derived facts strategies) in the number facts test. Some students (particularly those who were flexible/accurate) applied some number facts strategies to mental computation strategies. In the case of flexible/inaccurate students, using derived facts strategies in the test did not help them in mental computation, as derived facts strategies were not used in interim calculations – instead, *count* was often used.

Efficient mental strategies (e.g., *wholistic* and *aggregation*) were found to require good numeration understanding. Lower level alternative mental strategies (e.g., *separation*) also were found to require some numeration understanding (canonical and noncanonical). However, accurate/inflexible students, who tended to use the *mental image of pen and paper algorithm* strategy, did not require the same level of numeration understanding, although a threshold knowledge was essential for procedural understanding (e.g., canonical understanding). Inaccurate/inflexible students were found to have very poor numeration understanding.

The relationship of flexibility to understanding of number and operation was not so straightforward. Students who exhibited good understanding in number and operation were found to employ high-level strategies (e.g., *wholistic*). It appeared that both numeration and number and operation understanding was required for successful employment of the *wholistic* strategy. Research has found that an understanding of the effects of operation on number would be important for efficient mental computation (e.g., Reys, 1992). In particular, understanding how changing the addend and subtrahend affects the result of addition and subtraction examples is the basis of the ability to employ some *wholistic* strategies.

Similar to number and operations, the effect of metacognition on mental computation was mixed. In this study, metacognition was not directly related to either accuracy or flexibility, although accurate/flexible students showed evidence of metacognitive strategies, especially monitoring and checking. Research findings support a relationship with flexibility, that metacognition aids skilled mental computers (e.g., McIntosh, Reys, & Reys, 1992; Sowder, 1994). The reasons for this study not showing a clear relation between flexibility and metacognition might lie in the young age of the students and their lack of metacognitive knowledge (in particular, their unawareness of their metacognitive strategies). On the other hand, the students were able to verbalise their metacognitive beliefs (perceptions of their abilities).

Neither accuracy nor flexibility was found to be related to estimation. This finding was in contrast to the findings of Reys, Bestgen, Rybolt, and Wyatt (1982) and Heirdsfield

(1996). In the present study, even the most accurate/flexible mental computers did not exhibit proficiency in estimation. One reason could be the students were too young to have developed estimation strategies. Estimation is not part of Queensland's present Year 3 syllabus (Department of Education, Queensland, 1991). Heirdsfield (1996) found that, even in Year 4, most students with estimation strategies had developed them from out of classroom experiences.

Exceptional short term recall and retention were found not necessary for mental computation; however, threshold levels were necessary. These findings support those of Hunter (1978).

> *The expert (mental calculator) goes quite a way to meet these demands (of working memory), partly by the speed and quality of working, and partly by devising calculative methods which evade an excess of interrupted working.* (p. 343)

Of course, poor working memory resources might contribute to a poor knowledge base in long term memory and poor connections between this knowledge, resulting in the diminished performance of inaccurate/inflexible students. Further, these students scored poorly on the working memory tasks. Thus, working memory might be a stronger influence of proficiency in mental addition and subtraction than the evidence of this study showed. To check this requires a look at the different tests used in the memory component of the interviews.

The results for Digit Span Test indicated that, for most students (other than inaccurate/inflexible), the phonological loop could support retrieval of number facts from long term memory, and holding and rehearsal of interim calculations. However, inaccurate/flexible students did not have number facts in long term memory, so the phonological loop could not retrieve these. The results from the other tests indicated that the visuospatial scratchpad only supported strategies such as the *mental image of pen and paper algorithm*, which meant there was little evidence of the use of the visuospatial scratchpad for accurate/flexible students, even though it was expected that some numbers would be represented in some visual form. However, because of their age, the students using strategies other than the *mental image of pen and paper algorithm*, might have been unaware of their use of mental imagery (or so preoccupied with their strategies, that they could not remember using any mental imagery).

Conclusions

The study showed that students proficient in mental computation (accurate and flexible) possessed integrated understandings of number facts (speed, accuracy, and efficient number facts), numeration, and number and operation. These proficient students also exhibited some metacognitive strategies and possessed reasonable short term memory and executive functioning.

Where there was less knowledge and fewer connections between knowledge, students compensated in different ways, depending on their beliefs and what knowledge they possessed. Accurate/inflexible students used the teacher taught strategy of *mental image of pen and paper algorithm* in which strong beliefs were held. Combined with fast and accurate number facts and some numeration understanding, their familiarity (almost automaticity) with this strategy enabled the students to complete the mental computation tasks with accuracy. Working memory was sufficient to use an inefficient mental strategy accurately. The visuospatial scratchpad was used as a visual memory aid. The inaccurate/flexible students compensated for their poor number facts and minimal and disconnected knowledge base by using a variety of mental strategies in an endeavour to find one that would enable them to reduce the difficulty of calculation. Although their limited numeration understanding and memory (including executive functioning) were sufficient to support the development of some alternative strategies, these were not high level strategies. In particular, access to *wholistic* strategies was only partially successful. Finally, the inaccurate/inflexible students who exhibited deficient and disconnected understanding tried to compensate by using teacher-taught procedures (similar to the accurate/inflexible students), but they were unsuccessful, as they possessed no procedural understanding and also had poor working memory.

The importance of connected knowledge for proficient mental computation demonstrates the need for teaching practices to focus on the development of an extensive and integrated knowledge base. Students can and do formulate their own strategies, but do not always use them accurately. Therefore, students should be encouraged to formulate their own strategies but in a supportive environment that assists them to use strategies appropriately. Because of memory load, students should be permitted to use external memory aids (e.g. pen and paper) to assist mental computation. This has a second payoff in that efficient mental strategies are, at times, also efficient written strategies. By having students formulate mental strategies, they have to call upon number sense knowledge, thus acquiring connected knowledge while they develop computational procedures. This is in contrast to students using teacher-taught procedures, which require little connected knowledge.

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A STUDENT'S UNDERSTANDING OF MATHEMATICAL INFORMATION

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The possible causes of elementary and secondary students' difficulties with grasping arithmetic information embedded in a word problem are studied using the "decompose and reconstruct" experimental setting (modified from one type of message games) and theory of meaning making. Concepts of elementary datum and elementary image are defined for research purposes. Seven obstacles inhibiting a student's grasping of the word problem or making its grasping impossible have been identified and analysed.

1. Introduction and framework

In everyday life, numbers play many different roles (Verschaffel, De Corte, 1996, Freudenthal, 1983): to quantify (cardinal aspect), to identify an object's location in a sequence or in a group (ordinal aspect), to measure, to reckon, or to name things. The better the student's discernment of these roles, the more effectively he/she can use his/her arithmetic knowledge in everyday life and the better is his/her insight into the world of arithmetic.

The diversity of number notions in real world is most richly reflected in word problems. Many studies have dealt with the classification of addition, subtraction, multiplication and division situations (e.g. Vergnaud, 1983, Nesher, 1988, Verschaffel, De Corte, 1996, Verschaffel, Greer, De Corte, 2000). The effect of the format of a word problem (verbal or pictorial) on the solution to the problem has been studied e.g. by Minato, Honma, Takahashi (1993). A student's ability to pose (word) problems was studied by English (1997) before and after a carefully prepared teaching programme which brought to student's attention all aspects of word problems (their structure, context, semantic relations, their "critical information units", etc.). The readability factors in the ordinary language of mathematics texts for (second language) learner are investigated in Adetula (1990) and Prins (1997). Prins (1997) identified a variety of phenomena influencing readability and consequently a student's solution to a problem stated in words. The following are particularly related to our research: difficult vocabulary, text structure, obscure information (i.e. confusing information, culturally biased contexts, contradictory and senseless information).

We hypothesise that the core of the difficulties that (not only) Czech and Slovak students have with word problems lies in their inability to grasp them, i.e. to transform (to model) a situation described in words into the language of equations. A teacher trying to "teach" students to solve word problems often presents them with instructions how to transform a particular type of problem (e.g. word problems on common work, on age, on filling up a swimming pool, on movement, etc.) into equation(s). These instructions, however, replace, rather than promote real understanding. They may help a student solve a standard task but they do not enable him/her to create a clear and rich image of a situation.

We have been observing and analysing processes of students' grasping of situations described in words for many years. In this contribution, we shall present one type of such analysis, based on the atomisation of numerical information and the corresponding number image.

2. Methodology

2.1 Instructional Setting

In our research, the centre of attention is a set of students' semantic images of a number(s). Its starting point is our pedagogical experience – a database which we have acquired through classroom observation and/or experiments. The experimental setting used in this research has been designed as a modification of *an instructional setting* used in our experimental teaching for the development of a student's ability to grasp a mathematical text.

The setting, tentatively labelled "decompose and reconstruct", belongs among so called message games and is based on the following sequence of students' activities: student A gets a certain mathematical text, for instance a word problem, and he/she will decompose it into a set of elementary information units (by an elementary information unit, we mean a sentence or a phrase including one number); student B then tries to reconstruct the original text from them. For instance, this instructional context was used in a game based on the familiar game Chinese Whispers. Groups of four take part, each group has two "decomposers" and two "reconstructors". They work on their own. A teacher prepares text *T* and everyone is given the number of words of *T* and the number of words a decomposer can use. The teacher hands over to the student A a text which will undergo a series of transformations.

$$T \xrightarrow[A]{d} T_1 \xrightarrow[B]{r} T_2 \xrightarrow[C]{d} T_3 \xrightarrow[D]{r} T_4$$

Where T_i are texts, d = decompose, r = reconstruct and letters A to D stand for students.

Each team is evaluated according to the information value of their texts T_4, i.e. to what extent the information of text T_4 corresponds to that of the original text *T*.

A similar setting was investigated in the plenary panel in PME20 (Puig, Gutierrez, (eds.), 1996, pp 53-84). One pair of students was first asked to solve a presented word problem and then to write a message explaining their solution to their friends that have to solve a similar problem, *without using any number* in the message (later specified as "to write it in mathematics using mathsymbols"). The second pair of students was then asked to solve their problem according to the method described in the message.

2.2 Experimental Setting

The instructional setting "decompose and reconstruct" has been modified for research purposes and used in research in two ways. First, pairs of students took part in experiments during which one of them was given a text and asked to decompose it into elementary information units so that there was exactly one number in each unit. Then the second student was asked to reconstruct the original text. Experiments were recorded and observed by experimenters.

Second, experimenters chose fourteen concrete word problems from Czech, Slovak and Polish textbooks (some of them will be given below as examples) and decomposed them into a list of elementary information units. In some cases, the decomposition was done by other people as well, e.g. by students and teachers. During this process, the need arose for the specification of basic concepts, e.g. the concept of decomposition itself or the concept of elementary information unit. For instance, during an interview one student decomposed the following information "I have 25 crowns in my piggy-bank", which we considered to be elementary, into two elementary information units "I have a twenty-crown note in my piggy-bank" and "I have a five-crown coin in my piggy-bank". In other words, the quantity of 25 crowns, which was considered as one number by most students, was regarded to be two numbers.

To get a more complex picture, some of the analysed word problems were given to students and their solutions were recorded.

2.3 Definitions of Concepts and Our Assumptions of Understanding Texts

Karl Popper (Lorenz, Popper, 1994), in line with Bertrand Bolzano, speaks about three worlds: the World of Objects (mass and energy), the World of Culture (everything created by the humankind) and the World of Mind (everything which is present in the minds of individual people). In concordance with this view we will consistently distinguish between *information* which belongs to the World of Objects and *image*, evoked by the information, which belongs to the World of Mind. These two concepts are interconnected through a pair of projections:

grasping	information → image	
articulation	image → information	

The wide space of information will be narrowed down to numerical information embedded in the World of Objects. Such information will be called *a datum*. Let us specify the basic terms of our considerations even further.

By *an elementary datum*, we mean a statement which

(a) includes at least one known or unknown number (i.e. the question in the word problem will be considered an elementary datum, too),

(b) has an unambiguous and if possible also briefly described connection to the world of objects, and

(c) cannot be further decomposed.

For instance, the statement "I bought things for 18 crowns" is an elementary datum, while neither "16 is an even number" nor "Peter had one more crown" nor "in the classroom there are 5 girls and 11 boys" are elementary data; the first one has no connection to the real world, the connection of the second one is not clear (we do not know with whom Peter is being compared) and the last one can be further decomposed.

The concept of elementary datum has two components: (1) a number or numbers, (2) its (their) embedding in the world of objects. By grasping an elementary datum, *an elementary image* originates. The concept of elementary image has three components. Besides (1) and (2), it also includes its location in the mind of a concrete person.

By the *decomposition of text T* we mean a sequence of data D_1, D_2, ..., D_k such that each number (known or unknown) of text T is contained in exactly one of these data and the semantic meaning of a number in the text T and corresponding data D_i is the same. The inverse process to decomposition is the *reconstruction of the text*. The decomposition is considered to be good if it is brief and if someone else is able to reconstruct the original text or at least its mathematical layer from it.

We have seen that decomposition is a subjective activity – two persons can make two different decompositions of the same text. Moreover, the above characterisation can rarely help us to decide which of several decompositions made by teachers or students is the best. Yet, the given characterisation helps us both in analysing and describing the investigated cases.

The process of decomposition has brought to light several interesting phenomena which play an important role in a student's grasping of a mathematical text and may sometimes function as obstacles in this process.

3. Some Phenomena Elucidating a Grasping Process

In this section, we will describe seven phenomena identified as important elements in some grasping processes.

3.1 Non-verbal Information

By non-verbal information, we mean information whose main carrier is a picture, table, graph, scheme, etc. The importance of this kind of information is emphasised by the fact that pictorial information often appears at a pre-school and early school age and thus plays a key role in the process of developing a student's attitude towards mathematics. The example of non-verbal information is the picture in example 1 below.

Non-verbal information can be characterised with five criteria.

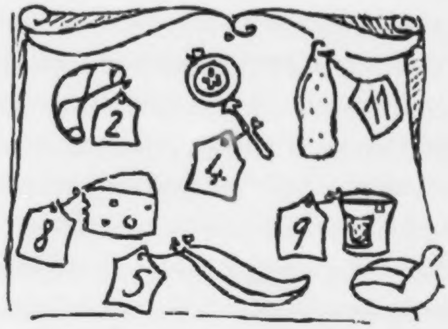
1. Adequacy: information falls into a student's experience.
2. Comprehensibility: information is presented in such a way that a student is able to create its image for him/herself.
3. Non-ambiguity: information does not allow for several different interpretations (see the next section).
4. Memorisibility: information as a whole and its individual parts contribute to its holding in a student's memory.
5. Motivation: information increases/decreases a student's personal interest.

What we call non-verbal information falls into the category of (external) representations which has been given a considerable attention (e.g. Verschaffel, De Corte, 1996) and it can be a source of serious learning difficulty as "such devices do not speak for themselves" and "their meaning must be constructed by the learner" (Becker, Selter, 1996) and as will be seen in the section below.

3.2 Vague (Ambiguous) Information

Example 1 (Kovácik et al, 1995):

I purchased three things for 18 crowns.
Draw them.



When asked by their teacher, both Eve and Mike said that a roll costs 2 crowns and cheese 8 crowns. It appeared that contrary to other students who gave various different interpretations of the picture Eve and Mike had the same image of the picture. This assumption proved to be wrong. Later when solving the problem, they disagreed. Eve saw the picture as a shop window offering six different kinds of goods. In view of this image, she included the case 5+5+8 (two bananas and cheese) in her solution. Mike, who understood the picture as a set of six different things, rejected this solution and gave only solutions in which each object was used once (like 2+5+11 = a roll, a banana, yoghurt). This illustrates the fact that a person's image is a multilayered concept. Two images which seem to be the same in one layer may differ in another. Thus we can say that the information in the picture is *vague (ambiguous)*.

Vague (ambiguous) information can be seen from two points of view. (a) From a mathematical point of view it allows for different interpretations than that of the author. (b) From the didactic point of view it allows for two different but meaningful students' interpretations. We consider the latter vagueness to be desirable at school. We agree with Byers (1998) that "we tend to react to every presence of ambiguity by attempting to remove it rather than by working with it" which is often the case of both the authors of mathematical textbooks and teachers in the Czech Republic. The rich discussion which began over Eve's and Mike's interpretations can be very informative because:

1. it improves a student's sensitivity to possible ambiguity in the presented information, it teaches him/her not to be content with a protetic interpretation (e.g. to use a key-word strategy solution (Verschaffel, De Corte, 1996)) but rather analyse it critically,
2. it shows students that the same situation can have several different interpretations and that the solution to any problem must begin with the specification of the situation (i.e. choosing one of the possible interpretations),
3. it helps overcome a widely held belief that a math problem has always one clear solution.

The prerequisite is that the teacher him/herself (1) finds the problem clear, (2) can see what different interpretations it can have, (3) is willing to monitor students' discussion.

3.3 The Word "About"

Example 2 (Repás et al, 1997): A bar of chocolate costs 12 crowns. A teacher bought 29 bars. He paid for the chocolate about .... crowns.

Example 3 (Cernek, Repás, 1998): My 10 steps are about .... metres.

In examples 2 and 3, the word "about" plays an important role. It indicates approximation. But while in example 3, it confirms the fact that there is no right solution to the problem, in example 2 there is an exact solution. In other words, in example 3 approximity is related to the result and in example 2 to the solving process. Students' solutions to the problems from examples 2 and 3 revealed four possible interpretations of the word "about":

1. I am to find an exact result and *round it off* (the answer was about 350).
2. I am to *guess* the result (i.e. $30 \cdot 10 = 300$).
3. I am to show how well I can *guess* the comparison of the lengths of "step" and "meter".
4. I am to *experiment* – I am to pace 10 steps and measure the distance. I can repeat the process and find the average.

The word "my" in example 3 shows that there is no given answer, that the result will be individualised. This phenomenon can be called **subjectivity of information.**

3.4 Comprehensibility

Example 4: The match ended with the result 3:2. In half-time, the score was 0:0. How many goals did the home team score?

As assumed above, the information is comprehensible for a student if he/she is able to create an image of it for him/herself. This phenomenon applies primarily to a student's image rather than to information written on paper. The same information can be comprehensible for one person and less comprehensible for another (e.g. Jane knows that one number means the number of goals of the home team and the second that of guests but she does not know which is which which became clear when she was asked to decompose the text) or totally incomprehensible.

3.5 Hesitation

By hesitation, we will mean a psychological state of an individual who is forced to decide a matter important for him/her but he/she lacks the necessary information.

Examples: (a) a student cannot remember whether 0 was assumed to be a natural number, or (b) if Pi=3.14 is an exact equality or not; (c) a student does not know if the formula for the area of a triangle $A=b \cdot h/2$ holds for the obtuse-angled triangle; (d) a student solves the problem in Example 3 and hesitates if 8 metres would not be considered by the teacher as wrong.

The examples (a) and (b) concern isolated facts which must be remembered and the reason for hesitation is memory failure. The third one is different. The student should be able to use his/her knowledge to resolve the problem, e.g. by relating the area of a triangle to the area of a quadrilateral. The last case is sad because the source of the student's problems is not his/her ignorance but fear that he/she would not guess the teacher's expectations right (see didactic contract – Brousseau, 1997). It is not a rare case in Czech schools. Consider the following example which highlights the difference between a mathematical and didactic conceptions of mathematics.

Example 5 (Kovácik et al, 1995): Ela took less than 17 steps. How many steps could she take?

The problem is in word "could". A mathematician can see no problem – the result can be any number between 0 and 16. However, the reaction of a student who believes that there must be a single correct answer to each problem can be different. He/she can be bewildered by this question.

3.6 Implicit information

Again, this phenomenon depends on the reader's experience. The space of implicit information can be classified according to two criteria:

1. *What* is hidden – a number, its meaning or relation.
Examples: (a) a number is hidden in the words "double-headed", "kilo" (meaning one kilo), "week" (one week or seven days), "goalless" (it hides either one number – no goal was scored in the match – or two numbers – guests scored 0 goal and the home team scored 0 goal);

(b) the meaning of a number is hidden in the label "the result was 3:2" (it is partly hidden for Jane above);

(c) the relation is hidden in the group of words "gross, nett, tare" as in the word problem "If gross weight is 58 dg and net weight is 35 dg, what is the tare?" (Demby, Semadeni, 1997).

2. The *carrier* of the implicit information – it can be a word, a group of words, a sign, a picture, a table, etc.

4. Conclusions

The "decompose and reconstruct" setting has been used in our experimental teaching for improving a student's ability to understand the text of word problems. It's research modification described above has been continuously elaborated. First, we try to standardise the techniques of analysis of material acquired through the "decompose and reconstruct" setting, second, we are looking for its other variants. The list of the seven given phenomena is being enriched by other phenomena and restructured.

A serious problem which has not been addressed in our research so far is how to apply this method in practice, how to convince teachers of its efficiency. In this respect, we direct our attention at university students – future teachers.

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Pathway between Text and Solution of Word Problems

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Solution of a word problem begins with the given text. The students' task is to discover the mathematical model that will lead them to the correct solution. We endeavored in our study to detect the text representation by analyzing the students' repetition of the text on the one hand and their mathematical solution on the other hand. While revealing the representation mechanism, we analyzed the performance of students who were successful, as well as those who failed in the mathematical solution of the problem. The findings demonstrate that the solvers have their own representation of the given text that affects both their retelling and the mathematical solution.

Theoretical Background

A solution of a mathematical word problem begins with a given text. The processes involved in reading the text, are linked on the one hand to text comprehension and discovering the formal mathematical model on the other hand. Comprehension in this context means building a representation of the textual information. According to Gick (1986) there are three major stages in solving a problem: constructing a representation of the problem, searching for a solution and implementing the solution. Our main interest in this study is the first stage, constructing the representation.

One method of revealing the construction of representation is to ask the students to retell the text they have read. The retelling can provide evidence of how the student has interpreted the original text. Kintsch (1986, 1994) and Verschaffel (1994), employing this methodology in their studies, found the source for subsequent erroneous solutions in the manner the solver represented the text. Thus, the subsequent erroneous solutions are reflected in the retold text. The solver, instead of retelling the original text, which may be too complicated for him, relates a simplified version of the story. For example, a relative quantity might become an absolute quantity.

Reading a given word problem, the student relates to the textbase in order to build a situation model (Kintsch 1986, 1994). To elaborate it to a mathematical model requires understanding of the situation described in the text and using mathematics to complete the missing data in the given text.

Similarly, Bilsky (1986) demonstrated that the context and the goal of reading a text affect the way the subject constructs the representation of the text. She found that the manner in which a subject views the text, whether a "math problem" or a "story", was decisive regarding the text representation. Similarly, Anderson and Pichert (1978) found that shifting perspectives on the same text (second recall) add additional information and recalls less irrelevant information, thus adopting a new perspective led subjects to invoke a schema that provided implicit cues for the different categories of story information.

Numerous evidence exists for schemas directing comprehension and representation Reed (1999), Marshall (1995). A schema provides a framework for integrating new information into old knowledge in order to construct a general structure for a variety of specific instances.

The purpose of our study was the verbal text and its representation. The method used was retelling. We endeavored in our study to detect the relationship between the text representation and the mathematical solution by analyzing the retelling. While revealing the representation mechanism we analyzed students who were successful as well as those who failed in the mathematical solution of the problem.

In analyzing the retelling of successful solvers, we also noted what Raney et al. (2000) wrote about the model of text repetition effects in which wording is represented in an abstract, context-independent manner, whereas the situation described by the text is represented in an episodic, context-dependent manner.

Method

The texts used in our study were three word problems, typically given in math classes, all of which lead to two-step solutions.

Forty-nine fifth and sixth grade Israeli students, who had already studied such problems, were individually interviewed in a single 45-minute session.

Task description:

Three word problems were presented to the students:

Problem No. 1: *I have a book with 320 pages. I have already read 80 pages of the book. How many days are needed to finish reading the book if every day I read 60 pages?*

Problem No. 2: *In the morning the seller distributed the roses equally into 6 vases. How many roses did he place in each vase if*

> *during the day he sold 120 roses and at the end of the day he saw that 60 roses were left?*
>
> Problem No. 3: *For the journey lunch-boxes were prepared for all participants. Each lunch box had 5 pieces of fruit of which 2 were apples and the rest were plums. In preparing the lunch-boxes 240 plums were used. How many participants received lunch-boxes?*

For each of the three problems each student was asked to read aloud the text (original text) of the word problem, to retell it (first retelling), and then to solve it. After the solution he was asked again to tell the story (second retelling). The complete record of the session was used in our analysis.

Method of analyzing the session records

Our initial analysis was performed on two levels: 1) Deviation of the repetitions from the original text, and 2) Correct or incorrect solution. Regarding deviation from the original text, we classified it into four major categories:

a) Changing the wording **without changing the schema** of the text.

> For example, in this category Yael retold problem No. 2 in this way: *There was a seller. He received roses and equally distributed them in vases. During the day a lot of people arrived and bought a lot of roses. Then he found out that 60 roses were left?*

b) Changing the order or the flow of information of the original text.

> *For example, in this category Michal retold problem No. 2 in this way: A seller sold 120 flowers and 60 flowers were left. The flowers (those sold and those left) were in 6 vases. How many flowers were there in each vase?*

c) Accurately retelling the original text.

d) Changing the text so that it describes a different schema.

> For example, in this category Joseph retold problem No. 1 in this way: *There is a book with 320 pages. Up to now I have read 80 pages, one day I read 60 pages. How many more pages do I have to read?*

Findings

In Table No.1 we can see the distribution among the above four categories of deviations from the original text in the repetition, and the correctness of the solution for each word problem.

Table No. 1: Deviation from the Original Text and Correctness of the Solution.

Problem	Correct Solutions			Incorrect Solutions	Not * Included
	(a) Changing the wording without changing the schema	(b) Changing the order	(c) Retelling precisely	(d) Changing the text into different schema.	
1	21 (43%)	14 (29%)	4 (8%)	8 (16%)	2 (4%)
2	17 (35%)	14 (18) (29%-38%)	2 (4%)	13 (26%)	3 (6%)
3	13 (26%)	14 (16) (29%-33%)	4 (8%)	16 (33%)	2 (4%)

* Due to any reason

We can see in Table 1 that very few students are included in category (c) in which the subjects retold the original text exactly. This means that most students engaged in some elaboration while retelling the original text, thus supporting the findings of Raney (2000) that the wording is abstract.

What is the difference between those who rephrased the text and correctly solved the problem and those who rephrased the text and erred?
Those who succeeded in solving the task, did not change the basic schema of the text. Their variations were of two kinds: (a) changing the wording without changing the schema, and (b) changing the order of the text. Those of category (a) usually added details to the description of the situation (episode) taken from their general world knowledge, which were not mentioned in the text.
For example there were students who told about the people who came to buy the flowers, or added some information about the flower-seller.
This also conforms to the findings of Raney et al. The situation described in the text is represented in an episodic context- dependent manner. Once the subjects grasped the situation they were ready to add their own details. On the other hand, frequently the wording was changed to express the original semantics.

Those included in category (b – of the correct answer) also preserved the text schema, but elaborated on the text in a way so as to help them detect the mathematical model,
Michal, who changed the flow of problem 2 (see explanation of category b above), immediately began to solve the problem and claimed: *"first*

> *I have to know how many flowers he had in the*
> *morning, the sold flowers and the left ones 120+60.*
> *Only then can I find out how many flowers he placed in*
> *each vase. I have to do 180:6=30"*

Six students belong to both categories (a) and (b). For example, Johnny in problem No. 3, rather than adding to the story some details from his world knowledge, also solved a part of the problem in the course of retelling it.
In this way he retained the problem's underlying structure while changing its surface structure.
Johnny's repetition was:

> *A member of the entertainment committee, or somebody else, I don't know exactly who, prepared the lunch-boxes for the journey the committee organized. In each lunch box they put 5 pieces of fruit of which there were 2 apples and 3 plums. 240 plums were needed to prepare all the lunch-boxes. How many children got lunch-boxes?*

While solving the problem he wrote 2 math expressions as follows:

"3 + 2 = 5 and 240 : 3 = 80"
plums apples fruits

He summarized *"80 children will get lunch-boxes"*.
He continued and said: *"Now I can find out how many apples were needed as well (80X2=160)."*

All those who solved the problems incorrectly, also consistently changed the schema of the text in their repetition (category d). As found by Kintsch (1986), the changes they made in the text fitted their erroneous mathematical solution. Moreover, when asked to give the second repetition (after the solution) they were again consistent in their story, The second repetition fitted their erroneous solution.
Changes in the schema were also of two kinds:

1. Those who entirely changed the mathematical model (e.g. from multiplicative to additive),

   For example: Joseph, whose repetition was demonstrated for category (d), changed the problem from two different structures (additive and multiplicative) to two similar structures (two additive structures). He continued to solve the problem by solving the expressions: 80 + 60 = 140, and 320 – 140 = 180.

2. Those who changed the relationships of the sets involved in the situation text (Bilsky, 1986). For example, Shay retold problem No.3 as follows:

   Lunch-boxes were prepared. There were 5 fruits in each lunch box, of which 3 were plums and 2 were apples. Shay continued to speak aloud while solving: 240:5=48 and said:
   "240 are all the fruit. Each child received 5 fruits".

   In this case the 240 plums were changed into 240 fruits, so that it would be reasonable to distribute 240 fruits - 5 fruits per each child, which was not the original question.

Discussion:
It is clear that the first step in the problem solving process is constructing a representation of the situation. This process does not take place in a vacuum, but is strongly affected by its context (Bilsky1986, Bransford and Johnson 1972).
Verschaffel et al. (1994) described the role of real world knowledge in the different phases of problem solving. Beginning with the initial phase of problem understanding, modeling that precedes the computation, and the final phase in which the result of the computational work is interpreted and evaluated. We also tried to trace the students' representation of the text read by asking them to retell the text. We examined their solution (thus, their modeling) and we noted their solution by asking them to retell the text again after their solution.

We found some students who retold the text precisely. According to Fletcher & Chrysler (1990): "*The most superficial level of representation, called surface memory, captures the exact wording of a text. This representation is viewed as a product of highly automated lexical and syntactic process.*" All those students solved the problem correctly. They probably found no reason to change the wording and they immediately saw the mathematical model, which led them to the correct answer. According to Fletcher & Chrysler (1990), in studies on reading, accurate repetition of the text is viewed to be the lowest level. In mathematics, we have thus far been unable to relate the precise retelling to a low level of understanding. In our sample, those who retold the text exactly did not support the claim that they were on a lower level.

Most students changed the wording while retelling the given text. We believe that this occurs while the student is constructing the situation model of the given textbase (Kintsch 1986). Fletcher & Chrysler (1990) add, *"the final level of representation is referred to as the situation model. This representation results from processes that integrate the information conveyed by the text with a reader's or listener's prior knowledge and produce the most lasting trace in long memory. The situation model corresponds to the equivalence class of all experiences that convey the same situation."*

We found evidence that most students do not "translate" the textbase directly into a mathematical expression. In order to construct the situation model from a given text, the student has to invoke qualitative considerations, which are necessary for constructing the situation in his mind. He frequently adds details not originally mentioned in the text but taken from his world experience (Nesher 1980). Anderson et al. (1983) claim that *"the content schema embodies the reader's existing knowledge of real and imaginary worlds. What the reader already believes about the topic helps to structure the interpretation of new messages about the topic"*. We found that all students in our sample who built a richer text by adding detailed information, did so because it was useful for them in order to construct a complete understanding of the text, and thus they correctly solved the problems.

Another path from the original text to correct solutions was found when students changed the original order of the text. This happened in problems in which the numbers given in the text were not in the same order they appeared in the mathematical expression. In this case the students changed the order so that the numbers appeared in their retelling in the same order they were supposed to appear in the mathematical expression.

All students who failed to solve the problems changed the text into another situation, mostly to a simpler one. These changes related to changing the text so that it described different mathematical structures. Kintsch (1986) and Verschaffel (1994) found this type of behavior in simple one-step additive problems. In our study, in accordance with theirs, the students felt comfortable with the changes they made. This was consistent even in the second repetition made after they already solved the problem. When they had to interpret and evaluate their solution, they did not feel that they were referring to another story.
Once again, we would note that solving a word problem is not a matter of a direct solution. Zwaan and Radvansky (1998) found similar findings in

relation to other expertise domains. In the path that leads from a given text to its mathematical solution, the situation model plays a crucial role. This model is constructed by the solver after reading the given text and adding interpretations based on his world experience.

We believe that our findings might have pedagogical implications, as teachers could gain some access to the students' internal representations by asking them to retell the text. This may be particularly helpful in tracing the source of erroneous solutions.

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DESIGNING SOFTWARE TO ALLOCATE APPROPRIATE DEMANDS ON MEMORY AND AWARENESS

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I begin by considering the division into necessary and contingent which exists within philosophy and discuss ways in which this can be applied to the mathematics curriculum (the divide having been re–named arbitrary and necessary). By looking at implications of this divide, I develop a pedagogic approach which clarifies when a teacher's role is to assist memory and when it is to educate awareness. I consider how this approach has informed the development of one of the dynamic geometry computer files in the package ACTIVE GEOMETRY.

Arbitrary and necessary

Within philosophy there is a divide made between those things which are logically true and can be derived in some way from other truths, and those which cannot be so derived and where some sort of choice is possible. The former are known as *necessary* and the latter as *contingent*. Kripke (1996, p36) expressed the divide as follows: *If* [something] *is true, might it have been otherwise?... If the answer is 'no', then this fact about the world is a necessary one. If the answer is 'yes', then this fact about the world is a contingent one.* In my application of this divide to mathematics education (Hewitt, 1999), the issue of whether the question *why?* can be suitably answered can be an alternative tool for deciding whether a 'fact' might be considered necessary or not. Within Nozick's (1984, pp140-141) discussion of statements concerning the principle of sufficient reason, he wrote:

> Let us state the principle of sufficient reason as: every truth has an explanation. For every truth *p* there is some truth *q* which stands in the explanatory relation E to *p*... When any other truth holds without an explanation it is an arbitrary brute fact.

Using a combination of Nozick and Kripke, if the question *why?* can be answered by providing an explanation which shows that this 'fact' could not have been otherwise, then this 'fact' is one which is necessary, otherwise it is arbitrary. I have chosen to use the term *arbitrary* rather than *contingent* since within education a student is rarely party to the moment when some people were aware of choice and had reasons for making the choice they did. So rather than being party to this choice, students are presented with the 'fact' in a text book, or by a teacher, some decades or centuries after the choice was actually made, which makes it appear arbitrary. For example:

> Student: What is the name of a 2D shape which has four sides the same length?
>
> Teacher: A rhombus.

Student: Why?

Teacher:…

Consider how you might answer the student's question before reading on.

Personally, I do not know why such a shape is named a *rhombus* I just accept the use of the name as a social convention within the English speaking mathematics community. I suspect there were reasons why this name was chosen at the time, but for me, working on mathematics today, I merely have to accept the term in order to communicate with colleagues successfully within the field of mathematics.

Walkerdine (1990, p2) wrote that *Saussure is credited with recognizing the importance of the fact that the relationship between the signifier and the signified is arbitrary; that is to say, conventional rather than necessary.* Names are signifiers and all names are arbitrary in my usage of the word. Names are the result of choices, as are conventions. Thus all names and conventions within the mathematics curriculum are arbitrary and as such are to be *accepted* rather than *understood* by a student. This is because there are no reasons for why any of these *have* to be how they are. Different choices could have been made which would have resulted in different names and conventions without affecting the mathematics of the situation at all. Only the language of description would have been changed. For example, there is no reason for why the x co–ordinate *must* be written before the y co–ordinate. It could have been the other way round and none of the mathematical properties concerning co–ordinates would change, only the way in which they are described.

I will now consider how a student may come to know the arbitrary. I am usually known as Dave Hewitt but I also have other names between Dave and Hewitt in my full name. What are they? *Consider this before reading on.* These other names are arbitrary and the only way in which it is possible for you to know them is to be informed by someone who already knows them, such as me, or perhaps an official document which has them written down. Students cannot know for sure how a co–ordinate is written unless they consult someone who knows. They can invent plenty of ways of writing co–ordinates, but if we expect them to learn what is a convention within the mathematics community, then those students will have to be told of the convention. This gives a clear role for both students and teachers: students need to be told, accept and memorise the arbitrary; teachers will need to inform and offer ways to practise the use of the arbitrary.

The name of the shape in Figure 1 may be arbitrary but there are other things about the shape which are not. The shape has certain properties, such as sides which are the same length, which I can work out and not have to ask someone else who knows. This is an example of something which is not arbitrary but necessary. There is an explanation which I can provide which convinces me that the sides *must* be the same length. There is no choice about the matter, it is a property which has to be so.

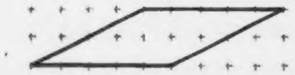


Figure 1. What properties has this shape?

Not everyone may be able to become aware of the property of equal length sides. A student who does not know about Pythagoras' Theorem may struggle to know for sure that the shape has this property. So the necessary is not available for *all* students to know as necessary. Whether students can come to know a property will depend upon the awareness they already have of the mathematics involved. If they do have sufficient awareness, then that awareness can become educated in the process of finding out that this property of the shape must be true. It is the role of a teacher to consider the awareness of students and make choices about what are appropriate challenges for certain students. Whatever the situation though, properties of shapes are not arbitrary and so I propose as a guiding principle that therefore they are not to be told. Instead, either particular properties are considered not to be within the awareness of the students and so will not form the focus of a lesson, or they are accessible to the awareness of students and so an activity needs to be provided to educate students' awareness so that they can come to know that these properties must be true.

	Student	**Teacher**	**Mode of teaching**
Arbitrary	*All* students *need* to be informed of the arbitrary by someone else	A teacher *needs* to inform students of the arbitrary	*Assisting* *Memory*
Necessary	*Some* students *can* become aware of what is necessary without being informed of it by someone else	A teacher does *not* need to inform students of what is necessary	*Educating* *Awareness*

Figure 2. When to inform and when not to inform.

Figure 2 summarises the differences between arbitrary and necessary. In particular, the arbitrary is in the realm of memory and a teacher's role is to assist students in memorising. The necessary, however, can be known through awareness and so a teacher's role is not to inform but to devise activities which help students to educate their awareness as I have just mentioned above.

The necessity of the sides being the same length in Figure 1 is based upon some information which is provided or assumed, such as the drawing itself, the assumption that the corners lie exactly on grid points, it is a square grid, that we are in Euclidean geometry, etc. These I describe as *givens* and provided certain givens are accepted, there are situations where there is underlying necessity which comes as a consequence of these givens. It is working on what is necessary where mathematics really lies. The arbitrary is for ease of communication and is not mathematics itself. Sadly, too much attention is often paid to the arbitrary when mathematics is to be found in the necessary. Richard Feynman (1988, pp13-14), a famous physicist of the 20th century, gave the following story:

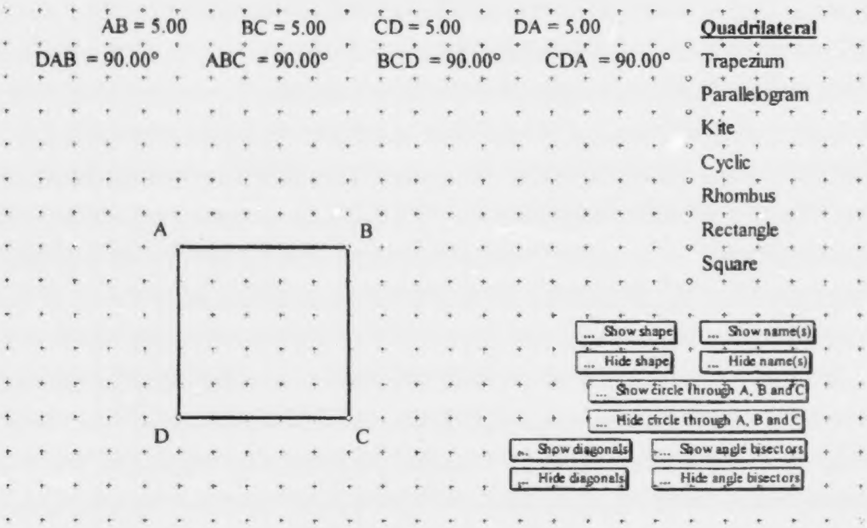
> One kid says to me, "See that bird? What kind of bird is that?"
>
> I said, "I haven't the slightest idea what kind of bird it is."
>
> He says, "It's a brown–throated thrush. Your father doesn't teach you anything!"
>
> But it was the opposite. He had already taught me: "See that bird?" he says. "It's a Spencer's warbler." (I knew he didn't know the real name.) "Well, in Italian, it's a *Chutto Lapittida*. In Chinese, it's a *Chung–long–tah*, and in Japanese, it's a *Katano Tekeda*. You can know the name of that bird in all the languages of the world, but when you're finished, you'll know absolutely nothing whatever about the bird. You'll only know about humans in different places, and what they call the bird. So let's look at the bird and see what it's *doing* – that's what counts." (I learned very early the difference between knowing the name of something and knowing something.).

The linking together of memory and awareness is often exhibited by skilled practitioners. Bird watchers attend to properties of birds and with the noticing of certain properties (size, markings, birdsong, etc.) comes the name of the bird, and likewise mathematicians attend to properties of the shape in Figure 1 and the word *rhombus* or *parallelogram* appears to come as if from thin air. Walkerdine (1990, p3) posed the question *how do children come to read the myriad of arbitrary signifiers – the words, gestures, objects, etc. – with which they are surrounded, such that their arbitrariness is banished and they appear to have that meaning which is conventional?* The arbitrary signifier *rhombus* only gains an (apparent) meaning when the relevant properties are associated with the word within a student's mind. As Saussure (1974, p75) said in relation to the signifier and signified coming together to form a 'sign': *Both terms involved in the linguistic sign are psychological and are united in the brain by an associative bond.* It is only when the name and related properties are brought together through association that the name stops having the sense of arbitrariness (in the usual English use of the word) and takes on an apparent meaning. This issue is developed within the next section where I consider some pedagogic decisions involved in the development of a software file.

An example of bringing together the roles of memory and awareness: the design of a dynamic geometry file

Dynamic geometry software, such as *Geometer's Sketchpad* and *Cabri–Géomètre*, has offered new ways for students to engage with geometry. The possibility of movement and the focus on properties, especially concerning geometric relationships through construction, has brought new pedagogic possibilities (for example, see Laborde, 1995, and Jones, 1998). Some files for *Geometer's Sketchpad* have been commercially available (for example, *Exploring Geometry with the Geometer's Sketchpad* from Key Curriculum Press) which generally take the form of demonstrating dynamically a mathematical property or setting up a mathematical construction where the locus of a point, for example, is to be explored. The emphasis on these files is on the mathematical content. The *Active Geometry* files available from the Association of Teachers of Mathematics (ATM) for either *Geometer's Sketchpad* or *Cabri–Géomètre II* consider ways in which parts of the UK National Curriculum can be directly addressed through dynamic geometry software without the need for students or teachers to know about construction techniques. I will consider one of these files, *Quadinc*, and discuss some pedagogic issues which underpin the design of the file.

Quadinc has a quadrilateral whose corners can be moved to any point on a square grid. As a corner is moved, so the properties of the quadrilateral change and names associated with the quadrilateral may also change as a consequence. For example, Figure 3 shows the quadrilateral ABCD. Such a shape is usually assigned the name *square*, however, it is also a *trapezium* (accepting an 'inclusive' definition for a trapezium), *parallelogram*, *kite*, *cyclic*, *rhombus* and a *rectangle*. All these names appear on the screen. If corner B is moved one grid space to the right then all the names disappear except for *trapezium*. Thus all the names associated with the particular quadrilateral appear on the screen and change dynamically as the quadrilateral is changed dynamically. At any given positioning of the quadrilateral, the associated names are displayed alongside. Thus the file provides the arbitrary – the names – by simply having the appropriate names displayed. However, the file does not attempt to explain or define any of the names. This is similar to situations that all young children face when they learn their first language. Words appear (are said), and appear within a context. The fact that certain words appear frequently (such as *no* or *yes*) means that a child has an opportunity to recognise them, and improve the way in which that child says the word through checking against the way adults say the word. The fact that words are said within context gives a child the opportunity of developing meaning for those words. Within *Quadinc*, mathematical meaning will come from a student considering the context within which the word *rhombus*, for example, appears. The power to abstract rules and meanings from examples is one which all children have used when learning their first language and which can be utilised in learning mathematics as an older child. Ginsburg (1977), Bruner (1960) and Gattegno (1971) have all made references to the powers that very young children use in learning their first language and yet these are rarely called upon by many traditional teaching approaches.



AB = 5.00 BC = 5.00 CD = 5.00 DA = 5.00

DAB = 90.00° ABC = 90.00° BCD = 90.00° CDA = 90.00°

Quadrilateral
Trapezium
Parallelogram
Kite
Cyclic
Rhombus
Rectangle
Square
A
B
D
C
... Show shape
... Show name(s)
... Hide shape
... Hide name(s)
... Show circle through A, B and C
... Hide circle through A, B and C
... Show diagonals
... Show angle bisectors
... Hide diagonals
... Hide angle bisectors

Figure 3. A screen from *Quadinc*.

Quadinc informs students of what is arbitrary – the names. It leaves to the students' awareness those things which are necessary – properties and relationships. In fact some properties are given, namely the length of sides and interior angles at the top of the screen. This has been done in order that attention can remain at the level of properties such as *this angle is the same as that* rather than attention having frequently to be taken away to the level of calculation such as the use of Pythagoras and trigonometry. There would be many students for whom the requirement to use Pythagoras, for example, to calculate lengths of sides would exclude them from knowing the lengths of sides since they may not be aware of Pythagoras or such calculations may be difficult to carry out. Choosing to provide the lengths and angles means that noticing properties such as equal length sides now becomes accessible for such students. Thus this is a pedagogic decision to make the task of finding properties associated with names more accessible when it may not otherwise have been or have required attention being taken onto a sub-task for a considerable period of time. For example, Ginsburg (1977, p164) wrote about a similar issue regarding the task of noticing commutativity when the sub-tasks of actually carrying out calculations take up so much time and attention for some students:

> Over the years, they work 6+7, 7+6, 7x6, 6x7, 7÷6, 6÷7. At first, the very act of calculation may be so difficult that they focus attention only on individual problems. They concentrate on getting the correct sum for say 6+7 and so they cannot notice that 7+6 yields the same

> result. Or it is so hard for them to do 7÷6 that they cannot see that 6÷7 gives a different answer.

If it is desired that students' attention is with the *results* of such calculations so that the issue of commutativity might be noticed, then asking some students to also carry out the calculations may result in attention being taken away from where it is required since the task of calculating may be sufficiently challenging in itself. For similar reasons, there is a button which can be double–clicked which will show on the screen a circle going through the corners A, B and C. It is necessary that such a circle exists (allowing for certain 'degenerate' cases such the circle having infinite radius) but providing it makes the property of *cyclic* become highly accessible since a student need only attend to whether the fourth point, D, lies on the circle as well.

There are many possible activities which a teacher may provide which help to focus a student's attention on to properties of the quadrilateral. Several are included within the activity tasks in *Active Geometry* of which a section brief section regarding the exploration of a rhombus below is given as an example:

> See if you can state a rule which makes a quadrilateral a rhombus, where the rule is *only* about:
>
> a) the lengths of the sides;
>
> b) the direction of the sides;
>
> c) the angles inside the quadrilateral;
>
> d) the diagonals of the quadrilateral (use the *Show diagonals* button);
>
> e) the angle bisectors of the quadrilateral (use the *Show angle bisectors* button).
>
> (Note: not all of these may be possible)
>
> Check each rule *carefully* to see whether *any* quadrilateral which obeys the rule *must* be a rhombus. One idea is to pass on your rules to someone else and see whether they can find a quadrilateral which obeys the rule but is not a rhombus. If they find one, then you need to think again about your rule.

Through such activities the names are practised (in order to assist memory) whilst attention is placed with properties and relationships (in order to educate awareness). The simultaneity of names and properties appearing and disappearing together helps develop an associative bond between the names and the properties of the quadrilateral. The appearance of a name indicates that associated properties are present within the quadrilateral. The student's role is to abstract from examples which properties remain the same when a certain name appears on the screen. Meanwhile the use of the name itself is being met and practised again and again.

Summary

Viewing the mathematics curriculum in terms of arbitrary and necessary can assist in clarifying the role a teacher has with respect to whether or not to inform students of certain parts of the curriculum. It also helps the way in which software is developed so that what is arbitrary is provided and is practised whilst working on the necessary. This way memory is assisted whilst awareness is educated. This contrasts with the use of tests as the main method of practice (where awareness is rarely educated) or treating properties as things which have to be memorised as well (thus placing a burden on memory as well as not educating awareness). Asking students to use memory for the arbitrary, and awareness for the necessary, means that students in mathematics classrooms can begin to use the powers they used so effectively as young children, such as the power of abstraction.

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MEASURING GROWTH IN EARLY NUMERACY: CREATION OF INTERVAL SCALES TO MONITOR DEVELOPMENT

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There is a need to measure learning and quantify growth. This is particularly important when looking at any curriculum developments and programs to improve teaching and learning. One off assessments (eg TIMSS) provide a basis for comparisons but do not inform curriculum, teaching or an understanding about the learning process. This paper reports on the development of a scaling process that allowed the assessment to be driven by a model of learning and mathematical development rather than the statistical process. The process then allowed the scales to be transformed to interval scales enabling growth to be investigated and comparisons to be made between groups.

When the Early Numeracy Research Project (ENRP) began at the start of 1999 there was a need to develop a comprehensive and appropriate learning and assessment framework for early numeracy. Drawing upon Australian and overseas research on young children's mathematics learning, the ENRP team developed a framework of the key "Growth Points" in the early learning of mathematics. For teachers these growth points, from a number of mathematics curriculum areas, can be used to describe children's numeracy development and to inform their teaching. For example, those in the Number area include Counting, Place Value, Addition and Subtraction, and Multiplication and Division. The process of development of the framework and its associated assessment interview is described more fully elsewhere (Clarke, Sullivan, Cheeseman & Clarke, 2000; Clarke, Gervasoni & Sullivan, 2000).

The purpose of the framework was to inform teachers in their planning and implementation of curriculum and to provide a basis for assessment which would enable children's mathematical development to be monitored. The assessment was to provide information for the research team on the development of aspects of numeracy, but it was also to inform teachers, enabling them to learn more about the thinking and development of individual children in their classes and to plan their teaching accordingly. Assessment, in the form of a 30-40 minute task-based interview, was developed based on this framework. The nature of this assessment is essentially different to the one off standardised assessments.

- It is developed from a structure of mathematical development rather than from a content base.
- It does not require all children to answer all the same questions but is rather structured so that students stop answering questions within a domain once there is a lack of success.

- It allows for a focus on the approaches the child uses as well as the answers given.
- Its purpose is to inform teaching as well as to provide information on children's mathematical development.

Data collected from interviews with over 5000 children in each of March and November in 1999 and 2000 have led to minor modifications of the framework and enabled monitoring of some aspects of children's mathematical development. To date, 4-6 Growth Points have been developed for each domain. It is possible that more Growth Points could be added, as the research and the interview protocols are extended to higher grade levels.

The ENRP is a large scale project looking at a whole school approach to school improvement in early numeracy. It is a developmental project with a team of researchers working with all teachers of early numeracy in the 35 trial schools. One aspect of investigating the development of early numeracy is to compare growth between students in the 35 trial schools which have been chosen to be representative of region, size and socioeconomic indicators and the matched 35 reference schools, at each of the grade levels involved. Another purpose is to study growth and inform teachers of the way children learn and their numeracy development. Such comparisons are obviously facilitated by the use of an interval scale of measurement. While the growth points were chosen to be important ideas and are not exhaustive but rather a set of sign posts, they were not chosen with any interval properties in mind.

One approach which has been used to look at underlying scales with interval properties in recent years has been Rasch modelling (Andrich, 1988; Pirolli & Wilson, 1998). The data from the interview are not the equivalent of students answering items on a written test as the interview within a domain ceases when errors are made so the children do not all answer the same set of questions, and, although some questions may be common, the full range is not appropriately sampled. Although Rasch models do allow for missing data, the cutting off of the questions once lack of success is experienced makes the use of Rasch modelling problematic.

The study of growth with such a large number of children and many domains would be enhanced by the use of an interval scale. Without an interval scale, growth differences between children are only clear when the starting points are the same. An interval scale would enable more appropriate analysis of such a large amount of data. This paper looks specifically at a new approach to the development of interval scales and reports some findings based on those scales.

Analysis

A large sample of students has been involved in the two years of the project – to date 8681 students from 68 schools balanced with respect to region, size and socioeconomic indicators. Not surprisingly, the frequencies of responses in each domain of the interview strongly suggest that the growth points do not form an interval scale. Figure 1 shows the distribution of Growth Points for the domain *Counting* in two successive years. The distributions are irregular, but consistent from year to year. The uneven distribution suggests that Growth Point 2 spans a wider range of performance than Growth Points 1 and 3 on either side. Similar patterns occurred with other Growth Points. The stability of these patterns over successive years indicates that they reflect a property of the measurement scale, rather than arising from peculiarities of the samples. Essentially, the implication is that the student development from Growth Point 1 to 2 may not be the same as that from 2 to 3, etc. While this is generally true of achievement test scores, it inhibits making comparisons between the growth of one group and the growth of another, unless they start at the same point. A key purpose of the ENRP is to demonstrate growth to teachers in a way that is meaningful to them, and to find how this growth is enhanced by the whole school approach and professional development that the project offers. For this reason, we have investigated adjusting the location of the Growth Points along a scale of achievement so that they represent achievement within a domain on a scale with interval properties.

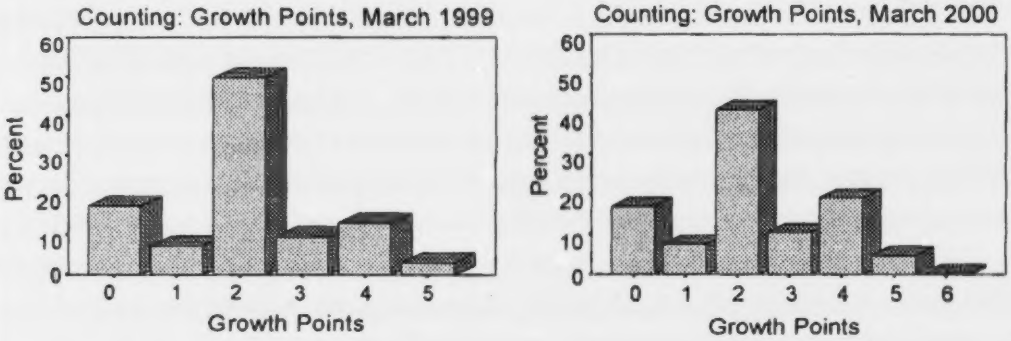

Figure 1. Distribution of growth points for the counting domain in March 1999 and March 2000.

Creating an interval scale

The major assumption that is made in the scaling process that follows is

> *that the nature of the distribution of learning in the population of children in the first three years of school is normal.*

The measurement of other attributes of a person such as height, with a cohort like this one, do approximate a normal distribution. As long as there is a continuum of learning in the domain, beginning before children commence school and extending beyond grade 2, the spread of children within the domain should approximate to a normal distribution. We can then regard the Growth Points as indicators of an underlying normal distribution, cut into slices of unequal width. The sample of students used is wide enough and representative enough that, given the above assumption, there is an underlying normal distribution of performance. It is not uncommon for researchers to assume a normal distribution for a measure, even when there is no particular reason to expect it. We emphasise that in this case, the assumption is not made lightly. The justification for this is that the sample is large, and that the schools have been selected to be representative with respect to geographic area, school size, and socioeconomic indicators. The test is whether scaling from two different populations yields similar sets of scaling points.

This adjustment of the location of the Growth Points along a scale was done by standardising the scores, then re-scaling them using the mean and standard deviation of the original set of Growth Points. With this process it is reasonable to expect similar scaling points from populations that differ in mean and standard deviation. In detail the steps were

1. We chose the median student to be representative of the growth point then calculated the proportion of students below, thus giving a cumulative frequency distribution. This means that below the median student were half the students on that growth point plus all students on lower growth points.
2. Assuming a normal distribution, we found the z-score of the median student for each growth point by using the probability from the relative cumulative frequency. The probability of being below the median student in a growth point is the number of students below (calculated in step 1) divided by the total number of students. This was then looked up on a normal distribution table to give a z-score for the median student representative of the growth point.
3. We translated this z-score back to a distribution with the same mean and standard deviation as the original data.

The Trial and Reference Group data from the March 1999 interviews were rescaled in this way. These are two large populations (n = 3639 and 1219, respectively) representing a wide range of geographic regions and socioeconomic characteristics. Table 1 shows the transformations, as applied to these two populations.

Table 1: *Scaling of growth points for Counting: trial and reference schools, March 1999*[a]

	Trial Schools		**Reference Schools**	
	N	**Scaled Growth Point**	**N**	**Scaled Growth Point**
0	588	0.287	254	0.256
1	242	0.984	112	1.002
2	1802	2.016	612	1.989
3	365	3.064	103	3.094
4	527	3.721	107	3.678
5	115	4.871	31	4.620

[a] Rows are shaded for n < 50. With so few people in the score range, the accuracy of the scaling may be problematic.

Comparison of the scaling from the two distinct populations indicates that it matters little which population is used to scale the Growth Points. For example looking at the scaled growth points for growth point number 2 in table 1 shows the scaled score for the trail schools to be 2.016, only just above the hypothetical growth point 2, while for the reference schools the scaled growth point at 1.989 was only just below. The difference between the two independent populations for this growth point was only .027. For 5 of the 6 Growth Points shown here in Table 3, the difference is less than 0.10 in magnitude, and the remaining one arises from a Growth Point with small n (less than 50, shaded in Table 3.) and has a difference of about 0.25. We assert that the process we have adopted yields results that are relatively invariant across populations, provided there are sufficient data at each Growth Point.

The scaling process yields scores that can be mapped onto scales with any size units. Given that teachers and others may still wish to interpret growth information and mean scores in terms of the Growth Point scales with which they are familiar, a case can be made that the most suitable mapping is onto a scale in which the two endpoints remain fixed, and the intermediate points are adjusted. Table 2 shows the results of such a scaling in the column labelled Scaled Growth point 0-6 Scale, using the entire March 2000 cohort, including the grade 3 and 4 children, (Trial and Reference schools) as the scaling population. The process followed was the same as the one above to give the growth points in the column headed Scaled Growth Point but with a fourth step included that stretched the scale from 0 to 6 while maintaining its interval nature. Again looking at the hypothetical growth point 2 in Table 2 the scaled growth point obtained by using the original process but the different population was just above at 2.08. One the re-scaling has been done to set the lower and upper growth points at 0 and 6 this scaled score becomes 1.83, just below the 2.

Table 2: *Scaling of growth points for Counting: trial and reference schools, March 2000*

	N	**Scaled Growth Point**	**Scaled Growth Point 0-6 Scale.**
0	983	0.24	0.00
1	444	1.07	0.82
2	2440	2.08	1.83
3	642	3.02	2.76
4	1320	3.74	3.47
5	492	4.92	4.65
6	69	6.28	6.00

Use of the scaled Growth Points

One of the ways in which the research data might be used is to compare groups. With some confidence in the interval properties of the re-scaled Growth Points, we are now in a position to reach conclusions about relative growth, even from comparison groups that have different starting points. Figure 2 shows growth on the *Counting* domain. The growth over the 1999 and 2000 academic years is apparent, as is the slower growth over the 1999-2000 summer break. Comparisons across cohorts show the gap between grade levels. In each case the graphs are in pairs showing trial and reference schools in a particular cohort. For example the upper two graphs on the left are the cohort of students that began in grade 2 in 1999 and in 2000 were in grade 3 and thus were not tested as they were no longer part of the project (a small sample was tested in 2000 but not as part of the main project). The two lines below these represent the cohort of students who began in the project as grade 1 students in March 1999 and were tested in Nov 1999 and in March and November of 2000. It can be seen that in the trial and reference schools grade 1 students were very similar at the beginning of 1999 but that the growth in the trial schools was greater during both 1999 and 2000.

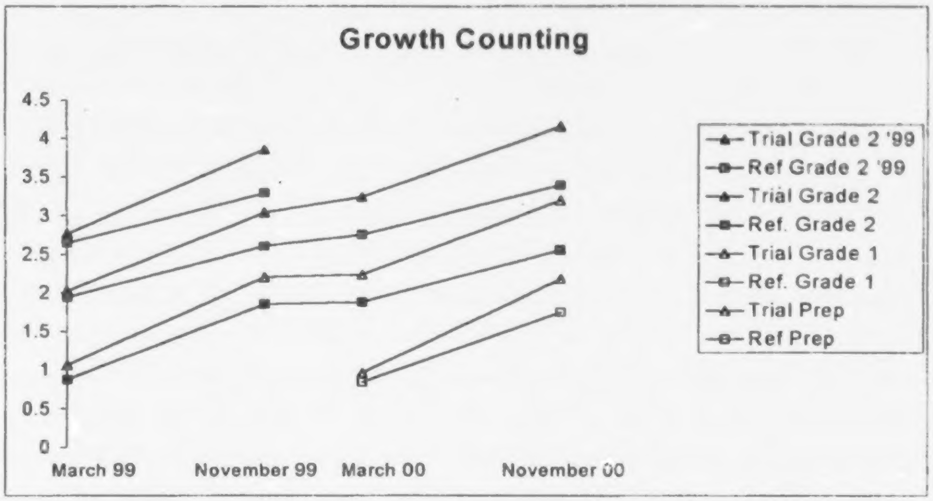

Figure 2: Growth in *Counting*, 1999-2000 in trial and reference schools

The interval scales can now be used to compare growth for investigating successful aspects of the program and to inform teachers of growth for their students and classes. The scales have enabled investigations such as the growth is typical in a year and the extra growth that occured as a result of the ENRP program provided in schools. Teachers involved in ENRP understand a scale of measurement framed in terms of Growth Points, and welcome information about achievement in this form. Until we were confident that Growth Points could be manipulated into an interval scale, we were unable to provide this information in a way with which we could feel comfortable. Because the transformation described previously is fairly close to the unscaled Growth Points, we can present information about means and variances in terms of scaled Growth Points, and remain confident that this information will not mislead users who interpret them in terms of understanding and skills possessed by their students. Because they are close this also means that we can refer to the growth points as whole numbers knowing that they are close to the scaled values. In terms that are readily understandable by teachers in the project, we can say that typical gains for students in Reference Schools are a little less than one Growth Point per year in *Counting*, but a little more than one Growth Point per year for students in Trial Schools. Very little of this gain occurs between November and March, a good part of which is summer holidays.

The differences found between trial and reference schools are statistically significant. Rowley and Horne (2000) reported an analyses of variance using the scaled March achievement as the covariate with the independent variables of schoolcode, grade level and ENRP participation (i.e. Trial versus Reference Schools). For all six domains used in both years, they found a significant ENRP main effect, indicating that the effect of ENRP on achievement can be detected, over and above that of prior achievement, school attended and grade level. A repeat analysis done with the year 2000 data supported the same findings.

Finally, the interval properties of the scaled Growth Points allowed us to test the applicability of particular models to the data. In this example, the models included, separately at each grade level, in logical-temporal order:

- The school that the child attended (as represented by the matched pair of schools, one Trial and one Reference school)
- The child's prior (March 1999) achievement
- The provision of ENRP Professional Development in the school.

Regression analysis allows us to ask whether each of these in turn produces a significantly better-fitting model than those that precede it. In essence, this tells us how well is achievement in November predicted by the school attended, achievement in March, and by whether or not the school participates in the ENRP Professional Development program. And finally, we can also see how much difference does the ENRP Professional Development makes by using the re-scaled growth points.

Table 3: *Regression coefficients for provision of professional development for Counting*

	1999	**2000**
Preps	0.256	0.385
Grade 1	0.376	0.381
Grade 2	0.466	0.481

Table 3 shows the unstandardised regression coefficients for ENRP Participation in the full model, which includes School, Prior Achievement, and ENRP Participation. These coefficients may be interpreted with some precision as the expected advantage in achievement, assessed in Growth Points, that students gain through their school's participation in the ENRP program. In general, students may expect to gain from 25 percent to 48 percent of a growth point per year more if their school participates in ENRP program.

Conclusions

In this paper, we have presented the construction of interval scales based on a set of protocols for one-to-one interviews of children in their first three years of schooling that can be used to assess their development on curriculum-relevant domains of mathematical learning. Because of the nature of interval scales, the assessments obtained are useful for assessing and charting growth.

Using the Growth Point Scales in this way, we have shown some representations of growth in achievement, and demonstrated a modest but consistent advantage in achievement from participation in the ENRP program. This rescaling process has provided data which has been used for comparisons to investigate differences between groups looking at the success of programs and, perhaps more importantly, to look at children's development in early years numeracy.

The scales for separate domains are referenced to Growth Points within each domain, but not to each other. Future use of the data will enable typical growth in the domains to be investigated. At the start of grade 2 before the project began, for example, students were typically at Growth Point of 3 in *Counting*, but just at growth point 2 in *Multilpication and Division*. Equally, a typical year's growth is approximately one Growth Point in *Counting*, but only half that in most other domains. While these comparisons are possible we also do not want to lose sight of the many other developments which are not measured by these interview based scales. The comparisons now possible are being used as one aspect of deciding on case studies to be carried out in 2001.

Acknowledgements

The main team of researchers involved in the ENRP includes D. M. Clarke (director), J. Cheeseman, B. Clarke, A. Gervasoni, D. Gronn, M. Horne, A.. McDonough, P. Montgomery, G. Rowley, and P. Sullivan. All of this team have been involved in the research presented here.

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PROGRAMMING RULES: WHAT DO CHILDREN UNDERSTAND?

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Abstract

We are working to build computational worlds where children can play, design and program videogames. Videogames represent rule-based systems which are expressible in non-textual animated ways. We describe the different ways children (aged 7 to 8 years) articulate a simple rule they have programmed themselves. The results indicate that rule expression is shaped by the type of prompt to make the articulation (to predict, describe or to explain), the narrative context of the game and the medium of expression (computational, spoken or written).

Background

This study concerns children's understandings of rules they have constructed by programming a computer, in the general context of building their own video games. There is much research into children's understanding of causal evidence and logical reasoning, much of it undertaken in the field of developmental psychology. Since the 1970's, evidence has accumulated that children rarely argue solely on the basis of universal laws of logic or domain-independent abstract rules. There are different interpretations as to why this might be the case: that children call up pragmatic reasoning structures derived from experience in context (Cheng and Holyoak, 1985), or that their arguments are derived from knowledge and the way it is structured (Ceci, 1990). How far children are aware of the logical necessity of even their correct conclusions is still a matter of debate.

Following a careful analysis of children's naturally occurring arguments, Anderson, Chinn, Chang, Waggoner, & Yi (1997) report that children tend to omit parts of the logic of their argument (premise, warrant or conclusion), to be cryptic when mentioning the known or obvious, and elliptical when expressing their position, giving no more information than was necessary. Nonetheless, they conclude that the children's arguments were logically complete: that is, the framework used was not inconsistent with deductive logic. The fact that everyday language is likely to be only a partial indicator of reasoning calls into question how children's reasoning might best be investigated. Anderson et al. report that children used words such as 'because', 'so', and 'therefore', in arguments which included *modus ponens*; so simple reference to linguistic pointers is no infallible guide to logical reasoning.

Aside from research on children's logical understanding, we briefly allude to a further strand of research which is an important source for our analysis. This concerns the literature on learning by designing, creating and debugging meaningful 'external' artefacts (the *constructionist* paradigm, Harel and Papert, 1991) and our own work (Noss and Hoyles, 1996). However, an assumption behind our work had been that the expression of complex ideas and its communication to a computer necessitated collections of symbols in the form of textual strings and it was this need for symbolic representation that was crucial for mathematical learning. This assumption is somewhat anachronistic, as we have begun to explore the potential of programming within a non-textual environment in which programs are created by directly manipulating animated characters and animations themselves are the source code of the language.

The study forms part of the *Playground*[1] project (http://www.ioe.ac.uk/playground) in which we are giving children (aged 6 to 8 years) the opportunity to construct creative and fun computer games (see also Harel, 1988 and Kafai, 1995) and at the same time, offer them an appreciation of — and a language for — the rules that underpin them. Broadly, we conjecture that through designing, building and playing their own computer games, children will think differently about ideas like causality and inference, and will find new, more formal and complete ways to express their thoughts about rules. Given the age of the children, it is obvious that we cannot expect them to build symbolic representations of rules. Rather we have based our work on *ToonTalk* (Kahn, 1999) where programming involves the 'training' of animated robots by example, and essentially without text. The conditions which determine subsequent performance of the actions at run time can be generalised or specialised after the training has taken place.[2]

Our research is framed by two questions: i. How is children's understanding and expression of rules mediated by a programming language in which the rules are available for inspection and change and ii. How far do children's descriptions of the rules they have programmed throw light on their understandings of causal reasoning?

The Meaning of a Rule

What do we mean by rules in the context of young children building their own computer games? As long ago as 1932, Piaget was interested in games as complex systems of rules and endeavoured to trace the development of the practice and consciousness of rules in young children (Piaget, 1932). He concluded (largely through an analysis of boys playing marbles), that between ages 7 and 10 years, children were unable to codify rules, although they were able to play games according to social conventions. A conscious realisation of the rules of their games and seeing that they could be changed, Piaget argued, only developed at age 11 –12. For Piaget (and Vygotsky as well), what changes over time is the explicit recognition of the rules of a game.

In our early interviews with children (aged 6 to 8) we asked them to tell us about their favourite games (both on and off the computer) and the rules of these games. We found most only described constraints ("you mustn't hit other children") and not the actual rules designed into the game. From our case studies of the children programming their own games, we have noted that children have indeed become more aware of rules[3], but also that they adhere to different types of rules that we have categorised as follows:

> *Player rules* – a regulation that must not be transgressed: "You must not hit the sides of the maze".

Player goal – a maxim or formula that is generally advisable, but not compulsory to follow: "Score as many points as you can".

System rules – a generalised statement that describes what is true in most or all cases: "When Theseus touches the minotaur with the sword then he will die".

Our interest focuses on children's understanding of the universal system rules that are hard-wired into the computer game (that is, built by programming). Clearly, any programming environment brings its own complexities: e.g. how the rule is expressed in the language, and how virtual 'phenomena' like "dying" are translated into programmable actions (like "disappearing")?

We have engaged in two types of studies to investigate children's understandings of rules: i. extended case studies tracing the evolution of both games and the children's understandings of the rules (see Goldstein & Pratt, submitted); ii. clinical task-based interviews which probe how children think about rules (in this case, conditions and actions), and the extent to which the expression and application of rules are mediated by the tools available (as well as what light this may throw on existing developmental sequences). The clinical interviews were based around tightly controlled, pre-designed tasks with specific aims (in contrast with the largely exploratory and open computer activities that characterised the case-study work). As well as interacting with the computer, the children, working individually or in pairs, completed worksheets. A researcher, who made metacognitive prompts to probe descriptions, predictions, and expectations, was present with each pair. All the interviews were audio-taped and transcribed. We report on one of the task-based interviews which we conducted with 9 children aged 7/8 years. The interviews took about one hour.

A Task-based Interview: The Cat and the Milk

The aim of the task was to probe children's thinking while programming a simple rule that involved a sequence of condition and action, 'if touching a particular object then do something'. The first part of the task involved constructing a rule with a warrant between a condition and action that had a common-sense explanation ("if a cat touches the milk it meows") followed by an application in a less realistic setting.

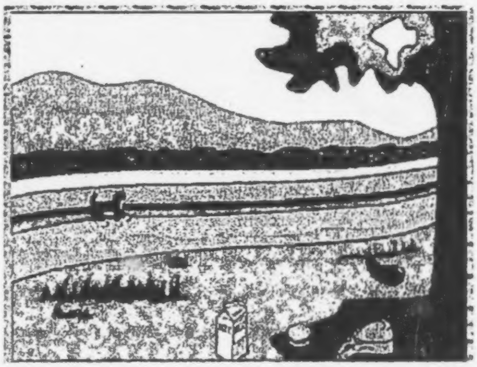

Figure 1: the *cat and milk* task

The children were given the scenario illustrated in Figure 1, in which the cat had already been programmed so that it could be moved about with the mouse. In ToonTalk, this means that a robot had been trained to move with the mouse and placed on the back of the cat. This piece of pre-written 'code' we call a *behaviour*: it can be saved on its own but is also transferable to any object, to make it move similarly. The children's task was to program a robot (to put on the back of the cat) so that if the cat touched the milk carton it would meow. After this, we wanted the 'move with mouse' behaviour to be transferred to the milk carton, so it could be moved (a fantasy element) while leaving the meowing behaviour on the cat.

The researcher's role was to ask the children to describe what they observed on the screen, to explain what was happening and to predict what would happen after programming the new rule. The following prompts were used as guides:

Description/explanation.	Imagine you have to describe what is going on the screen to a friend who cannot see it: What can you see? What is happening? Why is it happening?
Build the rule and describe it.	What can you see? Is it working? What does it do?(so a friend can know)
Add rule and predict what will happen.	What will happen? When will it happen? Why will it happen?
Use rules/behaviours in other contexts	What will happen? When will it happen? Why will it happen?

Specifically, the researcher asked the following questions:

After playing with the cat: "Why can you move the cat about?" *(explanation)*

After training the robot: "Describe what it would do" (*description*)

After putting the behaviour on the back of the cat: "Do you think the cat is going to meow or not when it touches the milk?" (*prediction*)

After trying out the new rule: "What is happening (when they move the cat)? Why?" (*explanation*)

After taking the move behaviour from the cat and giving it to the milk: "What do you think will happen when the milk touches the cat? (*prediction*)

After testing it out: "What is happening? Why?" (*explanation*)

Analysis

After the data was transcribed, we grouped the children's explanations of causality into two general categories. The categories were devised partly from theory (see Donaldson, 1986), but also as a result of a pilot study conducted in 1999.

Category A: Student-centred explanations

A1. Real World (Physical laws): using analogies of everyday events or real events, "it does that as it is a ball and balls bounce."

A2. Intentional World (Psychological laws): giving the characters motivations and intentions, "cats hate meat so they growl."

A3. Fantasy (Personal Imagination):

  i. shared narrative about the game, "the witch casts a spell on you so you die."

  ii. idiosyncratic.

Category B: Formal computer-centred explanations

These include explanations by reference to the logic of the system or according to the inevitable outcomes from programming. It is possible, however, to distinguish these explanations along a general-specific dimension:

B1. Explanations derived from prior knowledge of general computer systems.

B2. Explanations derived from knowledge of computer games.

B3. Explanations derived from general features of the Playground environment: "there is a behaviour on the back of the bear"

B4. Explanations derived from specific aspects of the programming environment: "the rule is programmed so the robot plays a sound when the bear is touching the cheese".

Results

We present the main types of response to each of the questions and illustrate them with quotes from the children. We also annotate each type of response with a conjecture which is informing our current observations.

Explaining what was observed on the screen: After experimenting with the scene, all the children were asked, "Why do you think the cat is moving?" Four children responded in a way that could be categorised as formal, but general since in ToonTalk all objects are moved by a virtual hand (B3). An example is:

I: Why can you move the cat about?
Joe: Because that's the main thing really.
I: What do you think makes it move?
Joe: Your hand.
I: What makes him move up then?
Joe indicates the mouse.

Another example from Hazel:

I: Why do you think the cat can move?
Hazel: Because you're moving the mouse so it moves the cat as well.

But the fact that neither chose to talk specifically about the behaviours on the back of the cat that made it move, maybe explains their later 'psychological' (A2) as opposed to 'formal' responses, for example:

I: Why can you move the cat about?
Joe: Because it's a creature.

In contrast, the responses of five of the children were categorised as formal/specific; they clearly made the connection between actions on the screen with programmed behaviours.

I: Why do you think the cat is moving?
Sophie: Maybe she already has the behaviour.

However, it was only on inspecting the behaviour in the course of the interview that most children were able to appreciate the mechanism involved. For example, having looked at the back of the cat, Sophie was more specific in her explanation:

Sophie: 'Cos it's got that behaviour 'I move with mouse'.

> Conjecture 1: Children need to link observations with behaviours as a first step to explaining the outcomes of the rules they have observed.

Describing a new rule: After the children had trained a robot to instantiate the rule, 'if touch milk, then meow', we asked them to label the robot so another child would know what it did.

Four children omitted the condition in their descriptions, as illustrated by:

I: First we have to type a label that describes what it does.
Alice: Meows!
I: Jane, is that what you want?
Jane: Meow!

Three children were unsure what to write and wanted to add comments like "this is a fantastic game", while the remaining two wrote complete descriptions. When the behaviour or rule was 'contextualised' by putting it in the back of the cat, *all* the children correctly *described it verbally* e.g:

I: If somebody ... found your cat, how would you describe it?
Sophie: It meows when... um... it touches the milk carton...

However, when *writing* a description, their labels varied from uninformative "Fofo" (2 children), to action-only "Meows" (3), to explicit/complete but contextualised, "When the cat touches the milk it meows" (2) to decontextualised and complete, "It meows when it touches the milk." (2). These disparities between what children programmed, what they said and what they wrote, was characteristic of other rule descriptions.

> Conjecture 2: Children are better able to articulate explicitly all the parts of a rule when it is programmed, and/or contextualised in a realistic situation, but the completeness of their expression varies with the medium.

Predicting the consequences of a rule as programmed: We asked the children to predict what would happen when the cat with its new behaviour was moved about the screen. Six children predicted correctly: e.g.

I: Before we play the game, when you move the cat around, what do you think will happen?
Sophie : When it touches the milk carton it's going to make a cat noise.
I: And when it touches the cake, what will it do?
Sophie: Nothing.
I: Try it and see... why is that, by the way?
Sophie: Because we've made a behaviour about the milk carton and not the cake.

Strangely, even after constructing their program and giving it the cat, the others were unable to predict what would happen with any confidence.

I: Now, without you having to press the space bar or do anything, do you think the cat is going to go meow when it touches the milk?
Hazel: No!
I: No, why not?
Hazel: Maybe we have to do a bit more stuff to get it to do it. [...]
I: Do you think that cat is going to meow when it touches the milk?
Both: No!
I: Why not?
Joe: Because we haven't ... we've done something wrong.

> Conjecture 3: Even after having constructed a rule explicitly and articulated it in natural language, children may be unsure of its consequences; that is not appreciate its inevitability[4].

Predicting the implied consequences: We then asked the children to put the 'move behaviour' on the back of the milk carton (instead of the cat) and predict if they will hear the meow sound when the milk touches the cat. *All* of the children were either completely certain this would *not* happen, or were very unsure about it:

I: If you move the milk so that it touches the cat, will you hear a meow sound?
Hazel: No, because we haven't told it to do that.
I: We haven't told what to do what?
Hazel: We haven't told the milk . . when it touches the cat, say the milk runs around, we haven't told it that when it catches the milk [cat? CH] *it says meow.*
I: We haven't told the milk that when it touches . . .
Hazel: We haven't flipped the milk over and when, so we haven't wrote when you touch the cat it will say meow.

It seems that the implications of the rule (that is, the symmetry of the formal relation of touching) did not 'transfer' to an unfamiliar situation of milk moving.

> Conjecture 4: Programming the rule correctly and seeing its consequences, does not necessarily lead to a formal understanding of all the rule's implications.

Explaining what was observed on the screen: After seeing what actually happened and hearing the meow when the milk was moved to the cat, all the children began to appreciate the symmetrical nature of touching, albeit often with an element of psychological causality:

Jane: I know why. 'cos the cat is touching the milk, and the milk is touching the cat.
I: When you move that milk to the cat, what happens?
Jane: Meowing.
I: Now, who's meowing?
Jane: The cat.
I: Why does the cat meow?
Jane: Because he likes the milk!

Conjecture 5: Playing with the rule and observing its implications, begins the process of appreciating the logic of the implied consequences of a new rule.

Conclusions

Overall when the mechanism of the behaviour is articulated and 'connected' to an object, children are better able to explain and predict what is happening in their game and come up with a rich mix of formal, narrative and psychological explanations. Yet even if children program a rule correctly, they may not express it completely in spoken or written language, especially in a decontextualised form. More interestingly, children may not predict the implied consequences of a formal rule they have programmed, particularly when the narrative does not 'make sense'. This suggests that the formal means of expression may not yet be fully integrated with verbal and written articulations. We have noticed that with more experience, children are able to integrate their various descriptions and to make simultaneous and multiple interpretations of what is going on – in terms of the narrative requirements of the game (the cat needs the milk to be healthy) and the formal requirements of the rules they program into the game.

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[1] European Union *Esprit* Grant No: 29329.

[2] Details of how ToonTalk works, its design principles and some applications can be found at http://www.toontalk.com/

[3] We should note here that there are many facets to a game other than rules, which are just as important in game construction and play, namely constraints, descriptive and declarative aspects.

[4] We recognise here that there may well be an element here of lacking confidence in their programming skills.

Investigating of The Influence of Verbal Interaction And Real-World Settings In Children's Problem-Solving And Analogical Transfer

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Abstract

Eighty fourth-grade children in mixed-ability dyads were randomly assigned to four experimental conditions: with and without real-world settings and verbal interaction with peers. Dyads of children were asked to solve two daily mathematical problems in the first phase. Three weeks later children solved three daily mathematical problems as delayed transfer tasks after completed the first tasks. The results show that verbal interaction accompanied with real-world settings situation is the most effective way to improve children's performance in problem-solving. Either verbal interaction or real-world settings helps children's analogical transfer rather than solving problems individually without any real-world settings. Moreover, children tend to mix addition, multiplication and counting to solve problems through verbal interaction with peers and tasks.

Introduction and theoretical Framework

It's plausible that a stronger connection between school mathematics and everyday mathematics will enhance mathematical competencies and learning (Streefland,1991). Therefore, one of the focal points within problem-solving research is the impact of real-world settings and how children apply their skills and knowledge through verbal interaction with peers to find respective answers in solving everyday mathematical problems.

According to Piaget's theory, the young children are constantly experimenting with objects, language and situations to understand more about the world (Rogoff, 1990). A child is supposed to be an active experimenter who discovers facts and relationships when s/he is presented with materials and situations that encourage the design of his own experiments. This will in turn lead to deeper and more long-lasting knowledge than will a rote memorization of facts presented by teachers or in textbooks. Furthermore, everyday mathematics should be thought of as a process of knowing in which the same activity (arithmetic) takes different forms across situations and occasions (Lave, 1988). Thus the practices and the problem context in which people engage are supposed to be important for problem solvers, especially for the young learners.

The goal of making mathematics meaningful to students is to connect their everyday life understanding, in particular where arithmetic is concerned, to formal learning (Freudenthal, 1991; Streefland,1991). Strategies and patterns of problem-solving, initially are supported by situations and contexts, which in the long run are superseded by abstractions. Therefore, even though the practices and contexts of everyday mathematics differs from contexts of school mathematics, it is expected that students are able to transfer their mathematical competence in everyday context to do similar arithmetic in school or test contexts. Empirical studies have found that people can carry out new procedures or solve novel problems that are quite similar to those on which they had previously learned (Vousniado & Ortony, 1989). This study will explore the effects of real-world settings on children solving problems and applying their knowledge from a previous setting to novel and formal mathematical problems.

According to a number of models in cognitive science, cooperative verbal interaction with peers is a fundamental component in cognitive processes that operates at a deep level of conceptual understanding, for instance analogical reasoning in solving mathematical problems (e.g., Huang, 1997), as well as mathematical operations skills and application skills (e.g., Fuchs, Fuchs, Bentz, Phillips & Hamlett, 1994). From the social cognitive point of view, when students interact with their peers, they are exposed to new strategies, terminology and ways of thinking about problems, which may in turn affect their problem-solving behavior. For children as well as for their social partners, engagement in shared thinking yields the opportunities for development of greater skill and understanding (Rogoff, 1990).

Although some research found inconsistencies and contradict challenges in peer verbal interaction (e.g., Orsolini & Pontecorvo, 1992; Webb, 1989), the usual findings from empirical support for the premise is that one-to-one discussion environments might remove many of the barriers that prevent students from asking questions and learning actively (e.g., Fuchs et al., 1994; Fuchs, Fuchs, Hamlett, Phillips, Karns, & Dutka, 1997). Studies showing positive interactions and learning between high- and low-achieving classmates have incorporated thorough training in how to interact constructively (e.g., Fuchs et al., 1994; Fuchs et al., 1997; Graesser & Person, 1994). Therefore, casing this study within the context of same-age, good-and-poor dyads, the researcher expected to examine the effect on cooperative verbal interaction in dyads accompanied with/without real-world settings in solving daily mathematical problems and delayed transfer tasks.

The purpose of this study is to examine how verbal interaction and real-world settings affect children's skills in solving daily mathematical problems and analogical transfer. Children's strategies in solving multiplicative problems through verbal interaction with peers and tasks were also analyzed. In light of previous work on the importance of verbal interaction and real-world settings, the hypothesis supposes that either verbal interaction or real-world settings would significantly influence children problem-solving as opposed to solving problems individually without verbal interaction and real-world settings. Furthermore, the other hypothesis is that there is an interactive relationship between problems-solving conditions and solvers.

Method

Subjects. Eighty fourth grade children participated in this experiment. They were selected randomly from a public school in Taipei city, Taiwan. Subjects had been taught and had experience in verbal interaction and problem solving with small groups cooperatively in daily mathematics courses. All subjects worked in mixed-ability pairs. Therefore, good and poor solvers in this study were determined based on the mean of subjects' achievement in the previous semester's mathematics course. If a subject's score was higher than or equal to the mean score, s/he was defined as a good solver. If a subject's mathematics achievement score was lower than mean score, s/he was defined as a poor solver.

Design of the study. There were two problem-solving phases included in this study. The first phase was focused on dyads of children in solving two multiplication problems. The second phase was focused on children's transfer of learning in solving three novel multiplicative problems. The contexts of problems designed in this study were derived from children's daily life and Taiwanese society. The item difficulty index and item discrimination index of the five problems had been tested from 97 forth grade students. They were randomly selected from a public elementary school in Taipei, Taiwan. The values of item difficulty index of these five problems were from .39 to .50. The values of item discrimination index of these five problems were from .58 to .96. The idea of problem settings and arithmetical structures of these problems are referred to the research of Franke (1998) and Ruwisch (1998). The mathematical structures contents of the five problems are listed on Table 1. The design, materials and procedures in two phases are illustrated as follows.

A. *The first problem-solving phase.* 1.Task problems. Two multiplicative problem-solving settings were provided. In the first problem setting, called "Class Camping", the children were required to buy items for a camping trip with 30 classmates. In the second situation, called "Lantern Festival", children were required to buy colored envelopes for pasting on three different sizes of posters for Lantern Festival. Children had to determine the number of packages of items needed for solving these two tasks and write down their answers on shopping lists. Both "Class Camping" and "Lantern Festival" problems have similar arithmetical structures but differing in contexts. The underlying mathematical contents of "Class camping" and "Lantern Festival" are numbers and area, respectively. 2. Experimental conditions. The problem-solving situations were designed differently shows, three problems in delayed transfer tasks have similar structural features but differing in superficial features.

C. *Evaluation.* Multiple evaluations (on a scale of 0-5) of written responses of each problem item were used. If a subject wrote the solutions on the shopping lists correctly, s/he was given a full score of 5 on each item.

Results

Table 2 shows the summaries of means and standard deviations of four experimental conditions in two problem-solving phases. A one between-subjects ANOVA indicated group comparison on performance. The results revealed that the difference among four conditions was significant, $F_{(3,76)}=7.02$, $p<.001$. The Tukey posteriori comparison analysis among the four experimental conditions suggested that there was a significant difference between I&D and I&ND, as well as I&D and NI&ND. There was no difference among I&ND, NI&D and NI&ND. The hypothesis is partially supported by the results. It suggests that cooperative verbal interaction with peers accompanied with real-world settings simultaneous would significantly improve children's performance in solving daily mathematical problems.

Table 2: Means and the standard deviations of four experimental conditions in two problem-solving phases.

	I & D		I & ND		NI & D		NI & ND	
	M	SD	M	SD	M	SD	M	SD
The first phase	45.70	4.28	31.70	14.20	36.10	12.68	29.25	14.79
Delayed transfer	59.45	14.83	53.65	21.45	57.95	9.36	39.50	23.31

Table3: Means and standards deviations of good and poor solvers in four experimental conditions in delayed transfer problem-solving phase.

	I & D		I & ND		NI & D		NI & ND	
	M	SD	M	SD	M	SD	M	SD
good solvers	65.70	7.42	67.40	3.75	62.70	6.41	54.60	13.43
poor solvers	53.20	17.95	39.90	23.17	53.20	9.68	24.40	21.44

In terms of analyzing good and poor solvers of the four experimental conditions in solving transfer task problems, the data was analyzed with a two (good Vs. poor solvers) and four (four experimental conditions) ANOVA. The results indicated that the significant interaction effect was not found with $F_{(3,72)}=2.56$, $p<.06$. The hypothesis is not supported by the results. However, a significant main effect of experimental conditions was found with $F_{(3,72)}=7.77$, $p<.001$. The results of follow up analysis and the Tukey posteriori comparison analysis for the four experimental conditions suggested children in I&D, I&ND, and NI&D conditions outperformed those in the NI&ND condition. Moreover, there was significant difference between good and poor solvers with $F_{(1,72)}=37.30$, $p<.001$. As shown in Table 3, good solvers demonstrated better analogical transfer performance than did poor ones.

Children's verbal interaction with peers in the discussion groups (I&D and NI&D) when they solved the "Class Camping" and "Lantern Festival" problems were collected and analyzed. Children's strategies used in solving problems were transcribed from videotapes and audio tapes. Two raters categorized children's problem solving strategies from the transcripts. The consistency between the two raters with the result of Kappa analysis was .57, $p<.001$. Table 4 shows the types of children's strategies in solving multiplicative problems.

Table 4: Children's strategies used in solving "Class Camping" and "Lantern Festival" problems through interaction with peers and tasks.

Strategies	Frequency	Percentage
1.Computation using multiplication and division directly.	27	34.6 %
2. Computation using addition and multiplication.	3	3.8 %
3. Computation using multiplication and subtraction.	9	11.5 %
4. Computation using addition, subtraction and multiplication.	2	2.6 %
5. Computation using addition, subtraction, multiplication and division.	2	2.6 %
6. Computation using counting, addition and multiplication.	20	25.6 %
7. Repeated addition.	11	14.1 %
8. Counting by ones or twos.	2	2.6 %
9. Drawing units and counting.	1	1.3 %
10. Calculation in head.	1	1.3 %
Total	78	100 %

Ten kinds of strategies were categorized from the children's peer interaction. Children's strategies used in solving daily mathematical problems are as follows: 1. Computation using multiplication and division directly. Around 34.6 % of the children determined the multiplier then applied multiplication algorithm as well as applied division directly to obtain the quotient. 2. Computation using addition and multiplication. About 3.8% of the children determined the needed amount and applied multiplicative facts and then added up the products. 3. Computation using multiplication and subtraction. About 11.5% of the children applied multiplicative facts to get the product then used the product to subtract the known number. 4. Computation using addition, subtraction and multiplication. Around 2.6% of the children used composite units in repeated addition and used multiplication partially, and then used the known quantity to subtract. 5. Computation using addition, subtraction, multiplication and division. About 2.6% of the children used the mixed computation. 6. Applying counting, addition and multiplication. About 25.6% of the children used skip counting by threes, twos or ones to determine the needed amount. 7. Repeated addition. About 14.1% of the children used a specific number as a unit to add up the numbers until the needed amount. 8.Counting by ones or twos. About 2.6% of the children accomplished computation by counting. 9.Children drew units or drew using a ruler on the given posters then counted the total squares to get the number of items needed. About 1.3% of the children used this strategy. 10. Calculation in head. About 1.3% of the children did not show any arithmetical strategies but stated the answer directly.

Discussion & Implication

Findings in this study clearly support the superiority of verbal interaction accompanied with real-world settings for children working on solving mathematics problems and analogical transfer. If either verbal interaction or real-world settings are provided separately, it is not sufficient to improve the children's performances particularly on their first time solving a specific problem. This may have been due to two reasons. First, a constructive discussion in the process of problem-solving was not found in all dyads. This was concluded from observation of peer discussion during problem solving. Some good solvers tended to solve problems on their own without verbal interaction with poor partners. On the other hand, several poor solvers tended to rely on their partners' explanation without questioning and analyzing. Second, children in the four conditions were able to deal with the items that represented figures shown on worksheets. Consequently, the effect of real-world settings did not significantly improve the children's performances. Although it was shown that the lack of either verbal interaction or real-world settings is sufficient to improve the children's performances on first time problem-solving, it did reveal quite a significant improvement on children's transfer performances.

With respect to the good/poor contrasts, the findings demonstrate the accuracy of transfer problem-solving was higher for good solvers rather than poor solvers. The good solvers seem to provide more procedural explanation, demonstration or checked work. Such instructional style is similar to tutoring and would in turn promote solvers' performances (Fuchs et al., 1994; Fuchs et al., 1997; Graesser & Person, 1994) as appeared in transfer superiority. It is possible that poor solvers tend to watch and listen to their partners with little opportunity to analyze and apply their partners' explanations. When comparing a good solver's in-depth learning with a poor solver's passive learning, it is not surprising to find that the poor solver's performance is inferior to the good one's.

In terms of the strategies applied in solving multiplicative problems, children seem to tend to mix addition, multiplication and counting to solve problems. A lot of fourth grade children might have been aware of the commutative principle of multiplication and the inverse relationship between multiplication and division. Some children counted each item one by one instead of count in multiples. Such perceptual counting is also found in lower graders(e.g., Franke, 1998; Ruwisch, 1998)

Results of this study suggest possibilities for improving children's problem solving. Effective training of peer tutoring and conceptual explanation during for each of the four experimental conditions. Subjects in dyads were assigned to the four conditions randomly. All subjects were provided with written and oral descriptions of the problems when they came in the experimental condition. The four experimental conditions are illustrated as follows. i. Items and Discussion (I&D): Children (n=20) were provided with real-world items and were encouraged to discuss with peers and work cooperatively. Children discussed cooperatively the purchase of the items in a fictitious store. ii. Items and No Discussion (I&ND): Children (n=20) were provided with real-world items but were not permitted to discuss with their peers. Children determined individually how to purchase the items in a fictitious store. iii. No Items and Discussion (NI&D): Children (n=20) were encouraged to discuss with their peers and work cooperatively without any real-world items. iv. No Items and No Discussion (NI&ND): Children (n=20) individually determined how to purchase the items according to the problems shown in the task without any real-world items and peer discussion. Videotapes and recorders were used to gather children's verbal interaction during solving problems. Analysis of variance indicated that the subjects' previous math achievement in the four conditions was not significantly different before the actual experiment began $F_{(3,76)}=.07$, $p>.05$.

Table1: The mathematical structures of problems in the first and transfer problem-solving phases.

	The first problem-solving phase		Delayed transfer problem-solving phase		
Problem Situation	Class Camping	Lantern Festival	Class Party	Tile Fitting	School Olympics
Items and objects in the problem	Each pack of items with different numbers of objects ($\boldsymbol{a}$)	Three posters of different area ($\boldsymbol{a}$) 2. Packs with a different number of envelopes($\boldsymbol{b}$).	Each pack of items with different numbers of elements ($\boldsymbol{a}$)	1.Three rooms of different area *(a)* 2. Packs with a different number of tiles *(b)*	Packs of colored jungles with different number *(b)*
Multiplication model	Equal groups	1. Rectangular array. 2.Equal groups	Equal groups	1. Rectangular array. 2. Equal groups	Equal groups
Mathematical Content	Number	Area	Number	Area	Number
Arithmetical structure	$X \times \boldsymbol{b} \geq 30$ Packs with $\boldsymbol{b}$ given as2,3,4,5, 6,7 or 8.	$X \times \boldsymbol{b} \geq \boldsymbol{a}$ $\boldsymbol{a}$ to be determined $b \in \{3,6,8\}$	$X \times \boldsymbol{b} \geq 18$ Packs with $\boldsymbol{b}$ given as2,3,4,5, 6,7 or 8.	$X \times \boldsymbol{b} \geq \boldsymbol{a}$ $\boldsymbol{a}$ to be determined $b \in \{3,6,8\}$	$X \times \boldsymbol{b} \geq \boldsymbol{a}$ $\boldsymbol{a}$ given in the situations as 30,23,16,9,or 40 $b \in \{2,5,7\}$
Superfluous information	One item	Size of envelope	Two items	Size of tile	One item
Full score	35	15	35	15	25
Value of item Difficulty index	.49	.39	.50	.44	.43
Value of item discrimination index	.96	.78	.92	.58	.75

B. *The delayed transfer problem-solving phase.* Three word problems were presented on worksheets as delayed transfer tasks and were tested three weeks later after children completed the first phase of problem-solving tasks. As Table 1 solving mathematical problems (e.g., Fuchs et al., 1994; Graesser & Person, 1994) are needed for increasing children's abilities to provide complete explanations and work in constructive, interactive fashion during instruction.

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EARLY MATHEMATICAL DISCOURSE

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This study analysed the language of three boys, aged 5, who were engaged in play that was judged to have a high degree of mathematical content. The boys were selected because of the diversity of their language. One was relatively taciturn, one used a great deal of language but of limited complexity, and one used complex language that appeared to describe and lead his thought and that of others. Analysis suggests that the last of these three has the skills to succeed at school while the other two will need help in expanding their language to describe and advance their concepts.

Introduction

Students' ability to discuss mathematical activities and express mathematical thinking is crucial in current methods for teaching and learning mathematics. Teachers rely on children's language to decipher their thought so that mathematics instruction can be built upon the children's existing. Teachers must pay attention to children's conversations (e.g., Gooding & Stacey, 1993; Solomon & Nimerovsky, 1999), their answers to tasks set by teachers (e.g., Brodie, 1999), and their self-talk. Yet there is considerable variation in the specificity with which different students discuss their mathematical activity. Gooding and Stacey explored the nature of discussion in groups of 10 and 11 year-old students who did or did not gain in understanding in cooperative problem solving. The unsuccessful groups talked less than did the successful groups and used a smaller proportion of explanations. Unsuccessful groups used hardly any explicit mathematical discussion while successful groups named and discussed key mathematical aspects of their problems.

While teachers at all levels of school need to understand children's thinking, teachers in children's first year of school may have the most difficulty in coming to an understanding of what their students know. It is hard for these teachers to make appropriate judgements because many students do not use the same terms as their teachers do, may use the same terms in different ways, or may be unable or unwilling to talk about what they know.

One guide to young children's concepts can be found in observing their play. A study by Ginsburg and colleagues (Ginsburg, Inoue & Seo, 1999) has demonstrated that mathematical thinking can be identified in a large portion of the play of 4- and 5-year-old children. Although this is important information, teachers have little time for an extensive analysis of individual children's play, and are generally dependent on children's language to understand their concepts.

The study by Ginsburg and colleagues involved an intensive analysis of the play of 80 children who were from lower, middle and high-income families in New York City and who attended pre-school facilities. The investigators videotaped individual children for 15 minutes during times designated as free play. Then each 1-minute period of these videotapes was coded for the presence of five mathematical concepts: magnitude comparison, enumeration, pattern and shape, spatial relations, and dynamics. Overall, these mathematical concepts appeared in about 45% of the 1-minute episodes. This was true for children from lower income, middle income and upper-income homes. Play with blocks, Lego™, or jigsaw puzzles was found to be particularly rich in mathematical concepts.

When these children were interviewed about addition and subtraction, differences in performance and underlying concepts were seen among children from different economic backgrounds (Ginsburg, Pappas & Seo, in press). In general, the upper income children showed a somewhat higher level of mathematical thinking and metacognition. This may have been related to their use of language.

The case studies reported here are an early step in a wider analysis of the ways in which children from this study use language in the course of their mathematical play. Understanding the nature and uses of language in free play may help teachers gain insight into children's mathematical thinking, foster their language development, and encourage language use in mathematics instruction.

In examining children's mathematical language, we considered the views of both Piaget and the Vygotsky. On the one hand, there is evidence (e.g., Jordan, Huttenlocher, & Levine, 1994) that children often have mathematical competencies they cannot express in words. As Piaget maintained, there is a sense in which thought precedes language. At the same time, Vygotsky (1978) discusses two ways in which language may facilitate intellectual growth: intrapersonal speech in which children use language to plan a solution or to control behavior, and interpersonal language to share ideas and to stimulate development, as when a more competent peer provides assistance in the zone of proximal development. In this study, we considered both Piaget and Vygotsky's perspectives. One purpose of this analysis was to see if patterns of language use could be seen that would form the basis for analysis of the rest of the sample and thus provide recommendations for teaching.

Case studies

Three boys, aged 5, were selected for exploration of mathematical language employed in free play. They were selected because they all demonstrated a high level of mathematical concepts in their play, but differed in their language. They were: (1) Nigel, an African American who was building a "roller coaster", (2) Franco, an Hispanic American who was building a robot and then a tower with Lego, and (3) Isaac, a European American who was helping to make a large jigsaw puzzle of a train. All spoke English. Franco's family spoke Spanish at home.

Table 1 shows the mathematical codes assigned to each minute of play observed for these three boys. Mathematics was coded in five categories: Magnitude Comparison (MC), Enumeration (EN), Pattern and Shape (PS), Spatial Relations (SR) and Dynamic or exploring the process of change (DY). Where more than one code is given, the first code is the one that was judged to dominate that episode. These codes are based on observations of behaviour and do not necessarily imply that any language was employed.

Nigel and Isaac were coded as engaging in play with a mathematical content in 14 of the 15 one-minute periods (93%), and Franco was coded in engaging in mathematical play in all of the 15 minutes coded (100%).

Table 1. Mathematical behaviour evident in the children's play

	Nigel	Franco	Isaac
Minute 1	MC	DY	MC, PS, SR
Minute 2		DY	PS, MC
Minute 3	PS	MC, DY, PS, SR	PS
Minute 4	PS	MC, DY, PS, SR	PS, MC, SR
Minute 5	MC	MC, DY, PS, SR	MC
Minute 6	MC, PS	MC, PS	MC
Minute 7	MC	MC, PS SR	MC, EN
Minute 8	MC, PS	MC, PS, SR	MC
Minute 9	PS, MC, EN	PS, DY	PS
Minute 10	MC, PS	PS, MC, DY	
Minute 11	MC EN, PS	PS MC DY	PS
Minute 12	MC, PS	MC DY PS	PS
Minute 13	PS, MC	MC DY PS	PS
Minute 14	MC, PS	PS MC DY	PS
Minute 15		MC, PS	PS, EN

Nigel was relatively taciturn in this episode. He spoke in only 7 of the 15 minute intervals, making a total of 14 intelligible utterances. Six (42%) of these utterances were judged to have a mathematical content. Each utterance identified as mathematical related to the type of block he needed or had found. Examples included "*There it is, two*" (judged to be self-talk, as no one else was in the vicinity, and enumeration) and "*Gimme one of these blocks*" (instruction to another boy, enumeration). His longest utterance involved a complaint to the teacher that someone had upset his structure: "*Ms M., look what he's done. You've broke my roller coaster, I don't like it.*"

Franco was talkative, and spoke in all of the minute episodes, making a total of 47 utterances, 18 (38%) of which were judged to have mathematical content (one containing two categories). A feature of his utterances was that he used many non-specific terms to accompany his play. For example while talking about adding blocks to parts of his structure he said, "*R, Lookit, I did it over there and I put over there and I put over there.*" He was employing concepts of position, but could not be understood by someone who was not present. Most of his utterances related to his building but he also exchanged comments about his father's bicycle and sharing. Most of his comments drew attention to his activity or structure. The following utterances came from minutes 14 and 15. Codes are given beside each utterance considered mathematical, to give a picture of the analysis.

"*Long bigger, long bigger, long bigger*"(while building his tower) [Description, MC]

"*I still more bigger, I still more bigger*" (comparing his tower with R's [Description, MC]

"*I put mine bigger. You see, I told you*" [Description, MC]

"*Wanna see mine is bigger?*" (to R) [Question MC]

"*No Matthew had it first.*" (claiming a block for M)

"*No, no Matthew, we are making robots. I'ma show you. Put yours next to mine*" (persuading M to join his Lego with his) [Instruction, SR]

"*Look Matthew, it's long bigger*" (showing the taller tower to M) [Description, MC]

Isaac was also talkative. He spoke in all of the 15 minute sessions, making a total of 61 utterances, 22 (36%) of which had mathematical content. When not commenting on the task, he sang, discussed a television cartoon, or made noises into the microphone. While Nigel's utterances tended to be the minimum that would get what he wanted, Isaac's were more complex, expanding ideas from one sentence to the following one. His self-talk included both repetition of a previous statement, "We have to take it apart and build it all over again", and reading aloud from the box "Over five feet long, thirty-eight [pieces], age four and up. Wow!" The following sample is from minute 1 of his transcript.

"*Guess we need a little help. Excuse me. Are we making pieces? I don't know. Are we making any pieces?*" (watching 3 children working on a puzzle)

"*Okay, that goes in the caboose*" (has joined in working on the puzzle) [Instruction SR]

"*And the caboose is in the back.*" [Description SR]

"*We're making a real big train, just like a real train actually.*" (taking more pieces from the box) [Description MC]

"*Looks like a real olden-day train. This looks like a real olden day train, for good*" (singing) "*just like I love you...this goes away*"

Judgements were made on what would be considered emergent mathematical language. For example, "one" could be classified as a pronoun or enumeration depending on the context. Time sequences were not classified, nor were iterations or the concept of 'broken' which could be a precursor to part-whole, although an argument could be made for including all of these as early stages of mathematical language. Mathematical language was classified by its function, as self-talk, instruction, questions, or comments. Table 2 gives the number of utterances with a mathematical content by function and mathematical code.

Table 2. Number of utterances with a mathematical content, given by function of language and mathematical categories covered

	Nigel	Franco	Isaac
Self talk	1 EN		1 EN & MC
Description	1 EN	8 MC 2 SR	5 SR 1 MC 3 PS 1 EN
Questions		2 MC 1 SR	1 PS
Instructions	2 MC 2 EN	1 MC 4 SR	5 PS 1 MC 1 SR
Word play			1 SR 2 EN
Total	6	18	22

Interpretation

The number of episodes of play considered mathematical was similar for these boys, as was the proportion of their utterances judged to have a mathematical content. The differences noted were qualitative. There was a marked difference in the variety of mathematical contexts that the boys spoke about and in the complexity of the structure of their language.

Nigel's limited use of language is interesting in itself, although it makes it difficult to judge his competence. His play covered three spheres of early mathematics, and he used language for two of these. His limited language did include evidence of specific nouns, modifiers, and logical connection between sentences in an utterances not classified as mathematical, that in which he complained to his teacher about someone breaking his roller coaster. However, his limited use of language, if typical, could make him like the students in Gooding and Stacey's groups who did not learn in cooperative groups.

Franco was not fully fluent in English. His play was very rich in early mathematical concepts, with a greater number of mathematical episodes being identified than for the other boys. While his play in this episode covered four categories of mathematical ideas, he used language to describe only two of these. The fact that he talked a lot is likely to be an advantage, as he was practicing putting the process of comparison into words. In this episode he showed a limited vocabulary for mathematical concepts, using pronouns for both processes and objects. While all three boys used pronouns when the object of their play was self evident, the other two also used nouns and modifiers. Franco only occasionally used a noun and used no modifiers. He used no words to describe either dynamic changes or pattern and shape.

Isaac's play covered four mathematical categories, and he used language for all of them. This sample gives many examples of his analytic thought. His sentences showed continuation of ideas that became more specific in succeeding utterances. The nature of his self-talk suggests that he will continue to generate ideas, so that his concepts grow with minimal external influence.

The language of three boys gives us initial insights that will be explored further with the full data. It appeared that descriptive language supported thinking for all three boys and, in Isaac's case, questions and instructions led both his activity and possibly thinking for him and for others. Nigel and Franco demonstrated more concepts than they talked about and there was less evidence that it led their activity. Although two boys used some self talk, it is difficult to judge the extent to which this may have led thinking.

The presence or absence of attributes that Gooding and Stacey found in the conversations of successful groups indicates the relevance of these factors in the language of young children. Studies of functional grammar suggest that the analytic nature and logical connectedness of language is the aspect that enables students to be successful in classroom discourse. The size of vocabulary is known to be a good predictor of success in school, and this is likely to be true in mathematics as well. All of these are factors that can be fostered in good teaching. We know that all of these boys exhibit mathematical concepts in their everyday behavior and it is important for teachers to be aware of what they know. Two of them are likely to need more help in using words to describe mathematical objects and activities and ask relevant questions.

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UNIVERSITY MATHEMATICS TEACHING – WHERE IS THE CHALLENGE?

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Data from university mathematics tutorials were analysed on characteristics of teaching using a construct 'the teaching triad', derived from analyses of secondary mathematics teaching. Elements of 'management of learning', 'sensitivity to students' (from both affective and cognitive perspectives) and 'mathematical challenge' were sought and rationalised with earlier manifestations of these elements. Meanings were derived which made sense in the university context, taking into account the particular culture of university mathematics teaching and expectations of tutors and students. Whereas at first there seemed little challenge on which to remark, a reconceptualisation of challenge allowed an alternative perspective to be offered. The paper reports on this perspective and looks critically at processes and strategies in teaching mathematics at this level.

Ways in which teachers offer and enable students to tackle appropriate challenges are important to students' engagement with mathematics and their development of mathematical concepts. Teachers' sensitivity to students' affective and cognitive needs is seen to be closely related to the effective nature of challenge[1]

The research reported here involves an analysis of tutorial teaching from a project – called the Undergraduate Mathematics Teaching Project (UMTP) - designed to characterise university mathematics teaching in first year tutorials[2]. Analysis has been done using *the teaching triad*, a construct deriving from earlier research into classroom mathematics teaching at secondary level (Jaworski, 1994) and used, subsequently, as a device to analyse teaching and as a developmental tool by teachers (Note 1). The triad has also been used to analyse the activity of mathematics teacher-educators in the professional development of mathematics teachers (Zaslavsky & Leikin, 1999)

Theoretical Background

The UMTP was rooted in theory relating to mathematical learning at university level: in particular the difficulties experienced by students at this level (e.g. Tall, 1991; Nardi, 1996). It is distinct in its fine-grained focus on tutor-student interactions and tutors' expressed thinking relating to these interactions. It draws on previous research into secondary mathematics teaching, with a focus on teaching activity, its relationship with students' mathematical activity, and the associated thinking of the teachers in their planning of interactions with students (Jaworski, 1994).

So, for example, research into university learning suggests that students have difficulty in conceptualising cosets of groups, and relating them to notions of conjugation in groups (e.g; Dubinsky et al, 1994; Burn, 1998). We have shown how one tutor struggled with his students' lack of understanding of these concepts, and ways in which he personally addressed them and constructed his teaching to overcome them. As a result of such analyses, we were able to offer a number of indications for a theory of mathematics teaching at this level (Jaworski et al, forthcoming; Nardi et al, in preparation)

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[1] See Jaworski and Potari (1998), Potari and Jaworski, forthcoming

[2] The research was funded by a grant from the Economic and Social Research Council (ESRC) number R 000 22 2688 It was conducted in collaboration with Elena Nardi (Univ. of East Anglia) and Stephen Hegedus (now Univ. of Massachusets). See also Jaworski, Nardi and Hegedus, 1999; *forthcoming*; and Nardi, et al, *forthcoming*.

Current analysis of tutorial approaches is using the *teaching triad* with three inter-related elements: *management of learning*, *sensitivity to students* (from both affective and cognitive perspectives) and *mathematical challenge*. Briefly, *ML* describes the teacher's role in the constitution of the classroom learning environment. This includes classroom groupings; planning of tasks and activity; setting of norms and so on. *SS* describes the teacher's knowledge of students and attention to their needs: the ways in which the teacher interacts with individuals and guides group interactions. *MC* describes the challenges offered to students, to engender mathematical thinking and activity, in tasks set, questions posed and metacognitive encouragement. These domains are closely interlinked and interdependent. Analysis. reflexively, categorises tutorial data in relation to these elements and reassesses the meaning of the elements with respect to the data.

Methodology

The methodology of the UMTP was reported in detail in Jaworski et al (1999). Briefly, from interview data, factual summaries, or *protocols*, were constructed that were then analysed, using tested coding systems, for their teaching characteristics. Commonly occurring codes were inspected against key episodes from the data, selected for their significance by the researcher gathering the data. Sections of data highlighted by the codes were analysed in fine-grained detail to seek insights into teaching issues including teachers' decisions in conducting of tutorials and their relation to the particular mathematics that was the focus of observed tutorials (largely analysis and abstract algebra, with some topology and probability).

Subsequent analysis of tutorial data followed that of interview data, by a production of tutorial protocols: each a summary of the activity of the tutorial. Coding of these protocols, led to identification of (a) typical patterns of interactions; and (b) instances where activity or dialogue might be characterised by elements of the teaching triad. Particular episodes were identified to typify examples of (a) and (b) and these were studied in greater depth from the full tutorial text.

Typical patterns of interaction

The lecture-tutorial culture in mathematics in this university requires students to tackle problems set in a lecture and give written solutions to their tutor for comments. Tutors' marking of the solutions forms the basis of tutorial activity

In analysis of activity in tutorials, the codes, *tutor explanation*, *tutor as expert*, *tutor questioning* were abundant. Tutor *explanation* [TE] was usually a straightforward exposition of some aspect of mathematics. Tutor *as expert* [TEx] included the offering of key methods of solution, or proof, or mathematical tricks or routines that are perceived to be part of what one tutor expressed as the "mathematical armoury" that students need. Tutor questions were mainly of three types: enquiring about students' thinking or difficulties [QS]; specific mathematical questions relating to the mathematics being addressed [QM]; and prompting or leading questions directed at filling gaps in an argument [QA]. The following episode represents these patterns. The codes are illustrated in the transcript, and used in the microanalysis of the episode in relation to the teaching triad. The mathematical focus is a question concerning the *orders* of elements of the group A_4. Students (two here) have shown difficulty in their solution to the question, and the tutor is highlighting aspects of the concepts that he perceives as important. Voices of the students are not distinct.

Episode 1: Words from tutorial transcript

1 T The first thing is what's the order of A_4? [QM]

2 S It's 12

3 T So if H is a subgroup of A_4; - we quite often write ≤ to mean subgroup of [TEx] - if H is a subgroup of A_4, then what can we say about the order of H? [QM]
[Tutor writes as he talks. Students observe, listen and respond.]

4 S It must be a factor of 12

5 T Right, so what are they [QA]

6 S 1,2,3,4,6,12

7 T And what result did you use to deduce that? [QS] Pause (10 secs) [Inaudible words omitted]

8 S Lagrange's Theorem

9 T That's the one. So we now know what sizes to look for: 1 and 12 are dead easy. Right? Order 1, H just has to be the identity. Order 12, it has to be all of A_4. Those are easy.
[Students make 'understanding' noises: ah, mm]
Order 2. A little while ago, you worked out that there was only one Cayley table for groups of order two. Another way to look at this is what orders of elements can you have? Well, in fact we've already proved this, haven't we here? [Reference to an earlier question and solution] 2 is prime; a group of prime order must be cyclic, so any of these would have to be generated by an element of order 2, and any element of order 2 will generate one of these. So the subgroups of order 2 are precisely *e* and an element of order two. [TE]
Three. The same argument works. [A few inaudible words omitted here]
Four Now what orders could elements have inside a group of order 4. [QM/A]

10 S 2,1,4

11 T Is there any element of A_4 of order 4? [QS] Pause (5 secs)

12 T No, there isn't. They come in one of three types
[Tutor continues with an explanation of subgroups of order 4, and moves on to subgroups of order 6. Student participation is of a similar degree to that recorded above.]

13 T So, there's no subgroup of order 6.
The two reasons that this is interesting are that first of all you *can* have several subgroups of the same size; the other thing is there may be a size which by Lagrange's theorem is *allowed* for a size of subgroup, but for which there is no subgroup. [TE] Neither of you wrote this but some other people did, they wrote, by Lagrange's theorem there *must* be a subgroup of size n for every n that divides the order of the group. And that's *not* true. Here was an example. Right. [TEx][3]

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[3] We might argue about distinctions between QM, QS, and QA, and between TE and TEx. It is hard to find clear distinguishing examples in one piece of dialogue. The insertion of the codes here is indicative only of their use and meaning.

A considerable part of tutorial activity can be characterised in the form exemplified above. Of course details change. Different bits of mathematics cause different problems for different students, and different tutors interact with their students in idiosyncratic ways. The pauses in the dialogue above are short compared to those of another tutor, who usually waits a much greater length of time (20-30 secs in some cases) for his students to say something; and much shorter than some tutors who wait hardly any time at all. "*Answering own questions*" is one coding category of tutor activity, in which a question is asked by the tutor, and almost immediately the tutor supplies the answer. It is a rhetorical form in which the question is an almost seamless part of tutor exposition of the concept. Tutors clearly want to offer their students good explanations of the mathematics they consider important.

After observing and recording this tutorial, the researcher interviewed the tutor, starting with a question about his tutorial agenda. The protocol reads as follows:

> **Researcher:** You mark their work. How does this form your tutorial agenda?
>
> **Tutor** It tends to be that all of them have made the same mistakes, so it helps to form a scheme of help, e.g. noticing that they all believed the converse of Lagrange's theorem to be true. Then, once this general scheme is formed, you have to tailor it individually, e.g. noticing that a particular student is writing left cosets for right and that gives her odd results. [TEACH; REC STU PRO;KNOW STU]

The bracketed words, e.g., [TEACH; REC STU PRO;KNOW STU] are part of the coding system used to analyse interview protocols. They characterise the associated dialogue as being about *planning for teaching*, *recognising students' problems*, and *knowledge of students*. These were some of the most commonly occurring codes in the analyses of *interview* protocols. They indicated common elements of tutors' thinking about planning and teaching of tutorials. In this case, the teaching focus, on Lagrange's theorem, is designed around students' misconceptions of the truth of the converse of the theorem, and particular errors in tackling associated questions. Tutor's knowledge of students' particular needs comes from marking their work, and seeing them in tutorials. An important part of the coding of both interview protocols and tutorial protocols, is a rationalisation of the two, providing insights into how a tutor interprets his thinking about teaching into practice in the tutorial.

Characterising elements of activity or dialogue using the teaching triad.

The teaching triad arose from teaching that engaged students overtly in *questioning* and *inquiry* in mathematics (Jaworski, 1994). A teacher's questions that challenged *sensitively* the particular thinking of a student were seen to be fruitful in enabling conceptual development. In micro-analysis, *harmony* between sensitivity and challenge was seen to be a characteristic of a successful teaching interaction. Micro-analyses were then scrutinised against a macro-analysis taking into account wider sociocultural issues of the learning environment (Potari and Jaworski, forthcoming).

In the episode above, at the micro level, management of learning, *ML*, is evident in the tutor's recognition of difficulties and planning of the focus of the tutorial to address these difficulties [*Interview protocol*]. Sensitivity to students, *SS*, (influencing *ML*) is seen both at an affective level, basing the focus in aspects of mathematics that are clearly relevant to students difficulties, and at a cognitive level, focusing on the example that will highlight clearly the concept that the tutor wants students to consider (converse of Lagrange's theorem). The tutor's questions invite student participation, but in some cases this is of a minimal degree [*statements 2, 6, 10*] , and in others, students seem unable to respond. The tutor often waits for a response [*7, 11*] and sometimes supplies the answer himself [*12*]. The dialogue seems to encompass little mathematical challenge, *MC*. Largely, questions seek students' knowledge of particular mathematical results and teacher exposition provides the substance of the interaction. Any challenge is left to the students themselves in making sense of concepts in their own personal study.

At a macro level, mathematical enculturation might be seen as a tacitly agreed basis of interaction. Students have met Lagrange's Theorem – one of them is able to name it in response to a prompt from the tutor. However, it is important mathematically that they perceive the difference between theorem and converse, both in terms of Lagrange, but as a consideration in theorems more generally. The tutor's choice of example might be seen as a clever management strategy, enabling students to perceive truth relations between a theorem and its converse. In this we might discern a (tacit) element of challenge. Students are being confronted with a challenge to their perceptions, and it is up to them to go away and make sense of it.

An alternative pattern of interaction

The episode that follows is chosen to show an approach that seems to incorporate *challenge* in a different form. Unusually the tutor here was also the lecturer of the Abstract Algebra course, so he had, himself, set the problems on which the students worked, including a question about quotient groups. In the lecture, a theorem had been proved and the lecturer had asked the student to prove for themselves its converse. This in itself is potentially a mathematical challenge for the students who tackle it. The tutor is working with two students, one of whom is ill (S1) and does not say much. The tutorial protocol and coding for the episode read:

> **Tutorial Protocol:** Tutor asks S2 a question. S2 explains at the board. Tutor offers advice and asks questions when S2 gets stuck. S1 offers suggestions. When S2 has finished his proof, tutor explains a quicker method he [the tutor]would have used.
>
> **Coding:** tutor questioning [QS/M/A]; student-led explanation [SE]; tutor-students interaction [TSI]; tutor as expert [TEx]; rapport between tutor and students [R].

Episode 2: Words from Tutorial Transcript
Conventions: () inaudible words omitted; … repetitive or irrelevant words omitted.

The episode transcribed here is 10 minutes of the tutorial.

1 T: You did the part of question 3 I did in lectures, but the part I left as an exercise, namely the converse, er, you forgot about (students laugh). [R]
One wants to show that for a congruence … - well its an equivalence relation, - that satisfies $g_1 \sim k_1$ and $g_2 \sim k_2 \Rightarrow g_1 g_2 \sim k_1 k_2$; and $g \sim k \Rightarrow g^{-1} \sim k^{-1}$
So its an equivalence relation that respects the group operations inversion and congruence … we're told that H is a normal subgroup of G and we're given this equivalence relation by $g_1 \sim g_2$ if and only if they represent the same coset. So lets try proving these two things.

Let's say we want $g_1 g_2 = k_1 k_2$ and this means saying that $g_1 H = k_1 H$ and $g_2 H = k_2 H$ then
this means saying that $g_1 g_2 H = k_1 k_2 H$ [On board is also written $g_1 g_2 \sim k_1 k_2$]
I said in lectures, although you may not remember, that the work had been done already
because the quotient group is well defined, you can actually multiply cosets together in a
well-defined fashion. … Why do you think those cosets at the end are equal? [TEx]

2 S2: Well for the things we're getting at the top –
from $g_1 H = k_1 H$ [T: yes] you can say that, for any h in H, $g_1 h =$ [SE]

3 T: Go on, go for it [tutor invites him to the board. S2 writes, talking as he writes] [R]

[The next portion of the tape is hard to follow as symbols are being written (evidenced by noise of chalk on board) but not necessarily articulated. The researcher's fieldnotes (FN) help with what was written. Reconstruction is as faithful as possible under these circumstances.]

4 S2: If you've got h and l in H, then this one tells us that $g_1 h = k_1 l$ (pause) for some l in H (S2
and tutor say this together) and same sort of thing for the twos, (he writes $g_2 h = k_2 l$)
(pause) [SE]

5 T: call it h' and l', well, call it l', you might want the same h as () (pause) [TE]

6 S2: erm (pause)

7 T: well what's a general element of the left hand side downstairs look like? [QS]

8 S2 erm that's gonna be $g_1 g_2$ and that's the same h [chalk on board and inaudible speech:
FN suggest he writes $g_1 g_2 h$ and wants to continue with $k_1 k_2$] [SE]

9 T: well, just expand - don't write $k_1 k_2$ now - keep expanding …what you had there – so we
can rewrite that can't we … [TE]

10 S2: well g_1 and g_2 can be combined to another g [SE]

11 S1: Can't you write as something else, can't you say $g_2 h = k_2 l'$ [SE]

12 T: Yes, go for that I'd say [R]

13 S2: Say that $k_2 l'$ equals (pause, hesitancy) can you say from that one, that is equal to er
(pause) k_1 (pause) before, that was only true cause that was in H [SE]

14 T: well you know now that

15 S2: [inaudible, but FN give us the following symbols: $g_1 g_2 h = g_1 k_2 l'$]

16 T: you haven't really written much up there yet that requires, uses, normality. The fact that
er H is normal means that $k_2 l'$ can be written in a different way. (Pause) [TE]

17 Together: T: $k_2 l'$ S2: some other thing - k_1 T: k_2 S2: Yes.

18 T: OK, so its $g_1 l''$, let's say, k_2 – and what is

19 S2: so that g_1 T; Yes S2: l'' from this Together: is $k_1 h''$

20 T: k_2, and then that is equal to [TSI] [R]

21 S: erm, same sort of thing, because of normality we can say that this is something else

22 T: oh, yes, you, why not go the other way

23 S2 depends what we're aiming for

24 T: well exactly, we're aiming for a $k_1 k_2$, so why not instead swap them round [TSI]

25 Together: $k_1 k_2 h'''$ OK [FN indicate that on the board now is the following:

$k, l, l' \varepsilon H \quad g_1 h = k_1 l \quad g_2 h = k_2 l' \ : \ g_1 g_2 h = g_1 k_2 l' = g_1 l'' k_2 = k_1 h'' k_2 = k_1 k_2 h'''$]

26 T: You did the very hard work of essentially proving that multiplying cosets of a normal subgroup is well defined. I'd have been perfectly happy if you'd assumed that. [TEx] OK? So you've got that out, and that is absolutely fine, but, I'd have been happy with this: You've got two cosets – that coset's the same as that, that coset's the same as that; so $g_1 H g_2 H = k_1 H k_2 H$, and these cosets multiply to give that [$g_1 \ g_2 H$] and these cosets multiply to give that [$k_1 k_2 \ H$] [i.e. $g_1 \ g_2 H = k_1 k_2 \ H$] and that's all I was expecting. [TE]

In the same way suppose g~k and gh=kh, so they have the same inverse in the quotient group and what is the inverse of this, we know it is represented by the inverse of the representatives [FN: writes $(gH)^{-1} = (kH)^{-1} \quad g^{-1}H = k^{-1}H \quad g^{-1} \sim k^{-1}$] so that would have done. So that was all really I wanted. What you proved was that multiplying cosets is fine when you have a normal subgroup; which is a good point to remember. [TE, TEx]

[Tutor now quickly explains the proof as he would have proved it] [TE]

27 T: So it's a bit simpler ... rather than getting lost in algebra. [TEx]

The episode, while largely managed by the tutor, splits into 3 parts: (a) [*1*] in which tutor creates the problem-solving environment, stating the problem, and clarifying its context and parameters; (b) [*2-25*] in which we see tutor-students interaction: one student, mainly, constructs a solution with support from tutor instructions [*5,9*], questions [*7,22*], comments [*12, 14, 16, 18*], prompts [*20, 22, 24*], and expert input [*16*]. At points tutor and student seem to think together [*17, 19, 25*]; and (c) [*26-27*] in which we see the tutor in expert mode, explaining and demonstrating. Part (b) seems rather different from the style of interaction represented in Episode 1. There is evidence of student involvement in activity and thinking. Although we see in this episode, as above, considerable *TE* and *TEx*, there are new elements. Students join in the discussion in a more active way, encouraged by the tutor. Parts of the tutorial are student-led, albeit with tutor participation. Interaction is less responsive on the part of the students and more generative. There seems to be rapport between tutor and students which implies a degree of trust built up through experience of working together.

The teaching strategy of inviting a student to the board is a familiar practice here. In one interview this tutor acknowledged it might feel threatening to a student, but that students had come to realise that he would provide helpful support.

> I do promise to help; or will help ... they actually know I'll start them off. They won't just be stood at the board and me twiddling my thumbs. I might, after a few seconds, like 30 seconds, or something like that, or perhaps even less if they're looking panicky, I would suggest, er. "Well, OK, write down, what's the first line? What's it mean to say that?"

So, the intervention of the tutor shown above is a considered approach to encouraging students' participation in the mathematics of the tutorial. As such, it seems to constitute mathematical challenge, sensitively handled, albeit a challenge to think publicly, rather than to tackle a particular mathematical question. *ML* in Episode 2 seems more sophisticated than in Episode 1 in that it incorporates a teaching strategy that not only addresses the tutor's mathematical issues, but does so in a way that the students are seen, overtly, to be involved in the thinking. There is evident struggle to express ideas, and present them in accepted forms. Tutor interjections might be seen as tutor imposing his own explanations or expertise, but alternatively they can be construed as supportive input (*affective sensitivity*), according to his own words. As they are generally pertinent to the immediate thinking of the student they are also sensitive in a cognitive domain.

Concluding Remarks

In our tentative theory of pedagogic development/awareness (Jaworski et al, forthcoming; Nardi et al, in preparation) Episode 2 is characterised at a higher level than Episode 1, indicating more sophistication of pedagogic awareness in addressing students' mathematical conceptualisation. However, a central part of the tutor role is seen widely as inducting students into appropriate ways of seeing and thinking mathematics. We see, above, examples of the various strategies that tutors use, which seem to form their pedagogical repertoire, and to be a practical manifestation of pedagogic thinking. Challenge to students is often implicit in the pedagogic approach, leaving its interpretation up to the students themselves. However, the second episode is more indicative of *harmony* in sensitivity and challenge. In previous research, harmony has been shown through teachers' questions being related to students' thinking, and through development of students' metacognitive activity [Note 1]. Here it is part of the teaching strategy (*ML*) that challenges students (sensitively) to address mathematical concepts. Further research might explore critically the relationship of harmony to students' conceptual learning.

My sincere thanks to Kate Watson for her contribution to data analysis

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Visualisation in Geometry : Multiple Linked Representations?

G. Kadunz (Klagenfurt) & R. Sträßer (Bielefeld/Gießen)

At a first glance, geometry seems to be the most appropriate field for visualising problems and supporting its solutions. The paper looks into this statement by analysing the role of multiple (sometimes: linked) representations, especially in computer environments like Dynamical Geometry Software (DGS). Using a prototypic example, the essay throws some doubts on the above optimism and gives reasons for a more sceptical evaluation of the role of visualisation in geometry and its learning.

1 The field of interest and related literature

Visualisation is a continuous field of interest within the PME community (see the plenary session by Dreyfus 1991, the plenary panel in 1992, pp.3-191ff of the PME16-proceedings, or the papers of the research forum in the 1999-proceedings, pp. I-197ff). "Imagery and Visualization" since long is a category in the "index of presentations by research domain" in the PME-proceedings. It is also a constant field of interest in the broad community of mathematics education research (see for instance the handbook edited by Zimmermann&Cunningham 1991).

At a first glance, geometry is an easy domain for studying visualisation because, traditionally, geometry is THE mathematical domain where imagery abounds. It is often described as the field where icons, imagery and their inherent relations are studied - but: research in geometry education seems not to have an accepted description of the role of visualisation in geometry. Even historically we find times when geometry relies on visualisation via numerous drawings which alternate with periods or authors widely refusing the use of diagrams (see e.g. the well-known case of Lagrange's introduction to the "Mécanique analytique").

2 The focus of the study and its framework

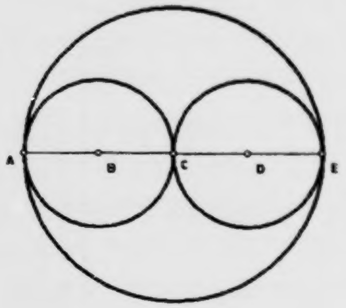
Within the field of visualisation and geometry, the paper tries to better understand the role of the variety of (sometimes, especially in computer environments: linked) representations of a geometrical problem in order to better understand visualisation in the field of geometry. The (non-surprising !) guess that this also throws some light on a more general idea of visualisation will not be treated here (for this, see Kadunz 2000).

The framework of the essay is the idea *that* -for the purpose of the study- *it helps* to distinguish between a human being (sometimes: a learner) and mathematical (in the essay normally) geometrical knowledge which are linked by external and/or internal representations (for this concept of representations see e.g. Goldin 1992). Consequently, the reader should not infer any special position of the authors with respect to epistemological questions (like for instance the adherence or degree of adherence to constructivism) from this study.

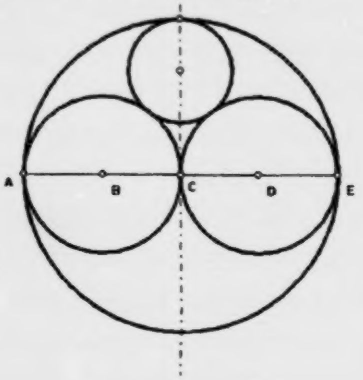
3 A prototypic example

3.1 The problem

Segment AE is the diameter of a circle with centre C, B and D are midpoints of the segments AC and CE. With respective circles around B and D through A and E, we come to a configuration represented in drawing 1a. How to construct a circle which only touches the circles around B, C and D (for a "solution" see drawing 1b).



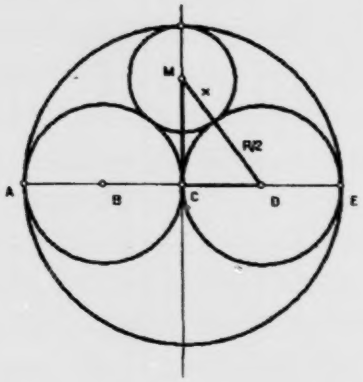
drawing 1a



drawing 1b

3.2 The drawing

Drawing 1a is an easy construction task with points, segments, circles and their names - the signs of standard elementary geometry. The fourth circle touching the first three circles may be arranged using ruler and compass or appropriate geometry software (like Dynamic Geometry Systems -"DGS"). A DGS-solution could place the centre M on the mid-perpendicular of segment AE (as a variable point on the perpendicular) and an additional point H in the intersection of the circle around C and the mid-perpendicular. Dragging M then arranges an appropriate circle with centre M through H to touch the circles around B and D.



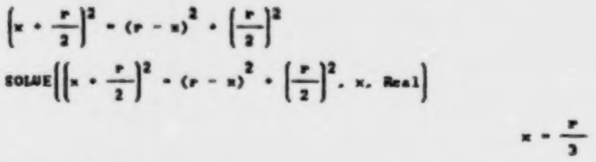
drawing 2

3.3 An algebraic solution

The problem solver can use drawing 1b as a plan for an exact construction: Because of symmetry, the centre M of the fourth circle must be on the mid-perpendicular to segment AE - giving rise to a numerical solution if the radius of the circle is known. As the circles around D and M have only one point in common, the intersection and points D and M have to be collinear - offering a right-angled triangle DCM (see drawing 2). With R as radius of the circle around C and x as radius of the circle with centre M, we come to the equation (*) below and its easy and simple solution of x=R/3.

$$(*)\quad (x+R/2)^2 = (R-x)^2+(R/2)^2$$

The algorithmic transposition of the formula could even be handed over to a Computer Algebra System (CAS) - and drawing 3 gives the commands for the CAS-"Derive".



$$\left(x + \frac{r}{2}\right)^2 = (r - x)^2 + \left(\frac{r}{2}\right)^2$$

$$\text{SOLVE}\left(\left(x + \frac{r}{2}\right)^2 = (r - x)^2 + \left(\frac{r}{2}\right)^2, x, \text{Real}\right)$$

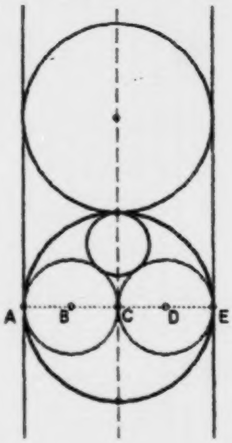
$$x = \frac{r}{3}$$

drawing 3

By concentration onto the algebraic solution of the equation (*), the original problem disappears. The equation does not tell its genesis and could be given as a drill-and-practice problem on equations without showing its geometrical origin. This is the price to pay for the use of the algorithmic solution - and only the unexpected result x=R/3 (only one solution of a quadratic equation - and a "simple" one!) may motivate the question why this special result had to be calculated. The algebraic solution does not offer a clue to an answer of this question and may come as a surprise because it can hardly be anticipated from drawing 1a/b. On the other hand, the algebraic solution by means of the Pythagorean theorem offers no clue or interpretation of this rather "simple" solution. Even the effective construction does not offer a hint to embed the solution x : R = 1 : 3 into a broader geometrical context. Nevertheless, it is easy to end the construction by using the mid-perpendicular of AE and a circle around A with radius 2R/3 for finding the centre and then the radius of the inscribed circle.

3.4 A geometric solution

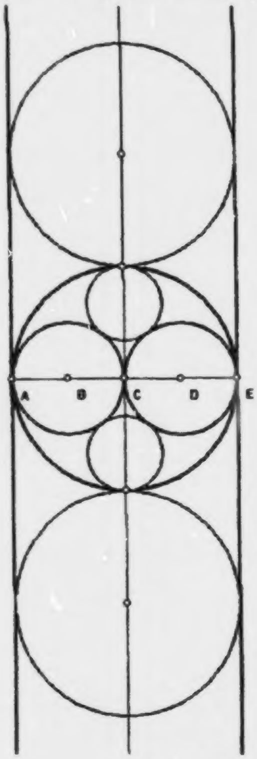
The solution using the algebraic information x=R/3 did not offer a geometrical interpretation for the solution. So we look for a way to cope with the construction of a circle tangent to three other circles. Within the range of elementary geometry, a productive method to cope with problems like these is the inversion of circles and we will use it to explore the problem - even if inversion is no more part of the elementary school geometry. With DGS and its power to group chains of constructions by means of a "macro"-definition, by means of modularising a construction, we have a chance to solve the problem (we will not elaborate on inversion because of the abundance of appropriate literature on this topic; see for instance Lindner, 1999, pp.336-341; Coxeter, 1989, pp.104-117).



drawing 4

Let us take the "large" circle with centre C as the circle of inversion. The circle around M and tangent to the other three circles is transformed into a circle tangent to the images of the other three circles. The images of the these three circles are the circle of inversion itself and two tangents (straight lines !) to the circle of inversion (!!). In short: as images of the circles around B and D we have perpendiculars f(B) and f(D) to segment AE through A and E because they touch the circle of inversion in A and E and go through the centre C of inversion.

Now the image of the circle C_M with centre M (a circle again!) has to touch these two perpendiculars and the circle of inversion. Its centre f(M) must be on the mid-parallel between f(B) and f(D). Consequently, the image of the fourth circle has the same radius R as the circle of inversion. The construction of it is very easy and the inversion of it immediately gives the solution C_M of our problem. Symmetry adds the other solution (as in drawing 5).



drawing 5

The only information lacking now is the value x = R/3 of the radius of the circle C_M - and one could even doubt the necessity of knowing the numerical value because the construction is already finished. But inversion can inform us about this radius with the following argument: The image f(C_M) has the same radius as the circle of inversion around C. If one additionally constructs a circle around C with radius 3R, this circle will (obviously!) touch the image f(C_M) of the solution of our problem. The image of this larger circle around C also has to touch the circle with centre M and has the same centre as the circle of inversion. If we consult the definition of the inversion, this image has to have a radius of R/3 - hence our solution as well.

With this, we solved the problem by means of geometry in a double sense: We found, we geometrically constructed the circle around M we looked for AND we deduced its radius from the characteristics of the inversion. Inversion was the key to our geometrical solution.

4 On visualisation in geometry

If we look back to the solution process, we can clearly distinguish three phases: At first we constructed an "empirical" solution using the drag-mode of the DGS, this offered a clue for the second phase, namely an algebraic calculation for the radius of the inscribed circle (and a consecutive construction of it). The third phase deliberately put aside this numerical/algebraic solution while inversion offered a purely iconic, geometrical solution with the length of the radius as an additional information from a careful analysis of the inversion.

4.1 Traditional representations

The succession of the different phases is clearly marked by the use of different types of representations: at first, the solution is sought in the "register" of geometrical, iconic representations (for the concept of "register" see DUVAL 2000), we "visualised" the problem. Unfortunately, this only leads to an empirical solution (the drag-mode solution), but also produces the drawing, the graphical representation which opens the door for the second representation: the algebraic, symbolic description of the problem. In some sense, the algebraic description hides the initial problem, but offers powerful means to solve it - namely: the transposition of equations - and a surprisingly and numerically simple solution. The effective construction in the iconic representation "degenerates" to a mere technical process. It was exactly the change of register, the change of representation which made possible a solution of the problem. The interplay of the iconic representation of the empirical "solution" and the symbolic representation of the situation (via equations) and its rule governed manipulation produced the exact and elementary solution of the problem.

But this change of representations (from iconic to symbolic registers), the algebraic solution of the problem and its iconic realisation have a severe deficit: The symbolic representation put aside and masked the initial problem, offered a general and effective solution - but no clue to understand this solution in terms of (synthetic, iconic) geometry. Having found a symbolic solution, we now looked for an iconic one, i.e. after the change from iconic to symbolic representation, we turned back to the iconic representation to "understand" our solution in terms of geometry. The driving force of this return to geometry may have been the simplicity of our algebraic solution (or the love of geometry and consequently a search for a geometrical solution). In all, the symbols of the algebraic result made us go back to icons and forced us back into geometry. And with the use of a somewhat more advanced tool like inversion, we came across a "purely" iconic solution and could even give a geometrical reason for the simple algebraic result.

4.2 Representations with DGS

In both "icons phases", the first as well as the third phase of the solution process, the special features of DGS were of special help: In phase 1 the drag-mode helped to produce the empirical solution. If we additionally go into technical details of Dynamic Geometry Software, we come across a representation behind the iconic representation on screen: invisible for the user, but nevertheless essential for DGS, there is a "second" representation inside every DGS "behind" the visible representation on screen. Parallel to the construction of the user, DGS save all input (like position, shape and measures of the constructed objects), including information on relations between parts of the geometrical construction. This representation is kept hidden from the "normal" user – but makes possible the drag-mode by the quick and iterated recalculation of the whole construction depending on the position of basic objects. The high calculation power of modern hardware enables the computer to produce a quasi-continuous movement on screen when varying basic objects of a construction. Following the functional dependencies of the other objects, the DGS produces an actualisation of the representation on screen following the new position of basic elements. The visible iconic representation is internally controlled by an invisible algebraic representation of relations. This algebraic representation in return obviously depends on the position values of the user input for basic objects. It is this power which – on the other hand – in some sense prevents the problem solver from searching for more general, geometrical means to solve the problem because the power of the algorithms of analytic geometry make the drag-mode so simple and comfortable. As a consequence, the user is kept inside the icons of geometry – the concepts simulated by the drag-mode remain blind, whereas the solution in phase 2 is conceptually void. Nevertheless, the surprisingly simple solution is a good reason to continue working on the problem.

For an effective realisation of the inversion in phase 3, the macro-functionality of the DGS is crucial. To state it in more general terms: The offer of a modularization of the solution by means of appropriate "macros" (like "inversion of a circle or straight line at a given circle of inversion") is crucial for an effective and swift construction in phase 3. Traditional paper&pencil constructions would be very tedious and time consuming (and be definitely out of reach in a normal school context already because of a lack of exactness). With macros, more general: a cognitively appropriate modularization of the construction, the iconic register becomes more manageable and flexible to use it in as tentative, but rule-governed way as we are used to when transforming equations.

So the third phase of the solution is characterized by a constant change between the iconic representation of the problem and its solution, represented by the drawing, and the reflections and theoretical concepts (mainly from the geometrical inversion) to further the solution process. The drawing is (re-)structured by means of conceptual entities ("modules") like "inversion of a line / a circle at a given circle of inversion". These modules are used to guide the continuation of the construction by "offering" new concepts/modules to advance the solution process. On the one hand, they offer a possibility to an economical construction, hiding those objects which are only intermediate states of a given construction (like a mid-perpendicular if only a midpoint of a segment is to be constructed). In addition to that, they give way for a look onto the overall structure of a complex construction, hiding those elements which conceptually are not needed for it. In some sense, they do the same job in a geometrical construction as algebraic expressions ("terms") do in algebraic transformation of equations. So they deeply influence the conceptual understanding and making of a (complex) construction. In our example, the modules act as links between the actual drawing and the geometrical theory of inversion, sometimes even motivating the use of additional modules. They are "only" heuristical tools which - in contrast to algorithms - offer hints how to continue the solution, they do not prescribe the next step (as would have to do algorithms). Within this complex, heuristical process, an iconic solution is embedded into the geometrical theory.

4.3 On visualization

From a more or less phenomenological angle, the first two phases of the solution are clearly different: the first one is marked by iconic, whereas the second phase heavily relies on symbolic manipulation. Icons offer an elementary heuristic to prepare for the second symbolic phase. Using elementary algebra, symbols produce a numerical solution, which can easily be transformed into an (iconic) construction. Following a simple, traditional visualisation concept, the first phase "visualises" the problem, while the second phase brings into being a non-visual, algebraic solution. Such a traditional concept of visualisation is inappropriate to fully understand the third phase, but opens a way to describe the interaction of "external" representations as opposed to "internal" representations (for the distinction cf. Goldin 1992).

We now concentrate on the third phase to understand the solution. On the one hand, we find constructions in the sense of traditional geometry. On the other hand and at turnings points of this phase, we interpreted our constructions by means of concepts from the theory of transformation geometry, especially the theory of inversion. We condensed parts of the drawing into a distinct "Gestalt", a module which could be understood as an instance of a concept from inversion theory. So we linked the iconic representation with a conceptual one, the construction was seen as an external representation of the concepts of inversion, which were the respective internal representation. And DGS-macros could additionally represent these concepts as simple software commands. Software offered (or could offer) an additional external representation of the theoretical concepts. The solution process was a constant to-and-from between external representations (on paper and/or machine-based) and internal, conceptual representations. Concepts grounded the solution process because they linked the solution with a geometrical theory – and led to the development of new icons and images which had to be interpreted within the theory of inversion to further the solution process. Looking onto the external, iconic representation had a heuristical function to decide which concepts were appropriate to bring forward the solution process. With a decision on the "next" concept to use, an algorithm - with necessity – defined the next iconic representation which in turn grounded the next heuristical step. This process of constant move between iconic and conceptual representations lasted until a complete and satisfactory solution was reached.

The interaction between the two representations (internal/symbolic and external/iconic and linked by the problem solver) can be taken as an enlarged concept of visualisation. This view of visualisation is characterised by a continuous interaction of perspectives, a constant change to be decided on using geometrical knowledge (and – in learning, especially school contexts: the support of a an external mediator, for instance a teacher). Visualisation is constant interaction of iconic, external and other (external and/or internal) representations.

5 Consequences for research

For (research within the field of) geometry, especially school geometry, the above view on visualisation can "explain" some difficulties: If change between different representations is a, if not ***the*** key to progress in a problem solution, the only type of geometrical representation, the iconic one, and its continuous use will not advance the solution process. Geometry as such, inherently, has to overcome a specific difficulty:

Already working in an iconic mode does not offer a chance to change representations to make available different registers to bring forward the solution process. Taken in a more constructive way, developmental research in geometry teaching and learning should deliberately further the change of representations, especially leaving the realm of geometrical, iconic representations and has to diligently analyse the consequences of such an effort.

More globally, research on visualization must look into both directions of change of representations - and not only analyze the change from other, especially symbolic representations to iconic ones. A perspective on visualization not only taking into account one direction of a necessarily two-way process to and from visual, iconic representations is needed to better understand the links between multiple representations, to better understand visualization.

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MEDIATING ARTEFACTS IN CLASSROOM INTERACTIONS

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This study uses an activity theory approach to explore artefact mediation within a middle high school lesson relating to the reduction of fractions. The first part of the paper discusses the main concepts, including the Bakhtinian concept of 'voice'. These are then applied to outline the activity system of a classroom - episodes of which are analysed in greater detail. Artefacts including the 'teacher voice', the expression 'simplest terms' and various strategies for calculation are identified. It is argued that the choices made by the students reflected the playing out of an hypothesised contradiction between use values and exchange values lying at the heart of the lesson.

The point of departure for this study is the belief that mathematical knowledge is dependent on both the mind and society (Gerdes, 1985; Bishop, 1988; Lave and Wenger, 1991; Nunes, Schlieman & Carraher, 1993; Bloor, 1983; Solomon, 1989; Cobb, 1994; von Glasersfeld, 1995; Walkerdine, 1988, 1994). A common finding/assumption associated with this view is that mathematical thinking is not valid by virtue of correspondence with what is factual (the so called Platonist view) - rather, its truth value depends on its coherence with what is already taken to be true and with what works. These concerns (coherence and utility) throw into focus the questions of how mathematical tools are generated and related to a problem situation and, if all knowledge is situated, how knowledge can be transferred from one site to the next. Related to these larger questions is the issue of how mediating artefacts facilitate the construction of mathematical knowledge within the local domain of a mathematics classroom. It is this topic which is the immediate focus for this paper.

In the next section a theoretical framework for the paper, drawing on activity theory (Scribner, 1997), is set out. Following this, the paper illustrates its main ideas with an examination of transcripts depicting interactions within a middle school classroom. Tensional qualities within situated artefacts are then discussed, and this leads to the conclusion.

'Mediating artefacts' in the processes of knowledge use

The idea of cognitive mediation develops from the work of Vygotsky. In *Mind and Society* (1978), for instance, he enriches the simple idea of stimulus-response processes by introducing a third element described by him as a second order stimulus or "mediated act". For Vygotsky, the mediating act "possesses the important characteristic of reverse action (that is, it operates on the individual, not the environment)" and in so doing "permits humans, by the aid of extrinsic stimuli, *to control their behavior from the outside*. The use of signs leads humans to a specific structure of behaviour that breaks away from biological development and creates new forms of a culturally-based psychological process" (italics in the original, p 40). In Vygotsky's work, signs are an instance of psychological tools, and in the work of later theorists (for instance, Leont'ev, 1981) these are subsumed into a larger category of cultural-historical artifacts associated with the physical, social and cultural context of knowledge production and use. What is especially interesting about Vygotsky's formulation is that mediating artifacts are not merely technical components of task performance - but they "transfer the psychological operation to higher and qualitatively new forms" (p 40). In other words, mediating artefacts transform knowledge in the course of their operation (Wertsch, 1985); thus, knowledge use and knowledge production are seen as interrelated on the Vygotskian view.

These insights have given rise to an 'object oriented' cultural-historical approach to human cognition (Leont'ev, 1981). In Engeström's analysis (1987), for instance, 'subjects' collaborate within an activity driven by a common motive or 'object'. Activities are mediated by tools, rules, and other social characteristics such as the community and division of labour specific to action. However, the object is characterised by internal or primary contradictions and these are played out across multiple domains of mediation leading to an expansive redevelopment of the object and subsequent re-mediation of the activity system.

In addition, because communicative events call up socio-historical processes and thereby express value positions as well as systems of belief and historical placement, an important artefact within an activity is the Bakhtinian concept of 'voice'. When applied to the classroom, for instance, this means we can sensibly talk about the "voice of the teacher" - and mean by this an artefact within the dialogue which both shapes and is shaped by activity. For Bakhtin, when two voices are in dialogue a "hybrid construction" of utterances is produced. By this he means that "an utterance that belongs, by its grammatical (syntactic) and compositional markers, to a single speaker, but that actually contains mixed within it two utterances, two speech manners, two styles, two 'languages', two semantic and axiological belief systems" (Bakhtin, 1991, p. 304, cited Wertsch, 1985, p 227; Engeström, 1995; Wells, 1996). Hybrid constructions such as these facilitate the trajectory of the object on its developmental path.

Looking at the data

In order to illustrate the ideas introduced above, I have selected four excerpts from a corpus of observed mathematics middle high school classroom interactions (Epidsodes, A, B, C, and D) obtained as part of a larger project (Clarke, in press). These involve the teacher, Mrs R and two students, M and D. Prior to the action depicted, Mrs R requires her class to do questions (a) - (e) of the following list taken from a work sheet entitled the 'Ratio pep test'.

Reduce to simplest terms

(a) 4/12 =
(b) 36/12 =
(c) 16/40 =
(d) 240/360 =
(e) 5:5 =
(f) ab:ab =
(g) $2^{1}/_{2}:4$ =
(h) 3x/2x =
(i) 6ab/3b =

As the class settles down to this task, she calls forward to the blackboard a number of students including M whom she says "like to work ahead" (also see A1 below). Here she illustrates certain algebraic techniques (see Episode A below) useful in undertaking items (f) - (i) which she then sets for this subgroup and sends them back to their desks to begin work. M takes up his position with D and the two commence to work collaboratively (Episodes B, C, D). In the course of this we see M and D checking their progress with Mrs R (Episode B) and making progress through the exercises (Episodes C and D).

Episode B starts with M concerned about his response to (h) which was '1.5'.

Episode B

B1 M: Oh, I've got to reduce to the simplest terms. [crosses out 1.5] Oh, OK, um. [pause, writes, works calculator] Alright. Shit. [pause, both work]
B2 D: [inaudible] here and speed.
B3 M: One point 5 - that's what we had [inaudible] [T talks with nearby student]
B4 M: Mrs R? Mrs R, do we simplify that as well? Those two numbers [not clear which referring].
B5 T: Yes, you do.
B6 M: OK.
B7 T: And you're right. [looking at M's work]
B8 M: OK.
B9 T: So you got one right and you got two right. [T then picks up a pencil of M's and ticks (i) and (h), respectively]
B10 M: And that one. [pointing to (f)]
B11 T: Yep. Ahhh, no because it's a ratio. Oh - I guess you could say yes. I'll accept that. Yep.
B12 M: Thank you. [T goes to A and L's desks]

Episode C occurs shortly after the preceding episode. In this M and D engage on the topic of what is the correct response to question (e) which asked for a reduction to simplest terms of 5:5.

Episode C

C1 D: Five to the ratio of five, is one.
C2 M: This was [inaudible], I don't know.
C3 D: One over one.
C4 M: No, we've got to reduce to simplest terms.
C5 D: Yeah, it's just one. The five over five one.
C6 M: It's just one to one.
C7 D: Yeah.

In analysing these episodes, the first step is to examine M's statement to D "no, we've got to reduce to simplest terms" (C4).

Already in B1 M's exclamation "oh, I've got to reduce to the simplest terms" coincides with expressions of hesitation, frustration and uncertainty(B1 - B5). This suggests that the voice used in C4 derives from elsewhere - I argue a hybridisation of the 'teacher' voice and the imputed voice of the text which required students to "reduce to simplest terms". In C4 the form of M's retort resembles the teacher's response given to M previously in B11 - "no, because it's a ratio". On this earlier occasion the teacher expressed a reservation concerning the legitimacy of the form of a response M gave to exercise question (f); in C4 M expresses the same kind of concern (albeit in relation to D's work), and does so in a similar way. In particular, the function of the voice for each instance (B11 and C4) is to progress problem solving attempts by redefining the problem into an alternative version of the question under examination. In B11 this was achieved by drawing attention to specific features of the question asked - the importance of ratio form. In C4 this was achieved by emphasising the salience of the need for "simplest terms". This shows that M's utterance in C4 ventriloquises the teacher's voice in B11. However, C4 also ventriloquises the imputed voice of the text for the text actually requires students to express in "simplest terms". Further, it is noted that M when in C4 M says to D "no, we've got to reduce to simplest terms" words crucial to his meaning are left out. From the context it is possible to tell that he means "No, we've got to reduce *the ratio* to simplest terms". Indeed, because he leaves the words *the ratio* out D is quite legitimately able to respond "Yeah, it's just one"(C5). It is concluded that, for M, the term "simplest" does not function strictly as a mathematical term, but as a tool used to think and perform tasks in accordance with the expectations of the mathematics lesson.

This example illustrates how M and D adopt and synthesise the vocal and textual media in which they are immersed; and use the artifacts so synthesised to reconfigure the problems they encounter in their progress towards accepted outcomes. These relations are expressed in the top triangle of Figure 1. Further, it is clear that the character of this kind of intersubjective knowledge was reflexive in that Mrs R and the students shared knowledge relating to the lesson and shared the knowledge that this knowledge is shared. In the section following, I propose to consider more closely the development of this intersubjectivity and, in particular, explore the issues of tensions among mediated artefacts.

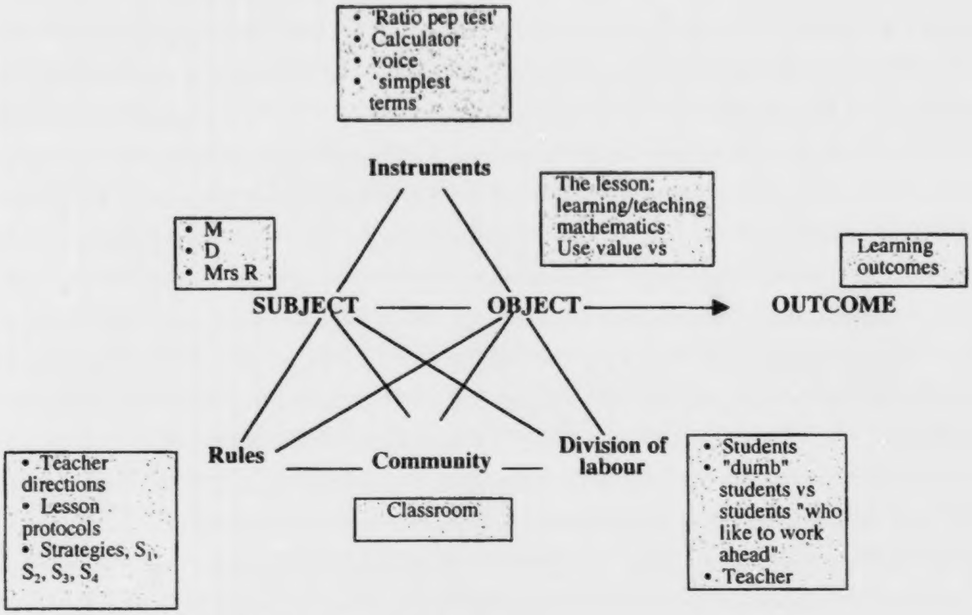
• 'Ratio pep test'
• Calculator
• voice
• 'simplest terms'
Instruments
• M
• D
• Mrs R
The lesson: learning/teaching mathematics Use value vs
Learning outcomes
SUBJECT
OBJECT
OUTCOME
Rules
Community
Division of labour
• Teacher directions
• Lesson protocols
• Strategies, S1, S2, S3, S4
Classroom
• Students
• "dumb" students vs students "who like to work ahead"
• Teacher

Figure 1: Model of the classroom activity system

Tensions among artefacts

Episode D occurs immediately after Episode C.

Episode D

[Question (a) reduce to simplest terms: 4/12]
D1 D: It's one third, the top one, then there's—that's one third as well. [pause] What's the forty one? [referring to question 1 (c) reduce to simplest terms: 16/40] Four over. [M writes 1/3]
D2 M: That's not right, the ...
D3 D: Divide it by four, it's, yeah. [M writes 4/]
D4 M: No. It's got to be two over five. [writes 2/5]
D5 D: Yeah, two over five. I knew that one. I was just [inaudible]. [writes 2/5]
D6 M: Oh sure you did, that's what they all say.
[Question 1 (d) reduce to simplest terms: 240/300]
D7 D: Um, two-forty, yeah, three hundred divided by two-forty.
D8 M: Hang on, twelve, so we'd have twelve ...[pause - writes 12/]
D9 D: No, that ... it's divided by it, right? That's one ...
D10 M: ... over four [interrupting - writes 4]. Four divided by ...
D11 D: ... or one third.
D12 M: ... it's going to be three over one. [writes 3/1]
D13 D: Why?
D14 M: 'Cause, it's just going to be the opposite. Look.
D15 D: Yes.
D16 M: Times twelve.
D17 D: 'Cause that's bigger number. Yep.
D18 M: Um. Um. Two forty over three hundred. Um, two four 'o' divide—no.
D19 D: Three hundred.

D20 M: Three hundred, divide two four 'o' —no. No, um. [Presses 300/4 = 75 on calculator] Seventy-five.
D21 D: Did you get twelve over. [pause]
D22 M: [Presses 240/6 = 40 on calculator] ... six ... forty. And that was one. [writes 40/75]
D23 D: Is it—yeah, it's eight over.
D24 M: Seventy-five divide eight doesn't work, no, can't be eight.
D25 D: It's um—try ten.
D26 M: Five. No, seventy-five, it's going to be five there.
D27 D: Yeah, five over—no, try five.
D28 M: By five, is fifteen.
D29 D: It's eight over fifteen.
D30 M: I think, hang on.

In this episode M and D exhibit evidence of four strategies (S_1, S_2, S_3 and S_4) relating to the task of simplifying fractions, as follows. In S_1 a fraction a/b is reduced to the form 1/n where n is the natural number for which either b = na or a = nb. D uses this strategy in D7 and D9 - it gives him the correct answer to (a) and a wrong answer to (b). S_2 provides a necessary condition for any reduced form: if a/b is reduced to c/d and if a>b, then c>d. This test enables M to state in D2 by inspection that D's response to (b) must be wrong (see also D14). Strategy S_3 is to divide both the numerator and denominator by a common factor (D3 - D5, and D8, D10 and D12), thereby determining a reduced fractional form. Finally, S_4 deals with the case where the natural number n as in S_1, does not exist. In this case factors of a, say n_1 and n_2, are sought such that $a = n_1n_2$ and there exists a natural number n_3 such that $b = n_2n_3$. The reduced fraction is then written as n_2/n_3. We see S_4 applied in D18 - D30. In D18 and D20 M carefully checks that strategy S_1 does not apply. Then he uses S_4 twice: first in order to reduce 240/360 to 40/75 (D20 and D22); and second, with D's help, in order to reduce 40/75 to 8/15 (D23 - D30).

Of these strategies, of course, S_2 is correct only in the limited cases to which it applies, and S_3 is a correct general strategy; S_1 and S_4 are both false. My focus in the following is on S_4. Where does come from? Preceding their work together, a number of other students including M who like "to work ahead" (Mrs R's words) is gathered at the board in the front of the classroom. Here Mrs R rehearses a series of algebraic manipulations working towards a cancellation algorithm involving algebraic expressions. Her method (a sample is illustrated below) involves an intricate construction of pronumerals and arithmetic operations involving factors.

Episode A

A1 T: Just working ahead a little bit? OK, guys I'm going to think of two numbers. OK? Oh, come on Alistair. Now stand over here because. See, Dale, we'll talk behind their back, you see. And so will L. Over this side guys, over this side. See if you can see. You stay, the small ones can stay here, you go over there. OK. All right, that's it, that's fine. Just squat down. OK. Now, I'm going to think of three numbers, right?, x is going to be 7, y is going to be 9, and uh, m is going to be 3. OK? Now I'm going to multiply x by 5. I would write it as 5x. OK? I'm going to multiply y by 5, how would I write it?
A2 Ss: 5y.
A3 T: OK. And I'm going to multiply m by 8.

A4 Ss: 8m.
A5 T: All right. Now, I'm now going to divide x by 5. Now what's going to happen if I do that?
A6 Ss: Be the same number.
A7 T: Ah. It's going to go back to the same number. All right. I'm now going to take this, um, I multiplied m by 8, I'm now going to divide it by m. What am I going to be left with?
A8 S51: Eight.
A9 T: Um. Is it?
A10 S52: Yes.
A11 T: Right, m is 3, 8 times 3 is 24, divided by 3, brings it back to 8. Do you notice that this one you told me brought it back to 7. Seven times—5 times 7 is 35, divided by 5 is 7. Good. And this one here you told me went to 8. Now can you see a pattern?

At the conclusion of this instructional episode, students are instructed to return to their seats and do the entire question set - that is, including those questions (f) - (i) she requested the general class to "cross out". Mrs R's figuring in this episode and M's construction of S_4 are comparable in their intricacy and in their emphasis on the multiplication and division of factors. Whether M takes more than these kinds of manipulations from Mrs R's work is unclear - nevertheless, in his choice of S_4, M exhibits a motivation to engage in a similar kind of exploration of factors relevant to the questions considered. It emerges that in his subsequent work with D, M expresses a stronger preference for the erroneous and more convoluted S_4 rather than the correct and apparently more straightforward, S_3. M seems to find the form of S_4 more appealing than the substance of S_3. What can explain this apparently anomalous situation?

My approach is to analyse the tensions among mediating artefacts within the activity system of the classroom, as set out in the application of Engeström's model of the activity system in Figure 1, above. At the heart of my argument is the assumption that there is a tension within the core of the lesson itself - namely, a primary contradiction between the 'use values' and the 'exchange values' of mathematics learning (Engeström, 1987, 1991, 2000). In the classroom, use value is associated with getting the answers correct; exchange value, with being seen by the teacher and other students as "liking to work ahead" (as Mrs R put it) or not being "a dumb person" (as the students put it). This primary contradiction is played out as a tension between the correct strategy S_3 on one hand, and the choice of the status conferring though erroneous strategy S_4 on the other. What this analysis shows is that 'the lesson' observed is more than merely a mathematics lesson - it is also a lesson about the social standing of mathematical knowledge, and about how critical value sometimes attaches to the forms of knowledge (such as are evident in S_4) rather than its substance (as in S_3). This paper therefore highlights the importance of recognising contextual and axiological variables when analysing cognitive events in the classroom.

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INFLUENCES ON AND FACTORS CHANGING TECHNOLOGY PRIVILEGING

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Based on classroom observations and interviews over two years, this study reports on the different ways two teachers incorporated use of a computer algebra system into their teaching of introductory differential calculus. Factors that contribute to successful CAS teaching were monitored: teaching approach, preference for representations, and use of technology. In the first trial the teachers adopted distinctly different ways of using CAS. In the second trial, these differences persisted, but they made some changes. In response to new knowledge about the testing, one teacher extended his focus on rules by teaching new CAS procedures. In response to new students, the second teacher reduced CAS use for routine symbolic procedures while continuing to use it to support conceptual understanding.

Background

It is generally acknowledged that teachers' classroom teaching practices are influenced by their underlying beliefs and knowledge about mathematics and mathematics teaching. Reporting on a professional development project during which teachers explored learning and teaching with computer activities, Noss and Hoyles (1996) monitored changes in the ways the teachers used Microworlds technology. These changes served as a 'window' on the teachers' mathematical beliefs and pedagogy. Noss and Hoyles observed that "there is a mutually constructive relationship between what teachers believe and what they do"(p.201). They maintained that while changing their pedagogy, "Teachers are not pushed arbitrarily by 'constraints'. Neither are they free agents"(p.201). The present study provides a detailed description of how such changes in teaching with technology come about and how they are linked to changes in teacher knowledge and beliefs.

Attempts have been made to classify teaching practices and to relate them to teachers' knowledge and beliefs. Shulman (1986) distinguishes between teachers' content knowledge (understanding and organization of knowledge of specific topics) and content pedagogical knowledge (knowing ways of presenting knowledge to students, including representations, and understanding what makes the learning easy or hard). These types of knowledge are interconnected (Even & Tirosh, 1995). Fennema and Franke (1992) describe other types of teacher knowledge; pedagogical knowledge (knowledge and planning of teaching procedures, behaviour management and motivational techniques), knowledge of students and knowledge of institutional constraints.

Teacher content knowledge impacts on teaching practice which in turn impacts on student learning. Jablonka and Keitel (2000) believe that teachers with highly developed content knowledge are "more flexible in structuring content for teaching and in discussing students' ideas"(p.120). Gutstein and Mack (1998), in a study of teaching fractions, show that a teacher's teaching decisions and practices were derived from the depth of her interrelated knowledge of content, pedagogical content, and of students.

Kuhn and Ball's (1986) model of teaching practice was commended by Thompson (1992), in a comprehensive survey of teachers' beliefs and conceptions, "as constituting a consensual knowledge base regarding models of teaching"(p.136). The model identifies four different teaching practices linked to particular beliefs about mathematics and goals for teaching, and characterized by related teaching styles.

> 1. Learner focused: mathematics teaching that focuses on the learner's personal construction of mathematical knowledge;
> 2. Content focused with an emphasis on conceptual understanding: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
> 3. Content-focused with an emphasis on performance: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures;
> 4. Classroom focused: mathematics teaching based on knowledge about effective classrooms. (Kuhn & Ball, 1986, p.2)

The relationship between teaching styles and student achievement for teaching with technology is a topic of current interest. For example, student success (i.e., improved conceptual understanding) on calculus courses taught with technology is attributed to the adoption of student-centred teaching practices within a constructivist perspective (Keller, Russell & Thompson, 1999; and others). Kendal & Stacey (1999 & in press) use the word *privileging* (originally used by Wertsch, 1990) to describe a teacher's individual way of teaching. It includes decisions about what is taught and how it is taught including: what is emphasised in the content (what is stressed and what is not stressed), what representations are preferred and ignored, the attention paid to procedures and concepts, rules and meaning, and how much is explained or left to students to work out for themselves. Privileging reflects the teacher's underlying beliefs about the nature of mathematics and how it should be taught. It is derived from an interplay of teachers' beliefs and interrelated knowledge sources (content, content pedagogical, pedagogical), is moderated by institutional knowledge about students and school constraints, is manifested through teachers' practice and attitudes, and is highly influential in student learning. Privileging has several components and in this study of the teaching of calculus with technology, we focus on teacher choices about:
(1) *Teaching approach* (evidenced by teaching method and style using Kuhn and Ball's classification above).
(2) *Calculus content* (evidenced by representations of differentiation taught).
(3) *Use of technology* (evidenced by the nature of use of the CAS calculator).

This study reports on the three aspects of each teacher's privileging during the teaching of an introductory calculus unit. It monitors the changes that occurred over the two years and explores the impact of new knowledge and a new situation on the changes in technology privileging, linking them to each teacher's beliefs and pedagogy.

Method

In two successive years, Teachers A and B volunteered to teach the same 25 lessons on introductory differential calculus course to their Year 11 classes (16-17 year olds). They were both experienced teachers of mathematics and had used graphics calculators in their classrooms for several years. They participated in the development of the teaching program that focused on numerical, graphical and symbolic representations of derivative and links between them (see Kendal & Stacey, 1999) and integrated the use of a CAS calculator (TI-92).

Observation of lessons during teaching trials 1 and 2. Half the lessons in the first trial, and every lesson in the second trial were observed and audiotaped. Teacher behaviour was closely monitored (e.g., time spent on each type of differentiation activity, the nature of every teacher-student interaction, and attitudes displayed towards calculus, technology and students) and a comprehensive checklist of 52 observations was completed immediately after each lesson. Finally, a privileging profile (with three components) for each teacher was developed. It consisted of teaching approach (style & manner), calculus content (the representations of differentiation they chose to teach), and ways the CAS calculator was incorporated into their lessons (frequency & nature of use). The nature of calculator use was classified as functional (primarily to get answers), pedagogical (primarily for learning) or neutral.

First interview after the first teaching trial. Nine months after the first trial and ten weeks prior to the commencement of the second trial, each teacher was interviewed separately, to identify their personal knowledge of differentiation. This also provided a basis for comparison with the privileging that occurred during the second trial. During the interview the teachers were asked to discuss their proposed solutions (and to predict their students' responses) to a set of problems most of which would be included on the students' tests six months later. A wide spectrum of teacher characteristics was monitored including: personal knowledge of multiple representations of differentiation; awareness of alternative ways to solve differentiation problems; preference for representation; attitude towards the CAS calculator; personal CAS calculator use; knowledge of students; awareness of subtle school pressures and explicit constraints; and evidence of teaching methods and teaching styles.

Second interview after second teaching trial. Ten weeks after the teaching second trial, a second teacher interview was conducted to substantiate the privileging identified by observation of lessons during the second trial. Teachers were asked to reflect about their teaching practices, particularly the way they had used the CAS calculator to support conceptual understanding of the concept of derivative.

The following results summarise observations from all of these sources.

Results

Teacher knowledge

Personal teacher knowledge of calculus (differentiation). During the first interview, Teacher A knew how to differentiate symbolically (using algebraic rules) and graphically (finding the gradient of the tangent at the point), and how to translate between the two representations. He nominated up to two different differentiation strategies to answer particular test items, reasoned successfully about numerical derivative (limiting value of a difference quotient), reasoned with difficulty about graphical derivative and was unable to reason about a symbolic derivative. Teacher B performed symbolic, graphical and numerical differentiation and translated between the three representations of derivative. He nominated up to four different differentiation strategies to answer particular test items, and reasoned successfully about numerical, graphical, and symbolic derivatives.

Overall, Teacher A displayed less depth and less integrated knowledge about the concept of derivative than Teacher B who displayed deep, holistic, and integrated knowledge. This was reflected in both trials by Teacher A's focus on teaching rules and Teacher B's focus on developing students' understanding.

Institutional knowledge. During both trials, the teachers were cognizant of the fact that although their students could use the CAS calculator for the trial tests, they would not be permitted on school examinations to be held in three months time and on official state school examinations in fifteen months time. They were also aware that the style of assessment on trial test 1 was similar to the official school examination (based mostly on the symbolic derivative) and whereas for trial test 2, the assessment involved numerical, graphical and symbolic derivatives.

Three components of teacher privileging

1. Privileging related to teaching approach during both trials

Teaching method. Teacher A mostly taught rules for procedures and during both interviews talked about routines to solve problems. In contrast, Teacher B emphasized understanding of the concept of derivative. He employed enactive representations and encouraged students to use visualization techniques. During the first interview he solved each problem several ways, explained his use of different representations and made sense of each answer. During the second interview he talked about conceptual understanding: "Getting the tangent idea through to them, what the gradient actually represents, what the derivative represents and the relationship between them - I think we've done very nicely with the calculator."

Teaching style. Teacher A lectured his students who were expected to copy down his lesson notes. In contrast, Teacher B orchestrated discussions between 'student and teacher' and 'student and student' and his student-centred teaching style fostered student construction of meaning. In both trials, Teacher A's teaching approach (which emphasised student performance and mastery of mathematical rules and procedures) is classified as *Content-focused with an emphasis on performance (using*

Kuhn and Ball's (1986) model). Teacher B's approach which emphasised conceptual understanding of content and student construction of meaning is classified as *Content-focused with an emphasis on conceptual understanding*. This involves the "dual influence of content and learner. On one hand, content is focal, but on the other, understanding is viewed as constructed by the individual" (p.15).

2. Privileging of calculus content

During the first trial and in the first interview, Teacher A focused almost exclusively on the symbolic derivative. However, during the second teaching trial, he expanded his use of representations to include graphical and numerical derivatives. This came about after the first interview during which he realized that the students' assessment would involve numerical and graphical differentiation, unlike the tests in first trial that were essentially symbolic. In consequence, during the second trial he decided to include the numerical and graphical representations of derivative.

During both teaching trials, Teacher B consistently stressed the symbolic derivative and involved the graphical derivative in explanations. Although during the interview he personally demonstrated ability to differentiate numerically (i.e. use a rate of change or difference quotient), he actively rejected teaching about the numerical representation in the second trial. He explained that this was because he believed that his students were a low attaining group and would not be able to cope with three representations of differentiation (i.e., he made changes because of knowledge of new students).

3. Privileging of technology

In the first trial, Teacher A linked the CAS calculator to an overhead projector and frequently demonstrated symbolic procedures to the students and allowed them to use CAS freely. He avoided using graphs. In the second trial, he taught his students the additional CAS numerical and graphical differentiation routines (described as *functional* use by Etlinger, 1974) and provided them with a step-by-step flowchart of corresponding CAS calculator procedures. In addition, he used the calculator to explain the links between the numerical and graphical derivatives (described as pedagogical use by Etlinger, 1974)

> I'd say, when you see these words it means between two points, and when you see this word that means at a point. . . [I am] giving them strategies. . . and we did it [used dynamic graphing program] to understand the straight line against the curve.

In the first trial, Teacher B used graphs freely but noticeably controlled student use of the CAS calculator for symbolic procedures. In the second trial, he actually reduced his functional use of the CAS calculator while maintaining his pedagogical emphasis on the symbolic and graphical links.

> It's [the CAS] good for discovery because it takes a lot of the hack work out of teaching for understanding but you still need to teach pen and paper skill. I think there are certain skills that the kids have to have, even if you can use the technology

> to do it. I think the kids have to have the [algebraic] skills as well, without the technology. I think that's essential for their understanding. It's not sufficient to just use the calculator; they have to have the understanding of what's behind it.

Table 1 below summarizes Teacher A and B's privileging demonstrated during the first trial lesson observations (reported by Kendal & Stacey, 1999) and the changes in their technology privileging that occurred during the second trial.

Table 1. *Teacher A and Teacher B's Privileging in Trial 1 and Changes in Technology Privileging that Occurred During Trial 2*

	Teacher A	Teacher B
	Privileging in First Trial	
	1. Teaching approach	
Teaching style	• Lectured students	• Orchestrated student centred discussion between 'teacher-student' and 'student-student'
Teaching method	• Used rules for routine procedures	• Promoted understanding of routine procedures and problem solving, used enactive representations and visualization
	2. Calculus content	
Preference for representation	• Preferred symbolic derivative	• Preferred symbolic & graphical derivatives
	3. Technology	
Functional use of CAS	• Strongly promoted use of CAS for symbolic procedures	• Restricted use for symbolic procedures
	• Disliked graphical procedures	• Permitted use of CAS for graphical procedures
Neutral use of CAS	• Checked by-hand solutions with CAS	• Checked by-hand solutions with CAS
Pedagogical use of CAS		• Used CAS for algebraic and graphical procedures to save time on activities that linked symbolic and graphical derivatives • Incorporated activities that stressed the links between the symbolic and graphical derivatives
	Changes in Technological Privileging in Second Trial	
Functional use of CAS	• Adopted additional CAS numerical and graphical differentiation procedures	• Reduced CAS use for symbolic procedures • Rejected CAS use for numerical procedures
Pedagogical use of CAS	• Incorporated activities that stressed the links between the numerical and graphical derivatives	

Conclusions and discussion

Teacher A's personal knowledge of differentiation was limited and the interviews revealed he believed his main responsibility was to help students pass examinations. During both trials, his teaching approach involved teaching rules and procedures using a lecture style of delivery. He focused on symbolic differentiation because he believed this was exact (stated in first interview). He appreciated the symbolic power of CAS, enjoyed using it himself, and encouraged his students to use it for symbolic procedures. During the first interview, which occurred between the two teaching trials, Teacher A realized for the first time, that the second trial assessment would involve multiple representations of differentiation. He responded to this "new" institutional knowledge by expanding his repertoire of calculator procedures to include numerical and graphical differentiation. In addition, his initial preference for the symbolic representation had an unexpected consequence - he gave a stronger emphasis to numerical differentiation. He showed his students how to use CAS procedures to substitute into functions to find ordered pairs and create a difference quotient calculation. This usually gives an 'excellent' approximation to the gradient of the tangent (and curve) and Teacher A led his students to believe it was exact. He also believed that with CAS, graphical differentiation was "exact".

Teacher B's knowledge of differentiation was deep and holistic and he believed that it was his responsibility to foster student understanding. During both trials, his privileging included teaching approaches that supported conceptual understanding using a student-centred style of delivery. He believed that the symbolic representation was the most powerful and useful for his students but he limited their use of CAS in order to prepare them for future examinations without it. However, to support student understanding, he adopted the graphical representation of derivative (gradient) using the CAS calculator. He also showed gradients enactively, and encouraged his students to visualize graphs of symbolic functions and derivatives. From the first to the second trial, Teacher B's privileging was essentially stable but he reacted to "new" knowledge about his students: that the second group was algebraically weaker. In the second trial, he totally rejected the numerical representation, believing they could not cope with three representations. He also reduced their opportunity to individually differentiate symbolically with CAS (but allowed pedagogical class activities) strongly believing they needed practice with by-hand symbolic differentiation to cope with future examinations.

Both teachers, in response to new knowledge, made changes to their technological privileging in ways that were consistent with their own beliefs and knowledge. These results are consistent with other research. For example, Tharp, Fitzsimmons, and Brown Ayers (1997) showed that teaching style tended to be unchanged by the addition of technology. In their study, after an initial attempt at more enquiry based learning using a graphics calculator, teachers with a rule-based (procedural) view tended to revert to their procedural style of teaching, whilst teachers who were not rule-based remained more likely to focus on student conceptual understanding and thinking.

The institutional constraints and teachers' knowledge or assessment of the needs of their students were important determinants of the change in privileging for both of our teachers from the first to the second trial. The constraints of the assessment system were important for both, although in different ways. Teacher A was more accepting of aims the research project, allowing students to use CAS and adding new techniques in response to new knowledge of expectations of students. Teacher B was always more concerned that students developed by hand skills and rejected some of the aims of the research trial for a group he assessed as weak.

As hand-held CAS calculators are now becoming more affordable and easily incorporated into the teaching of mathematics in secondary school classrooms, the issue of teachers changing their pedagogy for more effective teaching with technology is becoming increasingly urgent.

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EFFECTIVE TEACHERS OF SECOND LANGUAGE LEARNERS IN MATHEMATICS

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This paper presents an overview of findings of an ongoing project to identify teachers who demonstrate effectiveness in teaching mathematics with minority second language learners. The objective of the study is to better understand the processes of instruction that contribute to positive student achievement in the subject. The study rests on the assumption that the teacher is a critical factor in student learning given the teacher's role as a more experienced other and as the engineer of learning environments. Data for the study include classroom-based observations and videotape analyses of teachers from two different geographical sites. This discussion will summarize the common characteristics among the teachers. Findings strongly suggest that key elements in teaching cannot be taken for granted.

In many classrooms, in primary through secondary level schools, in various parts of the world, there are a growing number of students whose home language and/or more proficient language is other than the dominant language of instruction. These students are second language learners (SLL); they enter school needing to develop or improve academic proficiency (Cummins, 1981) in the dominant language of institutions, government, and power. In many contexts, these same students are members of groups who have a history of underachievement and underrepresentation in educational areas that are particularly associated with basic social prosperity and advancement. In countries such as South Africa, the educational issues and success of the majority of second language learners are strongly connected to the country's overall political well-being. In the United States, the ethnic and linguistic demographics of classrooms have significantly changed over the last thirty years; so much so that a teacher can no longer assume that all students will speak English or will be sufficiently proficient in all modes of the language to participate fully and equally with other students who are native English speakers. This discussion will focus on the USA context; however, as just noted, the findings and implications of this work extend to other contexts and other SLL groups.

The work presented here is primarily concerned with issues related to the educational achievement of Latinos, and specifically students of Mexican descent. In the USA, this group is disproportionately low-income, has one of the highest rates of non-completion of high school and has had a long history of underachievement in mathematics. Consequently, the improved schooling of Latinos, particularly in mathematics is a major concerned for the educational community. In the recent decade, there has been much discussion about the relationship among Latinos' predominant characteristic as English language learners, communication processes in classrooms, and mathematics achievement (Secada, 1992; Khisty, 1999). Indeed, there are clear and obvious connections between the clarity and comprehensibility of mathematical talk (both oral and written) in a second language and a student's ability to function with the talk to learn mathematics. In essence, a linguistically and culturally sensitive learning environment is highly relevant to second language learners' success in mathematics (Ortiz-Franco, Hernandez, & De La Cruz, 1999). We know generally what constitutes these effective learning environments. For example, connecting student experiences to mathematical problems makes them easier for second language learners to comprehend (Khisty, McLeod, & Bertilson, 199?). However, we still do not know enough about how teachers actually enact these supportive learning environments. A study was conducted to identify effective teachers of mathematics with second language learners, specifically Latinos, in order to better understand the classroom processes that seem to contribute to students' learning of the subject. This discussion draws upon part of the study and focuses on common characteristics found among a group of teachers. Two teachers who were found to be exceptional will be highlighted specifically by examples from their classrooms.

Theoretical Framework

The study is based on some key assumptions. First, that learning is a social activity; that higher psychological functions originate in human interactions and activity (Vygotsky, 1978). Second, these sociocultural activities are mediated through the use of cultural artifacts, tools, and symbolic systems, especially language, thereby implying a reciprocal relationship between the cultural and the intellectual. What is important here is that how and why these artifacts are used mediates how humans come to think with them (Moll, 2000). Third, sociocultural activity, or interaction, forms the context in which children participate and from which they appropriate tool use and cultural thinking. From this, we can assume that the "more experienced other" with whom children interact is highly relevant to their learning. The "more experienced other" is the person (be it parent, teacher, older sibling or peer) who initiates and assists the "less experienced" ones in learning; it is the "more experienced other" who provides the relationship between development and the cultural resources or tools that produce that development (Moll, 2000). However, this perspective does not suggest interaction or activity as an imitative set of dynamics. In former views of teaching, the teacher was seen as the subject and agent and the learner as the object and patient. On the contrary, if we observe normal human interactions, we can see that there is joint activity with teacher and students as co-constructors of the interaction. In summary, as Moll (2000) points out: "Social relationships are the key to the mental and personal development of individuals. The very mechanism underlying higher mental functions is a copy from social interactions" (p. 30).

Given the foregoing, the focus of the study was on the teacher, defined as the "more experienced other", and the primary person in the classroom from whom students would appropriate critical aspects of mathematics learning. It was further assumed that by observing effective teachers, it would be possible to identify key aspects of their instruction as they enacted them. Teaching is such a complex activity that often teachers are not aware of all of what they do or say as they instruct. Furthermore, there are times when there is incongruence between what one says one believes about instruction and what one actually does. Ongoing observations of teachers during instruction would more accurately capture the relevant elements.

Methods and Data

This work began several years ago in one middle-size city in the west coast region of the USA and was continued in a large urban area of the Midwest. Both areas have large populations of Latino second language learners, and many schools (at all levels) could be found to have enrollments that were very near 100% Latino. In both areas, a search for effective teachers of mathematics with Latino second language learners began with identification of schools that had high standardized test scores in mathematics. Principals in these schools then were asked to recommend teachers who had demonstrated consistent ability to develop students to do well on the tests in this subject. Once a pool of teachers was identified, they were initially observed to determine if they fit a profile of effectiveness. Effective was determined on two levels. First, did the teacher have a history of significantly moving students ahead academically, i.e., having students who scored at or above grade norm ? Did students with this teacher demonstrate significant growth in mathematics as measured by standardized tests? Second, did the teacher model instruction that was consistent with general notions of best practices? Interestingly, there were not many schools in either area that both had high populations of Latino second language learners and high test scores in mathematics. Generally, most teachers who were initially identified produced occasionally only small gains or had students who usually fell just short of a standardized norm.

A pool of five teachers was finally selected for observation. All teachers had a history of their students gaining two to three grade levels during the year with them. In all cases, most of the teachers' students entered below grade level and left, a school year later, one to two grades above norm on a standardized test. One teacher was at second grade, two were at fifth grade, one was at sixth, and one was at eighth grade. Each teacher was observed using fieldnotes and was also videotaped for at least twenty hours of mathematics instruction. Some observations were conducted on consistent days to capture a fuller development of a mathematical concept and the rest were done intermittently to capture different parts of the school year's teaching and learning. While other types of data were collected as supporting or elaborating documents, the videotapes are the primary source of data. The videotapes were analyzed for relevant patterns in the teachers' instruction without any *a priori* determined set of elements. Teachers were informally interviewed about their instruction when additional clarification or elaboration was warranted. This discussion focuses on common characteristics or elements shared by all five of the teachers.

Shared Characteristics of Effective Teachers

The five teachers of this discussion shared some striking similarities in their instruction of mathematics with second language learners. The teachers did not teach in the same school or know each other. Two of them were fluent in Spanish but only one taught primarily in Spanish. The different use of Spanish was due to school program demands and not to the teachers' belief of the importance of the primary language in learning. The other teachers only spoke English but used various methods to incorporate Spanish in the classroom in order to facilitate students' learning. All of them had ten to fifteen years or more of experience teaching with second language learners. Only two of them had a concentration of college coursework in mathematics or mathematics teaching. In what follows, I will describe the aspects of their teaching of mathematics that seem particularly relevant to students' success. Examples from two teachers in particular, one second grade and the other fifth grade, will be used to highlight some of these characteristics.

Writing mathematics. First, at a time when reforms in mathematics have emphasized connections between this subject and other disciplines (NCTM, 2000), very little has changed in terms of using writing to develop mathematical concepts and understandings. Yet writing is clearly a process that can support and advance student thinking. It helps to bring order out of chaos in one's thinking (Halliday & Martin, 1993). In these classrooms, writing mathematics was a constant and natural part of the mathematics curriculum. In the second grade classroom, the teacher used writing addition and subtraction problems to teach not only mathematical literacy, concepts, and skills, but also general emerging literacy skills. Practice spelling lists included mathematical terms and students learned letters, sentence structures, and conventions such as punctuation in the context of reading and composing their own mathematical word problems and explanations. For example, when students completed composing their problems, they brought their work to the teacher who sat a group of students' desks. She would read aloud each piece of writing and comment on both the fluency of the writing and the mathematics in the problem. She also edited the writing with language corrections and send the student back to make the necessary revisions. All writing was kept in a student's own writing book, and often, a student's piece of work would be used to demonstrate to the whole class some particular aspect of writing and the structure of language they all needed to learn.

In a fifth grade class, after some work on developing understanding of a particular concept, for example, finding the missing leg of a right triangle, students had to write to a fictitious person telling them how to do a problem using the concept (Chval, 2000). This writing was revised at least five times with each draft having dialogue-type comments from the teacher. These comments were usually in the form of questions from the teacher asking for clarification on specific aspects of the problem solving process. All comments from the teacher were detailed and clearly designed to guide the student's thinking and writing mathematically. It was not until the final draft that comments referred to issues of conventions of writing such as choice of words, spelling, and punctuation. Examination of samples of student writings over the several drafts strongly suggests how powerful writing and teacher's guidance through the process can be in learning mathematics (Chval, 2000).

Mutual support among students. Another characteristic among all the teachers was their development of mutual support among the students. All the teachers preferred to have students work with each other in pairs and encouraged the fluid movement of students such that, as needed, a student might move to join another pair to form a group of three or several students might form a larger group. Consistently, what determined the movement among students was the need to seek additional input on how to solve a problem or to get an additional check on the correctness of the answer. Sometimes a problem was so difficult that several students gathered to figure it out. Two things are particularly significant about this. First, the organization of students is different from what is often thought of or enacted for active learning and groupwork. Too often, teachers' understanding of groupwork means that students are organized into groups of three to five students who are to work together on a problem. The study's teachers felt that groups of three or more students hindered learning since mathematical problems were usually a task that could be accomplished individually thus not fitting the nature of groupwork. However, they did believe that good learning came from students talking together. Second, because of this belief, the teachers' focus became not the organization of students but rather students' sense of responsibility to help one another understand the mathematics. Therefore, a good deal of time was spent setting and reinforcing this culture. In the following example from the fifth grade teacher (Chval, 2000), we can get a glimpse of the development of this culture:

> Teacher: Alejandro wasn't participating because he never asked for help. So somebody over here. Anybody. You move around. I'm only one person. Move around quietly and ask each other. You can teach each other. Walk around. Help each other. I can't help all of you at the same time.

This kind of norm was not something taken lightly; in fact, establishing it was an integral part of the overall curriculum. All students were socially identified as having particular skills and expertise that could be tapped; at the same time, all students had to take initiative to help if someone else seemed to be having trouble. The organization of students into groups became secondary to the idea of mutual sharing of knowledge and willingness to help others learn.

High expectations. All the teachers also had high expectations for their students that were *actually* manifested in the curriculum. Much has been said over the last several years about the importance of high expectations for improving the learning of underachieving minority students (e.g. Dusek, 1985). However, too often this is a hollow belief with little evidence of it in teaching. This was not the case with these teachers. For example, the fifth grade teacher began the school year with problems revolving around a right triangle such as finding a missing leg when other sides are known. The year was spent on other themes in geometry such as measuring circles and rectangles. This is extremely unusual work for fifth grade minority students who too often start the school year one to two grade levels below norm--as was the case with this particular class. Many years experience conducting staff development with practicing teachers has demonstrated to this author that teachers tend to misinterpret the educational dictum of "beginning with what the students know" to mean starting with mathematical basics. Too many teachers would have decided that geometry was too advanced for the students and would have taught a remedial program. This teacher like the others in the study defined her task as supporting and guiding her students in such a way that they would grasp whatever curriculum she gave them. That the students were below grade level norm did not deter her from teaching an "advanced" curriculum. Interestingly, like the other teachers' classes, outsiders assumed these students were part of a gifted class (Chval, 2000). While the idea of high expectations for students seems like an outdated one, these teachers demonstrate that it is still highly critical for the success of minority second language learners in mathematics.

Contextualization of mathematics. The teachers in this study were all very skillful at contextualizing mathematics. They saw contextual support as a critical tool for developing students' comprehension of mathematical ideas and practice, and accomplished it through a combination of all of the following: drawings, concrete objects, stories that came from students, students' experiences, and presenting everything said in written form so that students did not rely on listening for learning. Students in all the classes spent a good deal of time constructing models such as making three dimensional geometric shapes or cutting paper to show operations with fractions. In one example, in order to reinforce the ideas of perimeter and area, the fifth grade teacher had students stand next to some grouped tables in such a way that their bodies only touched the perimeter of the tables. Then the students were asked to sit on the tables so that they did not touch the perimeter and touched only the area. Such physical demonstrations were a frequent activity to ensure that students both would not forget the ideas being taught and would have a physical image to support the ideas. It is easy to assume that contextualizing mathematics is a good thing but not critical to mathematics learning. These teachers believed just the opposite. Contextualization in as many ways as possible was deemed to be the key to their students' comprehension, and therefore their learning. Creating a highly contextualized learning environment is also a key principle of effective instruction for second language learners in general (Cummins, 1986). Therefore these teachers were integrating what they knew about learning in second language with their mathematics teacning.

Pedagogic talk. While some of the teachers used only English for instruction and others used both Spanish and English, they were all highly mindful, but not necessarily conscious, of their pedagogic talk. They used their talk such as probing questions and statements, both oral and written, as a tool for learning much like manipulatives. They all recognized that their students had thoughts about what they were learning but did not have words to describe those thoughts. Their task was to give the students the means to express their thoughts. However, the teachers believed this occurred by students appropriating what they frequently heard. Consequently, the teachers' talk was very deliberate but not unnatural; it also was very rich in terms of vocabulary, and students after a time could be heard using the same words and phrases appropriately. For example, the fifth grade teacher in one lesson used the word "congruent" over forty times but not in a repetitive manner. In the following excerpt, again, we can get a glimpse of how "congruent" is brought into students' environment and repertoire.

> Tch.: I have two congruent triangles here. Two equal parts, two exact triangles. I want only the area of my original triangle, ACB....[several teacher and student exchanges occur] Would you please read that, Julia?
> Julia: The triangle and its...
> Tch: Congruent
> Julia: Congruent [struggling]
> Tch: Look at that word everyone. Congruent. What does that mean? [students offer explanations which the teacher incorporates into her talk. Several exchanges later]
> Tch: They appear to be congruent to each other. I agree. They appear to be congruent. But this one and this are not congruent, are they?

All the teachers also were exceptionally skillful in guiding their students' thinking by the use of key questions, ones that would challenge and extend students' thinking. Overall, there were few lower level fact or knowledge questions. In this example, the class as a whole is reviewing how they solved finding the area of a rectangle. A student has offered only the correct numerical answer, and the teacher responds in a supportive manner: "I don't care what the number is. What does that number represent? What does it mean?" Such emphasis on asking for meanings punctuated all of the teachers' talk to an overwhelming degree. It should be noted that these types of questions served to both develop students' thinking

Conclusions

The purpose of this discussion was to highlight common characteristics of teachers of minority second language learners, teachers who had a clear record of being effective in significantly improving their students' learning of mathematics. The findings point to the importance of engineering supportive and linguistically sensitive learning environments, and that teachers' talk is critical to this process. However, it is not just any talk but talk that is carefully and meaningfully used as a teaching tool. It is talk that encourages students' positive appropriation of mathematics thinking and knowing. The teachers in this study also point to how important it is to integrate knowledge bases. In one way or another, they drew on principles of effective second language acquisition and literacy development, and incorporated these principles into their mathematics teaching. Observations of these teachers also suggest that it is important to ensure that teachers garner accurate images of innovations. These teachers did not adopt the too common version of groupwork that is often found in classrooms whereby three to five students are physically gathered to work together.

They, instead, recognized that the essence of groupwork centers around students talking with one another to share knowledge. This meant developing students to understand how important it is to support each other. Lastly, in essence, these teachers demonstrated that it is very viable to teach underachieving students as if they are gifted, that these students do not have to start at the very beginning because they did not learn the first time around. Ideas such as manifesting high expectations for students by teaching them advanced skills may be "old hat" in terms of research, but the effects of low expectations operate still in classrooms.

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LINKS BETWEEN INTUITIVE THINKING AND CLASSROOM AREA TASKS

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This paper investigates the links between intuitive area integration rules and classroom area tasks in students 9 to 13 years of age. The study used rectangular and near rectangular regions of varying areas and perimeters, as well as common classroom area tasks. Information Integration Theory and functional measurement procedures were used to reveal the students' intuitive additive and multiplicative rules. It was found that intuitive judgement rules are strongly linked to a students' responses to and success on common classroom area tasks.

Area is the most commonly used domain of measurement and the basis for many models used by teachers and textbook authors to explain computational strategies (Hirstein, Lamb & Osborn, 1978; Woodward & Byrd, 1983; and Baturo & Nason, 1996). It is also a concept that textbooks commonly used in Australia either fail to define or, often (for example, Blane & Booth, 1989), discuss with the apparent assumption that students already understand it.

This apparent lack of definition, and an emphasis on formulae, seems to be contributing to the documented confusion between area and perimeter (Kidman, 1999; Kidman & Cooper, 1996a; Outhred & Mitchelmore, 1996; Clements & Ellerton, 1995; Bell, Costello & Kuchemann, 1983; Hirstein, 1981; Hirstein, et. al. 1978; Bell, Hughes, & Rogers, 1975). In particular, as Kidman and Cooper (1996b) found, students have difficulty with the process of obtaining shapes' measurements, determining which dimensions to consider, and how to integrate these dimensions when calculating either area or perimeter.

The Information Integration Theory (IIT) method Anderson and Cuneo (1978) offer an excellent opportunity to explain the process of area concept development. IIT been widely used to identify the intuitive rules applied by students to integrate dimension information. In particular, recent studies have employed IIT to investigate students' perceptual judgement of area (Kidman, 1999; Kidman and Cooper, 1996b; Lautrey, Mullet & Paques, 1989; Silverman & Paskewitz, 1988; Schlottman & Anderson, 1994; Wolf, 1995); while functional measurement, the methodological counterpart of IIT, has allowed the diagnosis, in simple algebraic terms, "... of the rules which govern integration of information about perceived stimuli."(Wolf, 1995, p. 49-50).

In these studies, students have been provided with different rectangular shapes and asked to rate their area on a linear scale. The general consensus of these studies has been that students' judgements of area obeyed two-dimensional rules. They have also shown a transition from additive to multiplicative judgement rules. The expectation is that children will make the transition from an additive integration rule to the normative multiplicative integration rule at some stage between the ages of 5 and 12.

This paper describes and reports on an investigation to determine the link between students' intuitive judgement rules and their progress on common classroom area tasks. The investigation was based on the body of literature and the functional measurement methodology stemming from the work of Anderson and Cuneo (1978).

METHOD

The investigation used a multi-method design. The quantitative methodology of functional measurement was combined with the qualitative methodology of semi-structured interview. (A comprehensive outline of the methodology of the study, including how IIT determines area judgement rules, is provided in Kidman, 1997, and Kidman & Cooper, 1996b). The sample consisted of 36 students aged 9 to 13 years with an equal number of boys and girls and a range of mathematical abilities, one third each of above average, average and below average.

The instruments used were three experiments and an interview. The first experiment contained 16 rectangular wooden pieces painted to represent chocolate and with dimensions corresponding to the factorial combinations of 3, 6, 9, and 12 cm. The pieces were presented to students who were asked to judge the area of the rectangular pieces in relation to two end anchors. To obtain measures of the students' area judgements, the students were provided with a 19 point scale with twoend anchors, two 'special' pieces of dimensions 2.7 x 2.7 cm and 15.8 x 15.8 cm. The second experiment used 16 rectangular pieces identical in dimensions to the first experiment, but with a rectangular corner 'bitten' off producing a figure of equal perimeter, but less area. The dimensions of the 'bitten' off corner were all one third of the width and one third of the height of the rectangular pieces. The third experiment again used 16 rectangular pieces identical to the first experiment, with the exception that they had a semi-circular 'bite' out of one side producing a figure with less area but greater perimeter. The 'bite' was centred along one dimension with the radius of the 'bite' one third of the length of the shortest dimension. Throughout the three experiments, each student was quizzed as to the method they were using to rate each piece, they were asked as to whether they were aware of any changes they made to their method over the course of the three experiments, and diagrams were sought.

At the conclusion of the three experiments, the students were interviewed. Initially, the students' understanding of both area and perimeter were discussed. The students were asked to identify if they had employed either or both of these concepts to rate the chocolate pieces. To conclude the interview, the students were asked to complete the 5 classroom tasks shown in Figure 1.

The tasks were designed to cover a range of ability and instructional levels. All students were expected to be able to complete the task involving the congruent subregions as this should be among initial area activities presented to students 7 and 8 years of age. The task involving the diagram of a rectangle marked with 3 cm and 5 cm was designed to investigate the student's computational knowledge. The youngest students in the study were expected to be able to subdivide the region into a grid, while the older students were expected to be able to calculate the area using a formulae.

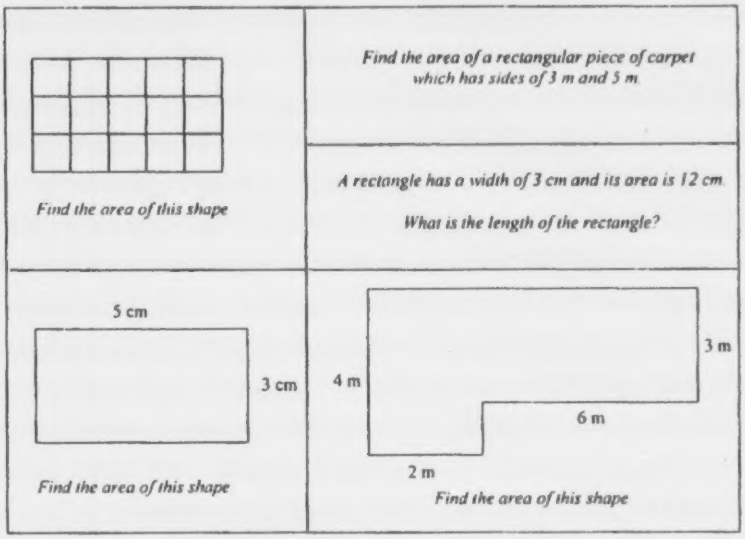

Find the area of this shape
Find the area of a rectangular piece of carpet which has sides of 3 m and 5 m
A rectangle has a width of 3 cm and its area is 12 cm
What is the length of the rectangle?
5 cm
3 cm
Find the area of this shape
3 m
4 m
6 m
2 m
Find the area of this shape

Figure 1. The 5 classroom tasks

The real-world word task was designed to determine if the "draw a sketch" technique (Department of Education, 1988, p. 196) would be utilised. All but the youngest of the students should have been familiar with this technique. Both the "draw a sketch" and knowledge of the area formulae were investigated in the other word problem. The remaining task of the L-shaped figure investigated the student's principled conceptual knowledge (Baturo and Nason, 1996) and problem solving approach.

RESULTS AND DISCUSSION

The functional measurement technique of the IIT method (Anderson &Cuneo, 1978) revealed both additive and multiplicative intuitive judgement rules were present in the sample of students (Kidman, 1997). It was found that a student could be categorised as being either predominantly additive (that is, the student intuitively had a perception of area where doubling the lengths of the sides of a rectangular region can be seen as doubling the area) or predominantly multiplicative (that is, the student intuitively had a perception of area where doubling the lengths of the sides of a rectangular region more than doubles the area) by noting the three judgement rules used by the student over the three experiments. Codes existed which were used to create the two categories. This is summarised in Table 1.

It can be seen that the predominantly additive category was composed of 16 students, from five different codes. It was not possible to determine a judgement rule for one student, in Experiment 1, due to intersecting locations on the factorial plot. The 20 students in the predominantly multiplicative category were lessvariant in their codes. 60% of the students experienced 'intra-individual' rule changes (for example, the removal of a corner in Experiment 2 caused the student to alter their judgement rule from additive to multiplicative.

It is interesting to note that for the students in this study, the proportions perceiving area as either predominantly additive or predominantly multiplicative is consistent across the age range (Kidman, 1997). Thus the differences between the ages was not as obvious as could be expected.

Table 1. Judgement rule codes and categories for Experiments 1, 2 and 3

Overall Judgement Rule	Judgement Rules			Number of Students
	Exp 1	Exp 2	Exp 3	
Additive	A	A	A	5
	A	A	M	7
	A	M	A	1
	M	A	A	2
	?	A	A	1
			TOTAL	16
Multiplicative	M	M	M	9
	M	M	A	5
	A	M	M	6
	M	A	M	0
			TOTAL	20

KEY A = additive judgement rule M = Multiplicative judgement rule
? = unknown judgement rule

The older students had had increased levels of instruction, but do not seem to have advanced much beyond the younger students. It is clear that many students, irrespective of age, are experiencing confusion between area and perimeter.

The two categories, predominantly additive and predominantly multiplicative, were linked with the strategies that emerged throughout the three experiments (a comprehensive description of the strategies is provided in Kidman, 1999), and from the completion of the five classroom tasks. These links are shown in Figure 2.

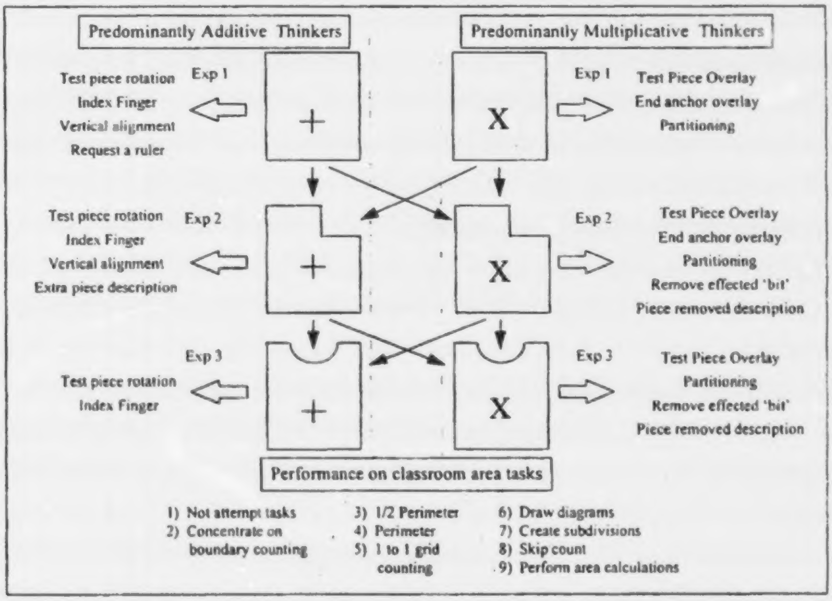
Predominantly additive thinkers. These thinkers perceive area to double when the lengths of the sides of rectangular regions double. Additive thinkers tend to think in terms of units of one (Steffe, 1992). 16 students displayed this thinking style. The actions and statements of these students throughout the three experiments revealed 4 strategies and 1 descriptor clearly linked to their additive thinking.

The *test piece rotation* strategy was particularly robust across the three experiments. In this strategy, the student would measure the outside edge of the rectangular region, by rotating the test piece so each edge was compared, around the end anchors.

The *index finger* strategy was also very robust across the three experiments. Initially the student placed an index finger adjacent and parallel to one side of the test piece. This is then repeated by moving the finger along to the next adjacent parallel position. The student repeated this procedure along the edge of the rectangular piece to the opposite side.

The *vertical alignment* strategy involved a rotation of the test piece prior to a rating being made. Each of the wooden pieces were presented to the students in a uniform manner, but some pieces were presented in a horizontal alignment (lyingflat ▭), while others had a vertical alignment (standing tall ▯). During Experiments 1 and 2, many additive thinkers rotated the test pieces when presented with a horizontal alignment so that it became vertical. One student indicated the need for the pieces to be presented "like chocolate on shop shelves ... like the way the wrapper would go" (Rhea, 10 yrs).

The *request a ruler* strategy occurred among additive thinkers in Experiment 1. Students commented prior to starting the experiment: "I need a ruler to do this" (Jack, 9 Yrs); "This can't be done properly without a ruler" (Jodie, 11 Yrs). Jodie insisted that "we always use rulers to measure. You see, without one you can't measure something". Jack (9 Yrs) explained that he needed to "measure the chocolate pieces to see if one was bigger or not". He wanted to measure the longest sides of each piece.



Predominantly Additive Thinkers
Predominantly Multiplicative Thinkers
Exp 1
Test piece rotation
Index Finger
Vertical alignment
Request a ruler
+
X
Exp 1
Test Piece Overlay
End anchor overlay
Partitioning
Exp 2
Test piece rotation
Index Finger
Vertical alignment
Extra piece description
+
X
Exp 2
Test Piece Overlay
End anchor overlay
Partitioning
Remove effected 'bit'
Piece removed description
Exp 3
Test piece rotation
Index Finger
+
X
Exp 3
Test Piece Overlay
Partitioning
Remove effected 'bit'
Piece removed description
Performance on classroom area tasks
1) Not attempt tasks
2) Concentrate on boundary counting
3) 1/2 Perimeter
4) Perimeter
5) 1 to 1 grid counting
6) Draw diagrams
7) Create subdivisions
8) Skip count
9) Perform area calculations

Figure 1. The links between intuitive thinking and classroom tasks

The *extra piece* descriptor of the Experiment 2 test pieces was quite prevalent among students employing predominantly additive thinking styles. Such students perceived the near rectangular region as being basically rectangular with an 'extra piece' added onto it, despite having been introduced to the second set of test pieces as being "identical to the first set of pieces except they have a rectangular corner 'bitten' off". It is possible that this view originates from the students' perception that the area has additive properties, and therefore the 'extra bit' is also added.

Predominantly multiplicative thinkers. These thinkers have the perception where doubling the lengths of the sides of a rectangular region more than doubles the area. 20 students displayed this thinking style. Such multiplicative thinkers have the ability to think of units of one, and about units of more than one, simultaneously (Steffe, 1992). The actions and statements of these students throughout the three experiments also revealed 4 strategies and 1 descriptor clearly linked to their multiplicative thinking.

The *test piece overlay* strategy was particularly robust across the three experiments. It involved overlaying the test pieces onto the large end anchor in a series of flip and slide transformations.

The *end anchor overlay* strategy involved the small end anchor being overlaid onto the test pieces using a series of randomised flip and slide transformations also. It was also particularly robust across the three experiments. Jenny (11 Yrs) claimed she "… counted how many squares, see these little squares [indicating the small end anchor], I want to see how many of them cover this bit [holding up the test piece]". Jay (9 Yrs) explained "it has to be the small square that you use because it doesn't change each time like the bigger ones do".

The *partitioning* strategy also occurred among multiplicative thinkers across the three experiments. These students were familiar with the method of partitioning both the length and width of a rectangular shape, and integrating these values using multiplicative reasoning. One of the students made imaginary marks along the two salient dimensions with her finger. She partitioned the width of the test piece into what appeared to be 1cm lengths, maintaining a mental count of the partitions. Upon completion of the width, her attention focused on the length of the test piece and she repeated the partitioning process, again with 1cm partitions, maintaining a mental count as she progressed. Other partitioning students explained that they imagined the chocolate pieces already divided into squares, counted the squares on two dimensions, and multiplied "… them together, like you do for area sums" (Phillip, 13 Yrs).

The *remove effected 'bit'* strategy emerged, through drawings, during Experiment 2 and maintained its presence through Experiment 3. Students using this strategy mentally removed the "effected bit". Phillip (13 Yrs) explained that when he worked with "the boards with nails in and rubber bands … and if my shape was wonky, the teacher told me to get rid of the crooked bit and work with the rest of it". As a result of his classroom geo board lessons, Phillip mentally removed the altered sections of the test pieces.

The *piece removed* descriptor existed for students who thought multiplicatively, where they perceived the basic shape as rectangular, but with a piece removed. Such students either remembered being told a piece had been removed, or they have a more wholistic view of the shape, than the additive thinkers.

Performance on classroom area tasks. Table 1 indicated that five students are truly additive thinkers as they only employed additive integration rules across all three experiments. For these students, their performance on the five classroom area tasks was characterised by 2 strategies -*failure to even attempt a task*, and if attempted, the student would concentrate on some form of *boundary counting*. The failure to attempt a task was partially characterised by students who simply said "pass" (Ben, 9 Yrs) presumably because they were tired or bored. Students who read a task a number of times but did not actually attempt the task were also included in this category. They were not able or prepared to offer a solution even when told a solution was possible; for example, "This doesn't have all the numbers, so I can't do it" and, after being told it was possible, "Next" (Anne, 12 years). Boundary countingincluded either grid line counting around the four sides of the figure, or a count of the spaces around the four sides of the figure.

Nine students were truly multiplicative thinkers as they only employed multiplicative integration rules across all three experiments. For these students, their performance on the five classroom area tasks was characterised by four strategies – *drawing diagrams, creating subdivisions, skip counting, and performing area calculations*. Two tasks were word problems. While both could have been solved without diagrams, six of the nine students successfully used diagrams. Two students (aged 11 Yrs) attempted to use diagrams for the carpet task but had difficulty deciding which measurements belonged on which dimension. Steven (11 Yrs) finally decided "it doesn't matter anyway, 3 times 5 is 5 times 3". The creating of subdivisions could be seen as an extension to the draw a diagram strategy. The majority of truly multiplicative thinkers automatically drew subdivisions on the L-shaped figure, however only a minority successfully found the solution. Computational errors caused two of the 9-year-old students to obtain an incorrect area. Skip counting in the task involving congruent subregions was also evident only among the truly multiplicative thinkers. The only students to obtain correct solutions for the classroom tasks, using multiplicative processes were the truly multiplicative thinkers.

The remaining 22 students (11 additive thinkers and 11 multiplicative thinkers) experienced 'intra-individual' rule changes. In the case of the additive thinkers, for one of the three experiments they thought multiplicatively. Similarly, for the multiplicative thinkers, for one of the three experiments they thought additively. This rule change may be the result of the students' having the ability to think of units of one, and of units of more than one, but not both simultaneously. The classroom strategies for the additive thinkers and the multiplicative thinkers in this group were the same. They attempted all tasks, and these attempts included calculations of *perimeter, ½ perimeter* as well as calculations of area through *1 to 1 counting* of the congruent subregions. Computational error also prevented these students from obtaining a correct area solution. For this group there was a lot of confusion between area and perimeter. The younger students particularly would calculate a ½ perimeter by simply adding the given dimensions.

CONCLUSION

This experiment extends the body of literature stemming from the work of Anderson and Cuneo (1978) by using non-rectangular regions. The findings confirm that two area judgement rules do exist, that an individual can alter their judgement rule, and that there is confusion between area and perimeter, possibly resulting from the student's inability to think multiplicatively .

The misconception of area of rectangles being dependent on the sum of the dimensions is fairly constant across the age range especially for students who experience inter-individual rule changes. Such students mayhave the ability to think of units of one, and of units of more than one, but not both simultaneously. Students making rule changes experience difficulties with classroom area tasks in that they confused area with perimeter. It may be beyond the student's ability to think multiplicatively for area tasks, irrespective of their academic backgound.

Students using additive thinking need to measure regions with a ruler or index finger and prefer a vertical alignment. Truly additive students tend not to attempt classroom area tasks. It is possible that such students do not have a workable method for some area problems, and as a result they choose not to attempt them.

Students using multiplicative thinking tended to use overlay strategies as well as partitioning, and some are capable of interpreting word problems and drawing diagrams to solve classroom area tasks.

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Teaching the limit concept in a CAS environment: Students' dynamic perceptions and reasoning

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By applying symbolic computation and graphics, we tried to enhance students' ability to go from visual interpretation of the limit concept to formal reasoning. While being taught the topic "approximations of functions by Taylor polynomials", the students analyzed the remainder and performed animations that illustrated its convergence. They used the Mathematica software for manipulating algebraic expressions and for generating a wide variety of dynamic graphics. We observed that the graphics produced by the animations were in a sense present in the students' minds even when the computer was turned off. Here we describe a situation in which the interaction with computer graphics helped the students overcome confusion caused by misleading images of the limit concept.

Introduction

This paper deals with the conceptual understanding of the convergence process obtained by approximating a function by means of Taylor polynomials. Central concepts in analysis such as limit and infinite sum are very much related to approximation theory. Therefore, by means of polynomial approximations, we tried to clarify the limit concept. We analyzed students' perceptions of the limit concept in the context of a computer-based mathematics laboratory program. For this purpose, we used Mathematica software (Wolfram Research), which permits symbolic computation, graphics, and animation. Special attention was given to using animation in order to visualize and analyze the dynamic process of convergence. Our research focused on the question to what extent did the use of symbolic computation and dynamic graphics actually help the students in the transition from their visual intuitive interpretation of the limit concept to formal reasoning.

This paper describes some research that examined an approach to teaching analysis at the high school level. The main topics taught were Approximation and Interpolation, from which we explored the issue of Approximation theory in connection to the limit concept. High school students learning at the highest level (Age 16-17, N=84) were involved in the research.

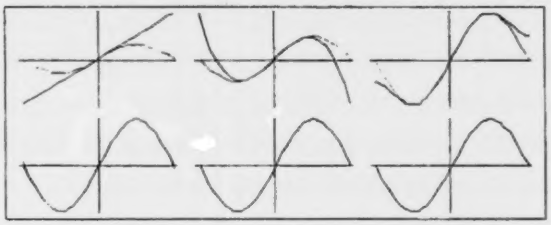
Theoretical Background

In studying students' perceptions of the limit concept, it is important to take into consideration the intuition of infinity. Our logical schemes are naturally adapted to finite realities. As Fischbein, Tirosh, and Hess (1979) observed, the natural concept of infinity is the concept of potential infinity. Openhaim (1986) noticed students' difficulties in grasping that the behavior of a sequence with regard to convergence is unaltered if we omit a finite number of the terms. Davis & Vinner (1986) noticed some unavoidable misconception stages in understanding the limit concept. In trying to understand the difficulties in learning the limit concept, Cornu (1981) described "spontaneous models" that pre-exist before learning the limit notion. Moreover, the definition of limit is formulated in terms of an unencapsulated *process* (given ε, an N can be found such that...) rather than being described explicitly as an *object* (Cornu, 1991).

In attempting to overcome such difficulties, Dubinsky & Tall (1991) proposed using computers in order to enable the students to make constructions on the computer screen leading to corresponding constructions in their minds. Li & Tall (1993) discussed three approaches to teach the limit concept: (1) a (formula-bound) dynamic limit approach, (2) a functional/numeric computer approach, and (3) the formal $\varepsilon - N$ approach. Monaghan et al. (1994) added a key stroke computer algebra approach. We suggest an additional approach: the use of animation to visualize the processes of convergence and to interact with the dynamic graphics. The "Calculus & Mathematica" course (Brown et al., 1991) and Devitt's "Calculus with Maple" course (1993) helped us in preparing the chapters on approximation by expansions. The reference to Euler analysis (Brown et al., 1990) was especially helpful. We used Mathematica for animating the remainder. For analyzing the results we were aided by Verillon & Rabarbel's article (1995) on cognition and artifacts. Assuming that cognition evolves through interaction with the environment, the authors studied the effect of accommodating to artifacts on cognitive development, knowledge construction and processing, and on the nature of the knowledge generated. They stressed the difference between the artifact, as a man-made material object, and the instrument, as a psychological construct.

The Teaching Experiment

The first author taught the students mathematics six hours a week, two of the six hours in the PC lab. The laboratory consists of 20 PCs, each equipped with Mathematica and a hardware system (called classnet) that permits transmitting the content of the screen of each computer to all the computers in the classroom. A pedagogical strategy in the experiment was to use the technology to follow great mathematicians' thought processes. For example, two different approaches were used to approximate a given function by polynomials: analytical and algebraic. In the analytical approach the notion of *order of contact* was introduced, and as an application the students were required to find the polynomials of degree 2, 3, 4,.... that have the highest possible order of contact with a given function at x=0. Mathematica helped the students to solve the relevant systems of equations. In the algebraic approach Taylor polynomials were introduced by using the intuitive idea of Euler: to express non-polynomial functions as polynomials with "an infinite number of terms". The students used Mathematica to *follow the original text of Euler* described in Euler (1988). Following Euler's "experimentalist" thinking, the students used his algebraic approach to represent infinite sums: they used Mathematica syntax in order to expand functions as power series, applying the method of undetermined coefficients exactly as Euler did. Both approaches converged to the coefficients of the Taylor series but each one has its own characteristics: the analytical approach describes the *process* of the different polynomials approaching a given function; the algebraic approach represents the polynomials with "an infinite number of terms" as an *object*. The students made a graphical representation of the results. By means of animation (Kidron, 2000 - Example 1), they were asked to “encapsulate” the process into an object. For example, Figure 1 shows a “dynamic” plot that illustrates the fact that in a given interval, the higher the degree of the approximating polynomials, the function f(x)=Sin(x) and the approximating polynomial are closer.



$f(x)$ and $P_n(x)$ for n=1,3,5

$f(x)$ and $P_n(x)$ for n=7,9,11

Figure 1 a "dynamic" plot of sin (x) and the approximating polynomials for $-\pi \le x \le \pi$

The animation permitted the students to see the dynamic process in one picture: they were also requested to stop the animation and observe the different steps of the dynamic picture. In the laboratory, the teacher demonstrated a full process, by means of animation, followed by a group discussion using the classnet. The students noticed that for x values nearer to 0, the function f(x) and the approximating polynomial $P_n(x)$ are closer. In order to clarify the meaning of “closer”, the teacher had the students analyze the remainder. The students were given the proof of Taylor's theorem at x = 0 and they computed the expansion of sin (x) around x=0 up to exponent 5. The error $(f(x) - P_n(x))$ - the remainder of Lagrange - is $\frac{f^{(6)}(c)\,x^6}{6!}$ for some c value between 0 and the current x value. The absolute value of the error as a function of x and c with $-\pi \le x \le \pi$, $-\pi \le c \le \pi$ was plotted. Because the c value in $-\pi \le c \le \pi$ that corresponds to the exact error is an unknown number, the students were requested to look at all pairs (x, c) such that $-\pi \le x \le \pi$, $-\pi \le c \le \pi$.

The following 3-dimension graphics (see Figure 2) represents the error (in fact, an upper estimate on the error) as a function of the two variables x and c.

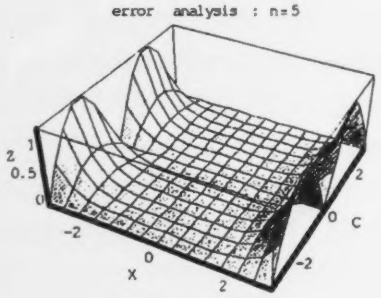


Figure 2 the error as a function of x and c

In this plot the upper estimate of the error is obtained, for example for $x = \pi$ and $c = \frac{\pi}{2}$. In the laboratory, the teacher demonstrated that in the example f (x)= sin(x): $\lim_{n \to \infty} R_n(x) = 0$, $R_n(x) = f(x) - (a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n)$.

Using animations of 3-dimensional plots in a fixed domain, the students saw with f(x)=sin(x), how the upper estimate of the error gets smaller when the degree of the approximating polynomial is increased (Kidron, 2000 - Example 2). One student raised an interesting question: Suppose that the degree of the approximating polynomial is fixed; could we obtain the animation with the domain as a variable? At this stage of the course the students could use Mathematica as a programming language to obtain the dynamic graphical output. The teacher encouraged them to construct a visual representation of $\lim_{x \to 0} R_n(x) = 0$ by animation, where the domain of $R_n(x)$ is variable (Kidron, 2000 - Example 3).

Research methodology and data analysis

The methodology adopted for evaluating the students' work and for research purposes was as follows: the teacher demonstrated an idea in the PC lab. The students were then asked to explore the idea by applying their examples with Mathematica. The teacher collected three types of data: (1) students' questions and remarks during the sessions, (2) the Mathematica files of the students' examples, and (3) written tests without the use of Mathematica. We present here a class discussion and some findings from a written test. The following class discussion demonstrates the way the students used Mathematica to interact with the dynamic graphics produced, in order to re-construct their knowledge of the limit concept.

The class discussion The task given to the students in the lab was as follows: Select a function f (x) and illustrate $R_n(x) \underset{(n \to \infty)}{\to} 0$. Most of the students dealt with functions similar to the example given in the class. Here we describe the class discussion that followed the presentation of an example by one of the students, Matan. His example was: f(x)= cos(2x) for $-\pi \le x \le \pi$ (see Figure 3). He animated (Kidron, 2000 – Example 4) the plots of the absolute value of the error $\frac{f^{(n+1)}(c)\ x^{n+1}}{(n+1)!}$ as a function of two variables x and c,

$-\pi \le x \le \pi$, $-\pi \le c \le \pi$

where n grows from 3 to 13

with step = +2

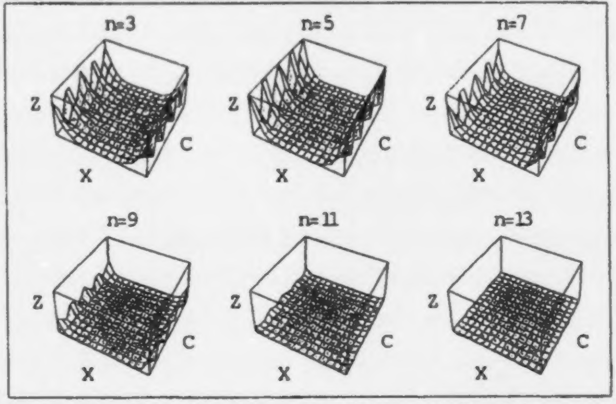


Figure 3 "animation" illustrating $R_n(x) \underset{(n \to \infty)}{\to} 0$ for f(x)=cos(2x)

In the example that was demonstrated in the laboratory, (f(x)=sin(x)), when n was increasing, the upper estimate of the error was steadily decreasing for every n. This was not the case in Matan's example, f(x)=cos (2x), as is seen for n = 5. We quote students' reactions:

Nimrod: *When the degree n of the approximating polynomial is increasing, the approximation must be better.*

Nimrod tried to explain what he meant by "the approximation must be better".

Nimrod: *In some place in the infinite they will be the same. I mean by "better" that when n is increasing the error is decreasing. It could not be that the error is getting bigger! Maybe the error is not getting smaller. I mean that maybe we cannot see it in the graph but the error is getting smaller all the time when n is increasing.*

We noticed some confusion in Nimrod's reactions. He did not expect that the error would suddenly increase for n=5.He attributed the surprising effect to some limitations of the graphics.

Matan: *f(x) = cos(2x) is an even function. An even function is expanded in a power series with even exponents. I should have given values to n that go from 2 to 14 with step +2 instead of taking n from 3 to 13.*

Matan connected this surprising effect to an irrelevant fact. He used Mathematica to check his conjecture. He chose even values for n but the surprising effect remained unchanged.

Hannah: *Let us look at the different graphs that produced the animation. They do fill the requirement that* $R_n(x)$ *approach 0 as* $n \to \infty$. *The problem is with the degree n = 3 and not with n = 5. From the fifth degree and all the degrees onwards we got exactly what we expected: the error becomes smaller as n increases. We say* $R_n(x) \to 0$ *if* $R_n(x) > R_{n+1}(x)$ *and this happened from a certain value of n onward. In Matan's example n>3.*

Tomer: *How could we know from which n=N the process begins?*

Nili: *Could it be that from a certain N the error will get smaller for a few steps and afterwards the error will get bigger? We could not find the N graphically. How could we know from which N the error becomes smaller all the time?*

Hannah: *Something disturbs me - if the accuracy* (ε) *is 0.8, for example, and you find N, for example, N=10 such that for all* $n \geq N$, $|R_n(x)| < 0.8$, *then you will not see the phenomenon we described for n = 3. This means that for every* ε *there will be the N that belongs to it. Maybe we will find N if we will compare* $\frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$ *with* ε *and we will look for the first n for which this expression is smaller than* ε.

The students gave other examples that demonstrated that $|R_n(x)|$ was not always decreasing. Motivated by these examples, the students searched for the N from which onwards, the absolute value of the remainder decreased.

The written test We were interested in two aspects:

(1) The students' ability to visualize the process described by the formal definition of limit, and (2) their ability to express the formal definition correctly. One of the written tests dealt with the notion of the limit, $\lim_{x \to 0} R_n(x) = 0$. The test checked the students' ability to connect the visual and the analytical aspects of the limit concept. The students (N=84) worked on the test without using Mathematica. We identified different ability levels of connecting the visual and analytical aspects of the limit concept.

Most of the students (81%) were able to visualize the process described by the formal definition of the limit and to translate visual pictures to analytical language: "We are given $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...$ find the appropriate $\delta_1, \delta_2, \delta_3, ...$". They had no difficulty in proceeding step by step through a discrete sequence of $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...$ finding the appropriate $\delta_1, \delta_2, \delta_3, ...$ and were aware that this process is infinite, probably in the sense of "potential infinity". "*You can always find a number which is smaller than the previous one, and so on infinitely*".... "*To every ε_n there is δ_n*". A smaller number of the students (68%) were able to express the formal definition: "*to every positive number ε, there is a positive number δ such that*...". Some of the students who failed to express the formal definition wrote: "*δ is not dependent on ε. ε is dependent on δ*". "*δ is not dependent on the error, since δ is fixing the error: the nearer we approach the point x=0 about which the function was expanded, the smaller is the error*". These students remembered the order in which they worked in the laboratory - beginning with domain and finding the error. The Cauchy's definition begins with ε ... It was difficult for them to reverse the order!

Discussion and Conclusions

The class discussion around Matan's example related to the concept of the limit, $\lim_{n \to \infty} R_n(x) = 0$. The surprising effect of the dynamic graphical feedback that Mathematica provided was very important for the students' learning experience. The students expected, as in the example demonstrated in the lab, that $R_n(x)$ will approach 0, steadily decreasing for every n. The unexpected effect of the little jump back when n=5 in Matan's example, f (x) = cos(2x), was stronger while observing the animation than in the static plots (Kidron, 2000 - Example 4). The contribution of such feedback to the learning process is particularly effective if it is surprising (Dreyfus & Hillel, 1998). The result could be a re-construction of the meaning of some mathematical notions. Mathematica helped the students to identify "that something is not going as they expected". They had to understand by themselves the cause of the confusion. The way the students used the dynamic graphical feedback enabled them to realize that the behavior of the sequence with regard to convergence is unaltered if we omit a finite number of the terms R_n.

The students used Mathematica to follow Euler's reasoning. In Euler's approach (Euler, 1988) infinite sums were represented as an *object*: the polynomials with an "infinite number of terms". The students used Mathematica also as a symbolic language to generate dynamic graphics, which enabled illustrating the convergence *process*. Using animation only to *visualize* the process of convergence was not enough in order for the *process* to become a *concept*, the concept of limit. In addition, the students had to *interact* with the dynamic graphics, to *have control over the dynamic representations*. Actions on the dynamic representations aided the students in developing their own reasoning. We could clearly see that the students' use of the artifact influenced the nature of the generated knowledge. In order to overcome their pre-conceptions of limit, the students were encouraged to further construct and re-construct their knowledge using the dynamic graphics approach to handle the limit concept explicitly.

We were interested in determining to what extent this re-construction of their knowledge helped the students in their transition from visual intuitive interpretation of the limit concept to formal reasoning. The class discussion around Matan's example enabled the students to modify the misleading idea that they could observe the approach $R_n(x) \to 0$ as n is increasing in the sense that $R_n(x)$ steadily decreases for every n. The class discussion paved the way to the formal definition of $\lim_{n\to\infty} R_n(x) = 0$ (beginning by ε, then finding N such that…). However, in the written test, only 2/3 of the students were able to write correctly the formal definition of $\lim_{x\to 0} R_n(x) = 0$. The dynamic graphics produced by the animations were present in the students' minds even when the computer was turned off. Some even remembered the order in which they worked in the lab (beginning with domain and finding the error) and had difficulties in reversing the order. To overcome this difficulty, additional tasks are being prepared for use in further applications of the program.

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TOWARDS A COMPARATIVE ANALYSIS OF PROOF TEACHING

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Abstract: *In this paper, the first results of a comparative study on proof and proving in geometry teaching are represented. Twelve eighth grade classes (approximately 14 year-old students) were observed in France and Germany, in order to analyse the impact of culturally embedded classroom practices on the teaching and learning of proof. Also, the differences in functions of proofs based on the observed French and German teaching practices are presented here. In particular two ideal types of mathematical cultures in classroom practice have been singled out.*

Results of international studies which compare mathematics teaching in different countries suggest that cultural diversity in mathematics education goes far beyond instructional methods. This research has outlined qualitative differences between the topics that were the focus of lessons and what students were expected to do as they studied these (Cogan and Schmidt 1999). Hitherto there have not been carried out any comparative studies concerning the teaching of proof, one main topic of mathematics teaching. However, Balacheff has been reflecting upon the idea that "ethnomathematical" questioning on the proof is just as necessary as its commonly accepted epistemological inquiry (Balacheff 1999).

In my comparative case study on proof and proving in French and German geometry lessons, differences in classroom practice and their implications for the learning and teaching of proof will be analysed. This study questions: *How do culturally embedded classroom practices influence the teaching and learning of proof?* For this I focus on the differences within proof forms and functions. The hope is that this will enrich the research in the proof and proving field from an inter-cultural as well as a classroom-cultural perspective.

Research in the Field of Proof and its Teaching

One main problem of the teaching of proofs is that students, once convinced of a statement's validity, experience so much difficulty understanding the importance of the statement's proving. In the following different approaches to this problem will be briefly represented and discussed.

On the one hand a series of learning situations, including proving exercises, have been created in order to stimulate among students processes of formulating conjectures and refutations (Balacheff 1987). Therefore, problems were designed to lead to controversy among students which should involve them into a process of proving (see further Boero, Garuti, and Mariotti 1996). In these approaches the social dimension of proof and proving is seen as essential for didactical concerns.

Moreover teaching practices of proof and forms of proofs being taught have been analysed from an epistemological perspective. Hanna and others criticised the over emphasis of formal proofs in everyday teaching (Hanna 1989; Wittmann and Müller 1990), which does not give students an adequate understanding of the meaning and function of proofs. Research in this field has made it clear that proofs have a variety of functions (de Villiers 1990), for example an epistemic function that is to understand why; a systematic function which is to relate different mathematical concepts already studied in class.

Further, proof conceptions of teachers and students as well as students' aptitudes in proving have been examined in several empirical studies (Healy and Hoyles 1998; Reiss and Heinze 2000). The latter having a focus rather on cognitive than on social aspects, even though including proving processes. These studies have given interesting insights into implicit proof conceptions of students, but they give little evidence in how far these conceptions are caused by the teaching of proof in mathematics lessons.

Research on argumentation processes gives a clue of argumentation formats in classroom situations, but is limited at the same time to the level of primary school (Krummheuer 1993; Schwarzkopf 2000). It might be interesting to see how argumentation formats in class might be linked to proving processes; analyses which have not been done yet.

All in all, in proof research few empirical studies exist on proofs and proving in everyday classroom situations (Herbst 1998). This means that we have little evidence showing in how far students' difficulties are due to everyday mathematics teaching practice.

Theoretical framework and methodological considerations

The present study refers to sociological and didactical research theorising everyday practices. These practices can be characterised as mostly determined by pragmatic aims and a general concern of effectiveness in action (Krummheuer and Naujok 1999). In these situations interacting partners typically presume common understanding and references, which allows to structure learning processes on the basis of habitual practices. Discourses are further marked by a high degree of implicity and indexicality, which means that statements often can be understood only within the context they have been stated. From a sociological perspective this has been described as a "pragmatic cognitive style" of everyday practices (Soeffner 1989).

Focussing on everyday classroom practices from this theoretical approach helps to analyse from students' and teachers' point of view what we might consider as learning or teaching obstacles. A focus on their prospect is inevitable as didactic research not only aims at offering alternatives to habitual practices but hopes changing misleading practices.

In order to analyse everyday classroom practices comparing these becomes an insightful tool. A comparative approach allows to single out characteristics of teaching practices in different contexts which could not have been reconstructed from single cases (Kelle 1994).

The methodological frame of the present study refers to the concept of "ideal types" developed by Max Weber (*Webersche Idealtypen)*, and describes idealised types of mathematics teaching reconstructed from classroom observations in France and Germany (Weber 1904 /1988). This means that typical aspects of teaching practices are reconstructed on the basis of the whole qualitative studies rather than on an existing empirical case.

Methodical design and data analysis

The study uses methods of qualitative social sciences mainly participating classroom observation – about 30 lessons were seen at 6 different classes of eighth grade students (14 year olds) in France, compared to 30 lessons at 6 German schools (same age group). Within the observed classes, two were from a French-German college, the others from ordinary classes of collèges in Paris and German Gymnasium and higher level classes of comprehensive schools in Hamburg.

With respect to proving on the one hand and variation of contents on the other, six teaching units concerning Pythagoras' Theorem and six further units dealing with similarity and special lines in triangles were chosen. Each lesson has been recorded, so that analyses of transcripts and blackboard drawings and writings could be possible later. Classroom observations have further been expanded by protocols. As a tool to support the data analyses the software Atlas-ti is used.

Analyses of the observed proofs are done from two perspectives which are supposed as complementary: firstly, content related analyses, secondly reconstruction of argumentations. This means that forms and functions of proofs are analysed with respect to their mathematical substance and not separate from it. For I suppose as Granger that form and content of proofs cannot be separated (Granger 1994).

Research on Pythagoras' Theorem by Fraedrich and on geometrical settings by Parzysz and Douady is referred to as a theoretical framework for the content related analyses (Douady and Parzysz 1998; Fraedrich 1995). Analyses of the structures of argumentation are guided by the theoretical work of Duval and Toulmin (Duval 1995; Toulmin 1958). Duval's theoretical analyses allow for a distinction between argumentation and proof by formal aspects, whereas Toulmins' scheme helps to work out different argumentation structures. Results of the latter analyses will be published elsewhere.

First results of content related analyses

Sorting proofs of Pythagoras' Theorem which have been observed in classroom situations in France and Germany showed four different forms of proofs: 1) Proofs based on comparisons of areas 2) Proofs based on calculations of areas 3) Proofs where applying theorems on similarities, 4) Proofs using visualisations of the theorem of Euclid, meaning $a^2 = pc$ and $b^2=qc$. Analysing these proofs makes evident two different interpretations of Pythagoras' Theorem: a statement about comparing areas as well as an assertion about relations of lengths, which is evidently not the same.

Taking into account that every type of proof favours one or the other interpretation of the theorem, it is surprising that in our observations tackling the meaning of Pythagoras' Theorem is not necessarily coherent with the proof done in class. When comparing French and German teaching, one has to notice that no differences concerning such inconsistencies of teaching could be found. Whereas there had been differences in the meaning, which has been assigned to Pythagoras' theorem, both interpretations – about areas and about relations of lengths - have been found in German mathematics lessons, but not in French classes. Further in German mathematics teaching all different types of proofs as systemised above could be found, while in French mathematics lessons only proofs of type one and two have been identified which correspond with proofs in curricula and in textbooks used in class.

By analysing the role of proofs within teaching it became apparent that introductory phases are essential for the teaching of proof in the observed German lessons, whereas phases of exercises play an important role for French teaching of proofs.

All observed German classes start with two or three lessons where students and teachers are engaged in a process of discovery of the theorem before proving it. In contrast to this in French teaching Pythagoras' Theorem is directly introduced to the students in the very first lesson of the teaching unit and proved with guidance of the teacher.

Having proven the theorem in French lessons, complex and sophisticated problems have to be solved by students at home as well as in class. These exercises require an application of different theorems and concepts, which have been studied in former teaching units, including Pythagoras' Theorem. Whereas in German lessons all exercises have been analysed as typical routine tasks which request simple application of Pythagoras' Theorem.

Two cases of German (the case of Nissen) and French (the case of Pascal) teaching have been chosen and shall be presented in the following, in order to illustrate typical characteristics of different teaching patterns.

The Role of discovery of theorems - the case of Nissen

Teacher Nissen starts her teaching unit on Pythagoras' Theorem with a calculation problem: the length of the rafter of a rectangular saddle roof is to be figured out, the width of the house given. At this time Pythagoras' Theorem has not yet been introduced, so that the students have to find a way of solving the given problem. Completing the figure by squares on the sides of the triangle and calculating the areas of these squares leads to a relation between the sides of the given triangle: $2a^2 = c^2$.

The study of the special case of a right-angled isosceles triangle is used to introduce the central idea of Pythagoras' Theorem in two lessons. This approach allows for consideration of the theorem from two different points of view: the aspect of areas and the relation of length of the triangle's sides. At the same time one of the most important applications of Pythagoras' Theorem, the calculation of length, is already treated in the foregoing lessons. Throughout in the course of the teaching unit, the insights gained in the special case are prolonged to the general idea of Pythagoras' Theorem. In contrast, the proof strategy used in the special case is not taken up any more. This might be easily explained from a mathematical perspective for the proof in the special case cannot be generalised.

In the teaching the first two or three lessons do have a special role, they lead students to discover the theorem itself and to understand the theorem in a double way, as a theorem about areas and as a theorem on the relation of lengths of the triangle's sides. Further the special case makes it possible for the students to figure out a proof on their own, or with little help from the teacher, for the central idea of the proof is as simple that the students can do so as well.

Proofs and problems initiating justifications - the case of Pascal

Having prepared the technical part of the proof by working on the binomial in the preceding lesson, teacher Pascal works out Pythagoras' Theorem together with the students by its proof : the area of a square with lengths a+b is calculated in two different ways: first by using the binomial, then calculating the sum of the inner square's area, that is c^2, and the areas of four right triangles $4 \cdot \frac{a \cdot b}{2}$. This takes about half of the lesson and is completed by exercises applying Pythagoras' Theorem.

The problems given to the students in Pascal's class after the theorem has been introduced can be divided into two types of problems: routine versus complex exercises. Whereas in the routine tasks Pythagoras' Theorem only has to be simply applied for calculating length, for example the length of a triangle's side, the complex problems cannot be solved without analysing the geometrical configurations and the use of other theorems and concepts, as properties of the circumscribed circle or similarities . Further, analyses of these problems have shown that different ways of solutions are possible, among those very elementary ones, these are solutions which are based on concepts and properties that have already been introduced at grade 6 (12 year olds).

The teacher insists on complete justifications, why the way a student worked out a solution and her or his use of theorems and concepts is legitimate. The proof of Pythagoras' Theorem, which has been produced in collaboration between students and the teacher, offers in a way a model how justifications should be structured and marks the level of rigour which is expected by the teacher. Problems and proofs of theorems are functioning as an amalgam where the responsibility of truth is given from teacher to students and back. Everyone is asked to justify by good reasons the validity of mathematical statements she or he has claimed.

Teaching styles and functions of proofs

In the observed German lessons it seems typical that the discovery of theorems is based on special cases and applied problems, where proof and different interpretations of the theorem are merging into one another. In this pattern of teaching proofs gain the function of assigning meaning to the theorem. For meaning is here that students understand applications and different interpretations of a theorem. This explains why no problems occur when the central idea of the proof used in the special case is given up for a general proof for the same interpretation of the theorem is prolonged.

Whereas different interpretations and applications seem to be essential for German ways of teaching proof, successful defence of claims of validity of mathematical assertions can be described as typical for the observed French teaching.

Proving in French teaching is seen as an activity which characterises the whole teaching and not just phases when theorems as Pythagoras' Theorem are proven. Every exercise, even those where students are not explicitly asked to prove something, has to be edited so that the given solution is justified. It has to be made clear what is considered as assumptions, which theorems, concepts and properties are applied. For meaning is here to state the conditions of a statement's or solution's validity. The proof of Pythagoras' Theorem, which has been done in collaboration but strongly guided by the teacher, serves as an exemplary scheme on how to organise one's thoughts. This pattern of argumentation, which is acquired through edited justifications for solutions of problems as well as through proofs of theorems, gives proofs the function of defining conditions of validity. The responsibility of justifying which of the already in class studied theorems and concepts are to be used and why shifts from teacher to students and back.

Conclusion

Comparing the role of proof in German and French teaching contexts has made aware of different teaching patterns and different functions of proofs. These functions of proof can hardly be interpreted as different levels of proofs, such as more formal and less formal proof types. It is the function of proofs which is different. In the observed German teaching the function of proof is to "understand why", whereas in French teaching it is important to "defend why" a statement is true.

Differences in proof functions might explain why a plurality of proof forms could be found in German teaching. Different types of proofs can be beneficially used for working out distinct interpretations of theorems' meanings. This might be regarded as a loss of time when proofs' functions are to give a model as to how results should be legitimated. In mathematics instruction where processes of "defending why" are typical for teaching and learning in general it is more important to allow time for students' own argumentation and proving activity. We may presume that this characterises distinct relations to knowledge and rationality as ingrained in culture.

It seems very interesting to see in how far these differences have an impact on the structures of argumentation in class. The argumentation analyses done so far point out that in general assumptions are made explicit in French discourse, whereas in German classes there might be more lenience for assumptions that rest implicit in argumentation. Formats of argumentation might remain implicit in German practices for sharing of meaning is more important than argumentation types.

Analysing "mathematics classroom cultures" from a comparative perspective gives a way to single out different teaching practices of proof. These differences might help to understand student's difficulties with proof from new perspectives. What effects do classroom discourses have on the learning of proof? Which functions do proofs gain through teaching and what impact does this have on students' conceptions of the proof and their aptitudes for proving?

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INVESTIGATING FACTORS THAT INFLUENCE STUDENTS' MATHEMATICAL REASONING

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We report on the responses of high attaining groups of fourteen-year-old students to one written algebra item and two written geometry items that formed part of a nationwide survey designed to test mathematical reasoning. Preliminary findings suggest that responses are influenced by topic (algebra or geometry), gender and familiarity, as well as by general mathematical attainment. Additionally responses to familiar (algebra) items appear to be subject more to the influence of textbook than of general mathematical attainment, while responses to unfamiliar (geometry) questions are more subject to variation between classes.

In this paper we report some preliminary findings from a written survey conducted in the first year of a three-year longitudinal study of mathematical reasoning. The aim of this research is to advance understanding of how students learn to reason mathematically by analysing their progress over time, and specifically to identify through large-scale longitudinal study, individual, school and teacher factors that are predictors of secondary school students' competence in mathematical reasoning[1].

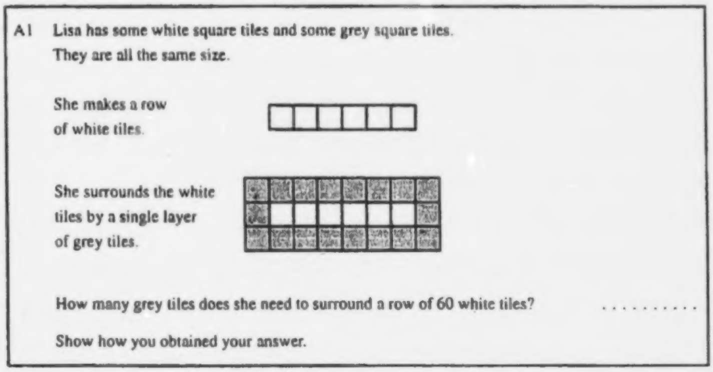
A 50 minute survey was administered in June 2000 to 2797 high attaining fourteen-year-old students from 63 randomly selected schools within nine geographical areas that spanned England. The items were iteratively designed and tested over a period of three months. The starting point for the construction of each item was an issue concerned with proving, followed by a trawl of the literature around this issue and a search for relevant tasks in the curriculum. We report here on students' answers to one open-response algebra item (A1), one open-response geometry item (G2a), and one multiple-choice geometry question (G3).

Frequencies for the sample as a whole are given as well as for four groups of students (P1, P2, Q and R) in order to illustrate some trends in the data. Groups P1 (N = 30) and P2 (N = 28) are parallel top mathematics sets from a non-selective suburban school, Q is a group of 25 of the best students selected from four mixed ability classes in a highly selective school, and R (N = 31) is the top set from an urban comprehensive school.

Students were given a Baseline Mathematics Test a few weeks before taking the proof survey, to provide a measure of their general mathematical attainment. This test consisted of 22 multiple-choice items selected from the Third International Mathematics and Science Survey (IEA, 1996). There were no questions on mathematical reasoning. The mean scores on the test, for the total sample and for groups P1, P2, Q and R were 15.3, 16.4, 16.2, 20.0 and 15.2 respectively. Thus the mean scores for P1, P2 and R were roughly similar to the mean for the (high attaining) sample as a whole, while the mean for Q was substantially higher.

Pattern spotting responses to a question about generalising a structure

Question A1 (shown in Figure 1) is concerned with generalisation in a setting (tile patterns) familiar to English students. (There is extensive work on a generalisation perspective to introducing algebra; see Mason, 1996).



A1 Lisa has some white square tiles and some grey square tiles. They are all the same size.

She makes a row of white tiles.

She surrounds the white tiles by a single layer of grey tiles.

How many grey tiles does she need to surround a row of 60 white tiles? ....

Show how you obtained your answer.

Figure 1: Question A1

As well as providing a numerical answer, students were asked to show how they had obtained their answer. Responses were coded into 5 broad categories (Table 1, below). We discuss just one category here, which we name pattern spotting. In this students obtained the incorrect answer 180 by applying a number pattern without recognition of the structure of the question (see Hoyles and Küchemann, 2000, for a description of the other categories). The diagram in question A1 shows 6 white tiles surrounded by 18 grey tiles. Students were asked for the number of grey tiles needed to surround a row of 60 white tiles. The incorrect answer of 180 grey tiles was found by deriving a (false) relationship, either between the given and required number of white tiles (there are 10 times as many, so there will be 10 times 18 grey tiles) or, less frequently, between the number of white and grey tiles (there are 3 times as many, so there will be 3 times 60 grey tiles).

Code 1	Incorrect answer (180); use of an incorrect number pattern
Code 2	Incorrect answer (eg 120); partial use of correct structure (eg doubles but does not add 6)
Code 3	Correct answer (126); use of correct structure in the specific case of the question with no indication of generality
Code 4	Correct answer (126); use of correct structure indicating its generality
Code 5	Correct answer (126); use of correct structure (expressed in variables)
Code 9	Miscellaneous incorrect answers (including no response)

Table 1: Response codes for question A1

It was not unexpected that some students would resort to using number patterns and thereby attempt to make an empirical, as opposed to a structural, generalisation (Bills and Rowland, 1999). Despite the drawbacks of such an approach (see Hewitt, 1992), the use of number patterns has been widely advocated in UK curriculum materials for many years, even in such stimulating materials as the DIME project (Giles, 1984). However, in such materials students are asked to produce a systematic list of numerical data from which to induce a rule. Here, for reasons as yet unclear, many students were attempting to generalise from a single numerical instance; that is, not only were they ignoring the spatial structure of the situation, but they were not testing their rule on other numerical data.

Figure 2 shows the frequency of the pattern spotting response for question A1. The solid black column shows that over one third of all the students gave this response (total sample, *N* = 2797). This is far larger than we had anticipated, and almost as large as the proportion of students who answered the question correctly (43 %).

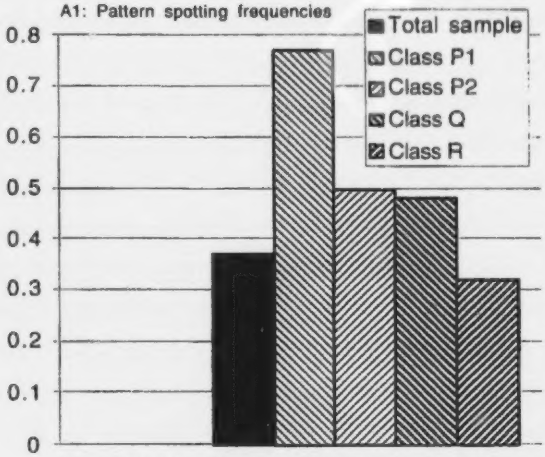
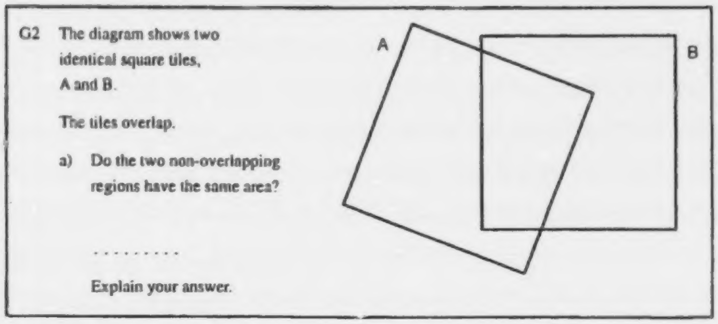


Figure 2: A1 pattern spotting frequencies for total sample and for four groups

Figure 2 also shows the responses for groups P1, P2, Q and R. We draw attention to two noteworthy features. One is the difference between classes P1 and P2. These are parallel classes from the same school, so the finding strongly points to the operation of teacher influences, though at this stage we do not know what these might be. A second interesting feature is the relatively high pattern spotting frequency (48 %) for group Q. The students in this group were selected for their high mathematical attainment within an already selective school. Many of those who gave this pattern spotting response to question A1 gave high level responses to other items on the proof survey (including the geometry item to be discussed below); many also scored highly on the Baseline Maths Test. Again, the reasons for the high frequency of the pattern spotting response for group Q are not yet known, though the textbook used in the school might provide a clue[2].

Responses to an item to distinguish perceptual from logical reasoning

Question G2a (Figure 3) is based on an item by John Gardiner (personal communication). We used it to investigate whether students use perception or logical reasoning in explaining their answers to a simple (but unfamiliar) figural question (see Lehrer and Chazan, 1998; Harel and Sowder, 1998).


G2 The diagram shows two identical square tiles, A and B.

The tiles overlap.

a) Do the two non-overlapping regions have the same area?

....

Explain your answer.

Figure 3: Question G2a

Students are presented

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with identical overlapping squares and are asked whether the two non-overlapping regions have the same area, and why.

Responses were coded into 3 broad categories, which are shown below (Table 2)[3]. For code 1 responses, students gave no reason or only mentioned the non-overlapping areas and deemed them to be the same (or not the same) because they looked to be the same (or not). Code 3 responses focussed on the overlap, with an argument along the lines: 'The squares overlap the same amount (and have the same area)[4], so the non-overlapping regions have the same area'. The code 2 responses were similar but with the area of the overlap treated in a specific rather than a general way; for example, students might state that if one third of each square overlaps, then the same amount, two thirds, of each square does not overlap.

Code 1	Correct or incorrect answer (Yes/No); no logical explanation
Code 2	Correct answer (Yes); logical explanation based on specific example
Code 3	Correct answer (Yes); logical argument
Code 9	Miscellaneous incorrect answers (including no response)

Table 2: Response codes for question G2a

Figure 4 shows the frequencies for each code, for the total sample (the solid black columns) and groups P1, P2, Q and R. Perhaps the greatest surprise was that over half the total sample gave a code 1 (perceptual) response. The relatively poor response of class R is also of interest. It matches that class's performance on another geometry item (question G1) where the students made an incorrect response based on perception rather than mobilising a simple geometric argument (see Hoyles and Küchemann, 2000). On the other hand, there is less difference between classes P1 and P2 than on question A1, and the performance of group Q is much as one might expect from their general mathematical attainment, with relatively few code 1 responses and relatively many code 3 responses.

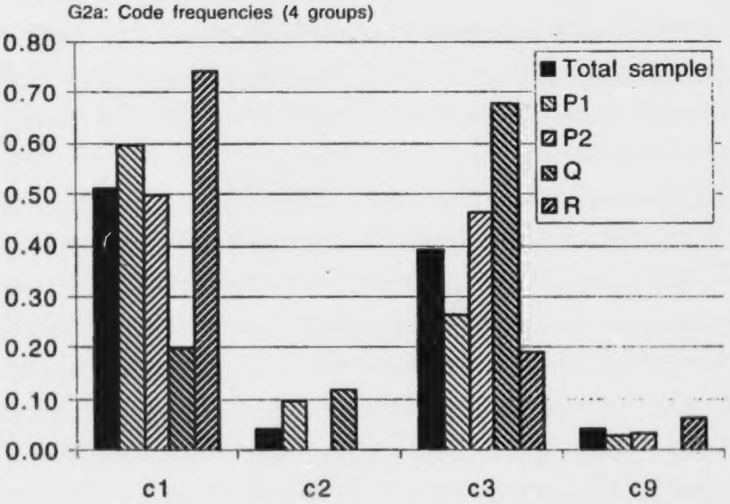
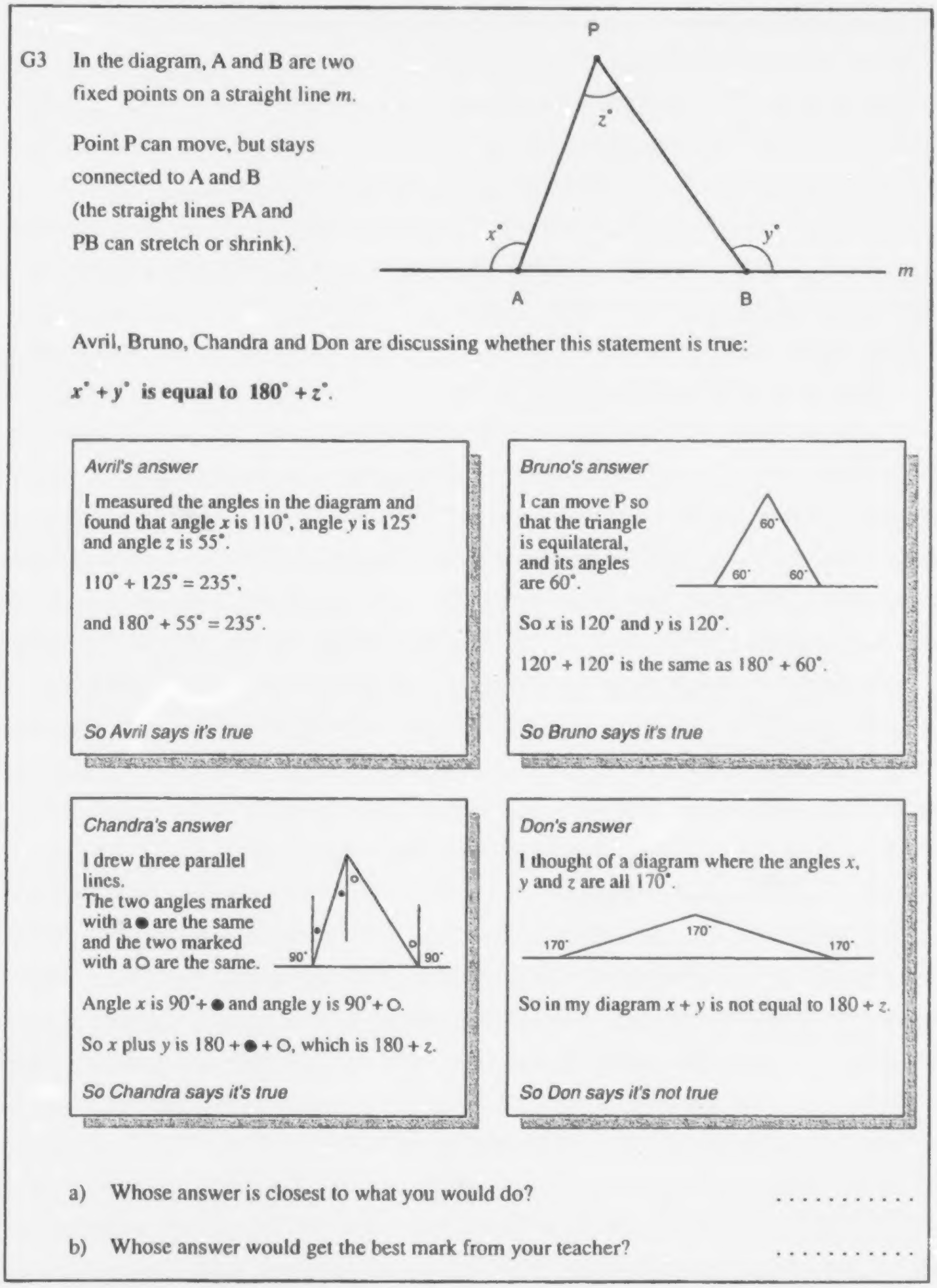


Figure 4: G2a code frequencies for total sample and for four groups

Choices of argument to explain a geometrical conjecture

In question G3 (Figure 5, below), students were presented with a mathematical conjecture and a range of arguments in support of it (options A, B, C and D). They were asked to make two selections from these arguments--the argument that would



G3 In the diagram, A and B are two fixed points on a straight line m.

Point P can move, but stays connected to A and B (the straight lines PA and PB can stretch or shrink).

P
z°
x°
y°
A
B
m

Avril, Bruno, Chandra and Don are discussing whether this statement is true:

$x^\circ + y^\circ$ is equal to $180^\circ + z^\circ$.

Avril's answer

I measured the angles in the diagram and found that angle x is 110°, angle y is 125° and angle z is 55°.

$110^\circ + 125^\circ = 235^\circ$,

and $180^\circ + 55^\circ = 235^\circ$.

So Avril says it's true

Bruno's answer

I can move P so that the triangle is equilateral, and its angles are 60°.

So x is 120° and y is 120°.

$120^\circ + 120^\circ$ is the same as $180^\circ + 60^\circ$.

So Bruno says it's true

Chandra's answer

I drew three parallel lines.
The two angles marked with a ● are the same and the two marked with a ○ are the same.

Angle x is $90^\circ +$ ● and angle y is $90^\circ +$ ○.

So x plus y is $180 +$ ● $+$ ○, which is $180 + z$.

So Chandra says it's true

Don's answer

I thought of a diagram where the angles x, y and z are all 170°.

So in my diagram $x + y$ is not equal to $180 + z$.

So Don says it's not true

a) Whose answer is closest to what you would do? . . . .

b) Whose answer would get the best mark from your teacher? . . . .

Figure 5: Question G3 parts a) and b)

be nearest to their own approach and the argument they believed would receive the best mark from their teacher. The question was deliberately couched in dynamic terms ("Point P can move …") to invite students to adopt a dynamic approach to the question. Fischbein (1982) suggests that such an approach can be an effective way of accessing generality and of gaining insight, and option C (Chandra's answer) is similar to an approach that he recommends for tackling the angle sum of a triangle. Frant and Rabello (2000) also suggest that a dynamic approach can be useful at an intuitive level and for forming conjectures, (though they seem to argue that a static approach is needed for a formal proof).

Two aspects of the students' responses are of particular interest. One is the marked difference between the choices for 'own approach' and for 'best mark'; the other is the difference in choice between girls and boys, particularly for 'own approach'.

G3		Own approach					
		A	B	C	D	other	total
Best mark	A	0.08	0.03	0.01	0.00	0.00	0.12
	B	0.09	0.10	0.01	0.01	0.00	0.22
	C	0.18	0.17	0.08	0.05	0.00	0.50
	D	0.03	0.03	0.01	0.04	0.00	0.11
	other	0.01	0.01	0.00	0.00	0.03	0.06
	total	0.40	0.35	0.10	0.11	0.04	1.00

Table 3: G3 - frequencies for 'own approach' and 'best mark' (N = 2797)

Table 3 shows the distribution of choices for the total sample. It indicates that by far the most popular choices for 'own approach' were A (40 %) and B (35 %), both of which are empirical arguments, with only 10 percent choosing the general argument, C. On the other hand, 50 percent chose C for 'best mark'. This response pattern also occurred with a parallel numerical question (A3) and it echoes the findings of Healy and Hoyles (2000) in their survey with 16-year-old students. (Not surprisingly, given the large number of students involved, the difference between the choices for 'own approach' and 'best mark' was highly significant: $\chi^2 = 1759.5$, df = 16, $p < 0.0001$)

Figure 6 shows the frequency of choices for 'own approach' for girls and boys in the total sample. It can be seen that the girls show a clear bias towards choice A with the boys showing a lesser bias towards C and D. These differences were also significant ($\chi^2 = 63.8$, df = 4, $p < 0.0001$). At this stage the reasons for the differences are

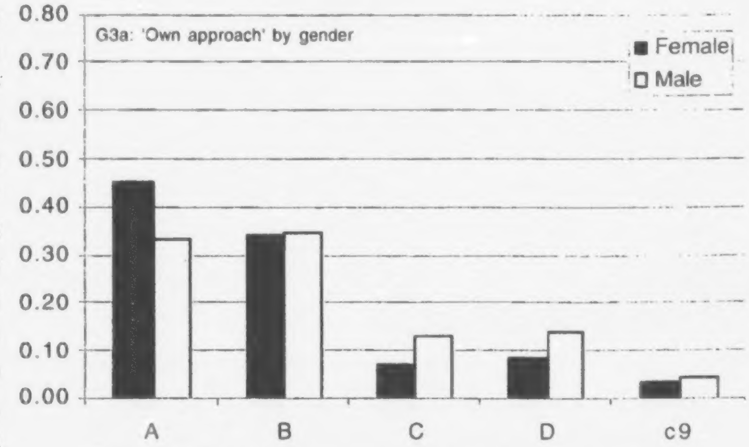


Figure 6: Question G3 – frequency of choices of 'own approach' for girls and boys in total sample

not clear, but they would seem to be worth investigating further. Similar differences occurred in question A3 and with some other items in the survey.

Discussion

These simple statistics suggest that: *first*, type of response to a familiar algebra item about generalisation may be related to general mathematics attainment but may also be influenced by teaching and textbook: *second*, that since geometry is given rather little emphasis as a context for reasoning in the English curriculum, 'better' responses in geometry (that is, introducing a logical as opposed to a perceptual explanation) may be more strongly related to general mathematics attainment than in algebra and also geometry responses may be subject to more variation (as in class R) due perhaps to teacher belief and interest: *third*, that even students who have had rather little introduction to proving have already developed two different conceptions of mathematical reasoning, in that arguments that they assess would receive the best mark differ from arguments they would adopt for themselves: and *fourth*, gender might also be a factor influencing response. Many of these results are similar to those reported following the analysis of a survey conducted in 1998 of older (16-year-old) students' conceptions of proof (see Healy and Hoyles, 2000). These suggestive findings will be investigated further: statistically using multilevel modelling (Goldstein, 1995) and qualitatively through interviews with students and teachers selected on the basis of the profile of individual or class response.

Acknowledgement

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Notes

1. Information about the study can be found on the project's website at www.ioe.ac.uk/proof.
2. The Year 8 book devotes several pages to number sequences, and these are presented in a fairly open way; however, the setting is nearly always purely numerical, rather than involving spatial patterns as in A1.
3. In a fourth category, not listed in the table, students obtained the correct answer by measuring, for example by imposing a square grid on the diagram and counting squares. Such responses were given by only 0.5 % of the total sample and by none of the students in the four groups under discussion here.
4. Code 3 responses were subdivided into those that did and did not make explicit reference to the fact that the squares A and B had the same area.

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Enhancing Spatial Visualization through Virtual Reality on the Web: Software Design and Impact Analysis

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Abstract: Rapid advances in the global information infrastructure herald a new era of computer-assisted instruction based on multimedia capabilities and online access. One of the most promising developments lies in virtual reality (VR), which allows for an immersed learning experience in simulated environments spanning 3 dimensions. VR can be used to merge images, video, animation, and text to provide a rich sensory environment. This capability can be deployed to support instruction in spatial visualization and geometric reasoning. The multimedia presentation in concert with the active involvement of the user can make the learning experience more comprehensible, enjoyable, and memorable. This paper explores some background issues relating to the use of VR on the Internet, and demonstrates the promise of the novel technology in the context of spatial visualization.

Introduction

In recent years, the rapid development of the Internet and multimedia capabilities has led to rapid innovations in the fields of science, education, business, and other domains. One of the most promising applications of the infobahn lies in education. A widely recognized strength of the Internet lies in its vast arsenal of data and documents (Pea and Gomez, 1992).

The potential of computer networks to revolutionize education has been widely acclaimed. Unfortunately, much like the earlier technologies of television and film, the envisioned promise of computer networks has remained largely unfulfilled (Solis, 1997).

Since the mid-1990s, a promising new technology has appeared in the form of virtual reality (VR) on the Internet. The technology allows for the simulation of 3-dimensional worlds on a 2-dimensional computer screen. Through a simulated control panel known as a dashboard, the user can explore the virtual environment in any direction: moving forward, backward, up, down, and sideways or spinning on the spot. Moreover, the interaction occurs in real time.

In the literature, several studies have reported on training materials to enhance spatial visualization abilities in general and spatial visualization in particular. To date, most of the programs have focused on the manipulation of physical objects. However, the use of tangible objects is subject to drawbacks such as procurement cost, storage space, restricted access, and mechanical failure. These limitations underscore the need for materials which transcend the constraints of temporal access and physical space. To this end, VR on the Internet represents a promising vehicle to enhance the learning environment.

The next section presents the background behind spatial visualization as well as VR on the Web. The material is followed by a case study in the form of an educational program to enhance spatial visualization. The concluding section presents some final remarks and directions for the future.

Theoretical Background

Spatial visualization represents a subset of spatial skills. The former has been described by McGee (1979) as "the ability to mentally manipulate, rotate, twist, or invert a pictorially presented stimulus object." (McGee, 1979, p. 893). According to one school of thought, mental manipulation is the primary task in spatial visualization (Ben-Chaim et al., 1988).

The importance of spatial visualization springs its relationship to most technical and artistic occupations including mathematics, science, art, and engineering. However, spatial visualization is not one of the standard components of the school curriculum. Rather, spatial reasoning is acquired informally through informal channels.

Even so, several studies of training programs to improve spatial visualization have been reported in the literature, in concert with various theoretical analyses and hypotheses regarding spatial visualization ability (Ben-Chaim et al., 1988; Battista, 1990; Battista & Clements, 1996; Lean & Clements, 1981).

Virtual reality on the Internet is a technology which allows for the fusion of multimedia files ranging from text to video. In addition, the technology offers a novel capability through its ability to present simulated 3-dimensional worlds.

VR on the Internet has been implemented through various formats. Perhaps the most versatile among these standards lies in the Virtual Reality Modeling Language (VRML). The language provides a relatively compact description of 3-D worlds which can be rendered or depicted using appropriate software.

VRML is a complement to the HyperText Markup Language (HTML), which specifies how information should be presented in a 2-D format on a computer screen. The complementary nature of HTML and VRML is illustrated by a program in which HTML is used to specify a window on a computer monitor. For this application, HTML could be used to partition the window into several frames, one of which might present a 3-D world specified through VRML; another frame might provide explanatory material for the VRML world through text and 2-D images specified in

HTML.

A third complementary standard lies in the Java programming language. Java is a general-purpose language which can be used to depict objects in HTML or control a VRML world. For instance, an applet is a small program written in Java which may be used for, say, providing an animation of a dog running across a 2-D scene whose overall structure is specified in HTML. In an analogous way, Java can be used to process information or specify complex relationships among objects in a 3-D world whose overall organization is specified in VRML.

The technologies of HTML, VRML, and Java provide a versatile vehicle for presenting information to students in multiple media formats. HTML can be used to lay out the 2-D presentation on the screen, while VRML provides a 3-D multisensory experience, and Java is used to control behaviors within and between the following interfaces: the 2-D screen, the 3-D world, and complex interactions with the user.

The present study involves the creation of software using VR to enhance spatial visualization skills, followed by an analysis of its efficacy. Differences in the performance among the students on a spatial visualization test were investigated, both before and after instruction using the software. More specifically, the study was designed to address the following questions:

- Does Web-based instruction in spatial visualization affect the attendant capabilities among students?
- Do the effects differ for spatial visualization instruction through virtual reality in comparison to simple text and graphics?
- Which sub-factors of spatial visualization are affected the most by the use of virtual reality on the Web?

Methodology

The study was conducted in fall 1999 at two girls' high schools in neighboring districts in southern Korea. The schools were the most prestigious in their respective districts, and were of equal caliber as measured by scores on a nation-wide entrance exam. Each of the two schools featured about 50 PC's in its computer laboratory, all of them with full Internet connectivity.

All students in the sample were members of tenth-grade computer classes taught by mathematics teachers. The two teachers, both male, had recently become digital enthusiasts and wished to introduce their students to the potential of the Internet.

Before embarking on the instruction, all the students were administered the MGMP Spatial Visualization Test to obtain a baseline and background information on their skills. The Test was developed by the Middle Grade Mathematics Project (MGMP) funded by the National Science Foundation (Ben-Chaim et al., 1988). Thirty-two multiple-choice items, each with five options, comprise the test. The test is an untimed test with 10 different types of items. The types of representations used are as follows: two-dimensional flat views, three-dimensional corner views, and a "map plan," which depicts the base of a building using numbers within squares to indicate the number of cubes to be placed on each spot. The test includes tasks such as finding either flat or corner views of "buildings," adding and removing cubes, combining two solids, or applying the notion of a "map plan".

After the pretest was given, all the students studied the software on the Web. The material required an average of about 2 hours of study, spaced over a period of approximately 2 weeks in December 1999. Then the posttest was administered immediately after the end of this period.

Case Study

A vital aspect of effective learning is the engagement of the student as an active participant. The use of virtual reality as a vehicle for education offers the following advantages.

- *Student interest.* VR offers a lively medium with a richer sensory texture than 2-D platforms such as books or television. The medium naturally attracts the student's attention. A motivated student is more likely to absorb and retain the material presented. In addition to the inherent attractiveness of VR, our case study elicits interest by presenting the target material in the form of a game.
- *Multiple media.* An amalgam of visual media helps to capture the student's attention and to develop a mental model of the material at hand. The technology of VR offers multiple visual formats such as imagery, animation, and video. Moreover, VR provides an integrated platform for sound and text as well as visual modes. These sensory models can be combined in synergistic fashion to convey concepts in a compelling way.
- *Learning by doing.* Many courses in the curriculum involve laboratory exercises in order to provide an immersive experience for the student. However, experimental facilities can be expensive to maintain, cumbersome to handle, time-consuming to run, and sometimes hazardous for the users. All these limitations can be reduced or even eliminated through computer simulations, while at the same time providing a realistic experience. These same factors have been prompting organizations around the world to rely on computer simulations to an increasing extent, ranging from business games to military training.

Our case study involved the development of an educational software to enhance spatial visualization ability. The software itself was implemented in VRML code. A snapshot of the primary module in VRML is shown in the left pane of the screenshot in Figure 1. By using the mouse, a student could rotate the virtual building along any axis, or "move around" to see any side of the building at will.

As shown in the figure, the VRML module was displayed in one of two key frames in the window. The overall format of the window was specified through HTML.

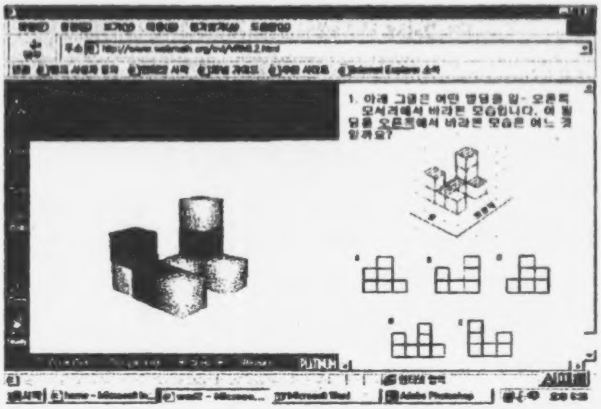

Figure 1. Spatial visualization software using virtual reality.

Data Analysis and Results

An analysis of the pretest data was first conducted in order to determine whether the group difference in spatial visualization was statistically significant. A summary of the analysis of variance on the pretest scores for the treatment and control groups is reported in Table 1. As anticipated, the intergroup difference was not statistically significant at the .05 level. The result indicated that the two groups were relatively homogeneous in their spatial visualization skills prior to instruction.

Table 1. Comparison of the pretest scores between the treatment and control groups.

	N	Mean	Standard Deviation	t	p
Treatment Group	36	25.55	4.39	1.163	.250
Control Croup	31	24.09	5.66		

An analysis of variance for the gain scores (posttest minus pretest) for the treatment and control groups separately is reported in Table 2. For each group, there was a significant overall gain on the MGMP Spatial Test. According to the t-test for two dependent samples, the t-value was -6.69 with p< .000 for the treatment group; and -3.36 with p< .002 for control group. Therefore, for each of the two groups, the difference between the pretest and posttest was statistically significant. One interesting outcome was that the standard deviation for the treatment group declined dramatically after the experiment. The result of this investigation showed that spatial visualization ability for each group improved significantly after the Web-based instruction.

Table 2. Comparison of the gain scores for the two groups.

	Treatment Group (N=36)		Control Group (N=31)	
	Pretest	Posttest	Pretest	Posttest
Mean	25.55	29.66	24.09	27.00
Standard Deviation	4.39	1.62	5.66	4.78
t	-6.69		-3.36	
p	.000**		.002*	

*p< .01

**p< .001

Table 3 presents an analysis the posttest results for the treatment and control groups. This comparison indicates the differential impact between software using virtual reality versus that without. The t-value using the t-test for two independent samples was 2.95 with p< .005. This result indicated that the instructional effect of the program using virtual reality was higher than that employing only text and images.

Table 3. Comparison of the posttest results for the treatment and control groups.

	N	Mean	Standard Deviation	t	p
Treatment Group	36	29.66	1.62	2.95	.005*
Control Group	31	27.00	4.78		

*p< .01

Table 4 presents the results of the pretest and posttest by item type for both the treatment and control groups. Item type 10 yielded a surprise: the difference between the treatment and control groups on the pretest was statistically significant at the .01 level. On the whole, however, spatial ability between the two groups did not differ significantly prior to the Web-based instruction intervention.

On the other hand, a significant difference in improvement due to the treatment was observed for item types 1, 2, 4, 8, and 10. There five item types dealt with the rotation factor, one of the sub-factors of spatial visualization. We may infer that the spatial visualization program using virtual reality was more effective than the one without. Moreover, VR software was most effective for enhancing the rotation factor in the spatial visualization task.

Table 4. The proportion of correct responses on the pretest and posttest by item type for each group.

Item Type	Pretest			Posttest		
	Treatment Group (N=36)	Control Group (N=31)	p	Treatment Group (N=36)	Control Group (N=31)	p
1	.89	.81	.117	.97	.86	.019*
2	.91	.79	.079	.93	.79	.032*
3	.92	.83	.489	.96	.95	.860
4	.98	.95	.330	1.00	.82	.003**
5	.70	.75	.441	.71	.66	.482
6	.87	.88	.866	.93	.93	.918
7	.83	.87	.274	.97	.95	.407
8	.63	.55	.277	.91	.79	.041*
9	.83	.69	.070	.94	.88	.212
10	.74	.55	.003**	.93	.84	.008**

*$p < .05$

**$p < .01$

Conclusion

The study investigated a number of issues relating to spatial visualization. The results were follows. First, Web-based instruction for spatial visualization was capable of improving the target skills. Second, the spatial visualization program using virtual reality was more effective than its counterpart composed solely of text and images. Finally, the Web-based program using virtual reality for spatial visualization was particularly effective for enhancing skills in three dimensional rotation.

The results indicate that certain materials which are difficult to teach in the mathematics curriculum can be taught effectively through suitable Web-based instruction. In particular, virtual reality is a versatile vehicle for enhancing spatial visualization, presumably due to its interactivity and dynamic display for the user.

The efficacy of Web-based instruction has been demonstrated through an educational program encoded in VRML. Virtual reality on the Internet provides an integrated environment for accessing the wealth of information being digitized all over the globe. Not only is access possible, but VR offers a multisensory vehicle for presenting information in a compelling way. The active role of the user in navigating a virtual 3-D space and the instant response of the system provide an immersive experience.

One promising direction for the future lies in autonomous learning capabilities. The maturation of machine learning techniques offers a means of developing intelligent tutoring systems which can tailor a presentation not only to the basic level of expertise for a particular student, but also to his or her changing level of knowledge. The learning capabilities may be implemented using techniques such as case based reasoning, neural networks, and induction.

Such intelligence may be implemented as a kernel of a smart system for multimedia presentations. A framework for such intelligent systems has already been developed (Bordegoni et al., 1997). The framework for intelligent presentations has been adapted to software agents and their embodiment as icons (Andre, 1997). The framework may also be tailored readily to the educational environment. We plan to investigate these and related topics in the years to come.

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CHILDREN'S INTUITIVE KNOWLEDGE OF THE SHAPE AND STRUCTURE OF THREE DIMENSIONAL CONTAINERS

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Abstract.
Research about children's knowledge of the shape and structure of 3-dimensional containers or boxes was conducted with 137 boys and girls in grades 1,2,3 and 7. The children had to build cardboard boxes from into which an empirical referent object had to fit. The problem solving processes were videotaped and analysed together with the children's spatial products. Grounded theory procedure was used to develop the followingcategories to describe the problem solving processes: spatial strategies, assignment of meaning, focus on influencing factors, measurement and planning and evaluation methods. The results showed age as well as gender differences and differences in the assignment of meaning which indicated that shape perception, as in contrast to object perception, is a minimum requirement for horizontal geometrisation of everyday 3-dimensional objects. Moreover, idiosyncratic representations of spatial aspects in childrens' drawings that were previously ascribed to lack of knowledge of conventions are suspected to have deeper conceptual roots.

Theoretical background

Liben (1981) proposed a model for the study of spatial development that distinguishes between three types of spatial representations, namely *spatial storage* (implicit spatial knowledge of which a person is largely unaware), *spatial thought* (spatial knowledge and processes such as imagery that one can reflect on and manipulate during problem solving) and *spatial products* (external products, 2-dimensional or 3-dimensional, that represent space in some way). These representations can be made of *specific spaces* (as in geographical or environmental cognition) or of *abstract spatial properties* (such as understanding of projective relationships). This study took children's *spatial storage* of boxes as the starting point and endeavoured to deduce aspects of spatial thought about *abstract spatial properties* by studying their *spatial products* such as their overt actions, verbalisations and the 3-dimensional boxes they built from cardboard.

Freudenthal suggested that geometry for young children should start in their living space and viewed the study of boxes as a sensible starting point for the geometrisation of 3-dimensional objects. Children should handle and study boxes through practical problem solving activities so that boxes can become mental objects for them, carrying properties of shape and structure that can be revisited on different levels (Freudenthal, 1983:228). However, Olson and Bialystok (1983:37) showed that children initially perceive objects without necessarily being aware of their spatial properties like shape and structure. In addition, Cassirer (1955) discussed the development of spatial thought and reasoned that the meaning given to spatial situations determines what is observed and represented. He suggested that spatial meaning and thought develop through three levels: from personal *mythical* meaning and thought, to *representational* meaning and thought, to *scientific* meaning and thought. Spatial representations in mythical space reflect the individual's imagination and personal interpretation (a box *is* a house in a play situation). In representational space greater objectivity is possible through the use of language to negotiate common understanding (a box can be *used* as a house, but is a box) of representations in different media. Understanding in scientific space requires the development of concepts that often contradict experience, for example that a point has no dimension and a line has no thickness. In order to develop scientific spatial thought, or geometric thought, many primary intuitions must be challenged. Van Hiele's theory of geometric thought emphasizes development from a basic level ability to observe and identify geometric figures as a whole based on visual appearance (Van Hiele, 1960). His theory does not deal with the meaning (mythical/representational/scientific) children assign to the (geometric) objects they work with. Recently, a more basic level of geometric thought than Van Hiele's first level was proposed, based on research indicating that young children do not attend visually to a figure as a whole but rather to some salient aspects of a whole figure (Clements & Battista, 1992:429). However, revisiting Van Hiele's earlier description of levels (1959:8) confirms that he did view thought on the visual level or Level 0 as *pre-geometric* thought. Where geometric thought is concerned with qualitative and quantitative aspects of *shape and structure* of spatial objects and situations, the *pre-geometric* spatial thought of young children in particular has not yet been described adequately with regard to factors like intent and assigned meaning (see also Clements & Battista, 1992:430). This study was designed to research children's primary spatial/geometric intuitions of and meanings assigned to the concept of a box.

Research design

Elements of a revised clinical interview (Ginsburg et al, 1983) were combined with the observation of small groups of children who were free to interact while they solved the problem. A revised clinical interview allows the provision of an empirical referent to the subjects to help focus their thinking. Their overt actions on the referent object are recorded as data and analysed.

The task. The task had to be difficult enough to elicit a series of spatial thought processes. For this purpose, a box had to be built on the basis of an *image* of a box, in the absence of an example. The task was compared to Watanawaha's DIPT classification (Clements, 1983:16) and assigned a (3,2,2,1) level of difficulty, which is the most difficult level described. In addition, the task was designed to be independent of the need for knowledge of conventions for drawing 3-dimensional objects, and conducive to self-evaluation by the children during the process of construction.

Each child received an A2 sheet of cardboard, a 30 cm ruler, a pair of scissors and a roll of cellotape. The children could choose an object (empirical referent) from a selection of articles small enough to hold comfortably in their hands. The children were requested to build a box into which the object that they had chosen could fit. They discussed their understanding of the word *box* and used hand movements and language to describe the shapes and structural properties of boxes they imaged. During the task giving phase it was stressed that simply wrapping the empirical referent object was not allowed. After a pilot study involving 31 children in different grades, the task was given to altogether 106 children in grade 1 (average age: 7 years), grade 2, grade 3, and grade 7. These children were used to tackling problems for which standard answers and procedures are not available. The children were videotaped while they solved the problem. Video data made possible the use of grounded theory development as analytic process, because the researcher could return repeatedly to the original data as the research question was refined.

Categories for describing the problem-solving processes

The children constructed boxes from cardboard according to two predominant strategies that can be described as a *whole to parts* strategy and a *parts to whole* strategy. Both strategies showed developmental variance in the degree of analysis and synthesis of the structural properties of a box. The most primitive *whole to parts* strategy that was identified was that of cutting a single piece of cardboard and folding it in half to produce a flat envelope. An equivalent *parts to whole* strategy was the cutting of two similarly shaped pieces of cardboard and cellotaping them together on top of each other to produce a flat envelope. Midway along a possible continuum of progressive analysis of parts and synthesis into a structured 3-dimensional whole, were the following variation of strategies: A *whole to parts* strategy that comprised of folding a strip of cardboard into three parts, one of which formed the base of the intended box and the other two forming vertical faces on the left and right sides of the base. The equivalent *parts to whole* strategy entailed the cutting of three similar pieces (typically rectangles) and cellotaping them next to each other to be folded as in the *whole to parts* strategy. In both cases the resulting holes (vertical faces) were covered with custom made pieces of cardboard that fitted the direct shape of the missing vertical faces. The most sophisticated strategies incorporated pencil and paper design processes and can be described as follows: The *whole to parts* strategy entailed the folding of four vertical faces, creating a base in the process (at this stage the box looks like a flat tray). Then measurement and drawing were used to determine the exact position and size of a vertical face at the back of an envisaged closed box. Cuts or incisions were made at the correct places to allow the one half of the flat tray to fold over the other half. These cuts created a new vertical face separating top and bottom, so that a typical box used by confectioners or take-away food providers for cakes, pies, pizzas, etc., was formed. The equivalent *parts to whole strategy* entailed the design of a standard net for a rectangular prism, which in most cases approached the shape of a cube. The most typical primitive box that the children constructed was a flat tray (fig. 1).

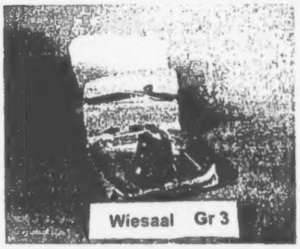

Figure 1: Flat tray

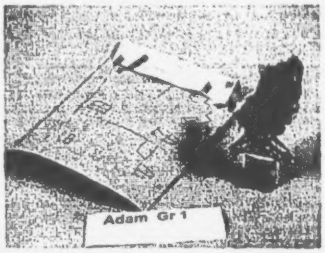

Figure 2: Envelope

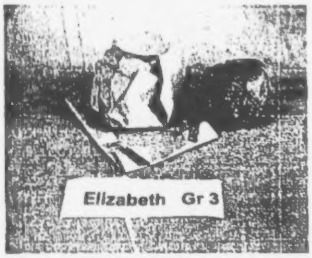

Figure 3: Wrapping

Factors that influenced the construction process. Three factors were identified.

1. The children's **focus** was either predominantly on the properties of the empirical referent *object* for which they had to build a box, or on the properties of the construction *materials*, or on a *mental image* of the box they wanted to make. The children who were able to integrate the demands of the referent object, the material and the image of the intended box, were successful and reached *integrated focus*. Some children's focus varied between object, material and image during the construction process. The children who focused on the object made boxes that were too small, and lacked structural synthesis of the parts into a 3-dimensional whole. Children who focused on the demands of the rigidity of the cardboard and difficulties with cutting and taping made boxes that were far too large and often diverged from their intended box. For example, triangular prisms were constructed non-intentionally, because gravity caused vertical faces to drop towards each other and the child would stick the top edges together in the position they came to rest. Children who focused predominantly on their image of a box made boxes that were far too large, but showed varied degrees of structural analysis and synthesis.
2. **Measurement** or the lack of measurement influenced the construction process. For the purpose of this research, measurement was defined as any intentional action to ensure fitting between parts of the box and between the empirical referent object and the box. Measurement strategies predominantly took the form of *manipulate and estimate* actions, where children handled the empirical object or the parts they constructed from cardboard to ensure fitting of various degrees of accuracy. Some children made no effort to fit parts intentionally while other children used their rulers to measure objectively with a standard unit.
3. **Planning and evaluation methods.** All children made use of kinesthetic imagery involving hand movements or manipulation of the parts of the box or the empirical referent object to plan and evaluate during construction. Excessive 3-dimensional modeling of parts of the box around the object was observed among children with object focus. They typically had to model the

position of each face in 3 dimensions to determine lengths of sides, orientation of faces and relative position of faces. Excessive 3-dimensional modeling usually vitiated the planning and evaluation process, since these children were unable to retain the structural relationships between the faces in 3 dimensions when they tried to tape the faces together in 2 dimensions. Excessive 3-dimensional modeling indicated ineffective imaging of the structural properties of the box.

Success. A successful box was defined as a container that provides a useful and appropriate (suitable) volume, has clearly visible faces that are integrated to a 3-dimensional whole and exists separately from the object for which it was made, retaining its 3-dimensional structure when the object is removed. Measured accuracy and closedness were not taken as criteria for success, since these aspects were not specified during task giving and lent themselves to estimation and interpretation. The boxes were classified according to shape and structure and the following variations were found: rectangular prisms (fig.1), triangular prisms, cylinders, envelopes (fig.2) and wrappings (fig.3). Clearly envelopes and wrappings were not successful boxes according to the criteria described above. In addition prisms and cylinders were judged unsuccessful when the faces were not differentiated clearly (*whole to part* strategies where fold lines were not intentionally constructed) or integrated properly (*part to whole* strategies where the edges of the faces were not joined intentionally).

Factors that caused failure. The following factors were identified.

1. Limiting effects of 2-dimensional thinking. Some children had difficulty in representing a third, vertical dimension. After 40 minutes of trying various methods, their boxes were flat envelopes. The following examples of acute limitations were found: Children who believed that a (perspective) drawing of a box or of the empirical referent object would produce a box when cut out (Daniel and Dale, gr 2); children who handled the cardboard as if it was a sheet of rubber and did not produce fold lines, resulting in the inability to form vertical faces (Elizabeth, gr 3); children who drew unfoldings (nets), but were unable to visualise the effect of folding these drawings into a 3-dimensional box (Adam and Lara, gr 1).
2. Limiting constructions in 3 dimensions. As already indicated in the discussion of criteria for successful containers, some children had difficulty in integrating the faces they constructed into 3-dimensional boxes. Three types of difficulties were identified. Children who lost the orientation of faces in relation to the whole when they had to tape the faces together in 2 dimensions (Yolande, gr 7); children who did not produce clear fold lines to separate the faces of the box (Hlubi, gr 7) and children who did not integrate the sides of the faces to form edges of a box. The lack of integration of sides was also evident while the children made the polygonal faces for their intended boxes from the cardboard. They would draw a rectangle with one side almost on the edge of the sheet of cardboard, yet would not use the edge of the cardboard as a side of the rectangle. In another case a child

(Shane, gr 3) used one rectangle as a template to cut two more rectangles side by side, the edge of the hole where the previous rectangle was cut out forming one side of the next rectangle. As he cut out the second rectangle, he taped the two rectangles together side by side and fitted them into the hole created by cutting them from the sheet of cardboard. He repeated the process with the third rectangle and ended with three rectangles taped together as if they had never been separate, fitting exactly into the hole they left in the sheet of cardboard.

3. Mythical meaning. Some children who had difficulty constructing their intended boxes regressed in terms of the meaning they had given to the task during discussion and made fancy objects instead of boxes. Examples of such objects, which the children themselves judged as not being boxes, were houses (complete with chimneys, doors and windows), mushrooms and bags. These terms were used by the children themselves when they talked about their products.

Discussion of the results

The results will be discussed according to age and gender variants.

Strategy. The choice of strategy shifted from *whole to parts* in grade 1 to *parts to whole* in grade 7, with the exception of grade 2 where *whole to parts* strategies (producing flat trays) dominated. In general boys used more *parts to whole* strategies and girls more *whole to parts* strategies. The girls in grade 7 that used *whole to parts* strategies used very primitive strategies, merely wrapping the empirical referent object or at most pressing in tentative fold lines after wrapping the object. The only boy in grade 7 that used a *whole to parts* strategy succeeded in fully explicating and representing the shape and structural properties of a box, producing a cake or pie box as described earlier. Most grade 7 boys used a *parts to whole* strategy that entailed the design of a net, while most grade 7 girls used a *parts to whole* strategy by which they assembled a net from cut out faces in a step by step fashion. This indicates that although the girls were able to fully analyse the shape and structural properties, they did not yet have a complete understanding of the synthesis of these aspects into a whole.

Focus. Children who followed *parts to whole* strategies showed predominantly image focus, while children who followed *whole to parts* strategies showed predominantly varying focus. Integrated focus occurred only among children who followed *parts to whole* strategies. Of grade 1 children, 50% showed varying focus, while the rest showed predominantly material focus, indicating a lack in skill with cutting and taping activities. Although 40% of the grade 2 children showed varying focus, image focus occurred most among the rest of the children. In grade 7, image focus and integrated focus occurred most and with the same frequency, indicating that skill in imaging has developed. The greatest difference between boys and girls was the degree to which they reached integrated focus. Only 9% of the girls compared to 28% of the boys reached integrated focus. On the other hand 19% of the girls compared to 9% of the boys showed object focus. As discussed earlier, integrated focus was a sure indicator of success, while object focus leads to too small and often unintegrated boxes.

Measurement. Most of the children who followed *whole to parts* strategies did no measuring. *Parts to whole* strategies were characterized by *manipulate and estimate* measurement methods. Objective measurement (with a standard unit) was only used by grade 3 and grade 7 children. More girls than boys in grade 7 made use of objective measurement, which may be an indication that accuracy becomes more important for girls than for boys in grade 7. On the other hand Piaget, Inhelder and Szeminska (1960:33) showed that visual estimates become more accurate once children are able to measure objectively, which may indicate that the boys trusted their estimated measurements and did not experience a need for more accuracy. Piaget et al mention the intuition that children have of vanishing volume between a 3-dimensional object and its 2-dimensional net, which results in nets that are too small to be folded into a copy of the referent object. A related intuition was noticed in this study, namely that children intuitively produced 3-dimensional boxes from 2-dimensional material that were far too large for their empirical referent objects, possibly in anticipation of extra volume needed in 3 dimensions.

Success. *Parts to whole* strategies were more successful than *whole to parts* strategies (84% compared to 45%). As can be expected, the success rate increased with age as the choice of strategies shifted towards *parts to whole* strategies. Across ages, with the exception of grade 2, the boys were more successful than the girls (see table 1).

Table 1. Success rate across ages (%)

	Grade1	Grade 2	Grade 3	Grade 7
Boys	64	18	80	100
Girls	30	78	53	84

In 57% of the cases failure was caused by the inability to produce vertical faces. This was also the reason for the dip in performance among grade 2 boys.

Meaning. As indicated before, some children (in grades 1 and 2) assigned personal mythical meaning to their boxes and constructed houses and other objects. It seems that the standard image of a box for most children was that of an open rectangular prism. From the strategies children used, two deeper representational meanings of the concept box can be deduced. On the one hand children assigned the meaning *a box covers an object.* This was evident from strategies that involved wrapping actions and favoured closure above structure. These children often constructed the vertical lateral faces of the box first and attached a base and a lid later. On the other hand children assigned the meaning *a box can receive an object* as was evident from the large number of flat trays and other open boxes that were constructed. These children constructed the base of the box first. The base served as a stable point of reference and therefore these children were better able to represent the structural properties of a box. Children who integrated the two meanings produced boxes with vertical sides high enough to cover the object and clearly visible structure in 3 dimensions.

Conclusion

This study showed that young children do not necessarily view boxes as geometric objects in the sense that they are aware of the shape and structural properties of such boxes. Moreover, they may be able to represent certain properties, noticeably properties of shape, in media like language, kinesthetic images and even plane drawings and yet they may not be able to construct a 3-dimensional model from 2-dimensional material. The development from object perception and the assignment of personal, mythical meaning to the perception and representation of abstract spatial/geometrical properties and relationships seems to require a transitional phase of shape perception and representation based on the assignment of representational meaning. This implies that the assignment of representational meaning and the accompanying perception of the shape of objects may be a pre-requisite for young children to start with geometry at an entry level. This study also showed that aspects of representation that are judged to be based on lack of knowledge of conventions in a medium such as drawing, may have deeper intuitive and conceptual roots. One notable example is the lack of integration of the edges of 3-dimensional objects and even the sides of 2-dimensional figures.

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LEARNING NUMBER THEORY CONCEPTS VIA INTERACTION IN COMPUTERIZED ENVIRONMENT

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This paper presents a case study of two 7th-grade students during their explorations in the computer-based microworld Geoboard. *In their explorations of the connections between geometrical shapes and the corresponding computer instructions, the students needed to use elementary number theory concepts. The study uses inductive and discourse analyses to trace the development of the mathematical discourse between the students. It demonstrates how the computerized environment supports the construction of concepts which are tightly related both to the characteristics of the learning environment and to the relevant number theory content.*

Introduction

Understanding elementary number theory concepts is fundamental for many branches of mathematics, yet it has received only minor attention in the research literature. Existing studies fall into three main categories: one, studies in which elementary number theory concepts are used to investigate other mathematical issues (Martin & Harel, 1989; Leron, 1985; Lester & Mau, 1993; Movshovitz-Hadar & Hadass, 1990); two, studies that explore the multiplicative structure of numbers through problem solving (Ball, 1990; Graeber, Tirosh & Glover, 1989; Greer, 1992); and three, studies that explore understanding of elementary number theory concepts by pre-service teachers (Zazkis and Campbell, 1996a; Zazkis and Campbell, 1996b; Zazkis, 1998; Zazkis, 1999).

In this study we use the qualitative methods of inductive analysis and discourse analysis to trace the construction of elementary number theory concepts by seventh grade students, engaged in computerized explorations of number theory concepts such as prime number, divisor, and greatest common divisor (gcd). Under the present space limitations we can only sketch the theoretical framework and hint at some of the results. For full details, see Lavy (1999).

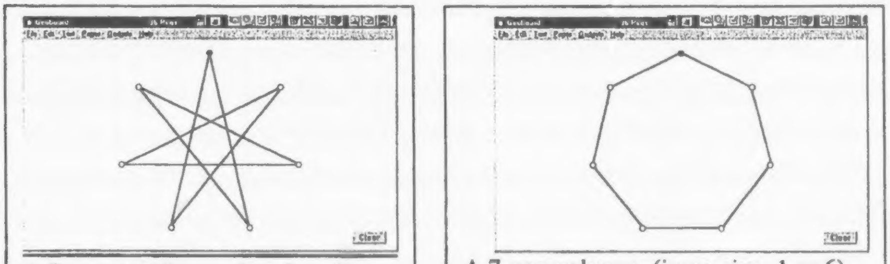
Methodology

The computerized environment under study consisted of MicroWorlds Project Builder (MWPB) – "a Logo-based construction environment which, in addition to retaining the expressive capacity of the Logo programming language, has a number of useful object-oriented features and facilities for direct manipulation which can be used to promote links between action- and conceptionally-based ideas." (Hoyles & Healy, 1997)

The *geoboard* is a mathematical package implemented on top of the basic MWPB environment. It consists of a simulated round board with pegs at constant intervals on its perimeter. The students can change the number of pegs by using the logo instruction: *newboard n*.

For example, *newboard 8* will create a 8-peg Geoboard.

A class of ten 7th-grade students met several times after school hours in the computer lab, and explored the effects of the instruction *repeat n [jump k]* on geoboards of varying number of pegs. The command *jump k* results in a line segment being drawn from the "current" peg to the k-th peg counting clockwise, which now becomes the new current peg. The statement *repeat n [command list]* results in the list being executed n times in succession. Each choice of specific values for n and k resulted in a screen display of a regular polygon or a star with varying number of vertices (Fig. 1). The students were encouraged to look for mathematical patterns connecting the input numbers and the shapes and the number of vertices of the resulting polygons or stars. These investigations led to the emergence in the students' discourse of concepts such as prime number, divisor and greatest common divisor (gcd).



A 7-peg star (jump size: 3 or 4)

A 7-peg polygon (jump size: 1 or 6)

Figure 1: Shapes obtained by the command *repeat n [jump k]*

We have chosen to concentrate on one couple, Noam and Jacob, whose actions and screen productions were captured by a Video camera. These students were selected because, more than all the other students, they tended to "think aloud" during their work.

The major part of the research data is the verbalized discourse, which took part between the students during the activities. The research data included in addition to the verbalized discourse, the screen pictures at every stage of the inquiry, the students' body language and every piece of written paper they produced.

The data were analyzed by two kind of qualitative methods: inductive analysis and discourse tools. Inductive analysis (Goetz & Lecompte, 1984) is a method which integrates between scanning the data looking for phenomenological categories, and successive refinement of them when confronted with the new events and interpretations. According to this approach, the investigator is the primary instrument of data collection and analysis. In keeping with this approach, there were no predetermined criteria or categories made.

To get a better understanding of the learning process, two discourse tools were applied: interaction analysis and focal analysis. These tools derive from the premise that in order to understand how students learn, we cannot separate cognition from socio-cultural influences: the way we express ourselves may throw light on how we understand. The research object is the discourse through which the pair under study interpret each other's ideas, thus gradually co-constructing their shared reality.

The first discourse tool is interaction analysis (Sfard & Kieran, 1997) through which we examined the interactions between discourse participants, with the help of a diagram constructed from the data according to a set of simple principles (ibid). The second discourse tool is focal analysis (Sfard, 1998), which is concerned with the discourse content and highlights additional aspects of the discourse between the students. Here the researcher interpreting students' utterances is trying to understand what students *actually see*[1] when they express their ideas.

Analyzing the mathematical discourse of the two students enabled us to get a better description of their personal learning profile, and to characterize more succinctly their individual contribution to the mathematical discourse. Triangulation of the three methods enabled us to obtain a comprehensive point of view of Noam and Jacob's learning process.

Results and discussion

Through a detailed analysis of the data scripts, we came up with four categories which were then refined in each iteration of the data review. We called the emerging categories *utterances types*, *number theory concepts*, *argumentation* and *rules*.

We used *utterance types* to categorize the different kinds of mathematical utterances Noam and Jacob used through the inquiry. These included general utterances like, "we need one [number] which has many divisors.

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[1] a real or imaginative action.

You understand why" (Noam); specific utterances like, "fifteen. It has a common divisor with twenty" (Jacob); and general uttrences masked as specific: "twenty four less the jump". Noam phrased this utterance when they were investigating the equivalence of *repeat n [jump k]* and *repeat n [jump n-k]*. He used the number 24 even though they were working with a 15-peg board, hence our conclusion that 24 (the "canonical" number of pegs in these investigations) stood for a general number of pegs.

There were two types of *number theory concepts*: the first type involved explicit geoboard notions: "With such primes, there will be only one simple polygon and all the rest will be stars". In the second type the number theory concepts appeared in a purely mathematical context: "Is twenty seven prime?"

Argumentation included the instances in which one student phrased an argument and tried to convince his colleague of its validity.

Rules included all the instances in the discourse in which the students tried to formulate mathematical rules.

The process of categorization helped us to characterize the learning profile of Noam and Jacob, to describe the concept development during the inquiry, and to understand the contribution of each of the students to the shared learning process.

During two sessions of inquiry, the students phrased 46 tentative mathematical "rules", most of which being related to two main concepts. Adopting the students' own terminology we called the first concept *n-star* and the second concept *common denominators*. In what follows we will describe the concept of n- star.

n-star is a star polygon with the same number of vertices as the number of pegs on the geoboard. The actual name used by Noam during the investigation was in fact *24-star*, but since he used this name for all kind of geoboards, regardless of their number of pegs, we thought the name n-star was appropriate.

After Noam and Jacob had been exploring the Logo instruction *repeat n [jump k]* on a 24-peg geoboard, Noam noticed that certain values of k yielded a 24-star. He said to Jacob, "Wait, let me try again, I want to see something here". He could not quite formulate yet what he saw, and he needed to check a few more values of k, and then he said: "I want to check something, ok? I think I reached some pattern in 24-stars of all the vertices." Jacob tried to formulate his interpretation to Noam's discovery but he seemed unaware of what was special about this star: "look, if you put a prime number, it will give you a star". At this point Noam made a correction: "no, not just a star, a 24-star". Jacob then rephrased Noam's idea by saying "star of all [vertices]", and Noam repeated, "yes, star of all, this is the pattern I reached". From this point on both students continue to use the new concept naturally in their discourse.

The development of the concept n-star reveals a process of *concept refinement*. At first the students use the utterance "a jump of prime number makes a star"; later they use the utterance "a jump of prime number creates an n-star"; then they refine their utterance further by saying, "a jump of a number that is disjoint to n [the number of pegs] makes an n-star"; finally, they use the utterance, "a jump of a prime number which is not a divisor of n, makes an n-star".

The construction and the refinement of the concept n-star has been achieved via a mutual effort of Jacob and Noam, and we will now follow this construction by identifying the specific contribution of each student to the process (Figure 2).

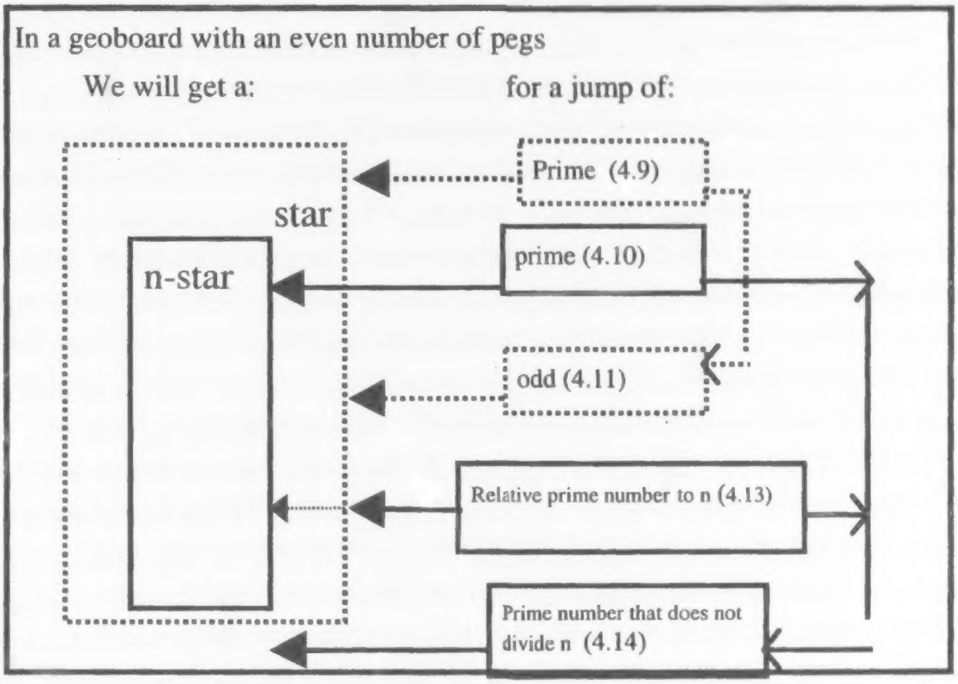
In a geoboard with an even number of pegs
We will get a:
for a jump of:
star
n-star
Prime (4.9)
prime (4.10)
odd (4.11)
Relative prime number to n (4.13)
Prime number that does not divide n (4.14)

Figure 2: Schematic description of shared construction of the concept n-star

Fig. 2 describes the construction of the n-star concept as a joint endeavor of Noam and Jacob. The broken lines refer to Jacob's part and the solid lines refer to Noam's part. In the rectangles on the right appears the size of jump and the in the rectangles on the left appears the geometric outcome – a star or an n-star. (The numbers in parentheses refer to the utternace number in the original data).

The first rectangle in Figure 2 refers to Jacob's utterance (4.9), that in an *even geoboard* (this is our own shorthand for a geoboard with an even number of pegs) a jump by a prime number will create a star. Few minutes later Jacob phrases utterance 4.11, that it takes an odd number of jumps to get a star. (It looks as though Jacob has regressed in his understanding at this stage of the inquiry: analyzing the rest of the verbalized discourse reveals that Jacob confuses between odd and prime numbers.) The second rectangle refers to Noam's utterance (4.10) in which he characterizes the star as a special case of n-star. This is Noam's discovery and he is the one who first coined the concept n-star. This utterance is a refinement of Jacob's previous utterance (4.9), in which he was not yet aware that this was not just any star, but actually an n-star. The fourth rectangle refers to Noam's utterance (4.13) in which he emphasizes the fact that the size of jump is a number that does not divide n. Although Noam used the word "star" in this utterance, looking at the preceding and following transcript, it is clear that he meant an "n-star" (hence the broken arrow in Figure 2).

The fifth rectangle refers to Noam's utterance (4.14) in which he connects 4.13 with 4.9 and arrives at the conclusion that in an even geoboard, a jump by a prime number that does not divide n creates an n-star.

We have used interaction analysis to characterize the learning profile each student brought to the discourse, and how it helped them to construct a shared meaning of the learned concepts and how it contributed to each one's individual learning. Applying interaction analysis to the discourse concerning the development of the concept n-star, reveals that Noam has a rich private channel which he uses in the inquiry process. Jacob, on the other hand, makes an effort to follow Noam's actions and tries to communicate with him. Noam's utterances are mainly concerned with mathematical interpretations that he makes for himself during the investigation, while Jacob tries to penetrate Noam's private discourse in an effort to take part in Noam's hidden processes. Significantly, most of the time Noam is the one who types the Logo instruction at the keyboard. When we pointed this out to Jacob he said, "he [Noam] is good at it". Most of Noam's utterance were completed or explained with certain screen images and came as reactions to Jacob's questions. Noam was busy with his personal discourse while he was trying to figure out the connections between the polygons or stars appearing on the screen and the Logo instruction he was typing. Jacob, on the other hand, helped Noam to work systematically and to run a more systematic investigation. Noam uses an "examples language" even when he talks about patterns. Jacob, on the other hand, tries to generalize even when his generalizations are sometimes incorrect. Many times during the discourse Noam's utterances are corrections to those of Jacob, for example, when Jacob says to Noam, "look, if you take a prime number it will give you a star", Noam answers, "no, not just a star, a 24-star". Since Noam find it difficult to express himself verbally, he relies on Jacob's utterances, and by using them as a basis and rephrasing them, they finally arrive at the correct conclusion. In fact, each student functions as an "expert", promoting his partner further within his zone of proximal development (ZPD). Noam, who is adept at handling the computer, helps Jacob understand the mathematical regularity of the geoboard. Jacob, on the other hand, pushes Noam into a more systematic investigation, which in turn leads Noam towards new directions to explore. The way Jacob formulates his ideas helps Noam articulate his own more clearly.

The n-star concept emerged from a synthesis of the geoboard learning environment (stars and polygons, the number of pegs n and the jump k) and the related number theory notions (primes, divisors, gcd, etc.) The construction of the n-star concept was the result of a process in which the students learned how to generalize from examples, create and refine hypotheses, and see mathematical patterns through the screen images.

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UNDERSTANDING TEACHERS' CHANGING APPROACHES TO SCHOOL ALGEBRA: CONTRIBUTIONS OF CONCEPT MAPS AS PART OF CLINICAL INTERVIEWS

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In this paper, we present a part of a larger study that examines changes in teachers' conceptualizations of school algebra. From a larger corpus of data, this paper examines data from two interviews with two teachers. Among other mathematical tasks the teachers were asked to draw a concept map for the concept "equation." Our hope was that these concept maps would help us identify teachers' conceptions of equation and how those conceptions fit inside a larger approach to school algebra.

Objectives/purpose

Within mathematics education, there is a strong assumption that teachers' "amount" of mathematical knowledge is a key component of their capacity to teach mathematics successfully. Quantitative attempts to support this assumption have used crude measures of prospective teachers' content knowledge and of the mathematical resources that they bring to teaching tasks and have not shown a positive correlation between the number of mathematics courses taken by teachers and their students' achievement (Begle, 1979). Thus, in order to examine this assumption more carefully in the future, it seems useful to develop better tools for describing and assessing teachers' mathematical understandings of the subjects they teach.

In science education, concept maps -- defined as a two-dimensional representation of relationships between selected concepts -- have been used extensively with a wide range of students, both as an instructional tool and as a mechanism for assessing the understandings of individuals (e.g., Novak, 1998). More recently, concept maps have been used in mathematics classrooms (e.g., Wilcox & Sahloff, 1998) and in research on student learning (e.g., Doerr & Bowers, 1999; Schmittau, 1991; Williams, 1998).

In this study, we will examine the strengths and weaknesses of concept maps as a tool for examining *teachers'* understandings of the mathematics that they teach. Are there reasons that concept maps might be especially useful in understanding how teachers conceptualize their approach to a topic? Or are concept maps limited in some principled ways when used to explore teachers' understandings?

In particular, we are interested in teachers' understandings of equations and the solving of equations. With the presence of new technologies, the algebra curriculum is in a state of flux; an extremely visible impact of the use of technology in school algebra involves using numerical and graphical representations of functions to identify solutions of equations. As a result, there is disequilibrium in the mathematics education community's views on relationships between algebra and functions (see, for example, Lee, 1996; Bednarz et al., 1996; or Lacampagne et al, 1995). Therefore, it seems especially useful to have tools to assess teachers' understandings in this area, as shared understandings of relationships between algebra and functions change.

Perspective or theoretical framework

Utilizing concepts from research on the psychology of student learning (Vinner, 1991), researchers interested in the understandings of high school teachers have studied their conception of function (see Cooney & Lloyd, 1993, for a review). As school algebra curricula have changed, some studies explore the conjecture that teachers' implementation of curricular topics related to functions are influences by the teachers' definitions of function. If that conjecture were true, teachers' definitions of function would shed light on their willingness and capacity to implement function-based curricula approaches to algebra. For example, Lloyd and Wilson (1998) focus on relationships between a teacher's implementation of the Core Plus curriculum and this teacher's conceptions of function. They suggest:

> "... because Mr. Allen was able to reconcile the Core-Plus approach to functions with the prominent features of his own conceptions of functions, the Core-Plus materials furnished a way for him to translate his understandings into new, but comfortable, pedagogical strategies" (p. 271).

We in addition suggest that it might be fruitful to create a construct "approach to school algebra" that indicates a teacher's overall orientation and to find tools to elicit and analyze such views.

Methods of inquiry

Sample: This research report describes a small piece of data from a larger study (Yerushalmy, Leikin, & Chazan, in press). We will focus on two in-service, high school mathematics teachers from a larger of sample of nine teachers. All of the interviewees in the larger study come from one U.S. school district. The district in which the interviews took place adopted a district-developed introductory algebra curriculum. This curriculum is built around the assumption of technological support, primarily in the form of graphing calculators. According to the criteria in Chazan and Yerushalmy (in press), this course takes a functions-based approach.

> "[It] initially emphasize[s] the interpretation of: letters as variables, rather than unknowns; expressions as the correspondence rules for functions; the Cartesian coordinate system as a space for displaying the results of calculation procedures, rather than the points in a solution set; and the equal sign as the assignment of a name to a particular computational process (f(x)=...) and as the indication of identity between two computational processes."

Interviews: Each interviewee was interviewed twice, once at the beginning and once at the end of the school year (1999-2000). Between the two interviews, the interviewees taught the district's curriculum and met five times during the year to discuss issues in algebra curriculum reform.

In the fall, interviewees were asked to compare and contrast different equations and to solve equations of different types. In the spring, the teachers were asked to discuss a set of statements about relationships between equations and functions and then relate these to their thoughts about teaching students what the solution to an equation is and how to recognize equivalent equations.

Besides these mathematical tasks, both in fall and in spring interviews, the teachers were given a table of concepts (e.g., variables, unknowns, statements about numbers, functions, ...) and asked to draw a concept map of an equation. Of course, interviewees also added their own terms. In the fall, if an interviewee did not know what a concept map was, the interviewer provided an example of a concept map for the notion of quadrilateral. In spring interviews, the teachers were asked to compare new maps they had drawn in preparation for the interview with their fall maps.

Throughout the interviews, the teachers were asked to talk aloud. All the interviews were videotaped and any written work done during the interview was collected.

Data presentation and analysis: We will analyze the fall and spring interviews to identify the teachers' *conceptions of "equation"* and whether they hold a functions-based or an equations-based *approach to school algebra* (Kieran, 1997; Chazan & Yerushalmy, in press). The discussion section will reflect on contributions of the concept maps to our understanding of the teachers' views.

Results

Peter: In the fall, Peter was just starting to teach the introductory algebra curriculum. He consistently connected the concept of equation with a concept of function when performing tasks and discussing his performance. But, different tasks highlighted two different kinds of connections. For example, Peter viewed the equation $2^x=x^2$ as a comparison of two functions.

> Basically the way I would think about it first anyway is that I would think about what this problem means is where does the output of this function 2^x the same as the output of this function x^2.

In contrast, when solving the equation $x^2/_4+y^2/_9=1$ he said:

> ... That one has the *y* in there and it doesn't look like we are comparing it to another function it seems like all those points in one relationship ... So I would say I could think of this one as a function just *y* equals [by isolating *y*]...

More generally, whenever *y* was explicitly a second variable, in an equation in two variables, or in a system of two equations in two variables, he continued to think about functions, by turning equations of two variables into functions of a single variable.

Overall, Peter's fall concept map suggests that he is thinking about a function-based approach to school algebra. Almost all of the terms used for construction of this map and connections between the terms can be associated with function based approach (e.g., letters as variables, inputs as candidates for solution sets, ...). Peter, for example, explained:

> ...When I thought of an equation ... the first thing that I saw was ... a function. What I really thought of when I meant function was I kind of think of that is relationship between variables. ... And form there spread out into the idea of variables, things that

> are changing that include the input and output … The input was restricted by the domain, and … part of the inputs can be a solution set.

One link seems not to fit with this interpretation, viewing an equation as a function. However in his discussion of the map, he connected it back to his two views of the connections between equation and function:

> I wasn't sure … if equation and function were exactly the same. … You could think of an equation as a function. …Equation I [also see] kind of built of function obviously kind relate to that. … I thought also that graph went of an equation as you can think of equation as the function itself.

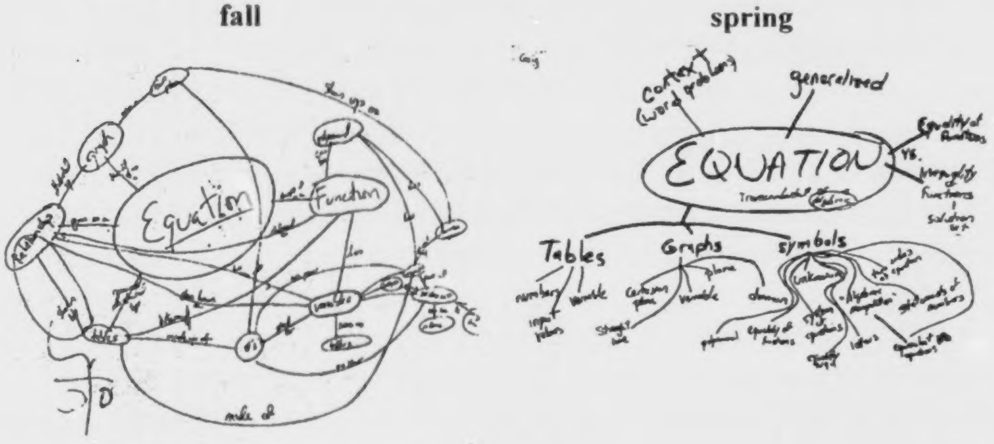

fall **spring**
Equation
Function
graph
make of
context (word problem)
generalized
EQUATION
Equality of functions
vs.
Inequality of functions
solution set
Tables
Graphs
symbols
number
variable
input values
Cartesian plane
variable
domain
straight line

Figure 1: Peter's concept maps

While Peter's fall map was not organized hierarchically, his spring map is more tightly organized and has hierarchical components. In his spring map, equations were defined as an equality of functions (as opposed to inequality). He also connected equations directly with tables, graphs in the Cartesian plane, and symbols "because equation can be thought of as a graph, …, or can be illustrated by table."

In contrast to the fall interview, in the spring, Peter thought of equations in two variables as constructed out of two functions of two variables. For example he expressed his reasoning about the equation $x^2/9 + y^2/4=1$ as follows:

> We are trying to find where that output matches up with this different relationship between x and y. So here [in the right side] x and y are not written here because it is a constant function of 12 … To make more sense I will write I am just thinking of these as $f(x,y)=g(x,y)$, that's how I am, interpreting those.

Peter felt that his ideas had not changed much from fall to spring.

> My thinking hasn't changed much… I [am] actually pretty surprised how this [spring map, which] is done after a lot of thinking about teaching similar to this [fall map, which presents] just some random thoughts I had 6 month ago…

Interestingly, he suggested that his teaching effected his spring concept map and even seemed to say that it was organized according to "how it [solving of equations?] was taught in my class in the last … three months or so…" However, his spring map was more complicated than that. In his words, it represented both how he thinks, how the introductory course is organized, and "normal" approaches as well.

> This [spring map] is kind of a combination of the way I think about algebra and the way that school curriculum thinks about algebra and how that probably different from the traditional view about algebra or what you see in normal Algebra-1 book.

He continued and clarified why he placed "unknowns" in the branch of "Symbols," something that could be taken as in tension with the rest of his map. Variables were related to equations that are built of functions; unknowns were associated with either equations as they are presented in standard textbooks or points of intersection of the functions corresponding to equations' solutions.

> I put unknown here, which is a little bit different… I could have put variable here too [under symbols] but traditionally when you are talking about symbolic manipulation which is usually the algebraic manipulation, which is traditionally taught, variable can be thought as unknowns. Which is different than how I think about it…. The reason I think the unknown fits under equations is because we are looking for the point now….

Bob: Bob was a second year teacher in the fall of 1999. In 1998-99, he taught at junior high school. For the new year, he was transferred to the high school. At the junior high school, Bob taught the introductory algebra curriculum. This curriculum seemed different to him than the way he had thought about algebra in the past.

Throughout the fall interview, Bob consistently compared two approaches to school algebra. He discussed solving each one of the tasks in the interview according to two different approaches. He provided explanations from two perspectives, continually referring to what he had thought "before" and what he thought "now:"

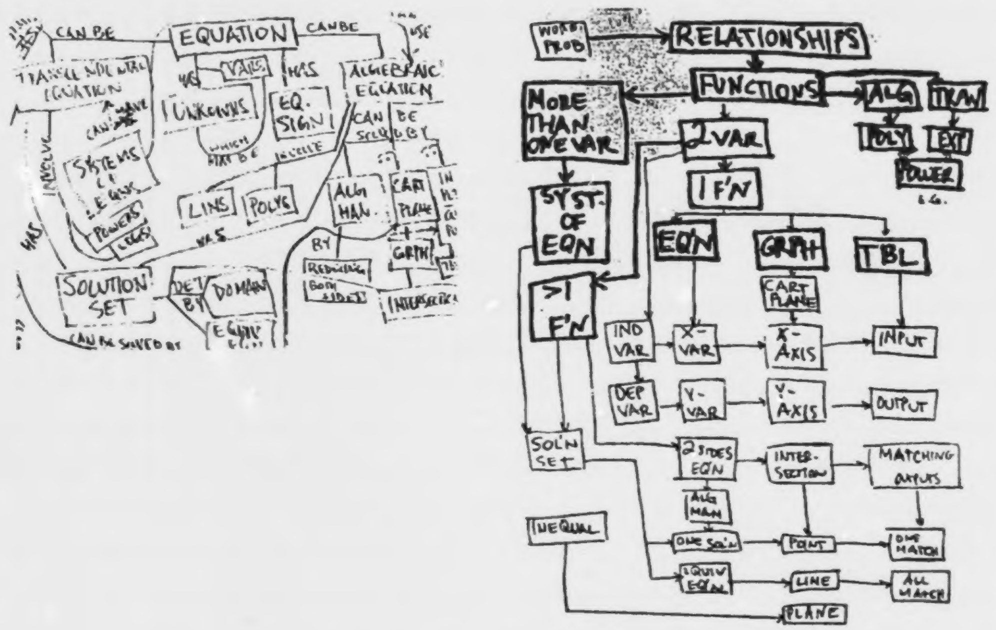
> Before it [solving equations] was, … much more of trying to manipulate and do operations to each side… Now I think of that more like this is one function this is another function they got to be equal somewhere.

However, when he solved equations, he seemed more comfortable operating in the "before" mode. For Bob $x^2/9 + y^2/4 = 1$ was an equation because it had an equal sign in it, an observation that fits his "before" mode. Then, shifting to his "now" mode, he discussed this equation in two variables as a comparison of two functions in two variables. At the same time, in discussing the graph of the solution set to this equation, he described the ellipse as "not necessarily a functional relationship." He concluded his reasoning about the similarities between this equation and an equation in one variable:

> Ah…it's still an equality um…this is in this has only one this [$2^x=x^2$] is really showing you only one variable this [$x^2/9+y^2/4=1$] is showing you two variables …It's [$x^2/9+y^2/4=1$] not necessarily a functional relationship except for again if we make that z it is in three dimensions.
>
> But, ah…similarly that their solutions would be values that if I'm solving this ($2^x=x^2$) I'm looking for the values and that makes this true. Solving this I'm looking for the values x and y that makes that true. So that will be similarity. So this [equation in two variables] is a curve and those ($2^x=x^2$) are curves.

Bob's fall concept map provides indications of two ways of thinking. First, an equation has unknowns that may be variables and it has an equal sign. But, it is not so easy to label his fall map as indicating an equations-based approach. For example, in discussing the map, he said:

> Both algebraic and transcendental [equations] can be solved by algebraic and symbolic manipulation, [by] looking at the Cartesian plane and [by] the graph and the intersection or tables of inputs and outputs... Solution set for either would be determine by domain and equivalent equations or [by] the intersection of the graph or where the tables ... have the same output.


fall

Spring

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EQN
GRAPH
TBL
>1 F'N
CART PLANE
IND VAR
X-VAR
X-AXIS
INPUT
DEP VAR
Y-VAR
Y-AXIS
OUTPUT
SOL'N SET
2 SIDES EQN
INTERSECTION
MATCHING OUTPUTS
ALG MAN
ONE SOL'N
POINT
ONE MATCH
INEQUAL
EQUIV EQ'N
LINE
ALL MATCH
PLANE

Figure 2: Bob's concept maps

Like his fall map, Bob's spring map includes both hierarchical and web-like components. In contrast, to the fall map, the definition that it provides for equations is less clear. Equation is under functions, along with graphs and tables. On the other hand, if one has "more than one function," it connects with "two sides of an equation." For him "An equation can frequently be made up of two functions then we can start talking about intersection points and solution set".

In the spring, Bob feels that both the organization and the content of his map have changed. Even with the ambiguity about the relationship between equations and functions, he spring map seems to reflect more explicitly Bob's "now" mode of thinking: "Here [in the spring], I started with relationships and functions, with equation as a part of function."

The rest of the spring interview supports these observations drawn from the concept map. When choosing a proposition that expresses a relationship between an equation and functions he, for example, said:

> I wouldn't agree with number 2: A function is a way to represent an equation. … Because I think of an equation as a subset of functions, as opposed to a function being a subset of equation… The function can be represented can be written out as an equation.

His interview comments revealed that he was referring to two views of equations. An equation can represent a function: "This [$y=3x-5$] is talking about a relationship: for any x value you are getting y values." This contrasts with his view of equations in one variable.

> I am trying to think, here [$x+2=3(x-1)$] we will be talking about two linear functions and we are looking at two linear functions and when does those two lines cross.

Equation in two variables bring forth a mixture of ideas:

> I can see that [5: $x^2+xy+y^2=12$] as … three variables three dimensions and then this is the value that you want I guess [if not in 3D] then I would have to refer back to what I said about 3 [$y+2=3(x-1)$] and 5 [$x^2+xy+y^2=12$] that this still representation a nonfunctional relationship. If x is supposed to be the independent and y the dependent or switch it around either way it is not a …function.

Discussion

Are there reasons that concept maps might be especially useful in understanding teachers' knowledge? One strength of concept maps is that they can indicate a conception of a concept as well as how this concept fits into a larger web, potentially revealing tensions in a teacher's thinking. For example, Bob's fall map suggests a definition of equation from his "before" mode and a web that has elements of his "now" mode.

Are concept maps limited in some principled ways when used to explore teachers' understandings? School algebra is a complicated domain to conceptualize. The teachers we interviewed knew different views (Bob's "before" and "now," Peter's own views and what appears in texts). There also is the question of how teaching appears in the maps. Does the way that teachers teach fit with how they think themselves? Which of these is being represented in the concept map?

As a result, we would certainly advocate the importance of discussion of maps with interviewees, as well as the combination of concept maps with other interview tasks. Without such supports, it seems very difficult to assess what one sees and to determine whether repeated use of concept maps over time reveals changes in perspective or representations of different aspects of a teacher's thinking.

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FROM A STUDENT INSTITUTIONAL POSITION TO A TEACHER ONE : WHAT CHANGES IN THE RELATIONSHIP TO ALGEBRA ?

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Abstract : *In this report, we present an on-going research about the professional development of pre-service secondary mathematics teachers. This professional development is analysed through their relationship with algebra. After presenting our theoretical frame and methodology, we describe the multidimensional grid of professional competencies in algebra that we have developed in order to analyse the professional relationships with algebra and their evolution. We face then our research hypothesis with the first results obtained through the analysis of questionnaires and interviews.*

I. INTRODUCTION

Various research studies have recently addressed the issue of teachers' professional development (see e.g. (Cooney, 2001) for a synthesis). Our on-going research belongs to this category. It focuses on the professional development of pre-service teachers (called PLC2) who, in France, have just passed the national competition called CAPES in order to become secondary mathematics teachers and are given one year of professional training in an IUFM (Institut Universitaire de Formation des Maîtres). Note that during this year of professional training, they also have one class in full responsibility, 6 hours per week. Research also focuses on one particular mathematical domain : algebra, as we hypothesise that reconceptualising mathematics in an important part of professional development, which includes changes in general epistemological views and changes in the perception of specific mathematical domains. The choice of algebra is motivated by the following reasons :

- algebra is a domain which poses severe learning difficulties and is the first step towards mathematics rejection for many students,
- algebra is a domain, all our PLC are concerned with as they have to teach algebra or pre-algebra in their full responsibility class,
- algebra is a domain which, a priori, they feel easy to teach. Algebraic thinking is for them a natural thinking mode and they have a great familiarity with the algebraic techniques they have to teach. Understanding pupils' difficulties requires thus from them a lot of efforts,
- algebra is a domain where naïve epistemological views are strongly inadequate,
- and, finally, this is a domain which has been extensively investigated by research. We have now a fairly good knowledge of the main difficulties pupils meet with elementary algebra, and research also provides engineering designs whose aim is

to tackle these difficulties more efficiently. These research results strongly inspire the short professional training they receive at the IUFM.

Through this research work, we want to better understand how the PLC's relationships with algebra change when they move from a student position to a teacher position, thanks both to the training they receive at the IUFM and their practical experience as teachers. We hypothesise that the different professional competencies which can be attached to algebra are not equally accessible, for reasons which have also to be understood and that a better knowledge of such differences could help to design more efficient pre-service training strategies. Conceiving efficient training strategies in the context of the IUFM is not an easy task, due to severe institutional constraints. For French secondary teachers, pre-service professional training is concentrated during the two years they spend at the IUFM. During the first year, they essentially prepare the very difficult national competition whose professional component reduces to the choice and presentation of a selection of exercises on a given mathematical theme, during the oral part of the competition. During the second year, the part of training officially devoted to the didactics of mathematics, different from one IUFM to another one, rarely exceeds 120h.

II. THEORETICAL FRAMES AND METHODOLOGY

Theoretical frames :

For this research, we rely on different theoretical frames. At a global level, we rely on the anthropological theory developed by Chevallard (1997, 1999) in order to analyse "mathematical and didactical organisations", the associated "mathematics praxeologies" (in terms of tasks, techniques, technologies and theories), and also the mathematical professional work of the teacher via its different "gestures". As regards the analysis of the teacher, we also refer to the frame developed by Robert (1999), relying on research in ergonomy in order to give account of the functioning of the teacher, seen as an actor in a "dynamical open environment". These theoretical frames lead us to hypothesise that professional development is a very complex process, that its understanding requires a multidimensional analysis taking into account and connecting the different professional gestures of the teacher, both in and out of the classroom. Of course, we also rely on educational research in algebra and especially on synthesis such as those produced by Bednarz, Kieran and Lee (1996) or Grugeon (1995). We hypothesise that professional expertise in algebra is a complex mixture of competencies specific to this domain and more transversal competencies which strongly intertwine in professional practices.

Methodology :

Taking into account our "problématique" and the nature of our hypothesis, we have chosen a qualitative methodology based on the triangulation on multiple sources of data and analysis which allow to take into account the multidimensionality of professional competencies and professional development as a process. It includes:

- questionnaires and interviews with students just before they take the CAPES competition, that is to say at the end of the first year at the IUFM,
- observation of the training process organised by the IUFM for the PLC2 as regards algebra,
- questionnaires taken by the complete cohort of PLC2 at the beginning and at the end of the academic year,
- and, most important, the following up of a group of PLC2 (6 in 1998-99 and 6 in 1999-2000), selected among volunteers, through regular interviews, videotaped classroom observations, collection of teaching material (personal teachers notes, assessment tasks, selected students' copy-books excerpts and productions).

Moreover, in order to analyse the whole data, we have built a methodological tool: a multidimensional grid of professional competence in elementary algebra (MGPCA), thanks to the results and categories coming from the research work used as a theoretical or experimental reference. We briefly introduce this grid in the next paragraph.

III The multidimensional grid of professional competence in elementary algebra

As stressed above, we consider that professional competence has to be analysed in multidimensional terms. It can be modelled as a multivariate function which shapes the decisions the teacher takes in his (her) different professional gestures, the way (s)he faces unforeseen situations, the discourse and analysis (s)he develops at a more reflective level. This professional competence, as regards algebra, relies both on transversal and specific competencies. The MGPCA focuses on the specific competencies and tries to describe potential underlying knowledge. We hypothesise that such knowledge influences the different professional gestures in a non-uniform way and that it cannot necessarily be made explicit. One ambition of the research is to explore the complexity of relationships between knowledge and competencies, to try to find the real influence professional algebraic knowledge plays with respect to other determinants of PLC2's behaviour, and to evidence some regularities which could help to better understand teacher's behaviour and improve training strategies. The grid is structured around three non-independent dimensions : the epistemological, the cognitive and the didactic ones. In the following, we synthesise the contents of knowledge which structure the grid according to each dimension.

The epistemological dimension :

Epistemological knowledge in the grid, is structured around, on the one hand, some important features of the historical development of algebra, on the other hand, the distinction between the "tool and object" dimensions of algebra (Douady, 1984).

As regards the first point, one essential epistemological characteristics is the complexity of the algebraic symbolic system and the difficulties of its historical development. Such knowledge can help to understand the difficulties met by present pupils. Another important point which arise from historical development is the extension and diversity of the algebraic domain. Such knowledge can support the change of epistemological views about algebra, allow to better understand the rationale for the progression in algebraic knowledge organised by the curriculum and eventually discuss this. We conjecture that, at the beginning of the academic year, our PLC2 are unaware of this complexity and tend to reduce algebra to the algebraic structures and theories they have been taught at university.

As regards the second point, the distinction between the tool and object facets of algebra allows to structure algebraic knowledge around two perspectives: a tool perspective where algebra is considered as a set of tools allowing to solve different kind of problems internal or external to the mathematics field, an object perspective where algebra is considered as a structured set of objects with specific properties, semiotic representations, treatment modes, which are studied for themselves (Grugeon, 1995).

We conjecture that PLC2, at the beginning of the academic year, mainly see algebra as a tool for solving problems which can be modelled in terms of equations and that, as teachers, they will tend to over-emphasise the work on algebraic techniques.

The cognitive dimension :

This component deals with potential professional knowledge about learning processes in algebra. We have organised this part of the grid around three main points linked to resistant learning difficulties evidenced by didactic research: the relationships between arithmetic and algebra, the symbolic system of algebra and the relationships between different semiotic representations used in algebra, the role algebra can play in the development of mathematical rationality. For this part of the grid, we especially rely on the synthesis made by Grugeon (1995) in her attempt to define a multidimensional grid of analysis of pupils' competencies in elementary algebra.

Algebraic knowledge builds on arithmetic knowledge but also against it. Algebra, when introduced, shares with arithmetic many objects and symbols but these have to take new meanings : for instance, equality has to become an equivalence sign in order to allow algebraic manipulations, letters have to take several different new meanings. Beyond that, arithmetic and algebraic thinking modes are of a very different nature, the first one belonging to the synthetic mode and the second one to the analytic mode. Understanding algebraic thinking modes also imposes semantic changes. Algebraic work contrarily to arithmetic work cannot only rely on external semantics, semantics has to take also an internal dimension proper to algebraic expressions. Other kind of knowledge deals with the specificity of the symbolic system of algebra, already evoked in the epistemological part, and the resistant difficulties it induces (for instance those resulting from the lack of closure). Learning difficulties can also be linked to the necessary interaction in algebraic work, as in any mathematical work, between different semiotic registers, especially here: the symbolic register, the natural language register, the numerical register and the graphical register, and the difficulties of conversion between non-congruent semiotic representations. Finally, we integrate in this cognitive part, knowledge about the role algebra can play in the construction of students' mathematics rationality.

As regards this cognitive dimension, we conjecture that, at the beginning of the academic year, PLC2 are not aware of the learning difficulties evoked above, but that, through their practice and discussions with pairs, they soon become sensitive to most of these, even if they are not able to interpret them in coherent and analytic ways as research allows to do, and thus react efficiently. As regards rationality, we conjecture that, due to the general educational French culture which over-emphasises the links between rationality and geometry at the expense of any other one, integrating this pole of knowledge will result more difficult.

The didactic dimension :

Knowledge relevant to these two first dimensions certainly influences the didactic and mathematical organisations, the teachers develop. But these are also shaped by what we will call here more specific didactic knowledge: knowledge of the curriculum, of the specific goals of algebraic teaching at a given grade, of possible progressions and activities for the teaching of algebra compatible with these and well adapted assessment tasks, knowledge of educational resources: textbooks but also publications from the IREM (Instituts de Recherche sur l'Enseignement des Mathématiques), websites (especially as regards the use of computer tools such as spreadsheets for the teaching of algebra), etc.

IV. SOME PRELIMINARY RESULTS

The first results we present here result from the analysis of questionnaires and interviews. In the near future, we will face them with the detailed analysis of the different elements of the PLC2's following up. This will allow us, we hope, to better understand the possible relationships between systems of knowledge and competencies. We summarise these results according to the three dimensions of knowledge structuring the MGPCA.

The epistemological dimension :

The first analysis tend to confirm the conjectures made about the initial PLC2s' state. Nevertheless, the training at the IUFM seems to easily destabilise the vision of algebra as a domain reduced to the field of algebraic structures and theories. For most students, boarders of algebra tend to blur or to become questionable, for some others a duality seems to install between two points of view: the mathematical and the educational, resulting in two distinct conceptions of algebra which coexist. As regards the diverse facets of algebra and the tool/object dimensions, the object dimension strongly predominates and this priority tends to resist to the training. Their initial vision of the different tool facets of algebra is limited and integrating a functionality of proof for numerical properties seems specially difficult. The strength of the cultural obstacle is especially visible here.

The cognitive dimension :

As conjectured, initially, the PLC2 are not aware of the difficulties of the transition between arithmetic and algebra. The short training they have (6h) seems to make them aware of the differences between the arithmetic and the algebraic processes for solving numerical problems. But they tend to see the algebraic method as the "right method" safer, simpler and more rigorous, which has to take the place of arithmetic methods, more intuitive and, at the same time, requiring more astuteness. As one of them expresses at the end of the year: "An arithmetic approach is more a reasoning out of standard ways. It is less framed. One cannot really say that there exists a method". PLC2 teaching grade 8, where the solving of problems which can be modelled by first grade equations is an important point in the syllabus, seem specially sensitive to this difference. Nevertheless, even at the end of the academic year, deeper interpretations of the arithmetic / algebraic cut are not spontaneously evoked by students, at the exception for some of them of situations where they are asked to interpret students' errors.

As regards the symbolic system of algebra, students are very soon aware that they have to face resistant learning difficulties. Some of these are helped by previous experience of giving private remedial courses. But we notice evident differences in the way the PLC2 interpret these difficulties and in the way they tackle these. Once more, the grade they teach in full responsibility seems to have an evident influence on their perception. Knowledge related to the internal semantics of algebraic expressions and the role this knowledge plays for piloting and controlling algebraic work seems the most difficult to internalise. What predominates in algebra for them is the existence of formal rules for transforming expressions and equations, and the algorithmic character of most elementary algebraic practices. This doesn't help them to pay the necessary attention to what becomes crucial in more complex algebraic computations. Once more, we see here the strength of a cultural obstacle equalling algebraic computation to computation without intelligence. As regards the connections between different semiotic registers, they identify difficulties in the mutual conversions between the symbolic register and the natural language but are not necessarily able to analyse these in more technical terms.

The didactic dimension :

As regards the building of a progression, textbooks and the local advisor the PLC2 have in the high school where they teach in full responsibility have a predominant influence. Generally, the PLC2 don't meet difficulties when deciding the different points they want to address, but they have more difficulties at identifying the real aims of the teaching of pre-algebra or algebra, at the grade level corresponding to their class or at foreseeing the time they will have to spend on such or such chapter. As expressed by one of them in the final interview: "At the beginning of the year, I tried to build a progression but… I don't succeed in knowing how many time I will stay on one notion. So, now I mainly try to see in what order and with what spirit I want to approach the notion".

The PLC2 also have some difficulties at taking into account the initial state of knowledge of their students when they come to teach algebra. Such a difficulty is stronger, as can be easily foreseen, at the beginning of the academic year. As expressed by one PLC2 in the first interview: "I have to say that I didn't give a lot of exercises on this topic because I thought that it was well known. At the beginning of the year, I wanted to arrive very quickly to algebraic developments and factorisations. The remainder was only matter of recall. I realised too late that it didn't work. I didn't choose to go back, but it was perhaps a wrong decision. My problem is that for some of these students, I can explain one thousand times, that doesn't change anything. I don't know how to do". This excerpt shows the PLC2 becoming aware of the problem, doubting about the adequacy of his initial strategy and, at the same time, we can understand up to what point he is deprived from means in order to tackle the problem. Visibly, spontaneously, he remains in a state of global approach, insufficient to analyse real students' difficulties and needs, and conceive appropriate educational strategies, beyond the explanatory strategy which is the sole to be evoked.

As regards the mathematical and didactical organisations, at a more local level, textbooks once more play an essential role, but most PLC2 don't only use the textbook selected by the high school for their class. Visibly, the work they did the year before, when preparing the CAPES, which obliged them to search and select exercises on specific mathematical themes by looking at different textbooks has created some "habitus". Two great tendencies emerge: in the first one, the PLC2 tries to cover extensively the diversity of types of tasks proposed by the textbook, this result in very few examples for each type ; in the second one, the PLC2 focuses on some selected tasks and one can see, through the progression along the exercises of the same type, a progressive attempt to play on specific didactic variables, even if this expertise mainly remains implicit. The same diversity occurs as regards the relationships between techniques and technologies (this word being taken here with the Chevallard's acception of discourse explaining or justifying a technique). Some PLC2 mainly stay at the technical level (including of course the description of the technique), other give real importance to the technological dimension. But, even at the end of the academic year, the elaboration of technological discourse seems to remain under their sole responsibility. The different data coming from the questionnaires and interviews show that there is a sensible evolution during the academic year: the lesson conception evolves, the relationship to assessment evolve, but each evolution seems to have its idiosyncrasy. As expressed by one PLC2: "Let us say that, every time, I have tried to ask myself more questions before introducing something new. A sort of a priori analysis. I ask myself what will be the students' difficulties. But it is difficult to foresee. And also, how to introduce that new thing. At the beginning of the year, I delivered the course more directly, now I always try to do some activity which prepares the notion". The interviews show the difficulties they have at analysing classroom situations, but we notice a great difference between the analysis they are able to produce personally and the analysis they are able to produce when working in small groups on the same activity designed for students or on classroom videotapes.

V. DISCUSSION

In this research project, we try to better understand the change in the relationships with algebra of French pre-service teachers, moving from a student position to a teacher position, and the development of an initial professional expertise in that area. We consider this development as a muldimensional and complex process, involving epistemological, cognitive and didactic changes. We are perfectly aware that what can be reached through this first year of professional training is necessarily very limited. These evident limitations make all the more important to detect possible germs for priming professional development. The results we have obtained up to now are certainly very partial but they clearly tend to show that some interesting and subtle evolutions can take place. All of these don't directly affect the design and management of classroom situations. At a first level, they seem more able to express in a priori and a posteriori analysis of classroom sessions, and more in a collective way than in an individual way. Results also show important differences in the accessibility of the respective parts of what can be considered today as professional expertise in algebra. We would like to add that the influence of the short didactic training offered by the IUFM will remain anecdotal if it is not properly echoed by a reflective analysis of PLC2's practices in the class they have in full responsibility. This is, both from an experiential and affective point of views, the fundamental object in the transition they live. There is no doubt that for the PLC2 who have been selected for the following up, we are in some sense in the optimistic conditions, taking into account the strong institutional constraints of this pre-service training.

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ANALOGY IN THE EXPLORATION OF REPEATING PATTERNS

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Problems that require examination of repeating patterns were presented to a group of 106 preservice elementary school teachers. A theoretical framework of reasoning by analogy was used to analyze students' responses. It was found that not only relational structure of problems but also their relative computational complexity influence students' ability to successfully map between familiar and unfamiliar tasks. Consideration of multiples and division with remainder were the main mathematical tools utilized in students' solutions, but in many cases students didn't demonstrate their awareness of the relationship between these tools. It is suggested that pedagogical approach should help students connect their understanding of multiples with their understanding of division with remainder and contribute to a more complete understanding of repeating patterns.

Patterns are the heart of mathematics. Students' ability to recognize or develop patterns is related to their ability to reason mathematically in general and to develop reasoning by analogy in specific (White, Alexander & Daugherty, 1998). In this study we examine students' strategies as they engage in tasks that invite exploration of repeating patterns in familiar and unfamiliar situations. What mathematical tools are utilized? Is there consistency in students' approaches? What influences their choice of a strategy? -- These questions are of interest in this investigation.

THEORETICAL FRAMEWORK: REASONING BY ANALOGY.

Analogical reasoning was explored generally in the context of science and cognitive science. Despite apparent relationship between analogical reasoning and mathematical learning (White, Alexander & Daugherty, 1998), only recently researchers in mathematics education became interested in reasoning by analogy in specific mathematical contexts. Reasoning by analogy in mathematics was explored mostly in considering word-problems in arithmetic and beginning algebra (Novick, 1995, English, 1998).

According to English (1998) reasoning by analogy in problem solving involves mapping the relational structure of a known problem (that has been solved previously, referred to as "source") onto a new problem (referred to as "target") and using this known structure to help solve the new problem. Analogy can be recognized in isomorphic or "almost isomorphic" problems. In our interpretation, isomorphic problems, also denoted by English as "completely isomorphic", are those possessing the same structure not only from a mathematical, but also from a linguistic perspective. They can be presented in (i) the same situational setting, or in (ii) different situational settings. We refer to problems as "almost isomorphic" when they have similar components, but are not "completely isomorphic", that is, (i) they possess the same mathematical, but different linguistic structure, or (ii) a source problem can be seen as isomorphic to a part of a target problem or vice versa.

Mapping the structure of the source problem to the target problem is the key aspect of reasoning by analogy. In order for a mapping to take place a relational structure of the source problem must be identified, as well as similar relational structure of the target problem. English (1997) suggests that failure to identify the relational structure of the source problem is a significant cause of children's impaired mathematical reasoning. Furthermore, successful solution of a target problem is not an immediate and natural consequence of a successful mapping (Novick, 1995). The solution procedure may need to be adapted or extended, especially in case of problems that are not completely isomorphic. Successful mapping is essential for recognizing the need for such adaptation. In summary, solution by analogy requires recognition/identification of a similar relational structure, mapping between source and target, and adaptation or extension. Each step presents its own challenges.

METHOD

Tasks

The participants were presented with three problems and in each case they were asked to explain and justify their solution.

1. What is the 177th digit to the right of the decimal point in the decimal representation of $0.\overline{76543}$?

2. A toy train has 100 cars. The first car is red, the second is blue , the third is yellow, the fourth is red, the fifth is blue and sixth is yellow and so on.

(a) What is the color of the 80th car?
(b) What is the number of the last blue car?

3. Imagine a toy train with 1000 cars, following the 7 colour repeating pattern

1- red, 2 - orange, 3 - yellow, 4 - green, 5 - blue, 6 - purple 7 - white

(a) What is the color of the 800th car?
(b) What is the number of the last blue car?

Each of these problems presents, either explicitly or implicitly, a repeating pattern (of length 5, 3 and 7 respectively) and asks to determine the attribute (digit or color) of this repeating pattern that corresponds to a particular place. A possible approach to the solution of these problems is to identify the *remainder in division* of "place" by "length" and assign the attribute that corresponds to the place of this remainder. For example, in question 1, the *remainder in division* of 177 by 5 is 2, and therefore the attribute corresponding to the 177th place is the same as the attribute corresponding to the second place, in this case digit 6. Another approach is to notice that every 5th digit is 3, therefore digit 3 appears in every place that is a multiple of 5; 175 in particular. Then the attribute of the 177th place is determined by *counting up from a multiple* . Mathematically, the two approaches described above are equivalent, however, as suggested by our data, they present a different cognitive difficulty and have a different explanatory power.

Participants and Procedure

Written responses were collected from a group of 106 preservice elementary school teachers, enrolled in a core mathematics course. It should also be noted that tasks similar to question 1, that is, consideration of digits in a repeating decimal pattern were discussed in class. Consideration of the remainder was among the possible solutions presented and examined by the participants. Therefore, question 1 presented a familiar situations, while questions 2 and 3 presented situations that can be describe as unfamiliar. Strategies used by participants to solve each problem were identified and recorded. The partial data analysis presented below focuses on inconsistencies in students' approaches when dealing with analogous situations. In analyzing students' solutions the researchers were interested in students' mapping between isomorphic problems as well as mapping and adaptation in cases that were not completely isomorphic.

RESULTS AND ANALYSIS

Isomorphic Mapping

(i) Same situational setting.

This mapping is represented within our instrument in the structural relationship between questions 2a and 3a (or 2b and 3b), where the only differences are in the numbers. Both the target attribute and the pattern length are larger in problem 3 than in problem 2. Most participants were successful in this isomorphic mapping. In most cases (79 of our of 106) the strategies presented by participants in problem 2 were exactly the same as the strategies chosen in problem 3. However, simply varying the magnitude of the numbers in the situation was enough to prompt 27 of the participants to change their strategies when going from one problem to the next. Carlo, for example, in dealing with problem 2a chose to attend to the *remainder in division* by 3 (the pattern length).

> *Q2a. When 80 is divided by 3 the remainder is 2 that means the 80th car will be blue because that is the colour of the 2nd car.*

However, when faced with the same situation in 3a he changed his strategy and *counted up from a multiple* of 7, the pattern length.

> *Q3a. Every 7th car is white, so car 798 is white because 7x114=798. Then 799 is red and 800 is orange.*

Why wasn't his strategy in 2a successfully mapped to 3a? A possible explanation is in the relative computational complexity of the task. While the *remainder in division* by numbers 3 can be easily identified mentally considering divisibility tests, the case of division by 7 is different. Therefore, Carlo's work can be seen as an attempt to relate the problem to pieces of easily identified components of prior knowledge, considering multiples of 7 as a point of reference and counting up from that point.

It is interesting to note that although division of whole numbers results in quotient and remainder, only the remainder provides useful information for these questions. Although many students felt comfortable discussing the *remainder in division* without pointing to the quotient, there were 12 participants who changed from this very consistent strategy. They felt compelled to explicitly state both outcomes of the division statement only when dealing with the larger numbers in question 3. This is further evidence supporting the claim that computational complexity of isomorphic problems is enough to prompt students to change their strategies.

(ii) Different situational setting.

Although question 1, and 2a have identical mathematical and linguistic structure, they present the student with a different situation. The variation of the problem from finding the 177th digit to finding the colour of the 80th car was enough to cause 21 of our students to change their strategy for dealing with the new situation.

Amanda displayed good use of the *counting up from a multiple* in question 1, but opts for *remainder in division* in problem 2a.

> *Q1 Since the repeating portion is 5 digits long, then every 5th digit will be a 3. So, 175 [digit] will be 3 and 177 [digit] will be a 6.*
>
> *Q2a. 80/3 = 26r2. Since the remainder is 2 then it will be the same colour as the second car... blue.*

Both of these strategies are equally effective. So effective, in fact, that it is surprising that Amanda would change from her initial one. Perhaps this is due to a refinement of her strategy as she maps from one problem to the next, as *remainder in division* can be seen as an improvement of the strategy of *counting up from a multiple.* Of the 21 participants that showed an inconsistent mapping in the face of a situational change, only six could be attributed to a refining of strategies. The other 15 produced blaring inconsistencies in their failure to map their procedures from one situation to the next. This is demonstrated in Jason's solution.

Q1. 177/5 has a remainder of 2. Therefore, the 177th digit will be 6.

Q2a. 80 is divisible by 2, and since the 2nd car is blue, the 80th car is blue.

Jason's strategy in 2a referred to as *multiple of the initial position*, will be discussed in a later section.

Almost Isomorphic Mapping

(i) Different linguistic structure.

Our instrument did not provide data representative of this type of mapping.

(ii) Target and source are partially isomorphic.

Problems 2b and 3b, are isomorphic to each other, but present a slight variation on 2a and 3a respectively. Problems 2a and 3a ask for an attribute that corresponds to a given place, while 2b and 3b ask to determine the largest place that has a given attribute. We consider the pairs 2a and 2b, 3a and 3b as almost isomorphic. An attempt to map a new problem (3b) to a familiar one (3a), is demonstrated in Cindy's work.

Q3a. 800/7 = 114 with a remainder of 2. So, the 800th car will have the same colour as the 2nd car. Orange.

Q3b. 1000/7 = 142 with a remainder of 6. Blue cars have a remainder of 5, so it must be car 999.

We consider this refinement to be an adaptation consistent with the *remainder in division* by the pattern length mentioned earlier. Alternately, the strategy of *counting up from a multiple* can be adapted to target a specific attribute (in this case colour) rather than a specific position, as exemplified by Bill.

Q3a. 798 is a multiple of 7, so it will be white. 799 is red and 800 is orange.

Q3b. 994 is a multiple of 7 (994 = 142 x 7). So 995 is red, 996 is orange, 997 is yellow, 998 is green, 999 is blue.

Of our 106 participants, 29 of them didn't rely on the "almost isomorphic" structure of the two problems and thus did not map their strategies from problem 2a to problem 2b (or 3a to 3b). In 21 of these cases this involved going from a *remainder in division* process in 2a to a *counting up/down from a multiple* in 2b; an inconsistency that can be viewed as a hierarchical regression in the development of students' schemes (Zazkis & Liljedahl, 2000).

Errors in mapping

In this section we mention several errors, that were more than single occurrences and therefore require attention.

Recognition of a repeating pattern involves recognition of its length as well as its attributes. Five students consistently identified incorrect length of these patterns. In each case the identified length was one more than the actual length. It appeared that they considered the place where the digit (or the color) repeated for the first time, and this served as a reference. Writing out the repeating pattern several times may help clarify this confusion.

Another error was caused by incorrect correspondence when considering multiples. In question 1, for example, searching for the 177th digit to the right of the decimal point in the decimal representation of $0.\overline{76543}$, Sally correctly recorded that

> *Q1. Every fifth place had the same digit, and therefore a digit in a place that is a multiple of 5 is always the same.*

However, this digit was identified as 7, which is the first digit in the repeating pattern, rather than the last one. So her reasoning continued as follows:

> *Q1. So 175th digit is 7 and 177th digit is 5.*

Sally, as well as three of her classmates consistently applied this incorrect mapping between the place and the attribute. Requiring students to list the correspondence between digits/colors and places explicitly should help in identifying the correct correspondence.

While the above mentioned "bugs" appear not difficult to "debug", the reasoning Jason used (discussed earlier) hints at a more profound misunderstanding.

> *Q2a. 80 is divisible by 2, and since the 2nd car is blue, the 80th car is blue.*

In this reasoning, Jason is assuming that every multiple of the initial position of the blue car shares that attribute. Monica, expressed the same reasoning even more explicitly.

> *Q3b. Since the blue car is in the 5th position all multiples of 5 will be blue.*

This logic remains a mystery to us. At first occurrence we were ready to dismiss such reasoning suggesting that these students memorized the ingredients that played a role in solving similar problems - multiples and remainders - and then mis-assembled them in their responses. It could be that students incorrectly generalized the observation that "every seventh car is white" (a statement which is true for our repeating pattern of length seven) to "every fifth car is blue" in the same repeating pattern. However, the fact that consideration of a *multiple of the initial position* occurred in responses of 15 different students, requires further investigation of this line of reasoning in future research.

CONCLUSION AND PEDAGOGICAL IMPLICATIONS

Reasoning by analogy follows the sequence of recognition–mapping–adaptation. We suggest that isomorphic mapping of a problem does not necessarily imply isomorphic mapping of a solution. Solution depends on computational complexity and often patterns with small and compatible numbers are not easily generalized. Therefore, adaptation may be required not only for problems that are not completely isomorphic, as suggested by English (1998) and Novik (1995), but also for completely isomorphic problems of different degree of computational complexity.

Although the claim that, for example, *996 leaves a remainder of 2 in division by 7* and the claim that *996 is larger by 2 than a multiple of 7* communicate the same mathematical relationship, these claims are not equivalent in terms of their explanatory power for learners. Consideration of *remainder in division* presents a unifying scheme that enables a student to deal with a large class of problems, including all of the problems in our instrument. However, students' ability to generalize this strategy and apply it for various questions is not well developed. When faced with unfamiliar situations students often prefer to invoke *counting up from a multiple* strategy. We found in these data, as well as in-class interactions among students, that a *distance from a multiple* is a reasoning that is applied by participants in a more spontaneous and natural way. This slip back towards the familiarity of multiples is an indicator of the lack of recognition of the invariant multiplicative structure inherent in division with remainder statements.

Our recommendation is that pedagogical approach to division with remainder should build on what appears as a natural tendency of students - consideration of multiples. Capitalizing on multiples, remainders can be considered as "adjustment" or "translation" along the number line. This approach would enforce the view of a remainder as indicator of partition of whole numbers, rather than simply an outcome of an operation, and empower the understanding of repeating patterns as well as the relationship between additive and multiplicative structures.

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THE DECA-BASED APPROACH: A MATHEMATICAL PROGRAM TO LEARN AND APPLY TWO-DIGIT NUMBERS

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This study examines the effects of an instructional program designed to promote students' understanding of two-digit numbers. The program enables students to identify the tens and ones in two-digit numbers and to add two-digit numbers of different levels of difficulty. One student was taught, using a multiple baseline experimental design to track the learning associated with teaching procedures during the process of schema acquisition. Results show marked gains in both the understanding and addition of two-digit numbers, generalization and expansion of these concepts in the classroom and at home, and a more positive attitude.

Introduction

This study, which is part of larger investigation, concerns one child, Jim, who at age 10 had still not learned to identify tens and ones or to add two-digit numbers. It uses a methodology rarely seen in mathematics education, although common in applied behavior analysis. This methodology tracks a student's learning with precision, so that the effects of teaching are clear.

The study is based on the theory, proposed by Fuson (1990), that children use particular schemas in developing whole number concepts, especially for multidigit numbers. She suggested that children begin with a unitary conceptual structure that links cardinal meanings of spoken and written numerals (five, 5) to perceived objects (5 dots). They extend this unitary conception to numbers over 10, in which the separate numerals (e.g. 15) have no quantitative referents in themselves. They then move on to develop further conceptual structures to accommodate the meaning of a decade with extra ones, a decade-ones conception. This is followed by two further conceptions, a sequence-tens and a separate-tens-and-ones conception. In a sequence-tens conception, single units of ten are formed within the decade part of the quantity. This is an extension of the unitary counting except that the sequence is now counted in units of tens (e.g. 10, 20, 30). The separate-tens-and-ones conception is built through experiences in which a child comes to think of a two-digit quantity as comprising two separate kinds of units -

units of ten and units of ones. Both kinds of units are counted by ones (e.g. fifty three is 53 - 1,2,3,4,5 tens and 1,2,3 ones) but the units of ten means that each digit in that position comprises a "ten-unitness". With this conception, the child understands that each ten is made up of 10 ones and can then switch to the unitary thinking of ten ones if that approach is required. This last conceptual level is required for additive decomposition as used by Gravemeijer, McClain and Stephan (1998).

A child's construction of the sequence-tens and separate-tens conception depends heavily on their learning environments, including that of linguistic structures (e.g. Fuson & Kwon, 1992; Miura & Stigler, 1987). Some languages are seen as having direct and systematic structures for labeling multi-digit numbers. For example, the Chinese number word for 51 is "five ten one". The direct correspondence of the spoken number word to the written numeral makes it easier to match the two forms of the number. This linguistic structure moves the learner directly into the separate-tens-and-ones conception. Combined with other cultural factors and experiences such as being taught with a strong emphasis on rote counting, this schema is constantly being reinforced for the learner. English has a much less regular structure, particularly from 11 to 19.

Fuson and colleagues trialed a teaching program to facilitate the separate-tens-and-ones conception with two groups of six-year-olds in a lower economic district (Fuson, Smith & Cicero, 1997). Results suggested that the program helped raise the performance of the participants to be substantially above that reported in other studies of children of higher SES and for older children. Their responses were reported to look more like those of East Asian children than of U.S children in other studies. The theoretical basis of this study was expanded for use with two-digit numbers (Fuson, Wearne, et al., 1997).

This case study builds on the theoretical framework of Fuson and colleagues. It follows the learning of one boy who started with a unitary understanding of two-digit numbers and follows his progress through to the addition of two-digit numbers using a separate-tens-and-ones conception. The question behind this research was whether or not an intervention program based on this theoretical framework could be shown to help a slow learner develop these conceptual structures for use in two-digit addition.

Methodology

Jim was a ten-year-old boy enrolled in a suburban primary school. He was referred by the school's special needs coordinator because of difficulties with his learning. The class teacher reported that he was unable to identify the tens and ones in a two-digit number even after an intensive year of remedial mathematics with a teacher aide. In a pre-intervention assessment with the, he gained a score on the Basic Skills portion of the KeyMath Diagnostic Test (Revised) of 5 years 10 months. An assessment with the WISC-III placed him on the 5th percentile with an overall score of 70-82.

The initial focus of the instruction was identifying the tens and ones in a two-digit-number (Fuson's decade-ones structure) as the boy had serious difficulties in this area. To accomplish this, teaching emphasized the use of the "deca-language" as well as the tens and ones value in a given two-digit number. In the created deca-language, numbers are named as in Chinese, so that 11 is called "ten one", and 71 is "seven ten one". Instructional charts associated with the deca-language program were prepared to provide him with an understanding of the separation-tens-and-ones concept. These are similar to standard 100s charts except that the single digits are at the bottom, so that it read, in part:

20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19
00	01	02	03	04	05	06	07	08	09

A multiple baseline across behaviors design (Barlow & Hersen, 1984) was used to examine the effects of the intervention on different areas of mathematical performance. This design clarifies the relationship between instruction and performance on measures that are not directly part of the instruction: in this case on 13 dependent variables used to assess this learning. The first set of three of these measured Jim's ability to identify two-digit numbers, the second set of six measured competence in horizontal addition, and the third set of four measured generalization in a different setting as well as vertical addition.

The researcher conducted all sessions during class time, three times a week for 8 weeks, in a quiet room within the school compound. Each session lasted about 30 minutes including the testing which followed the individual teaching.

Intervention

There were six phases in the intervention. These were a baseline phase, when initial measures of his performance were found through tests were administered during the session, four instructional or treatment programs and a maintenance phase when his performance was assessed but he received no special teaching. Treatment 1 (T1) used the deca-language to reinforce the separation-tens-and-ones conception. The theory was taught using different sets of charts as the basis of discussion. As Jim had demonstrated the ability to read and write two-digit numbers in English during baseline, the instruction focused on the ability to differentiate the tens and ones in a given two-digit number, using deca language. Treatment 2 (T2) focused on the addition component of the program. Again, the treatment was streamlined to meet Jim's needs as identified from baseline data. Therefore, there was particular emphasis on the addition of numbers 6-9 and of multiples of ten as the second addend. Treatment 3 (T3) focused on the addition of two-digit numbers as the second addend, both with and without regrouping. Treatment 4 (T4) focused on adding by visualizing a chart without it being present. Each treatment included a practice phase with a number of different questions that involved the use of the theory that had been discussed.

Figure 1 shows the number of test items that Jim completed successfully in each of these phases. The introduction of T1 was associated with an increased performance in reading and writing two-digit numbers in the deca-language. Results for the ability to read a given number in the deca-language (Deca read) show that he moved from a mean of 20% correct during baseline to 80% with the introduction of treatment. Similarly, Jim moved to 93.3% for writing a given

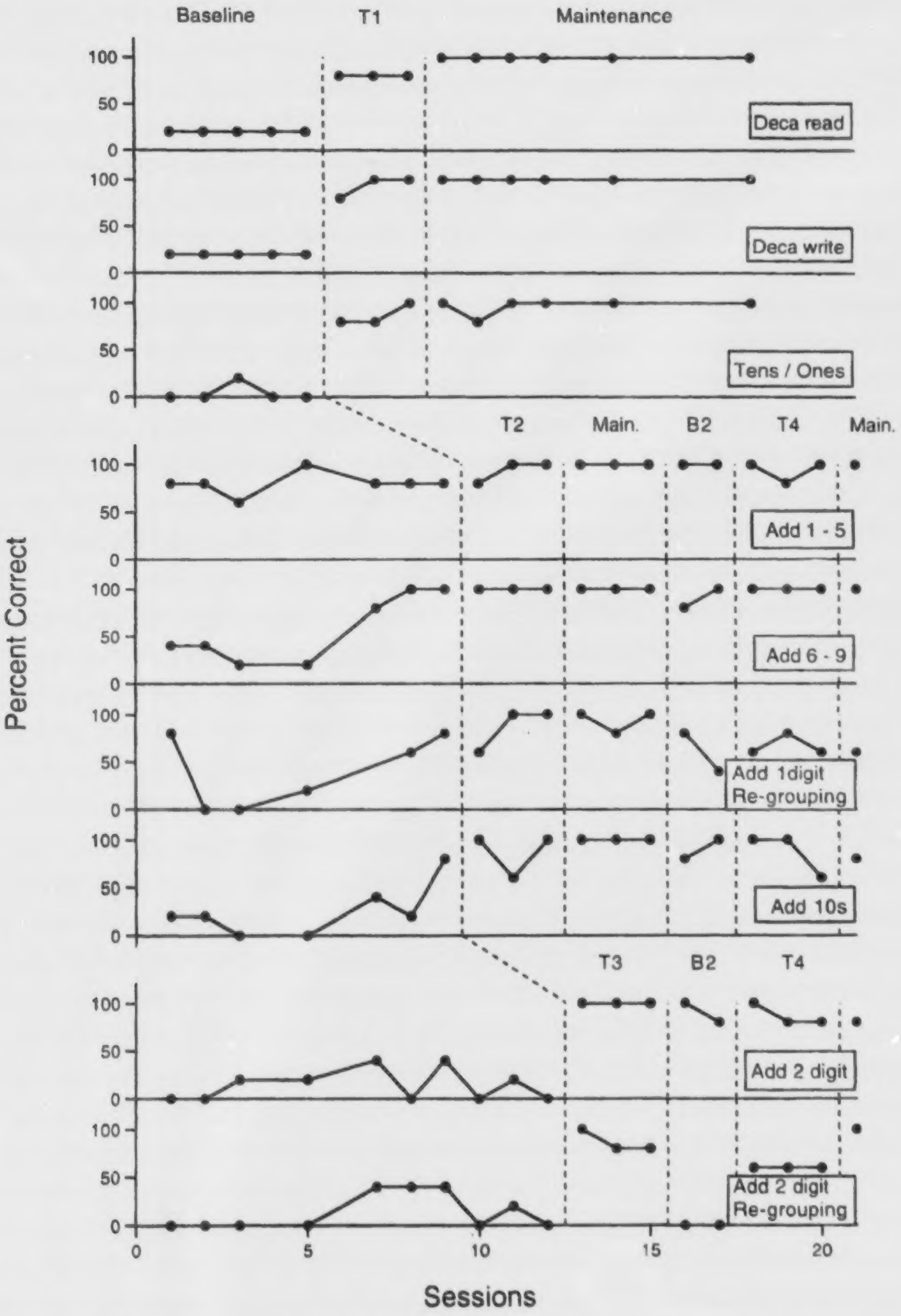

Figure 1. Percent correct of test scores for different measures across experimental phases

number said in the deca-language (Deca write). These scores improved further to 100% for both skills during maintenance and were still evident 4 weeks later.

With the introduction of T1, similar improvement in test results was obtained for the ability to state the tens and ones of a given two-digit number. During T1 Jim moved from a mean of 4% to 86.7% for the ability to state the tens and ones of a given two-digit number. These scores improved further to 100%.

For the next set of six measures, baseline data on addition suggest that Jim was able to add numbers 1-5 with reasonable accuracy. This performance was maintained throughout the treatments and maintenance. There was an improvement in performance for adding numbers 6-9 with the introduction of T2, from a mean baseline of 60% to 100%. This level of performance (100%) also occurred during maintenance. Although addition was not targeted in T1 the test scores suggest that T1 has had an impact on the adding skills of single digit numbers. This was also true for addition of multiples of ten, where addition moved from a mean of 25% (Sessions 1-5) to 66.7% (Sessions 7-9).

Treatment 2 also brought about a further improvement of test performance for the addition of single digit numbers with regrouping. Scores improved from a baseline of 42.8% to 100% during treatment, with a score of 93.3% during maintenance. For the addition of multiples of ten, the introduction of T2 was associated with an increase in performance from a mean of 25.7% during baseline to 53.3% during treatment and 100% during maintenance.

Treatment 3 focused on two-digit addition. This brought about a change from a baseline mean of 14% to 100% during intervention. This score remained high (90% mean) during the next phase without the use of the 0-100 charts. A similar improvement was evident at the onset of Treatment 3 with two-digit addition with regrouping. Test scores improved from a mean of 14% during baseline to a mean of 86.7% during intervention. However, test scores dropped to 0% on this during the next phase of return to baseline without the use of the charts.

During Treatment 4, charts were no longer available and Jim was asked to visualize them. This was done through an emphasis on mental mathematics. He was encouraged to visualize the previously learnt process of movement across the 0-100 chart. The visualization process was reinforced through a return to the tens-and-ones structure learnt during T1. Test scores in two-digit addition with regrouping improved from a mean baseline (B2) of 0% to a mean of 60% with the introduction of Treatment 4. This further improved to a score of 100% during maintenance.

Generalization was assessed through his ability to use these skills in the classroom. This is seen by his ability to state the tens and ones in a two-digit number, to demonstrate them with place value blocks for tens and ones and to add in horizontal and vertical formats. Initially he was correct only for single digit adding (31%). At the end of the 8th week he was correct on 87.5% of the items. Least progress was made in problems in a vertical framework. Therefore the last session (session 21) was used to apply his adding skills to the vertical format. This involved pointing out to the participant that the addends given in a vertical format consisted of two-digit numbers and had tens and ones that could be added in the same way as for addends in the horizontal format. A score for the test items in the vertical format after that session suggests that this knowledge was sufficient for the participant to then apply the adding skills to the questions.

Evaluation

In a meeting held 7 weeks after the conclusion of the program, the class teacher reported that Jim had further improved on the skills acquired during intervention. He was able to solve two-digit addition questions given in a vertical form as class work. She also reported that he was able to identify and write three-digit numbers. He was said to be much more confident in mathematics and was more involved in the class discussion. He was also more willing to take risks and attempt questions that were previously difficult. His parents reported a change in attitude towards numbers, so that he was more willing to attempt homework questions in mathematics and demonstrated confidence that his answers would be correct. Jim himself reported that he now felt that he could do more mathematics and did not struggle as much as in the past. He recognized that he was learning more and had made much progress since the beginning of the program.

These results support the hypothesis that the deca-based number system as introduced in Treatment 1 provided a scaffolding structure in enabling the student to separate the tens and ones in a given two-digit number. Using the skills that he had already acquired in adding single digit numbers (1-5), this number structure was then applied to areas in addition where he had previously struggled. With this knowledge, and further instruction (T2, T3 and T4) using the same number system in the addition of two-digit numbers, his results suggests a greater mastery of skills in addition, generalization into the classroom and an increase in confidence towards mathematics. These results support the hypothesis that the mastery of the separation tens-and-ones conception in the two-digit number will facilitate the learning of operational mathematics.

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Can interaction between inferior strategies lead to a superior one? The case of proportional thinking

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Abstract

The idea that interaction among peers leads to cognitive development is not new. However, this study focuses on an issue hardly been studied so far: this is the case when two children who do not succeed at solving a task by themselves, succeed to solve it when working in collaboration. A study by Schwarz, Neuman, and Biezuner (2000) showed that this phenomenon (we refer to as – "the two-wrongs-make-a-right phenomenon") is possible. It depends on the kinds of tasks and of interaction between students, though. In this paper we report on a research-study that investigated "the two-wrongs-make-a-right phenomenon" in the domain of proportional reasoning. We focus on: a) the cognitive gains of couples in which both students are "wrongs" in comparison to the gains of couples in which one is an "expert" and the other is "wrong"; b) The effect of hypothesis testing on the cognitive gains of the individual.

Scientific background

The role of a child's interactions with her/his age peers in the development of his/her cognition, has long been discussed both from a Piagetian and a socio-historical perspective (e.g., Hartup, 1970; Vygotski, 1986). This study is in line with these efforts. However, it focuses on a phenomenon that has not been systematically investigated, the cognitive gains of two "wrongs" interacting to solve a task they are unable to solve individually.

A priori, it seems dubious that when two wrongs interact, at least one of them reaches cognitive gains. At any rate, it seems obvious that in two other kinds of interaction, the wrong should gain more: In modeling studies (in which one child observes a more competent child) and in active interaction studies in which the "wrong" interacts with a more competent child (e.g., Botvin and Murray, 1975; Murray, 1974; Kuhn, 1972).

In this section we argue that a cognitive gain from interaction between two wrongs is theoretically possible. For this purpose, we analyze the factors suggested by researchers to be responsible for the cognitive development of peers in active interaction and in modeling studies. Two factors have been recognized, disagreement and being strategic (also designated as being able to give reasons/arguments for a specific solution or offering an operational solution e.g. Miller & Brownell, 1975). It was suggested that when two interacting solvers disagree, their cognitive gains originate not only from a pragmatic component, the disagreement, but also from the contradicting solution itself. This suggestion was confirmed by a study conducted by Doise, Mugny and Perret-Clermont (1975) in which students gained from being presented a contradicting solution by an adult, whether the solution was correct or not. In another study, Doise and Mugny (1979) showed that interaction with a less capable child proposing a contradicting solution led even the more capable child to progress. In the same study, Doise and Mugny showed that when interacting students used different strategies, they progressed, whereas when they used the same strategies, they could not. A key result obtained by Doise and Mugny in their study of interactions between children with different levels of competence is that if the difference of level between the two students is too big, low-level students did not progress. As for modeling studies, Botvin and Murray (1975) showed that observing peers expressing and defending different (counter)-arguments is responsible for cognitive gains. Also Kuhn (1972) showed that in modeling studies, the observer who disagrees with the modeler is in a situation of mismatch, and he actually converses with the modeler in a kind of "tacit interaction".

In addition to disagreement and to being strategic Doise (1978) has showed that hypothesis testing, that is, the manipulation of materials available in the task to check whether the solution found is correct, also leads to cognitive gains. However, Doise (1978) showed that the more competent student is often too dominant when undertaking manipulations, preventing the less competent student from making gains on the task.

Glachan and Light (1982), on the basis of their review of modeling and active interaction studies, hypothesize that "interaction between inferior strategies can lead to superior strategies or, in other words, two wrongs can make a right" (p. 258). Glachan and Light grounded their hypothesis on the analysis of the very conditions that peer interaction studies have found as affording cognitive gains: disagreement, being strategic, and hypothesis testing. Such conditions do not depend on whether the peers are right or wrong. It ensues then that the active interaction among two wrongs can lead to cognitive gains when such conditions are fulfilled. The researchers suggest that such an interaction can be beneficial because the differing strategies being pursued by the two children lead to the making of moves inconsistent with those strategies. "The child is thus led to (jointly) make moves which he would never otherwise had made, so that established inefficient strategies are disrupted. As a consequence of this disruption one or both of the children may see possibilities for better strategies. Interaction is thus envisaged as a destabilizing influence" (ibid.). In sum, Glachan and Light hypothesized that two wrongs can make a right if they have different strategies and have opportunities to resolve their conflict.

The "destabilizing influence" alluded to by Glachan and Light has, without any doubt, Piagetian roots. It relates to stages of development. But what about the process that turns wrongs to rights? The analysis of the studies on adequate conditions for cognitive development through peer interaction shows that cognitive gains are intermingled with the utterance of arguments and counter-arguments. For example, Miller and Brownell (1985) showed that non-conservers are more likely to yield to conservers because conservers produce counter-arguments (see also, Doise et al., 1975). Botvin and Murray (1975) have shown that in modeling studies, the cognitive gains of observing peers originated from listening at solvers expressing and defending different (counter)-arguments leads to cognitive gains (see also Kuhn, 1972). And indeed in subsequent studies (Murray, Ames & Botvin, 1977), it was shown that cognitive dissonance taking place in argumentation or in observing conservers giving reasons for their points of view was the main source for significant gains.

The studies on peer interaction and cognitive development we just reviewed are to be related to studies directly focusing on argumentation. For example, Kuhn, Shaw and Felton (1997) and Schwarz, Neuman, Gil and Ilya (in press) showed that peer interaction fosters argumentative reasoning. In these studies, the researchers observed the arguments as outcomes of dyadic interaction and showed that their quality increased as a result of the dyadic interaction. The researchers did not analyze the conditions under which peer interaction occurs, but it seems that at least the two first conditions, disagreement and being strategists were fulfilled.

In a recent study Schwarz and colleagues (Schwarz, Neuman & Biezuner, 2000) have experimentally identified the "two wrongs make a write phenomenon" in the domain of decimal numbers. In this experiment, weak high-school students with conceptual bugs regarding comparison of decimal numbers were invited to collaboratively solve problems. They had a calculator at their disposal. It was shown that in pairs of "wrong" students having different bug, at least one of the "wrongs" turned to be "right". In contrast, this effect was not identified for wrongs in pairs in which one was wrong and one was right. Analysis of several case-studies in this experiment showed that the calculator was used to test hypotheses and was important in the argumentative process that led to change. However, the conditions that led to change were not systematically manipulated. The object of the current research is to deepen the study of the "two wrongs make a right phenomenon". This phenomenon needs to be examined in new domains and under new conditions in order to lead to general principles concerning gains in peer interaction. In the present study we aimed at empirically checking the "two wrongs make a right" phenomenon, in a new mathematical domain – proportional reasoning. In light of the research-review on peer interaction, we studied the "two-wrongs-make-a-right phenomenon", under three working assumptions: (a) the two wrongs disagree, (b) they have different strategies, and (c) active hypothesis testing is made possible. We aimed at evidencing this phenomenon by studying the effectiveness of the interaction between two-wrongs and comparing it with the effectiveness of the interaction between one wrong and one right (expert).

Research Design

The choice of the mathematical domain and the specific tasks within this domain, has been made according to the following criteria: (1) there exist a diagnostic task measuring students' competence in the domain; (2) the number of strategies used to solve the task is limited; (3) The strategies are hierarchical; (4) The students are consistent in using the strategies. The topic of proportional reasoning and the "blocks task" (Harel, Behr, Post, & Lesh, 1992) fulfill these criteria. A detailed theoretical analysis of the blocks task appears in Harel et al. (1992). A short description of the task is given here.

Description of the blocks task: In Harel et al. the student was given a drawing representing two pairs of blocks (A, B) and (C, D). A and C are "built" of big cubes while B and D are "built" of small cubes. The tasks appear in nine configurations (a configuration is a set of two pairs of blocks). In A there are always less cubes than in C. The variables are the number of cubes in B and D. There are three possibilities: the number of cubes in B (resp. in D) differs from the number of cubes in A (resp. in C) by 1, 0 or -1. The students were informed about the relative weight of A and B. The three alternatives are A>B (A heavier than B), A=B and A<B. The students are then asked to decide about the relative weight of C and D (4 possible answers are always displayed: C>D; C<D; C=D; "It is impossible to decide").

Harel et al. gave the 9 configurations of the blocks task to 18 Grade 7 students. Three types of strategies were uncovered. Harel et al. showed that the match representation-strategy was not random and only six such pairs (called solution process) could be identified. From these findings, it appears that the blocks task lead students to adopt solution processes that attest their level of proportional reasoning. However, it appears that for none of the nine configurations proposed by Harel and al. it is necessary to adopt a solution process attesting explicit considerations concerning the comparison of two ratios.

For our study we have made some modifications: (a) in Harel et al. the cubes in the structures A, B, C, and D had different magnitudes, and sometimes students needed to infer that a small cube weighted more than a big one, a fact creating a physical "disturbance" regarding proportional reasoning. In our case all cubes had the same size. The different weights were represented by different colors; (b) the concrete models of the blocks were presented in addition to the drawing of the blocks; (c) Harel's configurations led students to opt for additive solution processes (yielding correct answers); in our configurations the difference between the number of cubes in the different blocks were various; (d) the students were provided with a balance to check their hypotheses.

Procedure

32 lower-level Grade 10 and 11 students participated in the study. The study had three phases: individual pre-test, pair-interaction, and individual post-test.

Pre-test: The students were given the 9 configurations, one at a time, and were asked to justify their answers. Five levels of strategies were expected:

S0: Guessing [included in SP1 in Harel et al.]. The student totally ignores configurations A and B and does not provide any justification.

S1: Visual Shape level [included in SP1 in Harel et al.] The student ignores A and B, however he/she refers to the difference between C and D.

S2: Additive proportional level [corresponds to SP2, SP3 and SP4 in Harel et al.] The student takes into consideration A and B. The explanations, however, are based only on the difference in the numbers of blocks among the 4 structures.

S3: Quasi-multiplicative proportional level [includes SP5 and SP6 in Harel et al.] The student's considerations are based on the proportional weight of a single block. However, he/she does not use it for a complete proportional reasoning.

S4: Multiplicative proportional level [new solution process] A full multiplicative proportional reasoning. S4 was not found in the students' responses to the 9 tasks.

After the pre-test, tasks 1,2 and 6 were excluded. Tasks 1 and 2 were found to be trivial and task 6 was identical to task 8. Tasks 5, 7, and 9 were chosen for the interaction phase, and tasks 3, 4, and 8 were used as the post-test.

The students' answers were scored according to two scales: (1) Competence: right – R=1, wrong – W=0; (2) Strategies 0 to 3: S0=0, S1=1, S2=4, S3=7. Since in each phase of the research 3 configurations were involved, the jump of three in the scores between S1 and S2 ensured that a student that used S2 once will score higher than a student who used S1 three times. It was found that students might use a high strategy and give a wrong answer or vice versa. The decision was to score strategies higher than competence.

Conflict Scale: The "conflict" between two students was defined as the difference between them in competence and strategy on tasks 5, 7, and 9 in the pre-test. From competence perspective there were 3 possibilities: R&R (the two students are right on the task); W&R (one wrong one right); W&W (the two are wrong). In the framework of this research theoretical focus, the following scores were assigned: R&R = 0; W&R = 1; W&W = 2, the highest score (2) was assigned to two wrongs. For each task the possible range was 0-2 and for three tasks it was 0-6. From strategy perspective the range between the students' strategies (S0-S3) was 0-7 and for the three tasks it was 0-21. The final conflict-score between two students was calculated as the sum of these two components.

Cognitive Gain: Each student was given a *cognitive score* for each task. The score was defined as the sum of competence (0,1) and strategy (0,1,4,7). The cognitive gain was defined as the difference between the average of a student's cognitive-scores in the post-test (tasks 3, 4, 8) and the pre-test (on the same tasks). We calculated also the cognitive gain on the basis of the interaction tasks (tasks 5, 7, 9) and the post-test (tasks 3, 4, 8).

The interaction phase: The students were randomly arranged in couples. The conflict-score of each couple was calculated. A minimum conflict-score of 1 was required. The couples that did not meet this requirement were placed again in the list and reallocated. All together we had 16 pairs. In the interaction phase each couple solved together 3 tasks – tasks 5,7,9. The interaction phase was done under two conditions, with and without balance, that concretize the possibility or the impossibility to check hypotheses. The students were encouraged by the interviewers to verbalize and explain their considerations and decisions. In each task, immediately after the couple had reached an agreement or agreed not to agree they were given the opportunity to check their solution by the balance. After the weighting process they could change their solution/s and explanations. All the sessions were videotaped. Each student got for every task two cognitive-scores, for performance before and after the weighting process. The sum of the three singles cognitive-scores served as the student's cognitive-scores.

Post-test: The students solved individually tasks 3, 4 and 8. The cognitive-score for each task and a total score for the post-test were calculated.

Results and Discussion

I. Students' performance in the 3 phases of the research are displayed in Table 1.

Research Phase	Average	S.D.
Pre-test	4.8542	1.702
Interaction I (before weighting)	5.0312	1.031
Interaction II (after weighting)	5.8854	0.721
Post-test	5.6458	1.54

The difference between the students' average cognitive-score in the post-test and the pre-test was statistically significant (t(32)=3.37, p=.002). The difference between the students' average cognitive-score in the interaction phase II and the post-test phase was significant (t(32)=4.09, p=.000). The difference between interaction phase II and interaction phase I was significant (t(32)=6.27, p=.000). These results show that interaction advances students' cognitive gains, it also shows that their performance during interaction phase II is higher than their performance in the post-test. However, analysis of the individual scores shows that in interaction phase II the jump in the cognitive-scores was due more to the competence component of the score than to strategy. Analysis of the individual scores shows that students who scored lower than their partners while working in pairs regressed more in the post-test than students who scored the same or higher than their partners.

II. The research main theoretical focus was "the two-wrongs-make-a-right phenomenon". Our assumption was that a couple of two "wrongs" would gain more than a couple of "wrong" and "right" (expert). The definition of a couple was done in the following way: An expert was defined as someone whose average cognitive-score in the pre-test was higher or equal to 6, and a "wrong" as someone whose score was lower than 6. The data showed that we had eight couples of two "wrongs", six couples of an expert and wrong and two couples of two experts. For the analysis the 2 couples of experts were excluded because of a possible roof effect. The difference between the pre- and post-test was statistically insignificant (t(28)=0.72, ns). However, the results indicated a tendency of a higher gain for couples of two "wrongs" (average of 1.313) in comparison to couples of expert and wrong (0.931). These results, albeit not statistically significant, support our theoretical assumption that when two wrongs interact the possibility for at least one of the wrongs to progress is real. Although the "wrong" student is exposed to incorrect arguments, the quality of the cognitive process is not inferior to the one that takes place when one of the interacting students is right. On the contrary, the results support our theoretical framework that indicates the possible benefits of such an interaction and the potential limitations of the other one. For example, expert students tend to dominant the interaction and thus to suppress counter-arguments to be raised.

III. Students' cognitive-gain as a factor of the conflict-score between the interacting couple is shown in Table 2.

Level of Conflict-Score	Average Cognitive-gain	Number of students
1-2	1.3	10
3-5	1.6	8
6	0.7	8
7-15	0.2	6

The results show that the maximal gain was obtained in the second level of conflict and the minimal one in the fourth level. These results support the theoretical assumption that a high level of conflict is not always beneficial. It may implies domination of one of the partners that prevents the interacting students from gaining while a moderate conflict, even between two "wrongs", may lead to significant cognitive-gains. However, these results should be re-checked in the light of a possible roof effect.

Summary

The paper reported on a study aimed at examining the "two-wrongs-make-a-right phenomenon" - where the terms "wrong" and "right" imply hierarchy of competence in a well-defined mathematical domain - in an empirical research. The mathematical domain in which this study took place was proportional reasoning. We argued that when certain theoretical conditions are fulfilled – disagreement, being strategic, and hypothesis testing – the quality of an interaction between two "wrongs" is not inferior to the one that takes place when one of the interacting students is right. The findings of the study support our assumptions. The first finding of the study is not surprising - interaction between peers yields cognitive gains. The second finding support our assumption that interaction between two "wrongs" generates cognitive gains. Moreover, the results suggest that interaction between two wrongs is cognitively more productive than interaction between wrong and right. The results also show that hypothesis testing effects the cognitive gain of the interacting couple. It was also interesting to see that the students' cognitive-gains reached their highest level immediately after the process of hypothesis testing and dropped in the post-test. However, this study is only the first step in the investigation and leaves many research-questions open. For example: What are the interaction processes between the two "wrongs" that lead to the cognitive changes of the individual? How wrongs turn to be right? What is the optimum level of cognitive heterogeneity? Will a student working on his own, while having the opportunity the check his hypotheses gain the same? These questions are the focus of our current research.

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CHILDREN'S STRATEGIES FOR NUMEROSITY JUDGEMENT IN SQUARE GRIDS: THE RELATIONSHIP BETWEEN PROCESS AND PRODUCT DATA

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Abstract

This study investigated the relationship between children's strategies, their accuracy and speed measures in a numerosity judgement task. Based upon a rational analysis of the task at hand, several strategies were specified in terms of their course of response times and accuracy. Several significant relations were found between pupils' accuracy data and the distinct strategies that were derived from their response-time patterns. These results support our rational task analysis and validate previous findings concerning numerosity judgement strategies.

1. Theoretical and empirical background

Many studies (for an overview see Siegler, 1996) have shown that children of a particular age use several strategies to solve the same cognitive task. This multiple strategy use allows them to adapt their strategies to inherent task characteristics such as problem difficulty as well as to situational demands such as the need to answer quickly or accurately. The larger the repertoire of strategies, the better one can adapt one's strategy choice in function of the task requirements.

In a series of studies (Luwel, Verschaffel, Onghena, & De Corte, 2000; Luwel, Verschaffel, Onghena, & De Corte, in press; Verschaffel, De Corte, Lamote & Dhert, 1998), we investigated the development of children's strategy use in a numerosity judgement task from the perspective of "strategic change" (Lemaire and Siegler, 1995). This theoretical framework distinguishes between four dimensions of strategic competence. Changes in any of those dimensions can yield overall improvements in speed and accuracy of performance: (a) acquisition of new strategies and abandonment of old ones, (b) shift towards greater use of more efficient available strategies, (c) improvement in the fluency and efficiency with with strategies are executed, and (d) increase of the adaptive nature of the choice among available strategies.

In all of our studies participants were asked to determine numerosities of blocks that were presented in a square grid structure (see Figure 1).

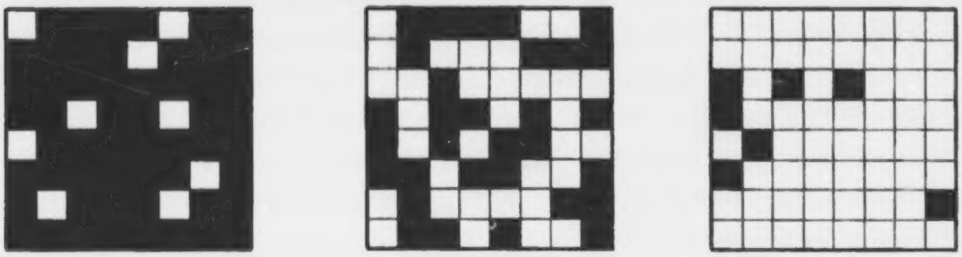

Figure 1. Three examples of stimuli from the experimental task.

Based on a rational task analysis and on the results of Verschaffel et al. (1998), three major strategies for solving this task were hypothesized. Application of each of these strategies depended on the ratio of blocks to empty squares in the grid. When there were few blocks but many empty squares in the grid the use of an *addition strategy* by means of which (groups of) blocks are counted (and added) was hypothesized. When there were many blocks but few empty squares, we hypothesized that participants would use a *subtraction strategy* in which the number of empty squares is subtracted from the (estimated or computed) total number of blocks in the grid (i.e., the anchor). When the number of blocks and empty squares is too large to be counted within the given time constraint participants will fall back on a rough *estimation strategy*, whereby the number of blocks is determined in a quick but imprecise way.

It was assumed that the addition and subtraction strategy were relative accurate strategies leading to, respectively, linearly increasing and decreasing response times with an increasing number of blocks. Due to its nature, the estimation strategy will elicit less accurate responses with relatively quick response times that lie within the same range. The assumed connection between the use of a particular strategy on a particular numerosity judgement item and the time needed to solve that item, led to four hypothetical response-time patterns (see Figure 2).

2. Method

One hundred and nine children participated in the study: 59 second graders (aged 7-8 years) and 50 sixth graders (aged 11-12 years). All participants were asked to judge different numerosities of blocks that were presented in square grids of three different sizes (7 x 7, 8 x 8, and 9 x 9) as accurately as possible within the given time constraint of 20 s. All pupils ran three sessions and in each session all possible numerosities of blocks in one grid size (i.e., 49, 64, and 81) were presented randomly. Children's responses and response times were recorded by the computer.

Half of the pupils was given information about the total amount of squares in that particular grid at the beginning of each trial. For instance, in the case of the 7 x 7 grid, children in the information condition were shown the number 49 at the beginning of each trial. We included this manipulation to investigate the effect of information on children's strategy use[1].

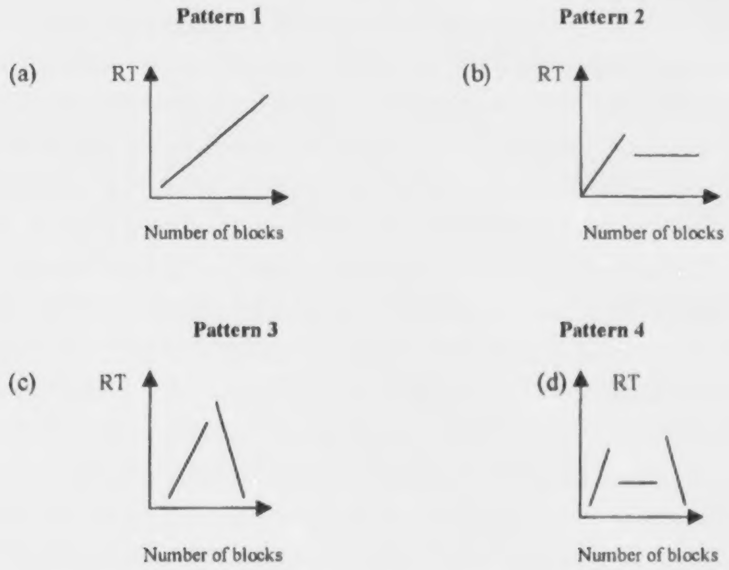

Figure 2. Hypothetical response-time patterns with (a) application of the addition strategy, (b) use of the addition and estimation strategy, (c) execution of the addition and subtraction strategy, and (d) application of the addition, estimation, and subtraction strategy.

3. Research questions and hypotheses

The results with respect to the effect of different task and subject variables are already reported elsewhere (see Luwel, Verschaffel, Onghena, & De Corte, in press). In the present paper we focus on several hypotheses about the relationship between the process data (i.e., the applied strategies) and the product data (i.e., response times and error rates).

Hypothesis 1: The accuracy of the addition and subtraction strategy will be considerably greater than the accuracy of the estimation strategy. Furthermore, there will be no significant difference in accuracy between the addition and the subtraction strategy.

Hypothesis 2: There will be a negative relationship between the number of trials on which the estimation strategy is used, on the one hand, and the accuracy of the pupil on the task as a whole, on the other hand. The more the less accurate estimation strategy is used, the lower the overall accuracy will be.

Hypothesis 3: The overall accuracy of children fitting Pattern 3 will be higher than the accuracy of pupils who fit Pattern 4, and both will be higher than the accuracy of participants with a type 1 or 2 response-time pattern. Using only the more accurate addition and subtraction strategies (Pattern 3) will result in more accurate responses than when the less accurate estimation strategy is used in addition to the other two strategies (Pattern 4). Moreover, because of the use of the subtraction strategy for the largest numerosities, both strategy repertoires will lead to a higher overall accuracy than the combined use of the addition and estimation strategy (Pattern 2) or merely applying the addition strategy (Pattern 1).

4. Results

Before we discuss the results with respect to each of the four hypotheses, we will shortly describe how we identified children's strategies. For this purpose we compared the individual response-time patterns with the four hypothetical response-time patterns in Figure 2 by means of two- and three-phase segmented linear regression models (Beem, 1993, 1999). These models look, respectively, for one or two change points in the data pattern and describe the relationship between the independent and dependent variable by means of, respectively, two or three regression equations. After having determined the number of segments in a data pattern by statistically testing for the presence of one or two change points, we looked for a possible fit with one of these hypothetical patterns by comparing the *b*-parameters in the different equations under two hypotheses. For a detailed description of this procedure we refer to Luwel, Beem, Onghena, and Verschaffel (2000). This method resulted in 77% of the second graders and 83% of the sixth graders fitting one of the four hypothetical data patterns. For reasons of clarity the data in all analyses were aggregated over grid size and type of information condition.

Hypothesis 1 was tested for each type of pattern separately. Due to an unequal division of the pupils over the different patterns (see Luwel et al., in press), we were only able to test this hypothesis for the second graders fitting Pattern 2, for the sixth graders fitting Pattern 3, and in both age groups for Pattern 4. Accuracy was measured in terms of the absolute deviation from the given response to the actual numerosity (i.e., the error rate). The error-rate patterns of the subjects included in the analysis were divided into two or three segments by means of the change points computed in the individual response-time patterns and for each segment the mean error rate was computed.

For Pattern 2, the mean error rates of the different segments were compared by means of a *t*-test for dependent samples. This test showed that the mean error rates of Segment 1 (addition strategy) ($M = 1.38$) were significantly smaller than the mean error rates of Segment 2 (estimation strategy) ($M = 11.78$), $t(24) = 2.67, p = .01$.

For Pattern 3, the same test did not show a significant difference in mean error rates between Segment 1 (addition strategy) and Segment 2 (subtraction strategy).

For Pattern 4, we conducted an analysis of variance with age as independent between-subjects variable and segment as independent within-subjects variable. This analysis revealed a significant main effect of age, $F(1, 94) = 7.06$, $p = .009$, and of segment, $F(2, 188) = 177.00$, $p < .0001$. Both variables were involved in a significant interaction effect, $F(2, 188) = 5.12$, $p = .007$. A posteriori Tukey tests revealed that in both age groups, Segment 2 showed significantly larger mean error rates than Segment 1 and 3 (*M*s: 6.05, 0.80 and 1.58 for the second graders and *M*s: 4.42, 0.72 and 1.21 for the sixth graders), all *p*'s < .0001. The mean error rates in Segment 1 and 3 did not differ significantly in both age groups. Furthermore, the mean error rates of the second graders in Segment 2 ($M = 6.05$) were significantly larger compared to the mean error rates of the sixth graders in that segment ($M = 4.42$), $p = .0001$, whereas we did not observe an age difference for Segment 1 and 3.

These findings confirm our first hypothesis: for Pattern 3 and 4 there was no significant difference in mean error rates produced by the addition strategy, on the one hand, and the subtraction strategy, on the other hand. Moreover, for Pattern 2 and 4 the mean error rates resulting from the use of the estimation strategy were significantly larger than the mean error rates produced by the addition strategy (and the subtraction strategy in the case of Pattern 4).

To test *Hypothesis 2*, we computed the correlation between the number of trials on which the estimation strategy was applied and the mean error rate of the pupil on the task as a whole. For both age groups this correlation was computed for pupils fitting Pattern 4.

This analysis revealed a significant positive correlation for the second graders, $r(41) = .60$, as well as for the sixth graders, $r(53) = .41$, all *p*s < .05, suggesting a negative effect of the use of the estimation strategy on the global accuracy.

For *Hypothesis 3*, we compared the mean error rate on the whole task for the different types of patterns by means of a *t*-test for dependent samples. For the second graders we could only compare Pattern 2 with Pattern 4. The *t*-test revealed that the mean error rates for Pattern 2 ($M = 4.64$) were significantly larger than the mean error rates in Pattern 4 ($M = 2.49$), $t(24) = 3.45$, $p = .002$. For the sixth graders, the only comparison was between Pattern 3 and Pattern 4. Results showed that the mean error rates in Pattern 4 ($M = 2.09$) were significantly larger than the mean error rates of Pattern 3 ($M = 1.01$), $t(44) = 4.42$, $p < .0001$.

5. Discussion

The present results confirmed our hypotheses in which we predicted a number of relationships between the strategies that were identified on the basis of the individual response-time patterns, on the one hand, and their accuracy data, on the other hand. These findings support our rational task analysis in which we specified the different types of strategies in terms of their expected course of response times and their degree of accuracy. It was shown that the addition and subtraction strategy were relatively accurate strategies, whereas the estimation strategy was a less accurate strategy. Moreover, this study also validated previous findings concerning numerosity judgement strategies.

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[1] A pilot study revealed that some children made large numerosity judgement errors due to an incorrect determination of the anchor. We wanted to investigate whether the provision of information about the anchor could prevent this kind of errors.

THE FORMULATION OF A CONJECTURE: THE ROLE OF DRAWINGS

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Abstract. *The study presented in this paper arises from a research project dealing with the process of solving open-ended problems in a geometrical context. The specific moment of conjecture formulating is taken into consideration and namely a particular behaviour concerning the process of drawing producing is analysed. The theoretical framework of figural concepts provides us with the starting point from which developing our considerations.*

Introduction

Many investigations on the role drawings play within geometrical activity have been carried out from different points of view. Duval (1995) focuses on the "traitements spécifiques au registre des figures et à celui d'un discours théorique en langue naturelle" (p.173) and highlights the differences between them. Other studies point out and inquire into the intimate relation that exists among drawings, figures and concepts (Fischbein 1993, Laborde – Capponi 1994).

Notwithstanding many questions remain to be answered and in particular the role drawings play in solving geometrical problems has to be deepened.

In this paper we present a study developed from a research project, still in progress, which intends to investigate on the solving strategies of open-ended problems (Maracci 1998). With reference to the theoretical framework of figural concepts (Fischbein 1993) we focus on the role that producing drawings plays in solving construction-problems.

Figural concepts and *satisfactory* drawings

According to Fischbein's theory of figural concepts (1993) when dealing with geometrical activity we are concerned with a mental construct which simultaneously and intrinsically possesses both figural and conceptual properties.

The perfect fusion between the two components of a figural concept seems to be only an ideal and extreme situation, indeed "what happens is that the conceptual and the figural properties remain under the influence of the respective systems, the conceptual and figural ones" (ibidem, p.150).

According to this theory a drawing is the material concrete representation of a figural concept, reflecting the tension between the figural and the conceptual component.

In previous studies (Maracci 2001) the hypothesis has been formulated that when producing a geometrical drawing students try to harmonize figural and conceptual aspects. The search for such harmony does not seem to be conscious, one could speak of a generic feeling of satisfation from students' point of view. Three factors have been pointed out which might characterize this generic feeling and make a drawing *satisfactory*:

- a drawing should correctly represent the geometrical situation into consideration, students' interpretation of the given situation and of the produced drawing should be consistent;

- a drawing should be recognized as sufficiently generic;
- a drawing should possess a good gestalt, it should satisfy the fundamental laws which control the basic processes of perception.

These conditions may appear and combine in many different ways.

Our research

As mentioned above our study derives from a more extensive research concerning the processes of solving open-ended problems in a geometrical context (Maracci 1998). Seventeen students (11th and 12th grades) selected from different scientific high school were involved, all of them evaluated by their teachers as medium - high achievers. These students were presented with 4 open-ended problems to be solved in individual videotaped interviews during which they were asked to think aloud.

The study presented in this report focuses on the specific moment of the formulation of a conjecture. The analysis of the transcripts of the interviews shows some different behaviours on students' part. In this report we will focus on the following one:

- after a period of investigation conducted with the aid of drawings, students get the correct conjecture and face the task of formulating it precisely;
- the conjecture is achieved with clear and explicit reference to one or more specific drawings;
- students verbally formulate their conjecture producing at the same time a new drawing very similar to those which they previously referred to.

With respect to such a behaviour the following question may be posed:

> when they seem to have elaborated the correct conjecture with reference to a satisfactory drawing, why do some students feel the need of producing a new drawing so similar, if not identical, to the previous one?

A first answer has been proposed by formulating the hypothesis that students may interpret drawings diachronically, i.e. they may consider each drawing with its history as representing a specific process of the problem solving session (Maracci, 2001).

Here we will concentrate on the particular case of construction-problems and try to face the question from a different point of view.

The only problem we will refer to in the following is:

Problem: *Given two equal segments, construct two equal triangles with a common vertex and having the two given segments as homologous sides.*

This problem presents some difficulties in the identification of a solution. In order to solve it one could reformulate the problem shifting her attention from the construction of the two equal triangles to the construction of the common vertex.

Conjecture formulating and drawing producing in construction-problems

A strategy to successfully approach a construction-problem may be assuming the required objects as given and then proceeding backward in the search for some characterizing properties from which getting the needed construction. In this case when formulating her conjecture one needs, first of all, to restore the correct logical order of the relations.

As far as construction-problems are concerned students are required to explicitly give a procedure, a list of operations to perform in order to actually carry out the construction. The conjecture students have to elaborate and formulate consists indeed of a series of instructions which may hardly be put in the form "if... then..." or in any case in a more concise form.

Correctly describing a construction is a really demanding activity: it requires both to control the global organization of the procedure and to assure that each step may be actually performed. Performing it on the paper might provide a very useful support and in fact many students actually carry out their construction.

Thus students are engaged both in the verbal description of the construction and in the production of a new drawing.

On the other hand the production of a drawing is a demanding activity too. In previous researches students' difficulties in producing and managing drawings in solving geometrical problems have been highlighted (Maracci 1998, Maracci 2001). When producing a drawing students try to reach a harmony between figural and conceptual aspects. Even if it is not consciously stated, the need of balancing correctness, generality and good gestalt permeates the whole process of production of a drawing.

Thus when formulating their conjecture students are involved in at least two really demanding processes: the verbal formulation of the construction itself and the production of a new drawing.

Silver (1987) pointed out that the "overwhelming number of processes to control" is one common difficulty students face in solving mathematics problems. In order to overcome such difficulty students might try to activate some *form of control* over the processes to be carried out.

Production of drawings as a means of control

We remarked that when verbally describing their construction some students produce a new drawing estabilishing a, more or less explicit, correspondence with those to which they referred in the search for the construction itself.

Our main hypothesis is that *the production of a drawing similar to previous ones is an attempt to activate a form of control over the processes of description of a construction and of production of a drawing.*

According to our hypothesis estabilishing such correspondence might provide a control at least at two different levels:

- in order to be a satisfactory representation a drawing has to combine and balance needs for correctness, generality and good gestalt. When producing a new drawing, making reference to an already made satisfactory one may assure students that they are really producing a satisfactory drawing.
- Moreover, with concern to construction-problems, producing a new drawing similar to previous ones may play a more specific role.
  The drawing, with respect to which the correct construction has been grasped, might represent the final configuration, the outcome itself of the construction.

Estabilishing a connection, a correspondence between that drawing and the new one might allow students to interpret the former as a preview of the result of the construction they are performing and describing.

If the construction is correct and it is correctly performed the estabilished correspondence has to persist up to the end of the construction itself and the resulting drawing has to be consistent.

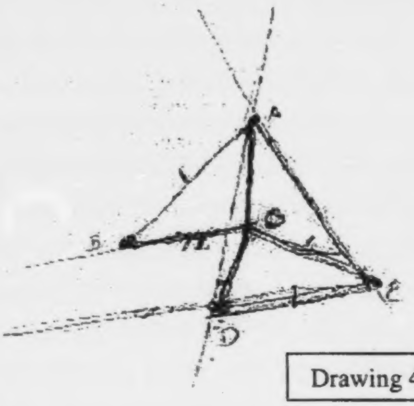
Extracts from problem solving sessions

In the following we present some excerpts from two protocols relative to the activity of formulation of a construction[1]. We shall analyse the excerpts in the light of the previous considerations.

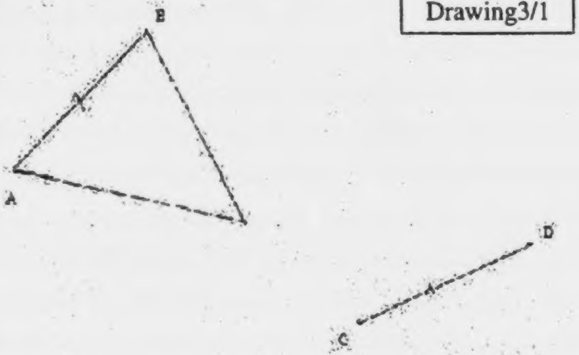
Barbara (12th grade, scientific high school)

What follows is the exact moment in which Barbara gets the properties characterizing the common vertex of the two unknown triangles. Drawing 4 is that with respect to which she has conducted the core of her exploration and it is that to which she is referring when grasping the correct construction (item 128)

128. **Bar:** there! I have to consider the perpendicular bisector of the segment... oh, there... I found it [...]. So I have to trace, to consider the segment AD - *she indicates the segment AD in drawing 4* - and to find... wait, - *she indicates segments AO and DO in drawing 4* - then segment AD, I found the perpendicular bisector of the segment by cutting it in two equal parts and tracing the perpendicular, then I know that point O has necessarily to belong to the perpendicular bisector of AD, wait - *she looks at drawing 3/1 and renames the endpoints of the two segments (drawing 3/2)* - I trace, - *she traces AD* - I trace... I consider the midpoint and trough it I trace - *she traces the perpendicular bisector of AD* - then, now I know that point O has necessarily to belong to the perpendicular bisector of this segment, but it has to belong to the perpendicular bisector of segment BC - *she indicates BC in drawing 4* - isn't it?... I trace BC and - *in drawing 3 she traces BC and its perpendicular bisector* - there ...



Drawing 4



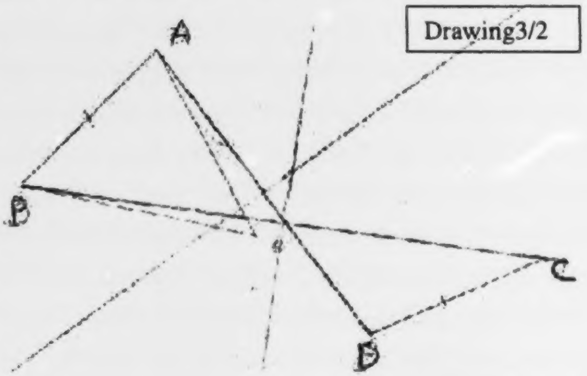
Drawing3/1

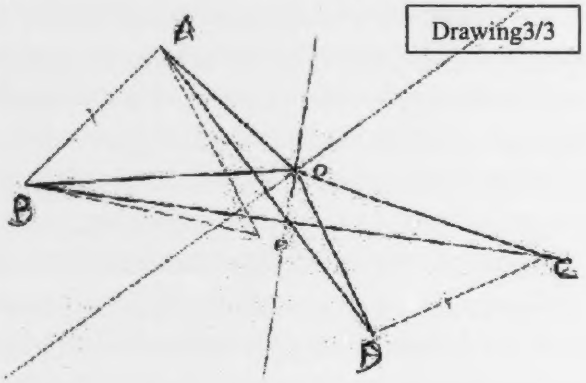
130. **Bar:** no, it does not work [...]

134. **Bar:** not, since... in this drawing - *she is referring to drawing 4* - I assumed these two segments equals - *she marks BO and CO* - but in this drawing - *she refers to drawing 3/2* - I'd actually get this one equal to this one - *she indicates (drawing 3/2) the segments AO and CO*[2]

136. **Bar:** point O should be here theorecally - *she labels by O the intersection point of AD and BC* - but it is here actually - *she indicates the intersection point between the perpendicular bisectors of AD and BC*

140. **Bar:** no, since... I'm really fool! - *she marks the correct point O and traces AO, BO, CO and DO (drawing 3/3)*


Drawing3/2


Drawing3/3

After having caught the correct construction, Barbara does not formulate it referring directly to drawing 4. She chooses to refer to another one. Her choice is indeed quite unusual: Barbara does not produce a new drawing from the beginning, she prefers to refer to an already made one (drawing 3/1). Drawing 3/1 provides her with a satisfactory representation of the initial configuration (i.e. the two equal segments); the presence of the triangle, related to previous investigations, seems not to influence the formulation of the conjecture.

Because of the mutual position of the two segments, drawing 3/1 seems suitable to be put in correspondence with drawing 4 and Barbara makes this correspondence complete by renaming the endpoints of the segments so as to conform them with drawing 4 (item 128).

While formulating her conjecture Barbara actually performs the construction (drawing 3/2), but suddenly she stops (item 130). At this point no one could tell what is wrong. In the following minutes Barbara continually shifts her attention from one drawing to the other.

In items 134 and 136 Barbara finally explains what causes her uncertainity. Trying to mentally anticipate the result of her construction, Barbara failed in positioning point O in her mind (it is only in item 136 that she labels point O in her drawing). Because of this mistake, the correspondence between the two drawing does not seem to be suitable any longer (item 130).

As it clearly appears (item 134), Barbara expected to get the equality between the sides BO and CO as a result of her construction. But what she gets, or better she thinks she is getting is the equality between AO and CO. Barbara's expectation derives from the fact that she is performing her construction making explicit

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reference to drawing 4. This allows her to interpret that drawing as a prevue of the outcome of the construction itself.

As soon as Barbara thinks that the estabilished correspondence fails, she has no doubt that some mistake occured, even if she is not immediately able to specifies which one.

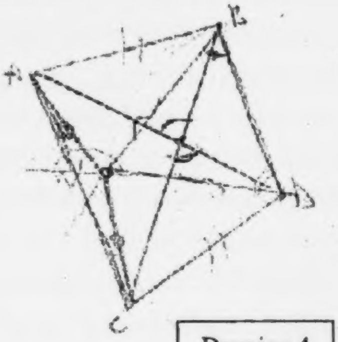
It takes some minutes before Barbara realizes what really happened (item 140) and can, so, successfully conclude her problem solving session.

This protocol gives an interesting example how estabilishing a correspondence among drawings might provide one with a means of control over both the production of a new drawing and the correctness of the construction itself.

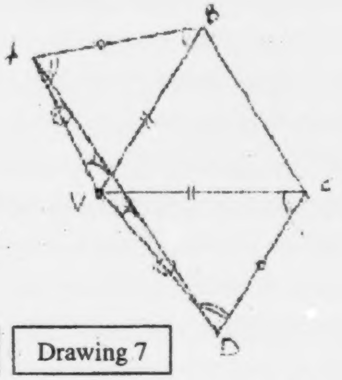
Let us remark a further aspect, at the end of the construction the resulting drawing (drawing 3/3) is quite similar to drawing 4: they share characteristics as the mutual position of the triangles and their shape. One might wonder whether such perceptive similarity plays the fondamental role in the correspondence between the two drawings. We could better discuss this question after having considered the next protocol.

Davide (12th grade, scientific high school)

Davide conducted his search for a conjecture mainly with the aid of two drawings (drawing 4 and drawing 7), and it is exactly with respect to drawing 7 that he grasped it. As Barbara did, he chooses not to refer to that drawing; he decides to produce a new one.

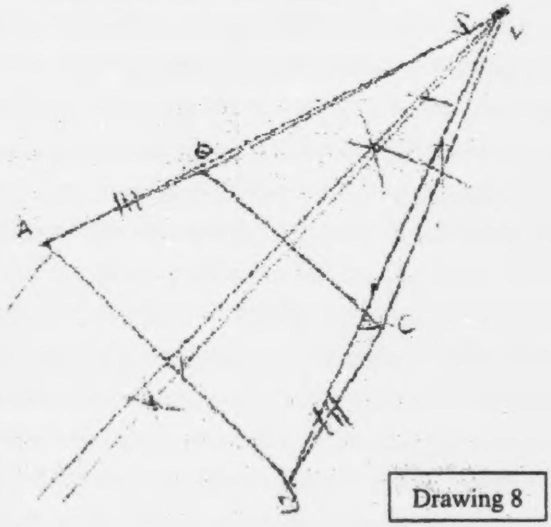


Drawing 4



Drawing 7

80. **Dav:** *he draws two equal segments (drawing 8)* - I know that these two angles… these two sides are equal – *he marks VD and AV in drawing 7* – in the same way these two ones – *he marks BV and CV* – so here, one can… however, I'm constructing it below – *he labels the endpoints of the previously drawing segments in conformity with drawing 7* - I mark the midpoint of the segments AD and BC – *he constructs the midpoints of the two segments* – these are... – *he indicates the perpendicular bisector of AD* – the geometrical locus of the points with equal distance from the endpoints A and D, […]



Drawing 8

> I construct another straight line which is… the geometrical locus of point with the same distance from the endpoints B and C – *he constructs the perpendicular bisector of BC and labels the intersection point by V*

We can notice that drawing 8 shares the same initial configuration with drawings 4 and 7, that is the mutual position of the segment AB. The core of the search for the construction has been conducted with reference to such configuration, it is evidently satisfactory to Davide, so that he decides to use it as the starting point for his new drawing. Moreover he makes the correspondence between drawing 8 and drawing 7 stronger by naming the endpoints of the two initial segments in the same way (we observed the same behaviour on Barbara's part (item 128)).

While verbally making his conjecture explicit Davide constructs the vertex V.

As we can see, the final configuration appears completely different from previous ones, notwithstanding this does not affect Davide's confidence about the correctness of his conjecture nor of his drawing.

We want to point out that despite the undeniable perceptive differences, the correspondence between drawings 8 and 7 persists in a deeper sense. Drawing 8 represents the segment BV equal to CV and the segment AV equal to DV, as they were represented in drawing 7. Even if they are "globally" different the two drawings represent the same "analytic" relations among the same elements.

On the other hand if we look back at Barbara's protocol we can observe that she never refers to how drawings appear globally. She explicitly refers to analytic relations among the drawn elements.

Consistently with the theory of figural concepts (Fischbein, 1993), students interpret drawings as reflecting figural and conceptual aspects, the correspondence students estabilish among drawings is not limited to perceptive aspects, it deeply involves analytic relations too.

Conclusions

Solving a construction-problem requires one to explicitly give a procedure, a list of constructing operations. Producing a drawing might provide a useful support for the verbal description of the construction. But verbally formulating a construction and contemporary producing a new drawing are two really demanding activities, in order to successfully manage them some means of control may be needed.

The analysis of the experimental data suggests that such a control may be found in relating the new drawing to that with reference to which the construction has been caught.

Estabilishing such a correspondence might assure that one is really producing a satisfactory drawing.

Furthermore previously produced might be interpreted as a models of the outcome of the construction itself.

Many key points remain to be clarified. Far from being exahustive, our research poses new questions to be answered. We should like to point out the followings ones:

- when problems different from construction ones are involved, is any

correspondence among different drawings estabilished? Which eventually is its role?
- might estabilishing correspondences among drawing play roles of control over other activities in the process of problem solving?

Students' difficulties in managing drawings have been highlighted in previous researches (Maracci 1998, Maracci 2001). We think that deepening which role drawings play in the process of solving geometrical problems might contribute to clarify such difficulties. In order to be able to plan specific didactical activities a further development of this research is anyway needed.

Notes

[1] The original drawings were scanned and processed by means of a computer in order to obtain, on the basis of the analysis of the videotapes, the way they appeared at each moment of the problem solving sessions. Numbers which designate drawings refer to the whole session. One of the drawings realtive to the protocol of Barbara (drawing 3) was processed in order to be allowed to follow its production step by step.

[2] Let us remark that at this point Barbara has not still labeled point O on her drawing (cfr item 136)

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SEMIOTIC MEDIATION FOR ALGEBRA TEACHING AND LEARNING

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Abstract. *In the theoretical framework of Vygotskij's Theory this report discusses the mediating function of particular tools available in a microworld. Following the analysis carried out by one of the authors, in the case of Cabri a new microworld has been set up and a teaching experiment carried out. Some results are reported and the theoretical notion of semiotic mediation discussed.*

Introduction

A long-term experiment concerning pupils' introduction to theoretical thinking in the algebra domain has been carried out over the last two years. Activities were accomplished in the field of experience (Boero, 1995) of "symbolic manipulation within a microworld": "L'Algebrista" (Cerulli & Mariotti, 2000) . As in previous experiences, (Mariotti, forthcoming b)), in this experiment the evolution of the field of experience is realized over time through social practices; in particular, classroom verbal interaction is made possible by means of '*mathematical discussion*' (Bartolini Bussi, 1998).

The didactic problem concerns the ways of realizing a theoretical approach to symbolic manipulation. A key-point is that of stating the "system of manipulation rules" as a system of axioms of a theory. The nature of the particular environment may foster the evolution of the theoretical meaning of symbolic manipulation. This is not really the approach pupils are accustomed to, on the contrary, Algebra, and in particular symbolic manipulation, are conceived as sets of unrelated "computing rules", to be memorized and applied.

In this report we shall limit ourselves to the discussion of some elements of the external context (Boero et al., 1995) and their functioning in the concrete realization of classroom activity. These elements may function as instruments of semiotic mediation (Vygotskij, 1978). The general basic hypothesis is that

> *meanings are rooted in the phenomenological experience, but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher.*

Our research project is aimed at studying the functioning of an artefact, when it is conceived as a potential instrument of semiotic mediation (Vygotskij, 1930-1978). In particular, we focus on a specific type of artefact : a microworld.

Semiotic mediation

Vygotskij distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*) (Vygotskij, 1978: 53). The use of the term *psychological tools*, referred to signs as internally oriented, is based on the analogy between tools and signs, but also on the relationship which links specific tools and their externally oriented (for the mastering of nature) use to their internal counterpart (for the mastering of oneself) (ibid.: 55).

Through the complex **process of internalisation** (Vygotskij, ibid.), a tool becomes a "psychological tool" and will shape new meanings; in this respect a tool may function as a semiotic mediator.

The following sections will be devoted to illustrating this theoretical hypothesis in the case of a set of particular tools (the microworld L'Algebrista) and the theoretical meaning of algebraic statements, that we shall briefly recall as 'Algebra as a Theory' (in the following AT).

Semiotic mediation in a microworld

A new research project was set up, along the same line and according to the same general hypothesis of a previous research project involving the Cabri microworld (Mariotti et al., 1997; Mariotti & Bartolini, 1998; Mariotti, in press). As far as Cabri is concerned, previous studies have focussed on the analysis of the specific elements of the microworld (dragging facility, commands available, macro …) as an instrument of semiotic mediation that the teacher can use in order to introduce pupils to a theoretical perspective. Taking into account the main results obtained from these studies, a new microworld (L'Algebrista) was designed, incorporating the basic theory of algebraic expressions. Similarly to what happens in Cabri in relation to Geometry, AT, as far as it is imbedded in the microworld, is evoked by the expressions and the commands available in L'Algebrista. As a matter of fact, L'Algebrista is a **symbolic manipulator** which is totally under the user's control: the user can transform expressions on the basis of the commands available; these commands correspond to the fundamental properties of operations, which stand for the **axioms** of a local theory. As a consequence, the activities in the microworld which produce a chain of transformations of one expression into another, correspond to a proof of the equivalence of two expressions in that Theory.

According to the Vygotskian framework, expressions and commands can be thought as external signs of AT and, as such, they may become tools of semiotic mediation (Vygotsky, 1978).

The main elements offered by L'Algebrista as instruments of semiotic mediation related to the theoretic aspects of algebra, are the following:

- *expressions in L'Algebrista* are signs of *algebraic expressions*; their intrinsic structure and their logic within the microworld refer to algebraic expressions as theoretical objects;

- *commands (buttons)* are signs of *axioms* and *definitions of an Algebra Theory*;
- *transforming an expression* into another using the buttons corresponds to *proving* that the two expressions are equivalent, the produced *chain of justified steps* corresponds to a *proof;*
- *new buttons* may be introduced using a specific command (Il Teorematore – i.e. Theorem Maker); such new commands become signs of *theorems*;
- *adding new buttons* to those already available is a sign of the meta-theoretical operation of *adding new theorems* to a theory.

Classroom activities develop constructing a parallel between the activities within the microworld and the evolution of AT. Transformations within L'Algebrista, via the use of the "buttons" available, correspond to proving equivalencies by using axioms in AT. Furthermore, as soon as a new equivalence is derived, i.e. it is proved, it can be introduced as a Theorem, and using the "Theorem Maker" it can become a new "button". The introduction of a new "button" is discussed collectively: thus the evolution of the theory occurs as a social enterprise.

In the following section we shall discuss a number of examples showing some aspects of the internalisation process related to the particular tools offered by the microworld, contributing to the evolution of the theoretical meaning of symbolic transformations.

Making conjectures and proving

The classroom activities consisted in alternated sessions in which the pupils were asked to solve problems inside and outside the microworld. Articulation between these two modalities constitutes one of the main characteristics of the activity design. As far as the analysis of the internalisation process is concerned, activities "without the computer" may highlight some interesting aspects. Let us consider the following problem, to be solved in the paper and pencil environment. Pupils are asked to compare three expressions and to find out which ones are equivalent; they are then requested to prove their conjectures.

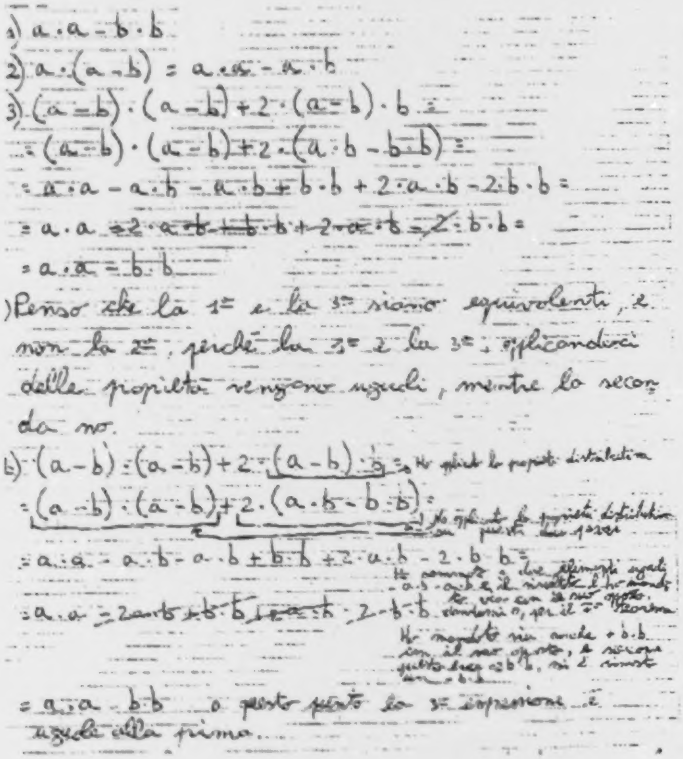
Consider the following expressions:

a∗a-b∗b **a∗(a-b)** **(a-b)∗(a+b)**

a) Which of them do you think are equivalent? Which do you think are not? Why? Can you prove this?

b) Analyse your proof and specify, for each step, whether you have used a theorem or an axiom.

Silvio (Fig. 1) starts to reduce the second and third expression into a form that makes it easier to compare them with the first. This part of the protocol looks like typical productions provided by the pupils when asked to *compute* (Italian: "calcolare"). In this case Silvio, although not requested to do so, uses his computing skills to produce a conjecture; nevertheless, his explanation shows that the properties of the operations, i.e. the *axioms* previously introduced, were used in the heuristic phase as *tools* to predict the equivalence.



1) $a \cdot a - b \cdot b$

2) $a \cdot (a - b) = a \cdot a - a \cdot b$

3) $(a - b) \cdot (a - b) + 2 \cdot (a - b) \cdot b =$

$= (a - b) \cdot (a - b) + 2 \cdot (a \cdot b - b \cdot b) =$

$= a \cdot a - a \cdot b - a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b =$

$= a \cdot a \cancel{- 2 \cdot a \cdot b + b \cdot b + 2 \cdot a \cdot b} - 2 \cdot b \cdot b =$

$= a \cdot a - b \cdot b$

) Penso che la 1ª e la 3ª siano equivalenti, e non la 2ª, perché la 1ª e la 3ª, applicandoci delle proprietà vengono uguali, mentre la seconda no.

b) $(a - b) \cdot (a - b) + 2 \cdot \underbrace{(a - b) \cdot b}$ = Ho applicato la proprietà distributiva

$= \underbrace{(a - b) \cdot (a - b)} + \underbrace{2 \cdot (a \cdot b - b \cdot b)} =$ Ho applicato la proprietà distributiva su questi due pezzi

$= a \cdot a - a \cdot b - a \cdot b + b \cdot b + 2 \cdot a \cdot b - 2 \cdot b \cdot b =$ Ho sommato i due elementi uguali $-a \cdot b - a \cdot b$ e il risultato l'ho annullato con il suo opposto, ottenendo 0, per il 1° teorema

$= a \cdot a - 2 a \cdot b + b \cdot b + 2 a \cdot b - 2 \cdot b \cdot b$ Ho annullato via anche $+ b \cdot b$ con il suo opposto, e [illegible] $-2 b \cdot b$, mi è rimasto $- b \cdot b$

$= a \cdot a - b \cdot b$ a questo punto la 3ª espressione è uguale alla prima.

Figure 1. Silvio writes:

I think the 1st and the 3rd are equivalent, but not the 2nd, because applying the properties they become equal, while the 2nd does not.

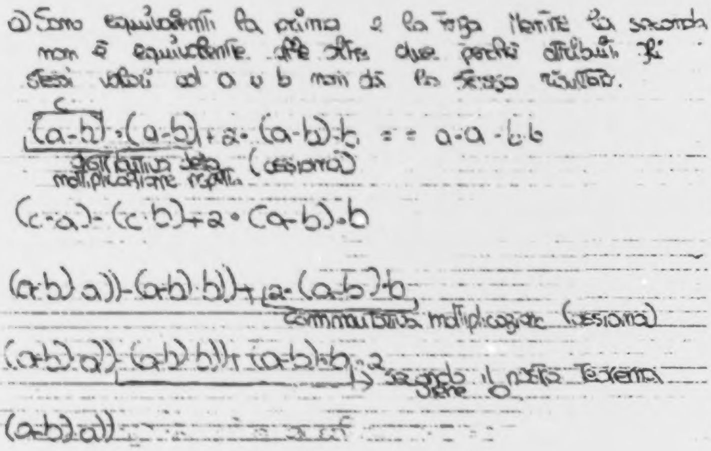
I applied the distributive property.

I applied the distributive property on these two pieces.

I summed the two equal terms *–a*b* and *–a*b* and I cancelled the result with its opposite obtaining 0 for the 1st theorem.

I cancelled also *+b*b* with its opposite and I obtained *–b*b* because it was *–2b*b*.

At this point the 3rd expression is equal to the 1st expression



a) Sono equivalenti la prima e la terza mentre la seconda non è equivalente alle altre due perché attribuiti gli stessi valori ad a e b non dà lo stesso risultato.

$\underbrace{(a-b) \cdot (a-b)} + 2 \cdot (a-b) \cdot b = = a \cdot a - b \cdot b$

distributiva della moltiplicazione rispetto (assioma)

$(c \cdot a) - (c \cdot b) + 2 \cdot (a-b) \cdot b$

$((a \cdot b) \cdot a)) - ((a \cdot b) \cdot b)) + \underbrace{(2 \cdot (a-b) \cdot b}$

commutativa moltiplicazione (assioma)

$((a \cdot b) \cdot a)) \underbrace{- ((a \cdot b) \cdot b)) + ((a-b) \cdot b) \cdot 2}$ → secondo il nostro teorema viene 0

$((a \cdot b) \cdot a))$

Figure 2. Marta writes:

The first and the third expressions are equivalent, whilst the second is not because giving the same numerical numbers to a and b the result is not the same of the other two.

Distributive property of multiplication (axiom).

Commutate property of multiplication (axiom).

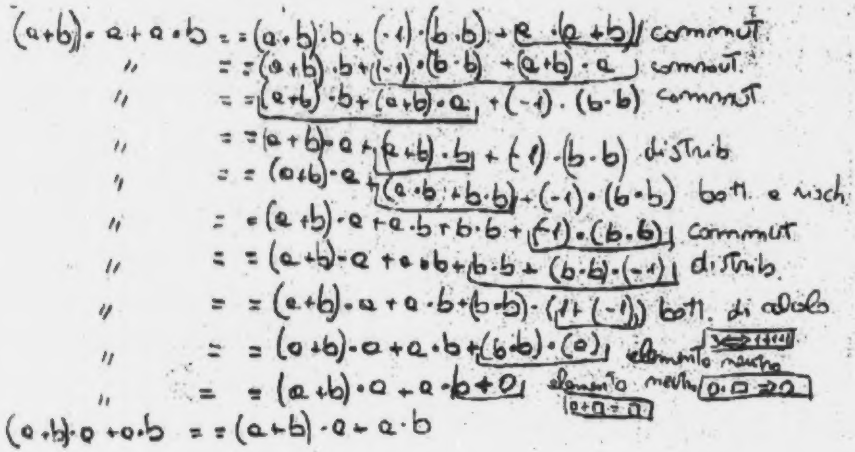
Accoriding to our theorem this is 0

Unlike Silvio, Marta (Fig. 2) does not use the properties of the operations to produce her conjecture: nevertheless, she produces a correct proof, specifying (as required), at each step, the axiom or the theorem she is applying. According to our hypothesis it looks as though the distinction, clearly defined within the microworld, has been internalised: axioms are represented by given buttons, theorems, once proved are introduced by pupils in the microworld, as new buttons. It is interesting to remark that, consistently with this history of the class community, Marta refers to the theorem, that she is applying, as "our theorem".

Signs derived from L'Algebrista

The basic point of symbolic manipulation concerns the use of operation properties as rules of transformation, which preserve the equivalence between expressions. Although perceived globally, manipulating an expression consists of successively transforming one single chunk at a time; consequently, identifying and treating the structure of an expression becomes fundamental, so as to foresee the effect of single transformations towards a specific global goal.

The representation of an expression within the L'Algebrista incorporates its mathematical tree structure, and this structure becomes explicit, thanks to the *selection function*. As a matter of fact, in order to act on an expression, one first has to select a sub-expression and then to click on a button: the corresponding command will be activated only if the selection is 'correct', i.e. compatible with the property to be applied. The selection function represents an external sign that in the actual interaction with the machine can be used as an external control on one's actions. Internalisation of this control seems to constitute a basic point in the construction of the theoretical meaning of symbolic manipulation.



$$(a+b)\cdot a+a\cdot b == (a+b)\cdot b+(-1)\cdot(b\cdot b)+\underbrace{a\cdot(a+b)} \quad \text{commut}$$

$$'' \quad == (a+b)\cdot b+\underbrace{(-1)\cdot(b\cdot b)+(a+b)\cdot a} \quad \text{commut.}$$

$$'' \quad == \underbrace{(a+b)\cdot b+(a+b)\cdot a}+(-1)\cdot(b\cdot b) \quad \text{commut.}$$

$$'' \quad == (a+b)\cdot a+\underbrace{(a+b)\cdot b}+(-1)\cdot(b\cdot b) \quad \text{distrib}$$

$$'' \quad == (a+b)\cdot a+\underbrace{(a\cdot b+b\cdot b)}+(-1)\cdot(b\cdot b) \quad \text{bott. e risch.}$$

$$'' \quad == (a+b)\cdot a+a\cdot b+b\cdot b+\underbrace{(-1)\cdot(b\cdot b)} \quad \text{commut}$$

$$'' \quad == (a+b)\cdot a+a\cdot b+\underbrace{b\cdot b+(b\cdot b)\cdot(-1)} \quad \text{distrib.}$$

$$'' \quad == (a+b)\cdot a+a\cdot b+(b\cdot b)\cdot(\underbrace{1+(-1)}) \quad \text{bott. di calcolo}$$

$$'' \quad == (a+b)\cdot a+a\cdot b+\underbrace{(b\cdot b)\cdot(0)} \quad \text{elemento neutro}$$

$$'' \quad == (a+b)\cdot a+a\cdot\underbrace{b+0} \quad \text{elemento neutro} \ \boxed{0\cdot\square=0} \ \boxed{0+\square=\square}$$

$$(a+b)\cdot a+a\cdot b == (a+b)\cdot a+a\cdot b$$

Figure 3. In the case of a comparison task, performed in the paper and pencil environment, the protocol (Fig. 1) shows that pupils use signs clearly derived from L'Algebrista, in particular the *selection function*, or the iconography of the buttons.

Lia's protocol is an example of an interesting phenomenon which can be interpreted as testifying an important phase in the internalisation process. Pupils may use the *selection function* as an external sign controlling the algebraic structure of an expression: Lia tries to prove that the two expressions are equivalent. At each step she underlines (*selects*) a sub-expression and transforms it using an axiom that applies. This behaviour, not explicitly required by the task, refers directly to the interaction between the user and L'Algebrista: Lia refers clearly to the buttons of L'Algebrista using the word *button* (It.: *bottone*) and reproducing the iconography (for instance, in the case of the *buttons of neutral elements* (It.: *elementi neutri*)[1]. At the same time, however, she refers to the properties using the expression commutative, distributive, etc.

Discussion

The instrumental function characteristic of the microworld, functioning as a control on the activity of transformation, is internalised, and contributes to the evolution of the theoretical meaning of symbolic manipulation.

A button provides the external control on the operations accomplished by the machine; the use of the button is interpsychological: it occurs externally, it concerns the interaction between the pupil and the machine; similarly[2] to a personal interchange between human beings, the machine reacts to the subject's action, autonomously from the subject's intention. This reaction may or may not be consistent with the goal of the subject but in any case it directs the activity towards the goal.

The button evolves into a sign and its functioning starts to be oriented internally, so that the button evolves into an intellectual tool regulating the symbolic transformation consistently with the goal of the task. In this respect Silvio's protocol is illuminating. Let us compare the two sequences of computation: according to the Vygotskian explanation, in the activities in the microworld the external sign, i.e. the button to be activated, has realized a break between the steps of automatic computing, allowing the pupil to become aware of the theoretical relationship between two expressions in the chain of computation. The written sign used by Lia and also used by Silvio and Marta is a temporary support control which testifies this evolution. (Similar phenomena of creating temporary sings have been described in Bartolini Bussi et al.1999; Mariotti, forthcoming b)).

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[1] 0+A ➔A: this button transforms an expression of the kind "0+A" into the expression "A", where "A" can be any expression. This button corresponds to the axiom defining the neutral element of the sum operator.

[2] Similarity with the case of human iteraction should not hide differences, which cannot be ignored. On the contrary, what we are interested in is to study the specificity of a machine, and in particular of computer devices, in the functioning of social construction of meanings. The limits of this paper prevent s full discussion which we are oblige to omit.

A final remark: as already said, our analysis does not take into account a basic point: the role of the teacher in the process of evolution of meanings.

The role of the teacher develops at the meta level, when guiding the evolution of meanings: it becomes determinant in a process of decontestulization required in order to redefine the role of "buttons", and "new buttons", outside the microworld. In fact, commands must be detached from their context and explicitly referred to mathematical theory.

Further investigations into the delicate role played be the teacher are required for a better and clearer description.

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ATTITUDE TOWARD MATHEMATICS: SOME THEORETICAL ISSUES

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Abstract

Research on affect has produced many meaningful results in the context of mathematics education. Nevertheless, the theoretical framework needs further development, in order to grant effective tools for observing, interpreting, and possibly modifying students' decisions in the context of mathematics activity. In particular the construct of attitude toward mathematics appears to be an ambiguous one. After a brief survey of some theoretical issues that are still open, we propose further questions involving the definition itself of attitude.

Attitude toward mathematics: an ambiguous construct

The attitude construct finds its origin in the context of social psychology (Allport, 1935), in connection with the question of foreseeing individual's choices in contexts like voting, buying goods, etc. The construct appears ambiguous from the beginning, but the research develops more toward the formulation of measuring instruments than toward the theoretical definition of the construct. Furthermore in these times the quantification is considered a warrant of the discipline's scientific nature. The instruments that have been produced have given theoretical and methodological contributes of great importance (such as those of Thurstone and Likert), but the attitude's "measurement" soon found itself facing the problem of identifying the possible variables.

In the field of mathematics education the construct gains renewed popularity with the reevaluation of affect in the learning of mathematics. This reevaluation, according to us, has two reasons, which are both very important:

-The needs to explain the failure in the problem solving context of individuals who possess the necessary cognitive resources. The studies in this field underline the role of metacognitive abilities (Schoenfeld, 1985; Zan, 2000) but also the influence of the emotional factors on the control processes (Borkosky, Carr, Rellinger & Pressley, 1990; De Bellis & Goldin, 1999).

-The mathematical activity itself, as described by important mathematicians, such as Hardy, Hadamard, Poincarè, is marked out by a strong interaction between cognitive and emotional aspects.

In the field of mathematics education there is a general agreement in seeing the affective domain as divided into emotions, attitudes and beliefs (McLeod, 1992). The agreement is not as unanimous in the definition of these constructs (Hart, 1989), and the studies on attitude, in mathematics education also, privilege measurement problems more than definition problems. With the development of research in this field, and especially with the increase of awareness of the affect's role in mathematics learning, the necessity of a theoretical framework has grown too. Attitude research in mathematics education has been criticized in several respects:

1. The construct of attitude appears to be vague and ambiguous.

Moreover mathematics educators often do not clearly describe the definition used in their own research (Hart, 1989).

2. The first attempts to measure attitudes seem "exceptionally primitive" (Leder, 1987), and "the driving force in much of this research seems to be the statistical methodology rather than the theory" (McLeod, 1987, p. 134). Kulm (1980) points out that: "The measurement of mathematics attitudes in the future should make use of many approaches, and researchers should not believe that scales with proper names attached to them are the only acceptable way to measure attitudes" (p. 365).

According to Hart (1989): "Beliefs, attitudes and emotions have been examined via scores on paper-and-pencil instruments and occasionally via individual student interviews. This view of beliefs, attitudes, and emotions might be called a black-box approach (...) The time and effort required to collect and analyze the data obtained from the think-aloud interview are much greater than the time and effort required for the paper-and-pencil instrument, but the information gained is a richer reflection of the student." (pp. 43-44)

3. Despite the fact that research lately has shifted from quantitative methods toward a multidimensional approach, including qualitative methods, like interviews, classroom observations, and essays, several problems have not been solved yet. One of these problems is linked with the relationship between beliefs and attitudes. Silver (1985) points out that: "(...) we need to investigate the relationship between beliefs and attitudes. Are all attitudes also beliefs; if not, then how do we distinguish those that are from those that are not?" (p. 256)

Independently from the chosen definition of attitude (whether implicit or explicit), the attitude observation instruments always include beliefs observation instruments. Thus the attitude research has to deal with problems that the beliefs research has highlighted, especially the mismatch between exposed beliefs and beliefs-in-practice: the beliefs that students declare are in the end definitely different from those that guide their solving processes and their behavior in general. A way of interpreting this mismatch is to distinguish the beliefs toward school mathematics from those toward abstract mathematics (Schoenfeld, 1989). If the measurement of attitude includes the observation of beliefs, this beliefs differentiation implies a similar distinction of attitudes.

4. More in general, some researchers think that the term "attitude toward mathematics" should be divided in several components. Kulm (1980) delineates the

objects and situations on which attention is focused for mathematics attitudes: mathematics content, mathematics characteristics, teaching practices, mathematics classroom activities, and mathematics teacher. Tirosh (1993) suggests the opportunity to describe students' beliefs, attitudes, and emotions toward mathematics in terms of "specific affects", namely their reaction and feelings toward specific mathematical topics, specific tasks and specific activities.

In conclusion, the problem of 'measuring' attitudes is often faced without an appropriate theoretical framework (McLeod, 1992).

Further theoretical questions

A deep analysis of the construct 'attitude' requires first of all a definition of the term itself. In the variety of definitions of attitude toward mathematics used in the different studies, we can identify two important typologies:

1. A 'simple' definition of attitude describes it as the positive or negative degree of affect associated to a certain subject. According to this point of view the attitude toward mathematics is just a positive or negative emotional disposition toward mathematics (McLeod, 1992; Haladyna, Shaughnessy J. & Shaughnessy M., 1983).
2. A more 'articulated' one recognizes three components in the attitude: an emotional response, the beliefs regarding the subject, the behavior toward the subject. From this point of view an individual's attitude toward mathematics is defined in a more articulated way by the emotions that he/she associates to mathematics (which, however, have a positive or negative value), by the beliefs that the individual has regarding mathematics, and by how he/she behaves (Hart, 1989).

According to us the acceptance of either definition brings up several and distinct problems.

Remarks on the 'simple' definition

-Apparently the absence of a connection with the cognitive aspects could be criticized in this definition. As a matter of fact many studies, explicitly or implicitly based on this definition, follow models (Mandler, 1984, 1989; Ortony, Clore & Collins, 1988) which emphasize the cognitive source of emotions: "Mandler's view is that most affective factors arise out of the emotional responses to the interruptions of plans or planned behavior. In Mandler's term, plans arise from the activation of a schema. The schema produces an action sequence, and if the anticipated sequence of actions cannot be completed as planned, the blockage or discrepancy is followed by a physiological response (…) at the same time the arousal occurs, the individual attempts to evaluate the meaning of this unexpected or otherwise troublesome blockage. (…) The cognitive evaluation of the interruption provides the meaning to the arousal." (McLeod, 1992, page 578)

-Even those who accept this definition, when measuring attitudes use questionnaires based on beliefs. In this way it is implicitly assumed that certain beliefs imply a positive emotional disposition.

For example a widely used item is "mathematics is useful". This belief is considered positive, assuming implicitly that it gives place to a positive emotion. As a matter of fact this implication is not natural at all, as the opinions and reported feelings of important mathematicians demonstrate (see for example Hardy, 1940).

Furthermore we agree with Gal and Ginsburg's remark (1994), about the attitude toward statistics: "Thus, a student's responses to items assessing usefulness-of-statistics issues might have little to do with feelings or attitudes towards statistics as a subject; instead they may only reflect on the student's vocational development (...) or knowledge about requirements or content of certain jobs."

-Accepting this definition, it is quite clear that 'positive attitude' means 'positive' emotional disposition. It is thus important to question *why* a positive emotional attitude is meaningful in the context of mathematics education.

The goal of promoting a positive attitude may have two reasons:

i) The attitude's influences on an individual's choices about mathematics courses to take. As Hart says (1989): "It is relatively clear that decisions about how many and which mathematics courses to take in middle school, high school, and college can be influenced by affective characteristics of the student (...)." (p. 38)

However it seems to us that to promote a general positive emotional disposition toward mathematics is not very significant, if this disposition is not linked with an epistemologically correct view of the discipline. In other words an affective goal of mathematics education is to promote a "view of mathematics as vibrant, challenging, creative, interesting, and constructive" (Silver, 1987, p. 57).

ii) The idea that a positive attitude is connected to success. As a matter of fact this connection is far from being clear. McLeod (1992) refers data from the Second International Mathematics Study, that indicate that Japanese students had a greater dislike for mathematics than students in other countries, even though Japanese achievement was very high. Ma & Kishor (1997), after analyzing the correlation attitude / achievement in 113 studies, underline that this correlation is not statistically significant: they explain this as caused by the inappropriateness of the observing instruments that had been used. According to us, on the contrary, this limitation is a natural consequence of the 'simple' definition of attitude. In fact it is not enough that the mathematical experience is generally associated with positive feelings: it is also important for such an experience to be meaningful. For example a distinction is needed between a child that likes mathematics because of the calculation involved and another one that likes it because of problem solving. Furthermore also negative emotions play an important role in mathematical activity.

Mc Leod, Metzger and Craviotto (1989) found that experts and novices have the same emotional reactions in problem solving situations, but differ in that experts are better able to control their reactions than novices. In particular a minimum degree of anxiety seems to be necessary to invest adequate resources in the task.

Remarks on the 'articulated' definition

-In this case both the emotional and cognitive dimensions are explicitly underlined. But too often the *interaction* between these dimensions is not properly considered. In particular it is important to consider beliefs together with the emotions that they elicit. For example the belief "In mathematics there is always a reason for every rule" is to be considered differently whether it elicits a positive emotion ("and I like this") or a negative one ("and I don't like this").

Similarly it is essential to distinguish emotions according to their cognitive source (Ortony et al., 1988). This suggests the opportunity to differentiate 'simple' emotions (associated with an individual's tastes) and 'complex' ones (associated with an individual's beliefs).

-According to the 'articulated' definition the attitude construct is multidimensional, so it can not be quantified with a single score. The possible alternatives are:

(i) Give a score for each dimension (beliefs, emotions and behavior). This is close to the original idea that Thurstone & Chave (1929) suggested. They pointed out that even if attitude is a complex construct (that can not be measured with a single numerical index) it can be measured using several indices. They underline the fact that the same process is followed in measuring physical objects (like a table). We find interesting their remark that the choice of the characteristics to be measured depends on the context.

(ii) Give up 'measuring' attitude, and describe it qualitatively with the pattern beliefs / emotions / behavior and the connections between them (it may be interesting to notice that in an article written recently by Ruffell, Mason & Allen (1998) the term 'probing' is used instead of 'measuring').

Regarding the dimension "beliefs", in our opinion a questionable aspect is the observation of single beliefs rather than of belief systems. For example the belief "Only those who are intuitive are successful in mathematics" does not give significant information about the individual, unless we know the belief that the individual has about his own intuition. Cooney, Shealy & Arvold (1998) underline this aspect regarding research on teachers' beliefs: "It is important to understand not only what teachers believe but also how their beliefs are structured and held" (p. 306). Grigutsch and Törner (1998) tried to carry on a research similarly focused on this aspect, analyzing the beliefs of expert mathematicians.

-In this case what a 'positive' attitude should mean is not clear: but referring only to the emotional dimension seems lessening to us. We think that two other possibilities are preferable. The first is to find out whether a typical pattern beliefs / emotions /

behavior of students who are successful in mathematics exists, and to define this pattern as positive. The second one is to define as positive the attitude typical of experts, which obviously brings up the problem of the existence of such attitude (Grigutsch & Törner, 1998). This rises other questions: recognizing if these two possibilities lead to two different definitions of positive attitude, and, if that is the case, deciding which is more useful.

Conclusions

Research in mathematics education has already accepted the importance of affective factors to interpret behavior of individuals who are involved in problem solving in the mathematics context. The research in this field has produced a substantial amount of knowledge. But for affect to be a real powerful theoretical instrument, a further analysis of the affective constructs is necessary. In each discipline, after a first period when the introduction of new concepts plays an important role in building up the theory and in the production of results (without considering how clearly and rigorously these concepts are used), a second period follows when the researchers feel the necessity of clarifying the nature of the concepts that have been used. The analysis of such concepts may create 'monsters' (i.e. ideas that apparently undermine the knowledge produced up to that moment), but this crisis is extremely fruitful for the further development of the discipline. This is what happened with concepts like those of function or continuity in mathematics history: monsters created by Dirichlet, Cantor, Weierstrass (i.e. discontinuos functions at every point) have, as a matter of fact, opened up new horizons in the field of mathematics.There is still another reason for this analysis. Because of its nature, mathematics education deals with constructs, theories and methods that are taken from other disciplinary contexts, like psychology, sociology, anthropology, etc. (Sierpinska & Kilpatrick 1998). The problems that researchers have to face, nevertheless, are exactly what pushes them to use such theories and constructs, and these problems are specific of mathematics education. As regard to the construct 'attitude', the research in mathematics education deals with problems that are different from those typical of social psychology, where the construct originated. Mathematics educators are interested not only in foreseeing the choices of the students, but in observing, interpreting and possibly modifying their decisions in the context of mathematical activity. In our opinion this difference is important, and is another reason that forces us to a deep reflection, and in particular to re-think the meaning and the sense of the terms involved.

Our aim with this paper is to contribute to this type of reflection.

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Encapsulation of a Process in Geometry

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Abstract: *The theory of encapsulating a process concentrates on "procepts" in arithmetic, algebra, and calculus (DAVIS, SFARD, TALL & GRAY et al, DUBINSKY et al). In this paper we will discuss the existence of procepts in geometry and we will give examples.*

1. Encapsulation of a Process

VINNER (in TALL 1991, p. 65ff) discussed the role of definitions in learning mathematics and how children may overcome the "conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition". Mathematical definitions mainly describe objects or a static view while the process of acquiring new insight often runs in parallel with activities or procedures or mental processes in time. Thus there are divergent roots to develop individual mathematical concepts and TALL & VINNER (1981) use the term "evoked concept image" to describe the part of the memory evoked in a given context.

These inconsistent views, an object on the one hand and procedures on the other, must grow together to form an appropriate mathematical concept (or rich and powerful "concept image" with the words of Vinner). PIAGET (1985, p. 49) has already pointed out that "actions and operations become thematized objects of thought or assimilation". This idea has become very powerful today to understand the development of certain concept images in mathematics education as a process of "interiorization" or "reification" or "encapsulation ".

"When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a *thing*. It, itself, is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. Its similarities to some other procedure can be noted, and also its key points of difference. The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun" (DAVIS 1984, p.29f).

Especially guess-and-test procedures - we think - are a valuable tool to develop such a "full range of perception", a rich "concept image", an efficient "relational understanding" (SKEMP 1978, MEISSNER 1979, 1983). In our case studies using the ONE-WAY-PRINCIPLE with calculators or computers we observed exactly that mental development as described by Davis. The children first learnt to press the correct sequence of buttons to solve the given problem. Then, when the problems change, the input must be guessed now to use the meanwhile familiar sequence of buttons to get a given output. An intuitive feeling developed how to select better inputs (MEISSNER 1987). By the use of the ONE-WAY-PRINCIPLE the children developed a relational understanding for multiplicative structures, for percentages, for growth and decay, and for the concept of function[1].

SFARD (1987) distinguishes also two kinds of mathematical definition, referring to abstract concepts as if they were real objects or speaking about processes and actions. "The structural descriptions seem to be more abstract. ... To speak about mathematical objects, we must be able to deal with products of some processes without bothering about the processes themselves." She claims (1987, p. 168) that the operational conceptions develop at an early stage of learning even if they are not deliberately fostered at school. In SFARD (1992, p. 64f) she identified a constant three-step pattern in the successive transitions from operational to structural conceptions: "First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired."

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[1] MUELLER-PHILIPP describes in her doctoral dissertation (1994) how the dynamic view (y = f(x)) and the static view (graph) grow together by guess-and-test and how children become able to switch from one view to the other.

SFARD calls these three components of concept development "interiorization", "condensation", and "reification", respectively. "Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps. Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity."

Since 1986 DUBINSKY and his colleagues (see TALL 1991, COTTRILL et al 1996) also studied the encapsulation phenomenon and they developed the APOS theory. They see three steps (A - P - O) to get mental objects (which then become part of a Schema **S**):

A Actions - usually step-by-step activities - are necessary at the beginning,

P By controlling and reflecting the action step-by-step consciously (interiorizing) the action becomes a **P**rocess.

O The process becomes an **O**bject when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations".

GRAY & TALL analyzed the duality between process and concept and came to a similar view. They consider (1991, p. 72ff) "the duality between process and concept in mathematics, in particular using the same symbolism to present both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema." They hypothesized that the successful mathematical thinker uses a mental structure "which is an amalgam of process and concept". TALL (1991, p. 251ff), reflecting the dual roles of several symbols and notations: "Given the importance of a concept which is both process and product, I find it somewhat amazing that it has no name. So I coined the portmanteau term "procept". In 1994 GRAY & TALL proposed the following definitions: "An ***elementary procept*** is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object. A ***procept*** consists of a collection of elementary procepts which have the same object." In TALL et al (2000a) we find a table of examples for symbols as process and concept.

Especially when discussing advanced mathematical thinking we can discover a lot of "procepts". DUBINSKY (2000, p. 43) lists such concepts, that have been treated on the use of APOS theory: "functions, binary operations, groups, subgroups, cosets, normality, quotient groups, induction, permutations, symmetries, existential and universal quantifiers, limits, chain rule, derivatives, infinite sequences, mean, standard deviation, central limit theorem, place value, base conversion and fractions"[2].

2. Procepts in Geometry ?

Studying the above mentioned references we miss geometry, at least "concrete", visual geometry. Are there no procepts in geometry? Is the process of learning geometry that much different from the process of learning arithmetic and algebra and calculus? Are there no procedures or processes in geometry to become objects on a procept level? Most of the work on the "encapsulation of a process to an object" concentrates on examples in arithmetic, algebra, and calculus. We do not know papers on examples in geometry.

One of the reasons might be that in many countries geometry is not in the center of teaching mathematics and therefore there is not much research on how children learn geometry. In German primary school books for example we have only about 5% of the pages with geometry topics. (And even less than 5% of the time spent on mathematics education in German primary school classes then really are used for teaching geometry).

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[2] For more details see http://www.cs.gsu.edu/~rumec/index.htm

Another reason might be that there still is a method of teaching and learning geometry which is similar to an axiomatic approach: We start with "definitions" and properties (line, point, circle, square, ...), discover relations and prove statements. Of course, it will be difficult then to discover (like in arithmetic) "processes which may produce a mental mathematical object". Then there also is no necessity in geometry for getting symbols which are used to represent either a process or an object.

TALL et al (2000b) formulate the hypothesis that there are three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) and that each of them is accompanied by a different type of cognitive development. They consider - before focusing on the growth of symbolic thinking - "briefly ... the very different cognitive development in geometry". There are perceptions of real objects initially recognized as whole gestalts and classifications of prototypes. Reconstructions are necessary to give hierarchies of shapes and to see a shape not as a physical object, but as a mental object.

STRUVE (1987) also analyses how concept images in geometry develop. He summarizes, that children in primary and lower secondary classes learn geometry like a natural science, they describe, explain, and generalize phenomena. Thus, for them, geometry becomes an empirical theory.

For the author of this paper it is a miracle that we in physics can use mathematical formulae and even complex mathematical theories to predict future events. We trust - but we cannot *prove* like in mathematics - that events will occur tomorrow in the same mode as yesterday when there will be the same conditions. There are big similarities between physics and empirical geometry: Given certain assumptions we can predict events - by the use of mathematical theories.

What does that mean for the theory of procepts? When we analyze in "3+2" possible step-by-step procedures of the children we also observe "empirical mathematics" with real objects. And like in geometry the children generalize and learn to predict future events. We trust, but we cannot prove, that "3+2" always "will be the same", but we (as mathematics educators or researchers) avoid speaking of a miracle by introducing "counting principles" (like axioms in geometry). In this view an elementary procept in the meaning of TALL et al (2000a) just is the shift from the empirical stage to the theoretical stage. Consequently, following these ideas there should be no fundamental obstacle to find procepts in geometry, too.

3. Pivotal: Tagging

When we look for procepts in geometry we first need activities or procedures. At the beginning they may be "experienced one step at a time". After practicing them for a while the user perceives "the overall patterns and continuity and flow of the entire activity, the procedure itself becomes an entity - it becomes a *thing*" (DAVIS).

TALL et al (2000a) distinguish procedure and process. For them procedure is like a specific algorithm. Using the example "4+2" there are lengthy procedures (as "count-all"), compressed into shorter procedures (like "count-on" or first "count-both" or "count-on-from-larger") or other techniques (i.e. "remembering known facts" or "deriving facts"). These different procedures all are used "to carry out essentially the same process in increasingly sophisticated ways".

We see the symbols used (4+2, 2+4, 6, etc.) like a tag to describe (or to evoke!) the according processes or objects mentioned. In general we think the symbols or tags play the same role as key words, they are tools to name or to recall a process or an object. They are used to abbreviate or to condense or to summarize the "evoked concept image" to get one single "sign" (for communication).

Symbols and tags serve like key words. In arithmetic/algebra/calculus we use letters a, b, c, d, ... and other symbols like +, %, dy/dx, ... to evoke concept images. But other key *words* like "six", "field", "parallel", ... or "□", "◊", ... or "⊕", "⊗", ... or "⊥" ... might do the same. Only important for mathematics education is the concept image evoked by that tag or symbol or key word.

TALL et al (2000b) point out that "symbols occupy a pivotal position between processes to be carried out and concepts to be thought about. They allow us both to do mathematical problems and to think about mathematical relationships". Important, there is only one symbol with a dual meaning. And we like to add, it is not important what type of symbol or tag or key word it is.

Thus we add explicitly the process of tagging or naming, that means communication is an essential part of developing procept images:

(a) carry out accurately the given one procedure/technique
(b) several procedures/techniques are possible, select one
(c) several procedures/techniques are possible, make an efficient choice ***and explain***
(d) carry out the process flexibly and efficiently, i.e. determine and select an appropriate procedure/technique ***and discuss possible alternatives***
(e) ***use an abbreviation (tags, symbols, key words)*** when discussing, arguing (the same "name" for both, the procedural and the conceptual aspect)

Table 1. Development of an (elementary) procept ***image***

4. The Spatial Procept "Net"

We will give an example from our classroom research (MEISSNER & PINKERNELL 2000):

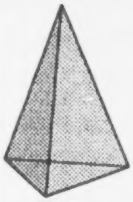


Fig. 1

A teacher showed a model of a three sided pyramid (Fig. 1) and asked the class: "What did the cardboard paper look like before I folded it to make this pyramid?" Friederike (age 8:2) drew a square and added three triangles to its sides (Fig. 2). She then showed with her hands how to fold the pyramid, pointed to the side of the square where there is no triangle, and said: "Then there's a hole, isn't there?"

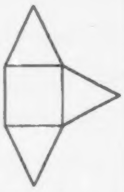


Fig. 2

Let us discuss and analyse this case in more detail. First, Friederike gets, probably without realising it, two contradictory stimuli at the same time. The key word "pyramid" leads to a square because all pyramids Friederike has known so far had a square as their base. However the given solid says that Friederike only needs three triangles. She compromises and gets the hole. Obviously she is familiar with a mental folding-up procedure, but she has not enough experiences to bridge the gap immediately.

But is it necessary at all to fold mentally before deciding if this drawing is a net of a pyramid? By looking at Fig. 2 experienced geometricians see that it does not represent the development of a solid. They can tell without actually folding the net. In their reasoning, the process of folding has been encapsulated to the static concept "development". Friederike, however, has some notion of "development" in which she still needs to carry out the process of folding explicitly, as her hands indicate. Thus we consider "development" as a procept in spatial reasoning.

This episode was observed before we systematically introduced activities in the classroom to draw developments for solids (pyramids, rectangular solids, houses): The children learned to make the net of a pyramid by placing a wooden model onto a sheet of paper and then repeatedly tilting it from its base onto one side and back to the base again, each side being encircled with a pencil. The resulting figure would be a star shaped net. Next, we have asked them to make the net of a rectangular solid. What we have experienced many times is that in strictly following the learned procedure they forget the solid's upper side and produce a net that would fold to an open box.

We then pointed onto the missing side of the given solid asking where this was drawn. Very often there was a laughter in the classroom and immediately the drawing was completed correctly. The procedure "development" they had acquired so far was based on an activity of what could be called "tilting from and back to the base". With the rectangular solid this procedure of "tilting" had to be revised by extending it. This mental change is typical for the development of procepts. Proceptual thinking also includes the ability to revise an encapsulated procedure to meet new demands (GRAY 1994, p.2). We saw a similar expanding of the procedures when we used solids with concave sides.

To draw the developments the children got a card board and a wooden solid. Some of the children just started tilting and drawing. Others first took the solid to find an appropriate starting position on the card board (by tilting without drawing) to make sure that their drawing will fit on the paper. Here the activity already becomes a flexible and efficient process.

The last lesson of that teaching unit (details see MEISSNER & MUELLER-PHILLIP 1997) started with an exhibition of about 20 different (plane) developments of buildings fixed with tape on the black board. There were only lines drawn where to fold later on (but not distinguishing if to fold inside or outside). The children (grade 3, age about 8 - 9) had to describe which net might become what type of building before they could choose one of the developments to verify their guesses. We are sure some of the children just identified simple nets without any mental folding. They just *saw* "that is a tower" (Fig. 3) or "that is a garage" (Fig. 4) or "that is a house" (Fig. 5). We think that, for them, a simple net had become an elementary procept.

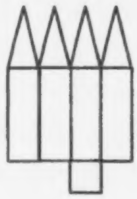


Fig. 3

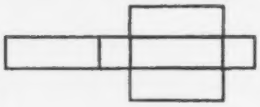


Fig. 4

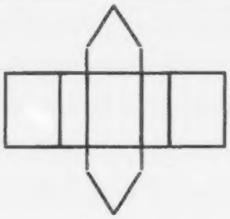
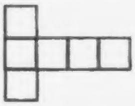


Fig. 5

But where is the symbol, one of the characteristics of a procept? We think the net itself is the symbol. The one interpretation of that symbol is a procedural one, "folding up". The other view is static, "this is ..." (an object).

Symbols of procepts follow syntactical rules. Also from this point of view there are reasons to take (at least simple nets) as a symbol. In the following we will demonstrate this view by comparing procepts from arithmetic or algebra with the procept "net".

A process is a set of procedures:
We can describe "6" by "4+2" or "5+1" or "3+3" or ... And we can describe "cube" by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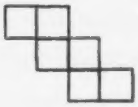
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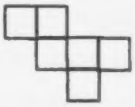
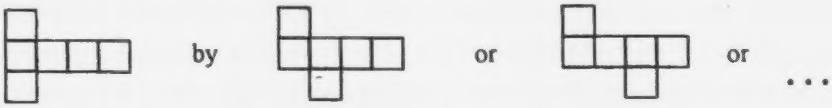

or ...

Each symbol belongs to a specific process:
This is true for "6" or "3^2" or "1/2" as well as for nets shown in figure 3, 4, or 5.

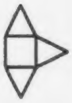
There are "syntactical rules" to transform symbols:

Replace “3+4” by “4+3” (3+4 = 4+3) or replace “2×8” by “8×2” (2×8 = 8×2) or replace “3×(4+2)” by “3×4 + 3×2” or ... We also can replace

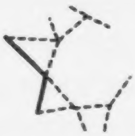


There are "syntax errors":

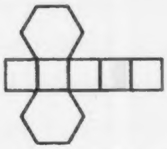
The notation of power does not allow symbols like “^2x” or “$_2x$” or “x_2” or “$_{2x}$” or “^{2x}”. Or the notation of addition does not allow “+2,4+” or “2,4,+” or ... The notation “net” does not allow



(missing side)



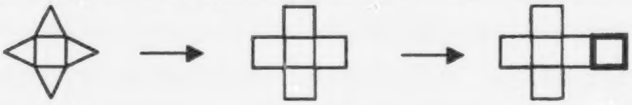
(neighbored lengths)



(number of side surfaces)

Procepts can be extended:

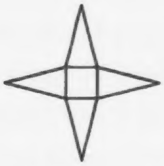
“3×4” (multiplication of integers) gets expanded to “3.5×6.9” (multiplication of decimals).



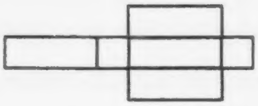
“tilting” gets expanded to a “conscious tilting”

Symbols can be variables:

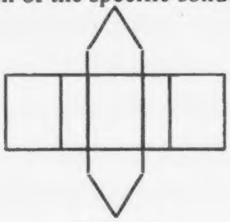
We use letters for variables in arithmetic or algebra. A “net” also may have the meaning only of a variable, i.e. by giving the spatial shape, but no geometric proportion of the specific solid:



“a” pyramid

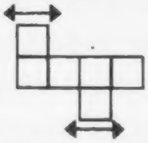


“a” rectangular solid

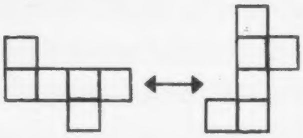


“a” house

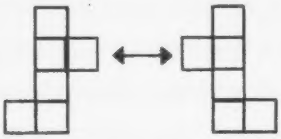
Symbols can be manipulated:



"shifting the base"



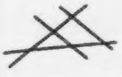
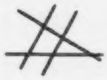
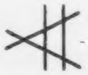
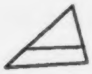
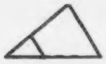
"rotation"



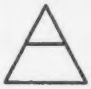
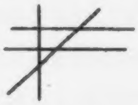
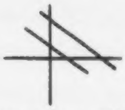
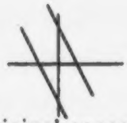
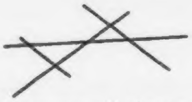
"reflecting"

5. Other Procepts in Geometry

One main theorem in geometry is, what we call in German the *Strahlensätze*. (At the PME conference in Hiroshima I was told that there is no key word for this theorem in English). A figure of four lines, where two intersecting lines cross two parallel lines, leads to three or four basic statements about the ratio of according lengths:

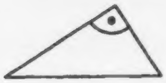
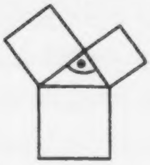






We think the symbol or tag of the Strahlensätze is one of the given figures. A proceptual thinking of "Strahlensätze" is only possible when we are able to regard the above figures as an entity (of related procedures). Then the procept "Strahlensätze" is encapsulated in each of these figures. The different types of figures can also be seen as manipulations of symbols according to syntactical rules. Some more manipulations may be the following:

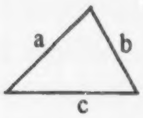
The last figures even indicate an extension of the original concept. Of course all these symbols also implicitly include variables: It is not important where the intersection point is in relation to the two parallels nor is the seize of the angle of the intersecting lines nor the width of the parallels.

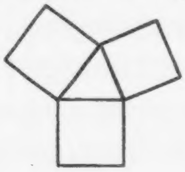
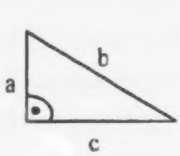
Another example is Pythagoras' Theorem. There are several types of tags:

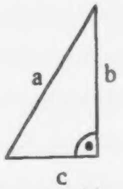



$$\boxed{a^2 + b^2 = c^2}$$

Often our students do not achieve a proceptual "pythagorean" thinking. They ignore or they do not see the property "perpendicular" or they have fixed mental images of how to name the sides of a triangle:





Also simple geometrical objects may lead to a procept. A keyword or a roughly drawn figure (of a triangle or a circle or a polygon) may provoke concept images of how to construct these figures, or of properties of these figures or of solids which these figures are part of. The roughly drawn figure might be the symbol to tag that procept with all possibilities of symbol manipulations or modifications as mentioned above. In this sense geometric drawings have a pivotal meaning. They may evoke a concept image of a single static geometric physical figure as well as of a complex procept.

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PROTOTYPICAL USES OF FUNCTION PRESENT IN SEVENTH- AND EIGHT-GRADE TEXTBOOKS FROM FIFTEEN COUNTRIES

Vilma María Mesa
University of Michigan

Conceptions of function at stake in problems and exercises in mathematics textbooks for early secondary students were investigated using Biehler's prototypical uses of function. Following a constant comparative analysis method that involved 2304 exercises and problems, five main categories were identified: rule, set-of-ordered-pairs, social, physical, and figural. Decontextualized uses—rule and set of ordered pairs—were by far the most frequent uses given to functions with contextualized uses receiving less attention. Such trend invites reflection about current practices associated with the introduction of functions in the early secondary school grades.

The evolution of the concept of function in mathematics as a discipline has followed an interesting path, changing how people understand mathematics (Buck, 1970, p. 237). The rapid changes that occurred in mathematics once the set-theoretical definition of the concept was introduced were echoed in school mathematics, generating difficult problems for the mathematics education community, problems that stimulated new lines of research (Eisenberg, 1991, p. 141). Research has provided descriptions of students', teachers', and prospective teachers' understanding of function, illustrating that their views are shaped by teaching practices, mathematical discoveries, and people's cognitive capabilities (Cooney & Wilson, 1993; Harel & Dubinsky, 1992; Sfard, 1991). The research has also suggested approaches in which less formal presentations are fostered, with technology playing an important role (NCTM, 1989, 2000; Tall, 1991). Textbook is an object that has received little attention from researchers on functions. Textbooks synthesize what is known about a concept from multiple perspectives: historical, pedagogical, and mathematical. As documents, they provide valuable information about the potential learning that could occur in a classroom. An investigation of textbook content is relevant not only to complement the set of views of function but also to help explain its relation to the difficulty of learning the concept. Obviously, what students learn from textbooks and the practicality of that learning are mediated by the school context, including teacher, peers, instruction, and assignments (Stodolsky, 1989). The textbook, as a source of potential learning, expresses what has been called the *intended curriculum* (the goals and objectives for mathematics intended for learning at a national or regional level; Travers & Westbury, 1989, p. 6), which implies that an analysis of textbook content becomes in some ways a hypothetical enterprise: What *would* happen *if*...? becomes the beginning of the inquiry. What would students learn if their mathematics classes were to cover all the textbook sections about functions in the order given? What would students learn if they had to solve all the exercises in the textbook? Would they learn what a function is? Would that learning work well in characterizing *function*? Such hypothetical questions leave open the space for different things to happen in reality: as said before, teachers mediate (and sometimes dismiss) textbook content. The answers to the questions, however, act as an *a priori* analysis, which help to determine the plausibility of different alternatives that could occur in classrooms.

Because practices within a country are usually quite similar, looking at more than one country offers the possibility of making contrasts to highlight aspects of the concept that are taken for granted within a culture. In this study I selected lower secondary school textbooks assuming that in grades 7 to 9 function would begin to appear explicitly in school mathematics. My interest in the findings of the Third International Mathematics and Science Study (TIMSS), and the availability of the textbooks used in TIMSS led me to choose those textbooks from participating countries written in a language that I could read. In the larger study I explored the *conceptions* (Balacheff, in press) of function suggested by the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS. In this paper I present the process by which the set of problems that put a conception at stake was operationalized and some associated results.

Theoretical Framework

Different situations generate different interactions between the subject (a person's cognitive dimension) and the milieu (those features of the environment that relate to mathematics), and in consequence lead to different meanings. The different interactions explain the coexistence of multiple knowings by a subject. Contradictory knowings can coexist, either at different times in a subject's history or because different situations enact different knowings. In both cases, what is isomorphic for the observer—probably the teacher—is not for the learner (Balacheff, in press). Balacheff has characterized conceptions as a quadruplet consisting of problems, representations, operations needed to solve the problems, and verification and validation activities needed to determine that an answer has been obtained and to establish its correctness. Different sets of problems require different representations, operations, and verification and validation activities that would correspond to different conceptions as described by an observer. The analysis of a conception as a 4-tuple of different but interconnected elements allows for a description of subtle differences in conceptions that otherwise could not be distinguished. However, the problems are at the core of the issue, once they are chosen, an observer can associate to them particular conceptions. I used Biehler's (in press) *prototypical domains of application*, to assist in characterizing the set of problems.

For Biehler a concept may have different meanings in different disciplines, and those meanings are determined by the differences in practices in each discipline. Three elements are constitutive of the meaning of a mathematical concept: the domains of application of the concept (its use inside and outside mathematics), its relation to other concepts and its role within a conceptual structure (a theory), and the tools and representations available for working with the concept. Using as an example the concept of function, he identifies the "prototypical ways of interpreting functions (prototypical domains of application) which summarize essential aspects of the meaning(s) of functions." These are *natural law* (e.g., a parabola as a representation of the curve of a cannon ball), *constructed relations* (e.g., to express a price depending on a quantity), *descriptive* (e.g., functions involving time-dependent processes), and *data reduction* (e.g., functions in statistics). He notes that the concept of causal relation has been abandoned in mathematics in favor of a "'functional relation' between two quantities (Sierpinska, 1992)... due to philosophical reasons [and] to pragmatic ones: If we have a 1-1 correspondence, we can invert the cause-effect functional relation to infer the 'causes' from the effects." The decision to invert the relation is rooted in the academic practice of mathematics; in disciplines such as physics, it might not make sense. Biehler's characterization of prototypical domains of application of function, that is, its uses, was instrumental for me in initiating a characterization of the problems in a textbook that eventually can be solved by the student. These different uses gave me a stepping stone to use in characterizing the problems needed to define the conceptions that could be elicited by textbook exercises. With these tools, I looked for answering the following question: What are the prototypical uses of function present in the seventh- and eighth-grade mathematics textbooks of selected countries participating in TIMSS?[1]

Method

The original sample for the study consisted of 35 textbooks from 18 countries chosen from the TIMSS data base according to the following criteria: the textbook was intended for 7th, 8th, or 9th grade; the textbook was written in English, Spanish, German, French, or Portuguese; and the textbook contained references to *functions, linear functions, graphing in two coordinates, graphing in the Cartesian plane, tables, patterns,* or *relations.* All the exercises, hereafter *tasks*, from such sections (for a total of 2304 tasks) constituted the corpus of data for the study. Each task received a 4-tuple code. The first code, P, identified the prototypical use of function present in the task. The second, O, contained all the operations that were needed to solve the task. The third, R, contained all the representations that were needed to solve the task. Finally, Σ contained all the activities available for the student to verify that a solution was obtained and that it was correct.

The development of the categories for coding each element of the quadruplet was accomplished in four steps. First, I selected one task from the first section of each textbook to analyze in depth (35 tasks). I worked each one, following as much as possible the textbook presentation that preceded the exercise section and developing categories for each element of the quadruplet. Second, I used the resulting categories to code the remaining tasks in all the first sections of each textbook, looking for new categories and refining the properties of each. I used the *constant comparative* method (Glaser & Strauss, 1967) in which I described the salient features of the categories for an element and at the same time looked for possible breaks or mismatches that could lead to the creation of a new category. This second step involved 518 tasks and resulted in 133 categories. Because there were so many categories, the third step consisted in merging categories within common groups, thus yielding a smaller, more manageable number of categories for each element. The final step was to test the coding system by having other raters use it to code tasks, which helped to further refine and validate the categories of the coding system. The final system consisted of 10 codes for uses of function, 36 codes for operations, 9 for representations, and 9 for controls. I report the results associated with the first element of the conception, the prototypical uses of function present in the tasks.

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[1] The larger study from which this paper is derived tackled the issue of *conceptions* suggested by textbooks (see Mesa, 2000).

Results

When working with the first set of 35 tasks, I found that Biehler's initial classification (natural law, constructed relations, descriptive, and data reduction*)* did not include tasks lacking a real context: namely, when the function was treated as a set of ordered pairs (e.g., "Represent in the Cartesian plane the relationship whose solution is given by the set $R = (x, y) \mid x, y > 0 \wedge x, y \in \mathbf{R}\}$"), when it was treated as a rule, when a pattern with numbers or figures was sought, or when there was a proportion involved. Biehler's categorization was not accounting for phenomena that are particular to mathematics (Puig, 1997). In addition, in some tasks that suggested relations that could be classified as constructed using Biehler's characterization, the content used geometrical definitions or principles (e.g., similarity), that suggested an additional category. The following task is an illustration of such cases:

> The slide projector puts a picture on the screen. The size of the picture changes as you move the projector. The picture gets bigger and bigger as you move the projector further away. When the projector is 300 cm from the screen, the picture is 120 cm high. Here are figures for other distances [a table with six values for distance and height is given].
>
> Draw two axes on graph paper. Mark the across axis from 0 to 500 and the up axis from 0 to 200. Label the across axis 'Distance from screen in cm'. Label the other axis correctly. Use the figures in the table [given] to plot points.
>
> (a)What do you notice about the points you have plotted? (b) Use your ruler to draw the graph through the points. (c) Use the graph to find the height of the picture when the projector is 350cm from the screen. (d) How far is the projector from the screen when the picture is 50cm high?

Such uses of function were labeled, *set of ordered-pairs, rule, pattern, proportion*, and *geometrical* respectively. Biehler's "descriptive relation" was renamed *cause and effect* and was used to characterize the cases in which the task dealt with physical phenomena not dependent on time. After the final coding step, a new category appeared, *graph*, which was used to characterize those tasks in which the relation was given by a graph in a Cartesian plane that did not have any marks (e.g., a graph of a function $f(x)$ is given; the student has to identify the graph that corresponds to $f^1(x)$). I kept a record of all the different instances of uses within each category. These examples of uses were crucial in fully characterizing the categories for uses of function (see Appendix).

To simplify the presentation, the uses that referred to physical phenomena, cause-and-effect relations, and time relations were grouped into a new category called *physical* to capture the character of these relations. Because they relate to human activity, data-reduction relations and constructed relations were grouped into a new category called *social*. Geometrical relations, graph-defined relations, and pattern relations were grouped into a new category called *figural*, to highlight the crucial role of images and patterns for defining functions in these relations. Rule and direct proportion/proportion relation were grouped together into the category *rule*. *Set of ordered pairs* was left as a separate category.[2] Table 1 presents the frequencies and percentage of occurrence of these categories. The results refer only to grades 7 and 8, yielding a sample of 1319 tasks, in 24 textbooks from fifteen countries.

Table 1: Frequency and Percentage of Prototypical Uses of Function

Uses	Frequency	%
Rule	556	42
Set of Ordered Pairs	319	24
Social	227	17
Physical	136	10
Figural	81	6

As the table shows, the most frequent uses were rule and set of ordered pairs. Only one third of the uses corresponded to those involving concrete contexts: namely, social, physical, and figural. Social uses were almost twice as frequent as physical uses, which suggests that at these grade levels physical phenomena in which functions can be defined do not play a very important role. Almost 25% of the tasks had a set-of-ordered-pairs use of function, which implies that such definition still plays an important role for introducing the notion. In contrast, the figural use of function accounted for only 6% of the tasks which implies that at these grade levels it is not a common practice to present functions based solely on mathematical phenomena. Six textbooks showed important differences from this pattern: in three textbooks (from two countries, about 5% of the tasks) there were no tasks with a set-of-ordered-pair or rule use (one textbook had only social uses and the other two had mainly physical uses). The other three textbooks (from three countries, about 8% of the tasks) had more than 40% of social and physical uses.

Discussion and Conclusion

One possible reason for the high frequency of rule uses in textbooks might be didactical: Because the idea of correspondence is so fundamental to the (modern) notion of function, and because the seventh and eighth grades mark the transition period from arithmetic to algebra, transformations of numbers by means of basic operations seem to fit the double purpose of defining valid functions—with a notion of correspondence as *transformation* or *constrained variation*—while at the same time linking known operations with the new idea of correspondence. In this way the burden of considering unrealistic situations in which the correspondence can be *arbitrary* (as is the case with the Set-of ordered-pairs use) is overcome. In other words, such uses of function are actually serving the didactical purposes of smoothing the transition from arithmetic to algebra and of introducing the idea of correspondence. The somewhat large proportion of tasks that with a set-of-ordered-pairs use of function may be due to authors' interest in keeping the textbooks updated mathematically; also, because textbooks change slowly (Farrell & Heyneman, 1994), the influence of the new math movement was still operating in this set of textbooks (with one exception, textbooks with copyrights in the 70s and early 80s contained more of such uses). The modest contribution of tasks with social or physical uses seem to be a consequence of reform movements that suggest the use of applications in presenting mathematical concepts. The difference between social and physical uses might be related to the complexities that the latter bring to the problems (Monk, 1992). One possible reason for the few instances of Figural use might be linked to the separation between geometry, arithmetic, and algebra in school mathematics curricula. The textbooks tended to contain separate chapters for geometry, and it seems likely that within those chapters, functions did not get much attention. The low frequency of geometrical uses could be also a consequence of the new math movement, which almost eliminated geometry from school mathematics in several countries (Ruiz & Barrantes, 1993).

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[2] The reorganization is not only a practical one; I wanted to highlight particular characteristics of the use of function, which make several categories look as equivalent. Different criteria would produce a different reorganization.

Vinner (1992), Norman (1992), and Even (1989) have documented that students, teachers, and prospective teachers view functions as defined by an algebraic expression that involves numerical variables only or as black-box machines (a rule use), with meanings that are contradictory with the formal Dirichlet-Bourbaki notion (a set-of-ordered-pair use). The results of this study suggest that these groups' understandings of function could have been expected, given the uses given to functions in textbook tasks. The characterization of problems and exercises of textbooks in this sample indicate that when functions are initially introduced to students, textbook authors tend to prefer situations in which the relation is defined trough transformation of an input to obtain an output. There is also a tendency to present a formal view of function as a set of ordered pairs in which the notion of arbitrary assignment is presented. Contextual uses of function do not seem to play a significant role in the majority of the countries. Such trend poses questions to us as researchers and as curriculum developers: is the same tendency observed in classrooms? Is this an appropriate strategy to follow for students' first encounters with the notion of function? To what extent is a more contextualized approach, as promoted by reform movements, more appropriate? How do we establish that appropriateness? As uses of function are just one of the constitutive elements of a conception, one can expect that these uses would not necessarily dictate what operations, representations, and control activities are enacted in a task. That is not the case, however (see Mesa, 2000). Tasks with rule uses tended to be associated with a very limited (in number and in requirements) sets of operations, representations, and controls. In contrast tasks with physical and figural uses offered the most variation, thus creating possibilities of enacting different meanings for functions but also illustrating that textbooks may promote the existence of several apparently contradictory views of function. Even though there are not clear answers to these questions, the actual situation should invite reflection on an intermediate strategy to follow if a variety of (uncontradictory!) meanings are to be promoted in the introduction of functions in early secondary school.

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Appendix: Characterization and Examples of Prototypical Uses of Function in Tasks

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Cause/effect

Used to code content that refers to physical phenomena other than time related and in which the behavior of one variable is an effect of the behavior of the other (it is a directional relationship). *Atmospheric pressure vs. boiling point, Density of water/ice vs. temperature; Hooke's Law*

Constructed

Used to code content that refers to "real life" situations other than cause/effect, time, data reduction, and geometrical. In these relations it is somehow arbitrary which variable is called dependent and which one independent. An interchange of the roles of the variables produces equally valid (for the context) relationships. *Number of goods (gas, phone calls, book, etc.) vs. number of other goods or vs. cost, Conversions*

Direct proportion/proportion relation

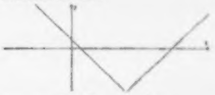
Used to code content where there is an explicit reference to a proportion or a direct proportion without context. *Fill a table in such a way that there is a direct proportion between the entries*

Data reduction relation

Used to code statistical situations; in situations involving two variables it may be possible to have more than one outcome for a given value of a variable. *Change of price of movie vs. year, Consumer price index vs. year, Diameter of sample of tree trunks*

Graph defined relation

Used to code content where the relation is presented in a graph whose two axes are neither labeled nor numbered.



Geometrical relation

Used to code content that refers to geometric figures and their characteristics. *Similarity; Height of a tower of cubes vs. number of cubes, visible or invisible faces, edges, and vertices.*

Pattern relation

Used to code content in which given a sequence the question is to find the general term (or an expression for the nth element) of the sequence. *Expression for triangular numbers, Number of sides of a polygon vs. number of diagonals*

Rule relation

Used to code content in which an input is transformed by certain procedure to obtain an output and in which a context is not provided.

All polynomial; rational, periodic; piece-wise; radical; step; trigonometric. Computer programming

Set-of-ordered-pairs relation

Used to code content where a list of ordered pairs is given or requested.

Any arbitrary pair assignment, Localization of points in a Cartesian plane, Relatives; numerical

Time relation

Used to code content that refers to physical phenomena where time is involved and the variable is treated continuously.

Speed) vs. distance; Speed vs. time; Distance vs. time.

ARGUMENTATIVE PROCESSES IN PROBLEM SOLVING SITUATIONS: THE MEDIATION OF TOOLS

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Abstract

Argumentative processes enacted in problem solving situations involve both concrete and discursive operations. An analysis of the processes of solution needs to take into account three main components: the problem, the agent and the context of solution (including all the tools available, other individuals and the situation for the devolution of the problem). When the context incorporates a dynamic geometry software, additional aspects related to the availability of dynamic tools need to be considered. The paper illustrates and analyses the role of heterogeneous tools, such as concrete tools and conceptual and theoretical tools in the process of elaborating a conjecture and proving it. Possible conflicts due to the static nature of the theory and the dynamic nature of the tools for exploration are discussed.

Introduction

Solving a mathematical problem requiring the elaboration and proof of a conjecture is a complex activity, made up of a number of phases and involving discursive, concrete and mental operations. The literature accounts for a number of studies focusing on the proving process and especially on the possible relationship between the production of the conjecture and the construction of a proof for it (Duval, 1992-93, Mariotti et al. 1997). Despite the undoubted differences between the argumentative and the proving discourse, a cognitive continuity between them seems possible, under specific constraints on the problem situation. An Italian research group (Boero et al. 1996, Mariotti et al.1997, Garuti et al.1998) has elaborated the theoretical construct of cognitive unity, that attempts to describe and analyse such continuity:

> **CU**: during the production of a conjecture, the student progressively works out his/her statement through an intensive argumentative activity, functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organising some of the previously produced arguments according to a logical chain. (Garuti et al, 1998)

For the purposes of the present paper, the cognitive unity (CU) construct will be considered as an analytical tool to interpret and explain some of the processes students engage in when striving to organise the informal arguments produced during the solution process, into a logical chain that corresponds to accepted mathematical rules. Concrete and discursive operations involved in the production and enchaining of arguments have a cognitive counterpart in mental operations for which they are the external signs. In order to trace the evolution of arguments toward a deductive discourse within the solution of a problem, I will set the issue in the mediated activity framework. From this perspective, in any problem solving situation three main components may be identified: the problem, the agent, i.e. the individual-acting-with-mediational-means (Wertsch, 1991) and the context of solution (including all the tools available, other individuals, be they peers or teachers, and the situation for the devolution of the problem). The study reported in this paper focuses on geometrical, open problems tackled by 11th and 12th grade students in a context including the dynamic geometry software Cabri-Géomètre (Baulac et al.,1988).

The mediation of tools: the notion of toolkit.

Within the context outlined in the previous section, the word 'tool' incorporates many different meanings and refers to both concrete and psychological tools. Drawing on the seminal work of Vygotsky, tools and signs (i.e. psychological tools) may be distinguished according to their function:

> The tool's function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to changes in objects. [...] The sign, on the other hand, changes nothing in the object of a psychological operation. [...] the sign is internally oriented. (Vygotsky, 1978, p. 55)

In some cases the distinction cannot be neatly drawn: within a certain activity, since some of the externally oriented tools may be internalised and function as psychological tools. The internalisation process, as well as the relationships among the tools used within a certain context, are complex and manifold. In order to describe them I have introduced the notion of **toolkit** (Mogetta, to appear) as an organised set of (both externally and internally oriented) tools that each individual develops and uses in a particular context. The idea of toolkit is meant to account for:

- *the diverse and manifold nature of its components*. Verbal or written signs, symbolic systems of notation, drawings, constructions and changes of configuration, dynamic manipulation through dragging, measurement and gestures may all be subsumed under the category of externally oriented tools, since the actions they mediate aim at changing the external, phenomenological world. Language, strategies of solution, theorems and definitions and all sorts of conceptual tools belong to the category of internally oriented tools, which shape and influence the mental processes enacted along the solution;
- *the relationships among the components, that may change and evolve continuously*. Mutual relationships among tools that are heterogeneous in nature may be developed, along the solution of a problem, either through a process of internalisation of externally oriented tools or through a joint use of different tools, that are necessarily re-interpreted (and, in some cases, re-conceptualised[1]) in a new context, as they may acquire a new functionality.

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[1] For instance, a student might use a theorem as an exploration tool in an initial phase of the solution, seeking possible properties of the configuration at hand, and later reuse the same theorem in order to justify the conjecture. In this case the same conceptual tool 'theorem' is functionally related to the phase of solution and assumes the value of argument when a proof is constructed.

The individual has a crucial role in the management of his/her personal toolkit: the evolution that occurs within the context of a problem situation is basically subjective and does not follow general rules. Wertsch (1991) refers to 'privileging' as a dynamic strategy to select one mediational means as more appropriate or efficacious in a particular socio-cultural setting. For example, in the context of the solution of a geometrical problem the choice of a theoretical tool, like a theorem, may be influenced by the exploration of the figure by means of the dragging tool (in Cabri): a particular configuration may be visualised and recall the mental image associated to a particular theorem. Nevertheless, in order for that theorem to function as a tool, the agent must have developed a modality of use for it. The basic idea is that any object or concept need to undergo a process of instrumental genesis (Rabardel, 1995, Verillon & Rabardel, 1995) in order to be fruitfully used for a certain purpose. Such process is described as turning an **artefact**, i.e. "the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task" into an "**instrument**, that is the artefact and the modalities of its use, as are elaborated by a particular user" (Mariotti, to appear). Schemes of use are individually developed and shape and organise the actions performed by an agent within any mediated activity.

Heterogeneous tools: is a harmonisation possible?

The re-organisation of tools within the specific context of the solution of a problem brings about a re-interpretation of previously acquired tools and the development of (possibly new) appropriate modalities of use for them. The process of re-interpretation is not necessarily successful, since there might be conflicts between the phenomenological and the theoretical worlds (Balacheff & Sutherland 1994), which coexist in the context where the problem is tackled. If the internal relationships among the elements constituting the toolkit mainly involve tools of the same type, and if the concrete and psychological tools are not related to an organised system of theoretical knowings[2], the world of theory and that of phenomenology can stay separated. Cabri is a microworld that incorporates the basics of Euclidean geometry as well as tools that allow a dynamic exploration and that give visual and conceptual feedback to the agent (Laborde, 1998). It may offer a good ground for the construction of a link between operations related to the phenomenological world and concepts and operations within the (Euclidean geometry) theory. Once they have been internalised, tools like dragging or even a generalised use of measurement, may control behaviour and shape the process of solution. As a consequence, two possibly conflicting situations might occur at the same time. On the one hand the mental processes based on an empirical approach and on visual evidence, that are spontaneously used by the agent, may be related to more rigorous and deductive processes and, consequently, the argumentative phase can end up in a proof. On the other hand, the co-existence of informal/empirical elements and formal/theoretical elements, can bring about cognitive conflicts that cannot be overcome spontaneously.

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[2] The term knowing has been introduced by N. Balacheff to account for the distinction between the French *savoir* and *connaissance*, which is not possible in English. So knowledge is the English for savoir and knowing is the English for connaissance.

The changes and evolution of the internal organisation of each individual toolkit during the solution of a problem are important for the argumentative process. When the concrete (externally oriented) tools are used in harmony with the theoretical (internally oriented) tools, the elements and information gathered along the conjecturing phase may be linked systematically within a structured argumentation that fits with the rules of the theory (Mogetta, to appear), according to the CU hypothesis. Things are not so linear in the actual solution of a problem: the intermingled use of tools of different nature requires a continuous re-organisation of the toolkit and an internal re-negotiation of the meanings previously attached to some of the tools.

The case of Andrea: when tools of different nature are not harmonised

The data analysis carried out for the study[3] reported in the paper attempts to show how the personal toolkit of an individual agent may change along the solution of a problem and how the relationships among the tools can evolve. Such changes might head toward a harmonisation of empirical/perceptual and theoretical aspects, or rather cause (and show) a cognitive rupture between the argumentative process and the actual production of a proof for the conjecture. This section illustrates the case of a student who does not manage to harmonise tools of a different nature, thus staying at an empirical and perceptual level in the production and enchaining of arguments. Andrea (12th grade, Liceo Scientifico), had been asked to solve the following problem in the Cabri environment:

Two intersecting circles C1 and C2 have a chord AB in common. Let C be a variable point on circle C1. Extend segments CA and CB to intersect the circle C2 at E and F respectively.
What can you say about the chord EF as C varies on circle C1?
Which is the geometric locus of the midpoint of EF as C varies on the circle?
Justify the answers you provide.

and talk aloud along the whole process, the interview being video-taped. Andrea starts off with a dynamic exploration of the problem, dragging point C around the first circle in order to observe the behaviour of chord EF. The conjecture of the constant length of EF is formulated on the basis of the visual evidence from the Cabri screen and a first attempt of explanation involving the fixedness of A and B is provided:

A: Yes, because ... EF always has to stay in the circle, the lines [CA and CB] *have a certain freedom of movement and not more than that and they have to go through those points necessarily. Hence when C is shifted in one direction* [he moves his hands showing the gap between the two lines as a rotating angle that does not change in width] *I don't know how to explain ... hold o, I'll think about it*

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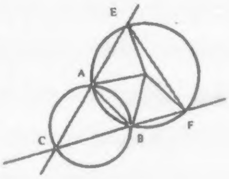
[3] The study has been conducted within a PhD project, carried out at the Graduate School of Education, University of Bristol and funded by the ESRC.

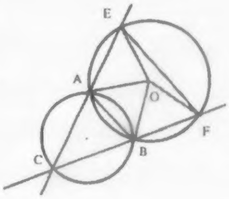
The rigid rotation of the lines through A and C and through B and C is perceived as the reason of the invariance of chord EF. The dragging of point C around the circle simply confirms that the intuition is correct and does not provide any other hint on the underlying geometrical reasons. The initial, purely exploratory, use of the dragging tool is partially modified when A. focuses on the quadrilateral in order to identify variable and invariant elements. It is the beginning of a dialectic between static and dynamic elements, respectively identified in constant angles and segments and in rotating triangles or lines, or a variable quadrilateral. Nevertheless, the basis of the actions is still empirical and the invariance is still sought at the perceptual level, with no reference to the possible geometrical reasons:

A: ... I was thinking about this quadrilateral EFAB ... to see it as many triangles and then ... I don't know... I try that [he is constructing segments to form a quadrilateral split into triangles with vertex at the centre of circle] *and all the triangles then are constant ... well ... they are all isosceles triangles*



The fact that the conjecture is not refined in correspondence with the operations carried out on the figure, makes Andrea stick to the empirical approach. The lack of a phase of unpacking for the perceptually based conjecture in terms of geometrical relationships with other elements of the figure brings about a persistence in a random exploration by means of the dragging tool, with a slow and continuous movement of point C around the circle. The multiple changes in the strategies adopted to explore the conjecture involve different conceptual and concrete tools, spanning from dragging to additional constructions, to the search for invariants within sub-figures. The idea of looking at segments and angles that stay constant as C is dragged around the circle, seems to suggest the possibility of using Carnot's theorem (known in the English mathematical tradition as the cosine rule)

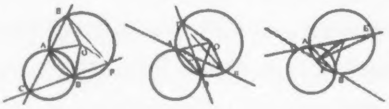
A: ... a cyclic quadrilateral, because as this angle here varies this other angle varies as well ... hence for Carnot's theorem ...[applied to triangle EOF] *... two sides and the angle in between ... a relationship for this one ... then a relationship for this other one with the angle ... this one is constant, this one is constant ... I can do it with Carnot's theorem!*
C: How?
A: To prove that EF is constant ... because it is enough to have two sides and the angle in between in order to find the length of EF ... the two sides are both r *and the angle is always the same...*



Andrea's use of the theorem is nearly circular: in order to find the length of EF and show that it is constant Andrea wants to use the fact that angle EOF is constant, which has not been proved, and which is strictly linked to the invariance of EF.

Although the theorem seems to have been conceptualised correctly, its use in this specific context is not recalling other theoretical results or making a link between the perceptual and the conceptual aspects of the problem. Andrea seems to "let" the theorem take up the cognitive burden in the phase of justification: the elements involved in the theorem's hypothesis are assumed to have the properties required simply on the basis of the perceptual judgement coming from the dragging test. The dragging function of Cabri might potentially be the tool linking concrete and conceptual elements of the toolkit: but the use Andrea makes of it, even when he uses dragging in combination with theorems, is not successful in this respect. Dragging is mainly used at the perceptual level and invariance is sought and conceived of as a visual property of the figure, which has some obvious underlying reason in the rotation of the lines under the constraints of the fixed points A and B. Theorems are kind of "imposed" upon the dynamic figure, but they are not linked up with the preceding argumentative discourse based on the dynamic exploration carried out by means of externally oriented tools, such as dragging and measurement:

A: ... constant... it is 360° ... I put x this one does not move ... but with Cabri I can measure them...
C: You can measure ... do whatever ...
A: [he measures the angles with vertex at O] *and I see that when C is dragged the angle stays constant*



Tools of different types come to interact and to be used jointly, but the lack of theoretical control by Andrea does not lead to a fruitful interaction. The argumentation produced as a justification for the invariance of angle EOF, used as a given in the application of Carnot's theorem, is empirical and draws heavily on the exploration through dragging, as a tool ensuring the general validity of a property observed in the continuous motion:

A: I know for Carnot's theorem that instead of finding ... AB, I may find the angle and then in a triangle the angle... this one is a still triangle, it does not move and therefore angles cannot transform with no reason, since it stays fixed ... [...] and there was the chord theorem... I do not remember how it goes ... the angle at the circumference ...

Along the whole solution process Andrea tries to use a number of theorems to justify his conjecture, without expressing it in relational terms: the invariance of chord EF is not related to the invariance of any other element of the figure. Although a theorem about chords is recalled, following the development of the previous reasoning around angles and segments, and, later similar triangles are sought, A. does not abandon his conviction based on the visual evidence. The conjecture, elaborated and formulated in terms of dynamic causality, remains linked to the phenomenological aspects of the problem; any attempt to justify it by means of theorems ends up in a list of results that do not correspond to the interplay of variable and invariant elements in the problems. The cognitive unity of conjecture and proof is thus broken, possibly due to the conflict between dynamic causality, pertaining to the phenomenological world, and static theory.

Discussion and concluding remarks

The analysis of the protocol presented in the previous section shows that the heterogeneity of tools within the context of solution of a problem may bring about difficulties in the elaboration of an argumentation and its possible evolution toward a proof. Andrea's personal toolkit includes externally oriented tools, such as dragging, measurement and additional constructions, as well as a number of theoretical knowings, exemplified by the use of Carnot's theorem, or the recalling of theorems about chords. The problem is that such heterogeneous tools are not harmonised: the joint use of dragging and theorems, for instance, is not fruitful because the dynamic, visual evidence of the (perceptual) invariance of elements under dragging is not interpreted in terms of the geometrical reason that would justify it within the theory.

Two main issues need consideration, also in view of future research: (i) the identification of possible reasons for the lack of harmonisation of heterogeneous tools and (ii) a deeper analysis of the features of dynamic geometry environments, with a particular focus on the issue of internalisation of the dragging tool.

As for the former issue, this paper suggests that a scarce theoretical control of the (concrete and mental) operations performed during the solution of a problem may end up in an unsystematic and nearly random use of theorems. The individual organisation of theoretical knowings may account for their status within the theory (hypothesis, premise, conclusion, derivation, postulate and so forth) as it has been conceptualised by the individual agent. Modalities of use of such tools necessarily reflect such organisation and require the explicit formulation of relationships among geometrical objects in terms of their status within the problem. When theoretical knowings are used in combination with tools of a different nature, they might be re-interpreted in the new context. On the other side, concrete tools might be used to refine the conjecture in relational terms and this might require the development of new modalities of use for them. In actual fact, when a conjecture is formulated in a compact form, on the basis of the dynamic, visual evidence of a property (e.g. the rigid movement of two lines seen as the cause of the invariance of a chord), a process of refinement is necessary in order to make explicit the (static) relationships among the involved objects. Often, the strength of the dynamic causality interrupts the refining process, thus provoking a cognitive rupture in the solution process. Results of the ongoing main study suggest a conjecture/hypothesis (Mogetta, to appear), with possibly strong educational implications. *The dynamic nature of the explorations carried out in the Cabri environment by means of the dragging tool, may conflict with the static nature of the theory (Euclidean geometry). Such conflict may bring about difficulties in linking the arguments elaborated during the conjecturing phase with those needed to construct a proof.*

Further studies are needed in order to test the hypothesis and to better characterise some of the features of dynamic geometry environments, in terms of the nature of the tools they make available to the individual-acting-with-mediational-means. One of the most important aspects to be analysed is the issue of the internalisation of the dragging tool. As suggested in this paper, the dragging tool has the potentiality to link up phenomenological and theoretical world. The point is to internalise it and use the dynamic variation of the figure in order to make a link between perceptual and geometrical invariance of objects.

Finally, more evidence is needed in order to establish whether and how an appropriate management of the individual toolkit each individual develops and uses in particular contexts, is linked to the idea of cognitive unity . My hypothesis, still to be tested is that *there is a tendency to establish, or re-establish CU once it is broken. In correspondence with possible different causes for the rupture to occur an appropriate management and organisation of the toolkit may help the agent overcome the rupture.* Further research is necessary to illuminate the issue and evaluate the actual impact of possible ruptures of the CU on the construction of a meaning of proving.

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TEACHERS' CLASSROOM INTERACTIONS IN ICT-BASED MATHEMATICS LESSONS

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Abstract *Do teachers' classroom interactions in ICT-based secondary school mathematics lessons differ from those in non-ICT-based lessons? This paper reports on data obtained from classroom observations of teachers who attempted to incorporate ICT into their lessons over the course of a school year. The analysis of the data does not indicate significant changes in teachers' classroom interactions in ICT-based lessons other than those that have quite straightforward explanations.*

Introduction

Do teachers' classroom interactions (roles, behaviours, communication) change when teaching and learning moves from 'normal' lessons to information and communications technology (ICT) based lessons? This question, while intriguing, requires clarification before an attempt at an answer can be made. The literature, however, often suggests a change in teachers' roles, e.g. Heid et al. (1990) state:

> In the implementation of computer-based laboratory explorations, the teacher must become a technical assistant, a collaborator, and a facilitator. ... the teacher will need refined skills as a discussion leader and as a catalyst for self-directed student learning.

This is a particularly strong statement chosen to make the point and the authors are clearly open to the charge of confusing an 'is' with an 'ought'. Farrell (1996) represents a more guarded statement:

> ...evidence suggests that teachers are holding on to the managerial roles while taking on some new roles (e.g. consultant, fellow investigator) when technology is used.

This paper reports on an analysis of videotapes of ICT and non-ICT mathematics lessons of 13 teachers. The analysis was part of a wider project and I begin with a brief overview of this project.

The wider study

The *Moving from Occasional to Regular Use of Technology in Secondary Mathematics Classes* project explored patterns of teaching and learning, teachers' preparation and use of resources, teachers and students' attitudes and teachers' confidence, over the course of one school year. 13 teachers made a commitment to move to 'regular' use of technology in the 1998/99 school year. Most had some experience using technology in their classes but none had made extensive use of technology before. Whilst I recognise the difficulties, if not the impossibility, of characterising 'ordinary teachers' these 13 were, in an everyday sense, ordinary teachers in ordinary schools. Project funds provided a financial incentive for schools/mathematics departments to be involved and ICT experts/enthusiasts who volunteered to be involved were excluded from the project. The 13 teachers were both the subjects of the research and teacher-researchers. Most of the teacher-researchers wrote of their experiences in a series of articles in the UK professional journal *Micromath* (volumes 14/3, 15/2 and 15/3, 1998/99).

'ICT use in mathematics classes' is a collective term for a diffuse range of software and hardware. In an attempt to focus on something common and manageable project work focused on using technology tools: spreadsheets, graphic packages and calculators and algebra and geometry systems. In an attempt to keep the project work as realistic as possible individual team members chose the tools they thought most appropriate for use with their classes. As these tools have widest application with classes studying algebra, upper secondary classes (14-18 year olds) were the focus of the project work (one project class per teacher).

Methodology

The project made use of a wide range of data collection and analysis tools but here I report only on aspects of data collection and analysis relevant to teachers' observed classroom behaviours. Apart from noting patterns in these behaviours *per se* I was interested in possible patterns of change over the course of a school year. For each teacher the lessons observed were with the same class, the project class. Our resources were not sufficient to observe every lesson over the course of a year. The maximum number of lessons it was considered feasible to observe and analyze was estimated to be four lessons per teacher, 52 lessons in total. It was decided that each teacher would be observed prior to starting ICT work. I refer to this as the 'base-line' lesson. Thereafter each teacher would be observed near the beginning, towards the middle and towards the end of the year in ICT lessons. With the exception of one teacher, where the last ICT lesson observation was not observed, this plan was carried out.

Lesson observation formats were discussed in project team meetings and the tool chosen for classroom observations was a modified version of the Systematic Classroom Analysis Notation (SCAN) (Beeby et al., 1979). It was decided that all lesson observations would be videotaped. A desire for consistency in records of lessons and considerable prior experience (often problematic) in videotaping lessons informed our practical decisions on how to videotape: the camera was stationary in a position that was as non-obtrusive as possible but which allowed the teacher and all the students to be seen; a remote cordless microphone was attached to the teacher; the camera followed the teacher whenever s/he was speaking. Although we were, subjectively, satisfied with suitability of the video recordings for SCAN analysis, there were some problems which should be noted: the visual image was centred on the teacher and students working away from the teacher were not recorded; the microphone picked up the teacher's words clearly but students' words were sometimes obscured; the camera did not always produce a clear image of written work or screen images which was the focus of discussion; technical problems in three of the 51 lessons resulted in much of the sound being unclear.

The remainder of this section describes the modified SCAN system and its standardisation and use in this research but first it should be noted that the length restriction on this paper prevent a full presentation the data. To avoid repeated reference to this fact I now collect together all curtailments, mergers and omissions.

- SCAN descriptors focused on student activity are not presented.
- A number of SCAN descriptors have been omitted.
- The SCAN descriptors explaining and facilitating have been merged in the Results section.
- SCAN data is presented for only three of the 13 teachers.
- Details of how the original SCAN was modified are not presented (these were not extensive).
- I have not subjected SCAN to criticism.

SCAN works simultaneously on three time scales – 'activity', 'episode' and 'event'. Lessons are viewed as a series of activities, e.g. teacher exposition, students working, teacher-student dialogue. Each activity is viewed as a series of episodes, e.g. coaching, explaining. Events sub-divide the episodes into social and linguistic categories, e.g. managerial, confirmation. I now provide a fuller description of activities, episodes and events referred to in this paper.

Activities: C - whole class exposition. Dn - dialogue, between teacher and a group of *n* pupils. D1 for one-to-one dialogue. D2/3 for teacher talking to a group of two or three students.

Episodes were essentially about what the teachers were doing, e.g. facilitating, explaining. We made a distinction between technological and mathematical foci in episodes. Thus two 'facilitatings', Ft and Fi, and two 'explainings', Et and Ei were introduced. Coaching, Co, or eliciting reasons/ideas from students was assumed to be mathematical and not technological.

The coding of the events were ways of describing what was happening on a small scale within an episode. It worked out that essentially each sentence or two was coded. The linguistic descriptors of the events were based on what the teachers were saying – an assertion, a; an instruction, i; a confirmation, cf. The questions, qi or qt, were the questions that the teachers were asking the pupils, not what the pupils were asking the teachers. (A confirmation, or rejection, was often the reply of a teacher to a pupil question.)

Question qualifiers described the level of the teacher's question, not the level of a pupil's question nor the general dialogue.

Nature or depth of question:

α - question requiring recall, single fact, single act, no processing involved

β - question of straight forward nature, putting together several facts.

γ - question extending of previous work involving new ideas

Situation or Level of guidance

1 - highly structured, close direction, small number of steps.

2 - some guidance, requires connections rather than selection.

3 - minimum guidance, open.

Coding were written into grids as below. The numbers denote minutes. 30 second blocks of time were the basic unit of analysis. Rows one and three were used to record, respectively, teacher and student, activity. Row two was used to record episodes and their linguistic descriptors and qualifiers.

		1		2		3		4		
T										
Ep										
St										

A research assistant and I spent over 20 hours coding 10 minute fragments until we reached 85% agreement. Thereafter the research assistant coded all tapes of lessons.

Results

Table 1 SCAN statistics for three teachers

	C	D1	D2/3	Co	EFi	EFt	a	i	cf	qi	qt	α1	α2	α3	β1	β2	β3	γ1	γ2	γ3	time
1	47	39		28	37		15	9	4	15		4				11					29
2	55	28	13	6	8	64			2	6	3	7				2					34
3	26	50	21		72	17	25	42	35	34	4	19			1	16	2				36
4	35	58	3		23	55	44	45	27	18	22	20			2	13				5	35
1	68	21		39	27		5	2	3	51		36	3			12					70
2	41	34		4	17	37	1	18	1	4	3	4				3					70
3	19	74		8	77		61	33	7	33		24	3		1	4			1		59
4	41	44	10		63	25	65	46	8	37	18	44	1			8				2	64
1	61	11		46	7		11	34		38		31				7					70
2	62	25		27	6	39	42	88	2	1	9	5				5					72
3	61	21			35	38	48	80	21	14	19	22	2			11					71
4	38	21			31	20	32	61	12	16	2	13				5					59

Table 1 shows SCAN statistics for the four videotaped lessons of three teachers. Column 1 indicates the lesson and the last column shows the length of the lesson in minutes. The codes for the other columns have been described earlier. Columns C to EFt represent percentage of lesson times. Columns $\alpha1$ to $\gamma3$ denote raw numbers of occurrences, e.g. teacher 1 in lesson 1 made 15 assertions.

Comparing the 13 non-ICT lessons with the 38 ICT-based lessons seven features stand out as markedly different. The figures below represent averages.

Table 2 SCAN statistics which show a marked difference over all 13 teachers

	non-ICT	ICT
(1) the percentage of time spent in teacher-whole class exposition (C)	48%	19%
(2) the percentage of time teachers spent talking to two or more students (D2/3)	28%	45%
(3) the percentage of time teachers spent coaching or eliciting ideas from students (Co)	19%	4%
(4) the percentage of time teachers spent explaining or facilitating mathematical ideas (EFi)	44%	29%
(5) the percentage of time teachers spent explaining or facilitating technological features (EFt)	0%	24%
(6) the number of assertions teachers made during lessons (a)	9	35
(7) the number of instructions (or initiating remarks) teachers made during lessons (i)	15	50

Discussion

I address issues relating to Table 1 and Table 2 before considering claims for and against changes in mathematics teachers roles, behaviours and interactions in ICT lessons.

While the project teachers had a number of common characteristics they also had many differences shaped by their attitudes to mathematics and to ICT, the ethos of their school and of their department and by their classes. The three teachers in Table 1 were selected to show differences whilst being representative of the 13 teachers. Teacher 1's project class were 14 to 15 year olds. The ICT lessons used a spreadsheet and a graphic package (principally for transformations). She used ICT almost every Tuesday of the project year. My own subjective reaction after her first observed ICT lesson was that she had imported her normal classroom technique to the computer room. Teacher 2's project class was an Advanced level class of 16 to 17 year olds. The ICT work focused on using *Derive* but spreadsheet and a graphic package was used as well. He 'blocked' his ICT work with the class, i.e. there were periods of intense use and periods of little use. He stated that he wanted to use *Derive* as a 'teach yourself' tool to break away from the norm of 'chalk and talk' with this class and he designed worksheets to assist him with this aim. Teacher 3's project class were 14 to 15 year olds who had a graphic calculator each during ICT lessons (but they did not take them home). As with teacher 2, ICT lessons were 'blocked'. ICT lessons observed were predominantly teacher led with the class imitating the teacher's key strokes. Again my subjective impression after the first observed ICT lesson was how similar his style was to his teacher-led style in non-ICT lessons.

My subjective impressions of similar styles in the first ICT and the non-ICT lesson of teachers 1 and 3 is, to some extent, borne out by a comparison of figures in the C and D columns for these two lessons of these two teachers. Later ICT lessons, however, show less whole class teaching. Against any effect or non-effect of ICT, however, it may simply be that mathematics teachers do less exposition later in the year as they become more familiar with a class.

The $\alpha 1$ to $\gamma 3$ columns of Table 1 are interesting but, perhaps, not surprising, because of the clustering in columns $\alpha 1$ and $\beta 2$. SCAN is not an instrument for discourse analysis, so statements about teacher-student discourse must be guarded. However, the very few $\gamma 3$ occurrences in ICT lessons must question any general claim that teachers become fellow investigators in ICT lessons. Where they do occur, however, they occur in ICT lessons and in later lessons (but again teacher-student familiarity over time could be a factor).

Table 2 shows some marked general differences between ICT and non-ICT lessons but, I believe, these can be explained in most cases in very practical ways. All 13 video-taped non-ICT lessons were of the form 'teacher exposition followed by students working on exercises'. The significant reduction in teacher-exposition in ICT-based lessons, (1), may be viewed partially as an organisational factor in that six of the teachers prepared their classes before they moved to the computer room. The percentage increase in time that teachers talked to two or more students, (2), largely reflects the fact that the availability of computers forced students to work with two or more to a machine. It is interesting to note, however, that even when students worked in pairs in non-ICT lessons the teacher talk was largely directed to one of the pair but in ICT-based lessons the teacher talk was largely directed to all students around a computer. The significance of the coaching figures, (3), lies in the relative absence of this in ICT-based lessons. I must admit that I cannot explain this but coaching in all lessons was interpreted as the teacher pointing out mathematical features without revealing the answer and a relative absence of this in ICT-lessons does not, to me, suggest that the teachers are acting as 'a catalyst for self-directed student learning'. The figures in (4) and (5) have obvious explanations in the ICT, or not, focus of the lesson (if you do not have ICT in your classroom, then you are not going to . The figures in (6) and (7) represent the average number of assertions and instructions teachers made. These averages conceal great variation over teachers and different lessons. One reason for the greater average in ICT-based lessons was an apparent propensity in ICT-based lessons for six of the teachers to move quickly around the class ensuring that technical problems did not slow work down, "copy cell B3 to D3".

There is thus a discrepancy, with many papers reporting "Clear changes in social behaviour and teaching methodologies could be seen by observation of the [computer] lessons" (Schneider, 2000) whereas, apart from changes resulting from quite straightforward reasons, this was not observed to any marked degree in the reported study.

One possible reason for the apparent discrepancy is that projects which report on changes in teachers' roles focus on teachers who are technology enthusiasts. The teachers in this project were volunteers who wanted to use ICT in their teaching, so there is likely to be something else at work here. One thing the project teachers had little prior experience of was of using ICT in their mathematics lessons. I suspect that the time factor of experience is important (as commented above, Table 1 shows that later ICT lessons involved less whole class teaching.). In this context time is not just time learning how to use ICT tools but time to get 'a feel' for how lessons will run by having tried things out. As Moriera and Noss (1995) put it:

> Developing a coherent pedagogical approach for learning with computational media is a far from trivial exercise. Time is an important factor ... it is more a matter of '*taking time to percolate*' than just to locate or create new working environments for pupils.

It must also be considered whether SCAN, or my use of it, is a reason for the discrepancy between my findings and other work. SCAN is certainly quite a clumsy tool for analyzing teacher-student discourse and I used it in quite a crude way to make frequency counts but every effort was made to ensure reliability and validity.

My final consideration in this section concerns innovation. A danger for researchers in the field of the use of ICT in mathematics classes is a tendency to believe that ICT innovation is somehow unique. Prestage (1996) made a study of teachers' perceptions of sequencing and progression as they implemented changes introduced by the UK National Curriculum. She claimed the teachers worked in three phases over time: (i) trying to accommodate the (assumed) givens without questioning, (ii) making sense of the (assumed) givens, (iii) trying to accommodate the (now personally interpreted) givens within their own frameworks for teaching and learning. My point in introducing Prestage's work is that there are similarities with my comments above on experience and time and that ICT innovation shares problems with other types of innovation. Cuban (1989), in a reaction to a computer symposium where claims were being made about changes in teachers' roles, drew parallels to earlier 20th century educational technology innovations (films, radio, television) and stated that "teachers teach the way they do simply to survive the impossibilities inherent in the workplace". Cuban may be overstating the case but all 13 project teachers saw their practice as supporting external curriculum and assessment criteria and felt a moral obligation to their students that ICT work had to support learning which would be assessed without ICT. Their innovative work had to support traditional norms. In such conditions it is hardly surprising that there were no significant changes in these 13 teachers' classroom interactions in ICT-based lessons other than those that have quite straightforward explanations.

The conditions and constraints that teachers work under point to the need for further research in this field. Like Cuban (1989) I do not think it is enough to expect that a machine will affect teachers' classroom interactions. Researchers and policy makers should be looking at how the curriculum and school structures might allow for new roles for teacher-student interaction – with and without ICT.

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NOVICES' AND EXPERTS' KNOWLEDGE ON STATISTICS AND RESEARCH METHODOLOGY

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Abstract

Many social science students in university constantly experience difficulties with research methodology and statistics courses. In this study we compared novice students', advanced students' and experts' knowledge on these complex and difficult domains. The results of the study refers to a tendency for novices, especially when they have had problems with mathematical subjects, to place the contents to emotional or other irrelevant categories which restricts or blocks their other cognitive activities on these subjects. There was a remarkable difference in the fragmentation of concept maps and explanations between novices, advanced students and experts. The novices were also not able to use sufficient representations to understand the concepts.

Introduction

Research methodology and statistics courses are constantly experienced as difficult by many university social science students (Filinson & Niklas, 1992; Forte, 1995; Lehtinen & Rui, 1995; Murtonen, 2000). When learning research methodology and statistics, students often face the situation when they for example may have some everyday experiences of statistical phenomena and implicit conceptions or "theories" of these phenomena, but they cannot relate these experiences to their studies. Their statistical knowledge may be composed of fragmented and isolated pieces, which do function at a sufficient level in specific situations, but do not connect the phenomenon into a wider context. Students may also try to understand the concept on the basis of the surface structure of the name of the concept. It is also possible that an integrated whole of statistical concepts is not possible to attain without enough operational understanding and experiences of the methods (Sfard 1991).

According to Chi (1992), conceptual change occurs when concepts are transferred from one ontological category to another. In Chi's model, the ontological entities belong to different ontological categories. This refers to an objectivist, Aristotelian ontological assumption that the categories exist in the world. In contrast, cognitive theories suggest that categories do not exist in the world, but in the humans' minds. According to Lakoff's and Johnson's (Lakoff, 1987, Lakoff and Johnson, 1999) embodied philosophy, categories are humans' way of behaving in the environment. On the basis of this theory we assumed that there might be several categories in humans' minds, which vary from person to person. There might also be emotional categories like "difficult things" that students form on the basis of their experiences. For example, if a student has experienced difficulties with mathematics, he or she might have placed the whole subject into a category of difficult or unpleasant things. Later, they may place statistics into this same category, because statistics reminds them of mathematics. This kind of a categorisation seems to function as an obstacle to other cognitive activities.

According to Núñez, advanced mathematical abilities are not independent of the cognitive apparatus used outside of mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought such as image schemas, aspectual schemas, conceptual blends, and conceptual metaphor. (Núñez, 2000.) Similarly, to understand statistical phenomenon students should be able to relate them to their everyday experiences and thoughts.

The aim of this study is to compare novice students', advanced students' and experts' conceptions of research and statistical concepts in order to find the major differences between them. We hypothesise that novices, especially when they have mathematical problems, do place contents to emotional or other irrelevant categories that block other cognitive activities, while those with more positive experiences do not do so. We also assume that novices' knowledge is more fragmented than experts' knowledge, and that they lack operational understanding and experience and because of that they try to understand concepts on the basis of the surface structure, and that they are less capable of using representations and metaphors to help understanding.

Method

This study consisted of two phases. In the first phase, a questionnaire was filled in by 31 education students in the beginning of a statistics course. Two questionnaires were used. The first one was a test of statistical content knowledge measuring the understanding of e.g. mean, deviation, correlation and statistical inference. Students were also asked to estimate their certainty in each of the tasks. The other questionnaire dealt with students experienced difficulties in quantitative research methods, attitudes on research and learning orientations. On the basis of these questionnaires, four students were selected for further research. Two of the chosen students succeeded well in the statistical test, were confident in doing the tasks, did not experience difficulties in quantitative methods, had positive attitude towards the methods and their orientation was deep and task oriented. They will be called the 'advanced students' because of their good success in the statistics tasks. The other two students had considerable problems in the statistics tasks, had experienced difficulties in quantitative methods and they were not confident in the tasks. They will be called the 'novice students'. They experienced problems in learning quantitative methods and they did not appreciate the methods. They were not deep oriented toward learning methodology but were more self-defensively oriented and less task oriented than the other two students.

In the second phase these four students were interviewed after the statistics course. We also interviewed two experts to be able to compare the students' answers to an expert view. The experts were psychologists who had been working as researchers for many years. All interviewees were female. The interviews were conducted in pairs on each expertise level. Both two researchers were present all the time. The interviews lasted from 1 to 2 hours. The reason for interviewing two students/experts at the same time by two researchers was to encourage a discussion between students and also between students and researchers.

The interview was about conceptions of scientific research and statistics. The interviewees were asked to explain what scientific research is and simultaneously to draw a concept map of scientific research. During and after drawing a concept map the students were asked questions concerning their attitudes and conceptions of different domains of scientific research and especially about statistics. Specific questions about statistics were asked on what they think that happens in a t-test and do they know what the p-value really stands for.

Results

A category of difficult things

In the questionnaires we asked the students about the difficulty of quantitative method courses. In order to confirm that we found the students we were looking for, we asked the students in the interviews about their experiences. We started with novice students:

> *Interviewer:* How do you experience research methodology as a subject to be learnt?
>
> *Laura:* It feels more difficult than other courses. It might be that when one specific subject is easy to learn, then this [methodology] is kind of a clump. It somehow frightens. It feels somehow foggy and difficult to learn.
>
> *Interviewer:* Does it include the whole research or just some specific domain?
>
> *Laura:* I cannot figure out the specific domains, but as a whole. Just the research - everything else feels detached from it.
>
> *Emma:* Well, at least statistics feels very difficult. It would be good to have a link from it to something more practical. It has now got a bit clearer, when I have been doing my practice work, but at the beginning it was really hard.

Laura's notion about the research being a clump seems to be a good example of a category of difficult things. She has no tools for managing research domain and she is also frightened about it. She cannot even name a domain inside methodology that is the most problematic. When Emma identifies statistics, Laura agrees with her. Laura refers to other study subjects which are comprehensible to her as independent domains, but methodology represents to her a domain that she cannot link to the other study subjects and she cannot understand the subdomains of research methodology. Emma mentions that things have become clearer when she has been working with her practise work, which refers to the importance of practise and operational activity in the elementary understanding of statistical concepts. The advanced students had a very different view on methodology and statistics:

> *Maria:* Well, if you think about statistics, you sure have to work on them, but I haven't had any problems that I couldn't have overcome. Rather, I would say it's refreshing, to have something else, something different. I have always liked mathematics, for to have something else, too.

> *Jenny:* And it is different from... if you think about our major subject in general, it's much about building up aggregate domains and understanding things, but here you have to learn also by heart what they mean and think how they are connected. It's not difficult. Maybe demands more work, but it's not more difficult.

In Maria's comment there is a reference to positive experiences with mathematics and she even talks about statistics as a 'refreshing' part of her studies. In Jenny's comment there is a reference to the difficulty of the subject but also a confident reassurance that she is going to work to be able to comprehend the things. She did not find the work impossible.

Concept maps

The concept maps of the student pairs and the experts were very different from each other. The maps are shown in Figures 1-3. The advanced students started to do the given task eagerly and they were confident about what they were doing. The experts asked if we wanted some specific kind of a map or can they just draw what they want (they were told to draw what they want). The novice students were worried about how they will do and they didn't know where to begin. They were asked to just start somewhere to write concepts on the paper. The concept map they produced was more fragmented than the others' map. It did not have as much content as advanced students' map and the concepts were just floating it the air. The map did not show much logic in the placement of the concepts. The advanced students did have a coherent structure that proceeded chronologically in the same sequence as ideal research. They drew first a small map of principles in science and then they were asked to think about practise also. They drew a different map of research in practise, but they said in the interview that these two could have been drawn in the same map.

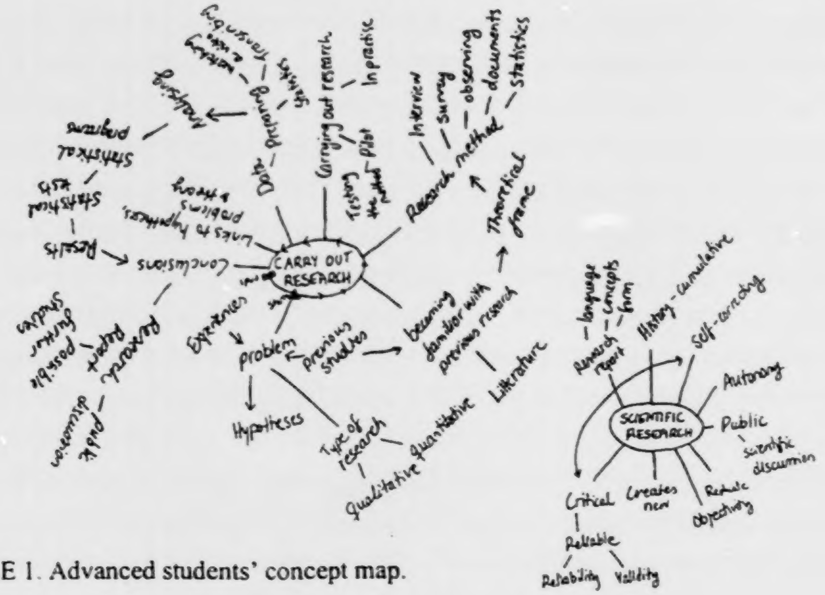
CARRY OUT RESEARCH
Experiences
Problem
Previous studies
becoming familiar with previous research
Hypotheses
Type of research
Qualitative
Quantitative
Literature
Theoretical frame
Research method
Interview
Survey
observing
documents
statistics
Testing the method - Pilot
Carrying out research
In practice
Data - processing
Conclusions
Results
Statistical test
SCIENTIFIC RESEARCH
Research report
language
concepts
form
History - cumulative
Self-correcting
Autonomy
Public
scientific discussion
Critical
Creates new
Refutable
objectivity
Reliable
Reliability
Validity

FIGURE 1. Advanced students' concept map.

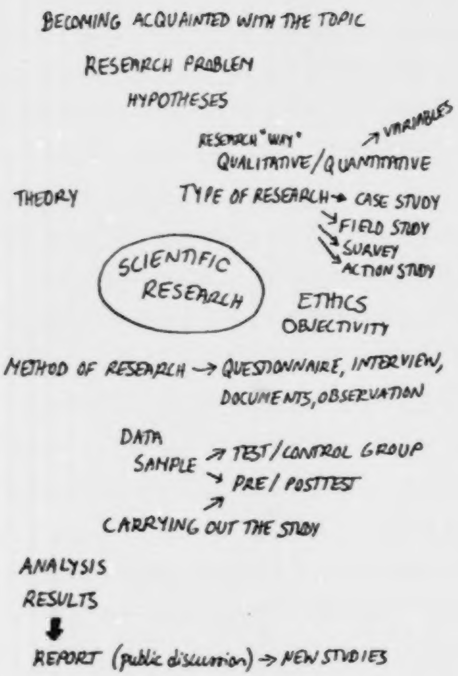
BECOMING ACQUAINTED WITH THE TOPIC
RESEARCH PROBLEM
HYPOTHESES
RESEARCH "WAY"
VARIABLES
QUALITATIVE/QUANTITATIVE
THEORY
TYPE OF RESEARCH → CASE STUDY
FIELD STUDY
SURVEY
ACTION STUDY
SCIENTIFIC RESEARCH
ETHICS
OBJECTIVITY
METHOD OF RESEARCH → QUESTIONNAIRE, INTERVIEW, DOCUMENTS, OBSERVATION
DATA
SAMPLE → TEST/CONTROL GROUP
PRE/POSTTEST
CARRYING OUT THE STUDY
ANALYSIS
RESULTS
REPORT (public discussion) → NEW STUDIES

FIGURE 2. Novive students' concept map.

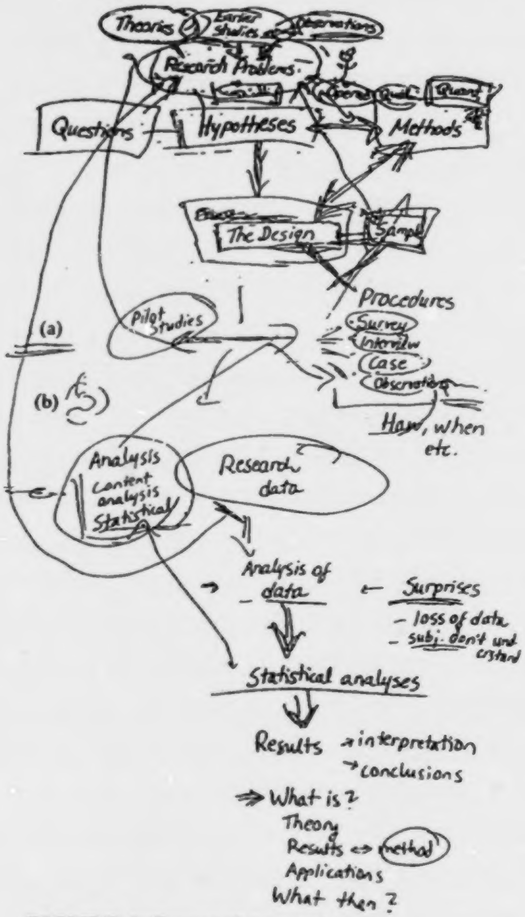
Theories
Earlier studies
Observations
Research Problems
Quali
Quant
Questions
Hypotheses
Methods
The Design
Sample
Procedures
Survey
Interview
Case
Observation
How, when etc.
(a)
(b)
Pilot studies
Analysis
content analysis
Statistical
Research data
Analysis of data
Surprises
- loss of data
- subj. don't understand
Statistical analyses
Results → interpretation
→ conclusions
⇒ What is?
Theory
Results ⇔ method
Applications
What then?

FIGURE 3. Experts' concept map.

The experts' concept map had many dimensions. They constantly talked about "interaction" between the subdomains and they also drew lines and arrows to describe the interaction:

> *Eva:* And these are, of course, constantly interacting with each other. You cannot draw a research like this. [Eva drew the pattern (a) in Figure 3]
> *Irma:* Yes, you cannot.
> *Eva:* Instead, it is something very complicated… [Eva drew the pattern (b) in Figure 3]

Although the advanced students did have lots of connections between the concepts in their map, they did not talk about the interaction between the domains. The experts mentioned that it is difficult to draw a figure of research, because it does not progress linearly. They ended up with a conclusion that research has to be organized somehow to be able to report it and thus it can be represented as a product. According to them, research description is like schemata that frame the research processes. Research report also gives opportunity to replicate the study, which in turn is an important tool for assessing the validity and reliability of a research. The experts noticed that research problems can rise from many different view points. They drew a researcher in the right upper corner of the concept map to point out that it depends on the researcher's individuality how he or she does the research. The student pairs did not mention the impact of an individual researcher.

While both of the student pairs always sought for a conception that they could both agree about, the experts did not consider it a problem to have a different conception about some issues. In the end they concluded that this concept map represented only a small piece of scientific research, which is an activity of a research community and some local research groups with their own activities are just a small part of the bigger unit. They thus saw research as a very wide concept, while students only saw it as one practical research project.

> *Eva:* And here we get to our beloved, research design
> *Irma:* I thought it goes here with these (shows the methods)
> *Eva:* Well, not quite… actually
> *Irma:* Well, for me it belongs there
> *Eva:* Ok, you could put it there if you were writing a section about research methods
> *Irma:* Yes, then it would be it's own section below the methods.
> *Eva:* For me this (research design) is a very important part. This is the core of the reasoning when we are building up the research logic.

The maps and the processes that the researchers saw when the students drew the maps suggest that the novice students did not have a clear conception about how research proceeds in practise.

Representations of statistical concepts

The novice students were asked if they are familiar with t-test. They said it was introduced superficially, but they did not know it very well. Then the interviewer asked if they knew what the p-value stands for. The students had just finished a statistics course, where they had studied the p-value, so they should have been familiar with it. We had the following conversation:

> *Emma:* I was just looking for the practice work, well, it (p-value) is a kind of, I mean, how it goes…
> *Laura:* … significant and almost significant…
> *Emma:* Well, that how they go, all the commas and nulls and others… I asked about it in a statistics lecture and the teacher tried to explain. She wrote this awful formula on the blackboard and explained that it is based on that and there is some theoretical thing in the computer and it comes from all of these… and … (laughing) it is not clear to me…
> *Interviewer:* Is it somehow mystical?

> *Emma:* No, it has been explained
>
> *Laura:* Yes, it has been explained that, how it goes... But, when you should explain it in the results... It is quite easy to look it from the papers, that what is significant and so, but, strictly speaking, I don't get it at all.
>
> *Emma:* Those certain numbers are in all of them, I mean that p is smaller than this and this, well, the significant is easy, but when it gives you all the numbers, then I cannot understand where these all numbers belong to.

The conversation above shows again how unconfident the novice students are about their knowledge and also how fragmented and fuzzy their knowledge is. When talking about t-test with the advanced students, Maria explains about two groups and simultaneously keeps her both hands in the air in front of her as to show two groups. The interviewer asks what the hands represent:

> *Maria:* Well, t-test reminds me about two groups
>
> *Interviewer:* Do you see some kind of distribution figures in your mind?
>
> *Maria:* No, I don't

When we asked the same question about t-test from the experts, they grasped pens eagerly and wanted to draw. They were, however, first asked to explain without paper and pencil. They showed similar hand representations as the advanced student did when she was describing the comparison groups. When allowed to draw, the experts drew two distribution lines partly overlapping each other. They had thus helpful representations about the asked test.

Discussion

The interviews showed that we succeeded to find with our questionnaires a pair of novice students, who had problems with statistics content knowledge and also attitudinal problems, and a pair of students who were good with the content knowledge and did not have attitudinal problems. The interviews about the difficulty of methodology referred to the tendency of some students to create a category of difficult things, a "clump", where they place all things they think that are not possible for them to learn. This kind of a categorisation seems to function as an obstacle to further cognitive activities. The novice students also called for more practices, which suggests that they suffer from the lack of operational understanding and helpful representations of the concepts.

The major difference in the concept maps of the interviewees was their state of fragmentation. The map of the novices was a static picture composed of fragmented pieces of external knowledge with hardly any connections between them. The map of the advanced students had more structural elements, connections between the domains and indications of a process' like knowledge, even some dynamics. There was, however a noticeable difference between the concept maps of the students and the one of the experts. The map of the experts formed an integrated whole of the research, which was clearly structured but simultaneously had the dynamics of the research in action. Besides the formal knowledge of research methodology there was also a vision of the important informal knowledge reflecting the experience of the experts. The experts had also clear ways of representing the given statistical concepts, while novices had hardly any indications of representations.

The most important finding of this study was the evidence of a category of difficult things, which the novice students had, but the more advanced did not have. We suggest that this kind of mental categorisation might be one of the serious challenges to the learning of statistical methods. In order to support the learning in this kind of a complex domain, deliberate teaching arrangements are needed to help student to reassign the "difficult" things into a category of "possible for me to learn" things.

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ON CONVERGENCE OF A SERIES: THE UNBEARABLE INCONCLUSIVENESS OF THE LIMIT-COMPARISON TEST

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Formal mathematical reasoning is often studied in terms of the students' conceptualisation of the necessity for proof, as opposed to empirical ways of reasoning; or in terms of the mechanics of the students' reasoning regarding specific proving techniques. Here, in the context of testing the convergence of a series in Calculus, we address one issue regarding the latter. In this, statements about convergence, for instance the Limit Comparison Test, are transformed by the students into statements about divergence in particularly problematic ways; in fact in ways that suggest a multiplicity of difficulties with mathematical logic and a resistance to the idea that, in certain occasions, convergence tests are inconclusive.

Transition to advanced mathematical thinking is often described as acknowledgement of and fluency with the abstract nature of mathematical objects and with formal mathematical reasoning (e.g. Tall 1991). Within upper secondary and university mathematics the latter has been studied mostly in the context of Proof: either in terms of the students' conceptualisation of a necessity for proof (as opposed, for instance, to intuitive, informal or empirical ways of reasoning (e.g. Coe and Ruthven 1994)); or it terms of the students' enactment of these conceptualisations (namely the mechanics of their reasoning, for instance regarding specific proving techniques such as Mathematical Induction (e.g. Movshovitz-Hadar 1993)). Studying the students' formal mathematical reasoning with the foci suggested in the latter studies is a particularly multi-layered, complex task as it involves a consideration of the conceptual difficulties within the specific mathematical topics, students' problem solving skills (a map of this complexity can be found in (Moore 1994)).

In this paper we wish to address one issue regarding the students' enactment of proving techniques. This is in the context of testing the convergence of a series in Calculus, a task encountered by most students in the beginning of the first year of their undergraduate studies. For this we will draw on data collected in a study of the transition from school to university mathematics currently in progress in one Mathematics Department in the UK. First however we outline the study and its methodology.

Methodology. This project is funded by the Nuffield Foundation and its initial phase (Phase 1: Calculus and Linear Algebra, October - December 2000) lasted 3 months (Phase 2: Probability will be January - March 2001). It is located within a series of projects that the first author has been involved in for several years (see Note 1) and its title is *The First-Year Mathematics Undergraduate's Problematic Transition from Informal to Formal Mathematical Writing: Foci of Caution and Action for the Teacher of Mathematics at Undergraduate Level.* It is an Action Research project (Elliott 1991) and can be seen as a natural descendant of its predecessors (see Note 2).

The aims of the study are: identifying the major problematic aspects of the students' mathematical writing in their drafts submitted to tutors on a fortnightly basis; increasing awareness of the students' difficulties for the tutors at this University's School of Mathematics; providing a set of foci of caution, action and possibly immediate reform of practice; and, setting foundations for a further larger-scale research project.

The study is carried out as a collaboration between the School of Education (where the first author is a Lecturer) and the School of Mathematics (where the second author teaches the first-year undergraduates) at UEA. The focus of the research, examining the students' *written* expression, has been identified as a worthy domain of investigation in Projects 1-3: these studies examined the students' development of mathematical reasoning in the wider context of both *oral* and *written* expression - the latter merits further elaboration and refinement and has also been highlighted by teachers of mathematics at university level as an aspect of the students' learning that calls for rather urgent pedagogical action (e.g. Nardi 1999).

This is a small, exploratory data-grounded study (Glaser and Strauss 1967) of the mathematical writing of the students in Year 1 (60 students in total, 16 in the second author's tutorial class). Phase 1 was conducted in 6 cycles of Data Collection and Processing following the fortnightly submission of written work by the students during a 12-week term. Phase 2 will be conducted in two such cycles. Each 2-week cycle consists of the following stages:

- Beginning of Week 1: Students attend lectures and problem sheets are handed out.
- Middle of Week 1: Students participate in a Question Clinic, a forum for questions from the students to the lecturers.
- End of Week 1: Students submit written work on aforementioned problem sheets.
- Beginning of Week 2: Students attend tutorials in groups of six and discuss the now marked work with their tutor.

- End of Week 2: Data Analysis Version 1, towards Data Analysis Version 2.

The second author, who is also a tutor and is responsible for collecting and marking the students' work, carries out an initial scrutiny of the students' scripts and composes Data Analysis Version 1 (see Note 3): this consists of a Question/Student table where each student's responses to (a selection of) the problem sheet's questions are summarised and commented upon. The focus of her comments is quite open and covers a large ground of regarding the content and format of the students' writing. In an appendix to this table she produces rough frequency tables that reflect patterns in the students' writing and informal commentary by the tutors who teach the rest of the 60 students. Following a detailed discussion of Data Analysis Version 1, the first author produces Data Analysis Version 2, a question by question table where the major issues are summarised, characteristic examples of the students' work are referred to and links with current literature are made. A large part of these discussions revolve around the exchange of ideas and expertise. Examples of this exchange include: the communication of the second author's experiences as a tutor and a mathematician as well as her observations of the lectures and the Question Clinic, her consultation of other tutors and lecturers involved with teaching the students in Year 1; also her introduction to relevant findings from mathematics education research and educational research methodology.

Version 2 is then available to the other tutors for further informal commentary (we intend to introduce more formal strategies of evaluation in subsequent projects). An outcome of the discussion on Version 2 across the cycles will be a set of Macro and Micro Points of Action - a brief reference and examples of these can be found in (Nardi and Iannone 2000).

In Phase 1, by the end of the 12th week, 6 sets of data and analytical accounts as described above were produced. On completion of Phase 2 (Easter 2001), we intend to organise a Departmental Day Workshop to disseminate and discuss our results and also cultivate opportunities for extending the project towards an implementation of our Action Points.

Formal Mathematical Reasoning in the Context of Convergence of Series. The data we wish to draw on here originate in Cycles 5 and 6 of Phase 1 and concern the students' responses to Questions 5.4(2) and 6.1c(iii) below. The students answered by using the *Limit Comparison Test: Assume that for sequences* $a_n>0$ *and* $b_n>0$ $\lim_{n\to\infty} a_n / b_n = c \in \Re$. *Then: if* $\sum b_n$ *converges then* $\sum a_n$ *converges.*

Question 5.4:

(4) For each of the following series, decide whether or not it converges. State carefully any tests for convergence that you use. (1) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$. (2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$. (3) $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$. (4) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$. (5) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

Question 6.1c:

(c) Decide whether each of the following series are convergent (justify your answers and make clear which results you are using):

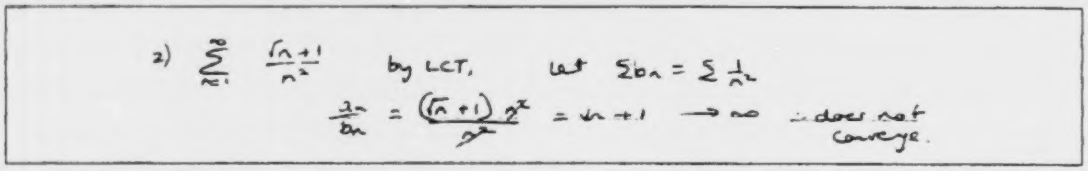
(i) $\sum_{n=1}^{\infty} \frac{1}{n!}$;
(ii) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$;
(iii) $\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+11}$.

In fact both series are convergent: 5.4(2) by the Comparison Test (compare, for example, with $2/n^2$) and 6.1c(iii) by the Limit Comparison Test for $b_n = 1/n^2$. In both occasions a substantial number of students used the Limit Comparison Test in particularly problematic ways. In the following we exemplify the students' responses and reflect on ensuing matters.

Regarding 5.4(2), out of 16 students only one concluded that it converges (by comparison with $\sqrt{n}/n^2$); one stated it converges but provided no justification; one stated it converges but provided an inscrutable (unintelligible scribble on the draft) justification based on the Ratio Test; and 5 did not attempt it at all. Here we are concerned with the remaining 8 responses, 4 of which involved the use of the Limit Comparison Test and 4 involved the use of the Comparison Test. In doing so we hope to illustrate one deep-seated difficulty with formal mathematical reasoning in the students' thinking.

Regarding 6.1c(iii), which was in the Problem Sheet of the following fortnight, results were better but still alarming: out of 16 students 5 did not submit any draft or did not attempt the particular question; 6 applied the Limit Comparison Test successfully and one used the Comparison Test successfully (comparison with $3/n^2$); one attempted but left incomplete and inscrutable use of the Comparison Test. Here we are concerned with the remaining three responses which involved the use of the Limit Comparison Test. As above the focus of our concern will be on the students' reasoning processes.

Here is one of the four responses to 5.4(2) that involved the use of the Limit Comparison Test, Hazel's - the other three were identical:

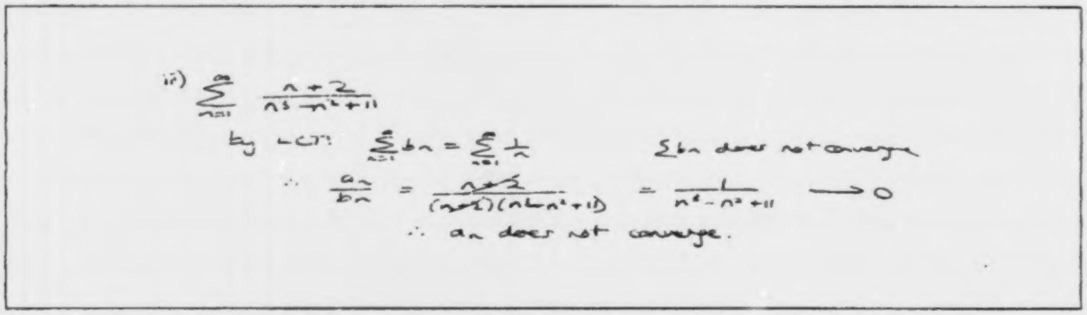


2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$ by LCT, let $\sum b_n = \sum \frac{1}{n^2}$

$\frac{a_n}{b_n} = \frac{(\sqrt{n}+1)\cancel{n^2}}{\cancel{n^2}} = \sqrt{n}+1 \longrightarrow \infty$ ∴ does not converge.

Hazel's interpretation of the Limit Comparison Test in this case seems to be the following: *if* $\sum b_n$ *converges but* $\lim_{n\to\infty} a_n / b_n = \infty$, *then* $\sum a_n$ *diverges*. In fact the test is inconclusive in this case.

Despite cautionary remarks in the Question Clinic and the following tutorial, Hazel's interpretation of the Limit Comparison Test is still problematic two weeks later. Here is her response to 6.1c(iii) - again identical to those of her peers:



ii) $\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+11}$

by LCT: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ $\sum b_n$ does not converge

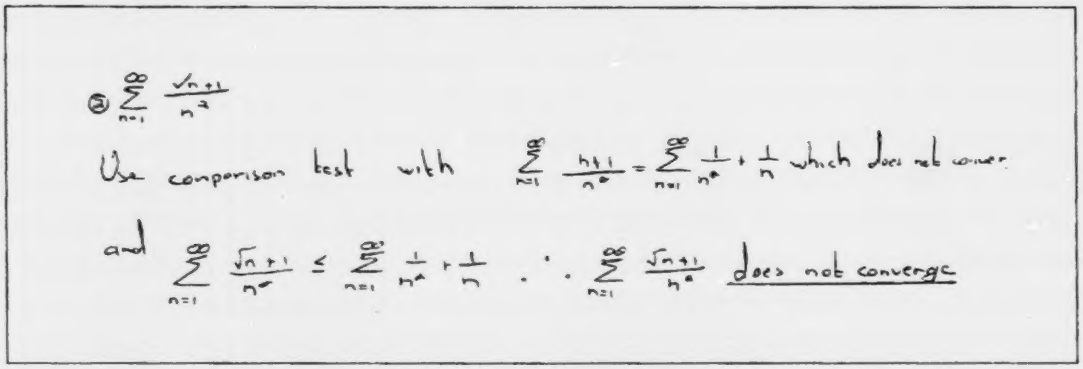
∴ $\frac{a_n}{b_n} = \frac{\cancel{n+2}}{(\cancel{n+2})(n^4-n^2+11)} = \frac{1}{n^4-n^2+11} \longrightarrow 0$

∴ a_n does not converge.

This time her choice of $\sum b_n$, the harmonic series, *does not converge and* $\lim_{n\to\infty} a_n / b_n = 0$. As a result she concludes that $\sum a_n$ *diverges*. Again the test is inconclusive in this case and the student ought to have pursued an answer via a different test.

What we wish to bring attention to here is the students' ***resistance to the idea of a test's inconclusiveness***: what appears to be the case in Hazel's (and her peers') work is that, once a theorem has been selected for testing the convergence of a series, in this case the Limit Comparison Test, it *must* provide an answer. What escapes the students is that, for the Limit Comparison Test to provide an answer, that is the convergence of $\sum b_n$ to imply the convergence of $\sum a_n$, all conditions must apply: a_n and b_n must be positive AND $\lim_{n\to\infty} a_n / b_n$ must be real. If either of a_n or b_n is not positive, or the limit is not real, then the convergence of $\sum b_n$ cannot imply the convergence of $\sum a_n$. But not implying the convergence of $\sum a_n$ is not equivalent to implying its divergence (as Hazel's 5.4(2) response seems to suggest). Similarly, if either of a_n or b_n is not positive, or the limit is not real, then the divergence of Σb_n cannot imply the divergence of Σa_n (as Hazel's 6.1c(iii) response seems to suggest).

Resistance to the occasional inconclusiveness of the tests was evident in the responses to 5.4(2) of a substantial number of students who attempted to use the Comparison Test: *Assume that for sequences* $a_n>0$ *and* $b_n>0$ *there exist positive constants N and c such that* $a_n<cb_n$ *for* $n>N$. *Then: if* Σb_n *converges then* Σa_n *converges.* Here is a characteristic response, Nicolas':



② $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$

Use comparison test with $\sum_{n=1}^{\infty} \frac{n+1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n}$ which does not conver

and $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n}$ $\therefore$ $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$ does not converge

Nicolas' interpretation of the Comparison Test in this case seems to be the following: *if* Σb_n *diverges and* $a_n<b_n$ *then* Σa_n *diverges*. (Note also that in his draft it is $\Sigma a_n<\Sigma b_n$ that he has written, not $a_n<b_n$, but we leave this issue - dealt elsewhere (Nardi 1996) as tendency to handle series and Σ as finite sums - aside for the moment). The fact is that the test is inconclusive in this case. Again, what escapes the students is that, for the Comparison Test to provide an answer, that is the convergence of Σb_n to imply the convergence of Σa_n, all conditions must apply: a_n and b_n must be positive AND $a_n < b_n$. If either of a_n or b_n is not positive, or the inequality does not hold, then the convergence of Σb_n cannot imply the convergence of Σa_n. Also, if the conditions hold but Σb_n is divergent, then the divergence of Σb_n cannot imply the divergence of Σa_n (as Nicolas' 5.4(2) response seems to suggest).

Underlying the students' attitude seems to be a desire for closure and completeness: according to this, a convergence test *must* cover the ground of possible responses to a question; the contingency of absence of such an answer is unsettling and therefore students feel it needs to be avoided at all costs. Albeit, instead of seeking an answer via the employment of a different convergence test (lack of flexibility in switching modes of pursuing an answer is documented in the problem-solving literature, e.g. in (Schoenfeld and Hermann 1982)), they tend to 'mutate' the proposition at hand (here: the Limit Comparison Test) towards a convenient expression that serves their purpose.

Here the 'mutants' include the versions of the Limit Comparison Test from which the alleged divergence of the series in question can be deduced and are ostensible reversals/negations of the proposition in the Test (difficulties with the action of negation have been documented e.g. in (Barnard 1995)).

This desire for closure and completeness is not uncommon at all: it seems to be located neatly side by side, for example, with the students' greater fluency and gravitation towards a sense of symmetry in reasoning in general. Examples: the students' far swifter handling of 'if and only if' statements as opposed to 'if' ones - as the literature on the students' difficulties with *modus ponens* suggests (dating back in the 1970s e.g. with O'Brien's work (1973)); of equalities as opposed to inequalities. The latter is documented e.g. in (Anderson 1994) and in parts of our data, sampled elsewhere (Nardi and Iannone, in preparation), where, for example, the students' handling of a proof by Mathematical Induction was hindered by the fact that the statement-to-be-proved was in the form of an inequality (hence the proof for P(n+1) could not be constructed from the assumption that P(n) is true as straightforwardedly as in the case of an equality).

NOTES

1. These projects are:
   Project 1: a doctorate (Nardi 1996) on the first-year undergraduates' learning difficulties in the encounter with the abstractions of advanced mathematics within a tutorial-based pedagogy
   Project 2: a study of the tutors' responses to and interpretations of the above mentioned difficulties (e.g. Nardi 1999), and,
   Project 3: UMTP, the *Undergraduate Mathematics Teaching Project* with Barbara Jaworski and Stephen Hegedus, a collaborative study between researchers and tutors on current conceptualisations of teaching as reflected in practice and their relations to mathematics as a discipline (e.g. Jaworski, Nardi and Hegedus 1999).
2. We tend to think of this study as Project 4, not only for its obvious thematic links with the previous projects but because it carries further the methodology of partnership (Wagner 1997) and materialises what was an underlying intention in Projects 2 and 3: the involvement of the mathematician as a reflective practitioner and her engagement with Action Research.
3. In the conference presentation we intend to demonstrate and discuss samples of the Problem Sheets, Versions of the Analysis (1 and 2), Extracts from the Data and the List of Micro and Macro Action Points.

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OBSERVATIONS ON THE NATURE OF QUIET DISENGAGEMENT IN THE MATHEMATICS CLASSROOM

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Research often focuses on disaffection in the mathematics classroom as evident in disruptive behaviour, absenteeism or special needs: thus it ignores a group of students whose disaffection is expressed in a tacit, non disruptive manner, namely as disengagement and invisibility. Ignoring this often large group implies that the mathematical potential of these learners may remain defunct. We have been awarded a 12-month research grant to study quiet disaffection in secondary mathematics classrooms, to uncover the reasons for student disengagement from school mathematics (1) and suggest re-engagement strategies (2). Here we focus on (1): we review relevant literature, discuss methodological constraints and introduce a preliminary set of themes that have emerged from our initial classroom observations.

In the UK, as well as other countries, an increasingly smaller percentage of students appears to be pursuing the study of mathematics at upper secondary level and beyond (ICMI 1998). Evidently the number of students who pursue mathematical studies affects the supply of teachers for this core National Curriculum subject area. The students' choice is seriously influenced by their attitudes towards and performance in mathematics which in turn are deeply shaped by their school mathematical experiences (Johnston 1994), and, in particular, by the mathematics teaching they have experienced in school (Dick & Rallis 1991).

Since the publication of the Cockcroft Report in 1982 attitudes seem to have slightly improved (Brown 1999) but performance is at unsatisfactory levels as evident in recent international comparisons (Jaworski & Phillips 1999). In sum, given the strong links between attitude, performance and choice of further study and career, research on attitudes towards mathematics at school level, and in particular on disaffection in the mathematics classroom, is essential.

Disaffection is defined often in research as disruption or truancy (Elliott 1997) and disaffected students are often seen as a subcategory of students with special educational needs (e.g. Tattum 1986). In these studies two major theoretical perspectives seem to frame current discourse on the origins of disaffection: cultural transmission theory (e.g. Reid 1987) (disaffection as faulty socialisation into local and familial cultures: so, for example, regular school non attendance is accounted as parentally condoned absence) and process theory (e.g. Cooper 1993) (disaffection as a result of the experience of schooling). Research however also suggests that schooling can compensate for faulty local and familial socialisation and thus can reinforce or ameliorate culturally transmitted attitudes (Reynolds and Sullivan 1979). Therefore curriculum and pedagogy can be employed towards modifying student attitudes.

Moreover, recently emerging perspectives view disaffection as rational choice rather than deviant behaviour (e.g. Dorn 1996). These works suggest that a pathology of absence from school can be studied in terms analogous to a pathology of presence: in a world outside school which offers increasing access to knowledge that is independent of adult authority, education through schooling may seem less and less relevant (e.g. Schostak 1991). In attempting to explain what actually motivates students to attend school and conform to its conventions, Schostak and others contend that it is not curricular provision but an unthreatening environment for self discovery and development that maintains school attendance.

This new perspective implies a modified definition of disaffection beyond truancy and disruptive behaviour that includes the quietly, invisibly disaffected (Rudduck, Chaplain & Wallace 1996): those with low engagement with learning tasks, those who perceive these tasks as lacking in relevance with the world outside school and their own needs, interests and experiences, those who routinely execute but do not get substantially involved with the tasks. These students attend school but often underachieve. Re-engagement of these learners is then of strategic importance and the role of curriculum and pedagogy in this is central.

Quiet disengagement is a relatively under-researched type of disaffection. Our study aims at examining students' experiences of quiet disaffection in the mathematics classroom and at suggesting re-engagement strategies. It thus intends to highlight the needs of an often large group of learners whose mathematical potential may at the moment remain inert. This integration of cognitive and affective perspectives on mathematical learning, namely one that merges the study of students' attitudes towards and achievement in mathematics, has been highlighted in the relevant literature as a potentially fertile ground for research in an area where traditionally the distinction between cognition and affect has been dominant (McLeod 1992). Arguably this distinction has been counterproductive as studies of mathematical cognition have tended to miss important characteristics of performance as they failed to gather crucial data on students' affective responses. Furthermore studies of performance, unlike affective studies, have had a stronger influence on curriculum development and teacher education, and an integrated perspective is likely to enhance the influence of findings relating to affective issues.

Since the early 1990s there has been a growing realisation that the classical divide between cognition and affect in mathematics education, traced back in 1956 and Bloom's two volume Taxonomy of Educational Objectives, is not particularly helpful. In fact the interrelatedness of the two domains emerged as early as the 1960s (e.g. Simon 1967). Nowadays non cognitive predictors of performance (House 1995) are seen as pertinent in studies of learning: beliefs, attitudes and emotions towards mathematics are an inextricable component of general mathematical performance (Reynolds & Walberg 1992; Wong 1992; Jones & Young 1995; Ma 1997; Hensel & Stephens 1997) as well as particular mathematical skills (e.g. abstract mathematical thinking (Iben 1991); problem solving (Kloosterman & Stage 1992, McLeod 1993)). Reflecting tendencies in the general literature on disaffection, various studies address the relationship between attitude and performance as a function of the individual's self concept (Jones & Smart 1995; Maqsud & Khalique 1991; Williams 1994; Norwich & Jaeger 1989; Norwich 1994; Skaalvik 1994) as well as of the students' experience of mathematics teaching in the classroom (e.g. the role of 'interesting' class activities (Schiefele & Csikszentmihalyi 1994); the role of teachers' attitudes towards error making (Brown 1992)). In general, disinterest in mathematics generated by certain pedagogical approaches seems strongly linked with underachievement (Boaler 1997).

Non-mathematically specific research (e.g. Keys and Fernandez 1993) suggests that it is likely that, as students proceed to the later years of their schooling, they often become more disenchanted with the education process. In their work 'teaching and learning practices' ranked highly in the students' questionnaire responses to what made them positive towards school and school work. In considering implications for mathematics lessons, the students expressed a general preference for 'working with their friends', 'making' and 'discussing things'.

The above resonate with the findings in Jo Boaler's comparison of two schools with different approaches to mathematics teaching (1998): in the first school, which used a traditional text book approach, despite being 'repeatedly impressed by the motivation of the students who would work through their exercises without complaint or disruption', the students' three most frequent descriptors of mathematics lessons were 'difficult', comments related to the teacher and 'boring'. Students believed that mathematics just involved memorising and routine execution of rules. In the second school which used an open-ended project approach despite having 'very little control, order, and no apparent structure to lessons' students were expected to be responsible for their own learning and the three most frequent descriptors of mathematics lessons were 'noisy', 'a good atmosphere' and 'interesting'. Elsewhere (1997b) Boaler discusses also gender related differences on the same issues.

In the study mentioned earlier, Keys and Fernandez refer to disillusionment with and dislike of school; lack of interest and effort in class and homework; boredom with school and schoolwork; dislike of certain teachers or types of teachers; resentment of school rules; belief that school would not improve career prospect; low educational aspirations; low self-esteem and poor academic performance, as factors associated with disaffection or disengagement. They also discuss the concept of motivation as intrinsic (arising from interest in the subject being studied) or extrinsic (depending on the availability of external rewards). Norwich (1999) adds to these reasons two more categories: identified (e.g. recognition of the importance of mathematics) and introjected (e.g. parental pressure). In his work, introjected reasons were the stronger influences on satisfactory learning and behaving whilst intrinsic reasons were the stronger influences on unsatisfactory learning and behaving. This substantial reciprocal relationship between attitude towards and achievement in mathematics has been made in another recent quantitative study in the United States (Ma 1997) with the three attitudinal measures being 'Importance', 'Difficulty', and 'Enjoyment' and with 'Achievement' as the outcome. Significantly Ma contends 'making difficult content easy to learn is barely enough to improve mathematics achievement. It is more important to ensure that difficult mathematical content is presented in an interesting, attractive and enjoyable way'. And: 'It is inappropriate to assume that high achievers in mathematics have few attitudinal problems.'

Our study originates in the first author's previous involvement with a study of disaffection in secondary education and the second author's previous school-based research and teaching experience in the area. Results from the now concluding study that the first author has been involved in indicate that there is a wealth of evidence specific to mathematics to be explored with regard to this form of disaffection. Therefore research which offers an extension of this study and addresses this rarely explored, but significant, topic is timely.

Methodology. Participants of the research are mathematics teachers and students based in 3 Norwich schools, involved with the previous study (e.g. (Oakley 1999)). This previously established contact and willingness to participate (all schools were approached but our selection was based on school response, pilot lesson observations and timetable constraints). The field of the research are Year 9 mathematics lessons. This is a one-year project and is funded by the Economic and Social Research Council (Award No R000223451).

We are currently completing Phase 1 (October 2000 - January 2001). The second author observes students in mathematics lessons in which the participating mathematics teachers are involved and, also through consultation with the teachers, is now engaged with identifying a group of mathematically disengaged students.

In Phase 2 (January - April 2001), this extensive observation of the mathematically disengaged students will be supplemented with interviews of the observed students (these will be interviews of the whole class cohort in groups of approximately two to five students to avoid the implication of the observed students noticing their 'singling out' for observation): these will be semi-structured interviews in which the researcher will draw the students into an exploration of particular classroom incidents (Disengagement Incidents) as well as their general attitudes towards mathematics and its teaching. This process will be supported by occasional interviews with the teachers and supplemented quantitatively by an attitudinal survey administered to the students.

The researcher keeps fieldnotes of the lesson observations - the interviews will be audio-recorded. She then passes her fieldnotes on to the project director, the first author, who annotates them with comments of a substantive and of a methodological nature. This commented upon document is the Observation Protocol and there is one such document for each lesson. In this document there is preliminary identification of Disengagement Incidents. This process is carried out on a weekly basis so that the researcher's technique is constantly informed by these comments. Also in a weekly meeting we discuss the researcher's response to these comments. As an example, in the Appendix, we provide an Observation Protocol from one lesson (a preliminary analysis of the Disengagement Incident in this lesson is available in (Nardi and Steward 2000). Also: see Endnote for the plan regarding our conference presentation).

A note on methodological constraints. During Phase 1 and the seven weeks of classroom observation, we encountered, and partly resolved, several obstacles of a methodological nature: while engaging with identifying the 'subjects' of our study, the quietly disaffected in the mathematics classroom, or, to use another term that has grown to be of common use amongst us, the 'invisible' ones, we may have not 'seen' them; in other words they are perhaps the ones that are not present in the researcher's fieldnotes. The occasional hesitation of the researcher to approach some of these students was due to her concern that, by focusing on them she would actually render them 'visible' or her focus would actually result in re-engagement because the students would believe this is what she wants or expects to observe. We are currently resolving this by trying to foster an image of the researcher to the children that is completely dissociated from that of a teacher or that of an assessor of their work or some sort of ‘authority’ figure. This has been a complex task given this researcher's long-term experience as a teacher and the internal struggle between her identity as a teacher ('insider') and her identity as a researcher ('outsider') (Elliott 1991). Another methodological constraint is a certain transience in the nature of invisibility that we observed: over the seven week period we have identified episodes of 'invisibility' for some students who, in a following lesson, sought help from their teacher or peers and appeared engaged with their mathematics. Of course there have also been a number of students that are 'permanently invisible'. We intend our fieldwork to be as inclusive for both groups as possible. Finally, there seems to be a type of invisibility *as far as engagement with the mathematics is concerned* that is, paradoxically, deeply embedded in extreme, disruptive behaviour; in other words disruptiveness seems to cover up feelings of inadequacy towards the mathematics. Often these loud students *choose* not to engage in the mathematics and the manifestations of this can take various forms - for these students it is very apparent, for 'invisible' students it is not; yet the underlying reasons for their disengagement may be the same. We feel that our observation and interviewing techniques need to allow the evidence from these 'visible' students to inform the main body of data.

Phase 1 Preliminary Themes. Within Phase 1, where examples of students' lack of engagement in tasks they are expected to do has been observed we have sometimes been able to attribute these examples to a number of tangible reasons. We stress that this attribution is tentative. We offer this list of themes but, for want of space, the grounding of these themes to the data and the relevant literature (e.g Norwich's extrinsic reasons for engagement have repeatedly appeared in our observations) is omitted here but is available in current (e.g. Nardi and Steward 2000) and in-preparation publications: **(i)** students may react positively to a change to the normal routine of their mathematics lessons especially if these are textbook based however if the format of this change is too different and students are not given enough time to adjust then, what is assumed to be a re-engagement strategy, may potentially disengage them even more or disengage a different group (this does not include puzzles but includes the often perceived as strong on re-engagement activities such as 'investigations', where the otherwise welcome open-endedness of the task can be puzzling) **(ii)** Students can be re-engaged through using information technology and, in particular, subject specific software. However if students rarely use computers in mathematics, are unfamiliar with the software or the software is not well written this has been observed to lead to frustration and a lack of interest **(iii)** the impact of setting, and in particular of placing the students in a set that is beyond/below their ability, as well as the teacher's perceptions and expectations of individual students, is significant **(iv)** a pedantic, procedural or mechanistic approach to mathematics teaching (and/or a teacher personality with these features) has a clearly alienating impact on the students despite the teacher's aspiration for clarity and precision **(v)** conceptual difficulties within particular topics as well as certain uses of mathematical jargon have an immediate effect on the student's emotional response to the task and ensuing engagement. Extension and further refinement of these themes is to follow in the subsequent phases of the study.

ENDNOTE: In the conference presentation we intend to **discuss samples** of the Observation Protocols, the Disengagement Incidents and Associated Interview Extracts in order to **exemplify** the Analytical Themes that will have emerged in a more elaborate and refined manner from Phase 2.

APPENDIX: An Observation Protocol

Lesson [name of school]1.2: 01/11/00 Wednesday, Period 4, Time: 1.25 – 2.25	
Researcher 1 Field-Notes	**Researcher 2 Comments**
I sat at back opposite the door. In class also (male) student teacher from university (not introduced) and support teacher for the Portuguese speaker. Students carry on work on circles. Teacher began by recapping work on algebra in Year 8. On board: " $4r = 16$, $r = 16/4$ [several students called out 4], $r = 4$, $4t = 22$, $t = 22/4$, $r = 5.5$ ". Teacher: "If anyone has problems with that now is the time to say so". NOT SURE MOST STUDENTS WOULD OWN UP IF THIS WAS THE CASE.	Speculative comment about the students' response to teacher's general calls for assistance
On board " $\pi \times$ diameter = circumference, radius 4.2. Teacher asks what the diameter is. Hannah answered 8.4, " *pi x 8.4 = circumference*". *Students asked another question "π x diameter = 42". Verbally "what's the* diameter?". Boy at front + Zebedee + Anna put up hands. On board: "diameter = 42 / π" Verbally: "you divide by π". Hannah: "I don't understand". Teacher: "I'll come and talk to you". Hannah: " I never understand". Girl on table near me asked someone else who also shook her head. Teacher on board: "a circle with circumference of 72.6cm, the radius of this circle to the nearest *0.1* cm is π x diam = circ, π x diam = 72.6, diam = 72.6 / π, [diam = 72.6 / 3.14]. Teacher asks girl at back to work this out using her calculator. Teacher wanted all the digits shown on the calculator. Even though girl tried to give answer accurate to 0.1cm. "You **do** have to write down the whole calculator display", teacher says. "diam = 23.12101911, radius = *23.12101911* divided by 2 = *11.5605... = 11.6 to nearest* 0.1cm". *Teacher asked who didn't have a* calculator. A girl owned up – was asked how good her long division was. Anna who had moved to back table near me said: "I really don't understand". Teacher: "But you always say that". Teacher went to help Hannah. Teacher came back to Anna – the whole table seemed to listen.	Evidence of disenchantment with own performance. Evidence of teacher trying to achieve too much: learning on circles etc. AND decimal place (and how calculators deal with decimal place)? Evidence of teacher dismissing student's call for help or of teacher encouraging student to overcome usual hesitance about her performance? Anna is not an invisible child in this sense because her potential disenchantment is visible.
I WASN'T SURE ANY OF THEM UNDERSTOOD WHAT TO DO. *WHEN THE TEACHER WENT AWAY THEY DISCUSSED THE FIRST QUESTION - THEY WERE NOT SURE WHAT TO WRITE DOWN FROM THE QUESTION - THEY TRIED TO FOLLOW THE RULES FOR SETTING OUT ON THE BOARD. THEY HAD LOST SIGHT OF THE PROBLEM – THEY WERE BOGGED DOWN WITH DETAIL – NO FLUENCY IN SOLVING THESE PROBLEMS HAD BEEN DEVELOPED.*	On Researcher 1 comment: Speculation or based on evidence?
I sat next to Jade, Charlotte and Ellie. I talked to Jade who seemed to confidently start the first question but then couldn't believe she could do it. *She didn't want me to sit next to her.* Jade: "I'm no good at maths. I've never been any good at maths". Me: "What about last year or in Middle School?". Charlotte: "It was alright last year with Mr. J". Jade: "He didn't make you work". Jade gave up completely when she thought her two friends had a different answer to her – looked at her watch and sighed. Charlotte and Ellie were too bound up with setting out of problem – *they seemed to lose sight of what they were doing*. Jade approached by student teacher – body language suggested that she just wanted to be left alone. She could not believe she had done it right especially when it was pointed out that only the diameter was needed in this question. And she had worked out the radius. She started drawing margins in her book and writing down question numbers – *avoid maths activities*. Charlotte and Ellie started to understand what they had to do. Charlotte was more vocal than Ellie but Ellie seemed reasonably quietly confident. At 2.10 the teacher stopped the lesson to explain rounding. Charlotte and Ellie could do rounding. Charlotte (to Ellie) **"Oh no she's going to explain it all"**. Teacher made students go to the back of their books to write. On board: "**When asked to correct to n decimal places you always look to the next digit to see whether it affects the nth decimal place. It will only matter if the value of that "n + 1" decimal place is 5 or more in which case you add one to the nth decimal place,** then e.g. 1.989898...'. Ellie and Charlotte got on with the exercise. *THIS INTERRUPTED THE FLOW OF THE LESSON, LOST THE FOCUS OF CIRCLES AND WAS AN UNINTELLIGIBLE PIECE TO WRITE FOR ME LET ALONE YEAR 9 STUDENTS.* Student teacher went back to Jade: "I really don't like maths". Me: "What do you do when you don't understand?". Jade: "Wait till she's free and ask her". Me: " Do you put up your hand?". Jade: "Sometimes or otherwise I ask them (friends on table) – she's too scary". Teacher gave answers at end of lesson at which point everyone gave up. ***IS MATHS JUST ABOUT ANSWERS?*** The 3 girls talk about their next lesson. Ellie: "He (another subject teacher) sent me out for saying his jokes are rubbish. " *c.f. maths where she never says anything to the teacher.* Jade:" I don't like any teachers".	**DISENGAGEMENT INCIDENT: Distilled (after 'irrelevant' facts are removed or summarised), this episode is perhaps the first here to directly address the development of disengagement (Jade starts from a confident point but then her confidence deteriorates?) and a certain denial from the student on tackling it.** **END OF INCIDENT**
IS THIS AN EXPRESSION OF DISENCHANTMENT OF SCHOOL IN GENERAL NOT JUST MATHS IN PARTICULAR? *AT THE END OF THE LESSON I SPOKE TO THE TEACHER WHO SAID SHE HAD HAD TO ASK A GIRL'S NAME BEFORE SHE COULD GIVE HER A CREDIT - AN ACKNOWLEDGEMENT THAT SOME STUDENTS ARE INVISIBLE AND UNKNOWN EVEN AT THIS POINT IN THE TERM.* ***MY IMPRESSION IS THAT TEACHER'S PEDANTIC APPROACH AND EXCESSIVE ATTENTION TO DETAIL IS INHIBITING LEARNING.***	Teacher's assistance in identifying invisible children is invaluable. However bringing their attention to potentially invisible children their attitude towards them maybe will change. *Will this affect our data?* Conjecture about the impact of teaching style on learning
Researcher 2 'Preliminary Verdict': Disengagement Incident. Also disperse evidence.	

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CHANGES IN ENDORSEMENT OF AN INTEGRATED LEARNING SYSTEM OVER A PERIOD OF THREE YEARS

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This paper extends previously reported research on the long-term use of an integrated learning system (SuccessMaker) that delivers mathematics activities randomly in the form of electronic worksheets and progresses students through various levels as success improves. Quantitative and qualitative changes were noted in the schools' endorsement of the ILS over a period of 3 years. However, most schools still endorsing the ILS in 2000 had predominantly transmission/absorption models of teaching and learning.

An Integrated Learning System (ILS) is basically a collection of tasks, presented as electronic worksheets, divided into a range of *topics* (e.g., numeration, multiplication, fractions). The manufacturers of Integrated Learning Systems endorse their products as tools to "develop and maintain mathematical skills, and to develop problem-solving skills" (Computer Curriculum Corporation, 1996, p.1) and as tools to diagnose student difficulties. In order to achieve these goals, the ILS in this study (*SuccessMaker*) presents "a mix of dynamically distributed exercises appropriate to a student's functional level, providing feedback and tutorial intervention when necessary and representing mathematical concepts through highly visual exercises" (Computer Curriculum Corporation, 1996, p.1). It delivers courses to each student individually, manages all student enrolment and performance data, and designates which tasks have to be completed. Its management system provides the means for teachers, laboratory managers, and administrators to organise the use of courses and to monitor student progress.

SuccessMaker is a closed system where curriculum content and learning sequences are not designed to be changed or added to by either the tutor or the learner (Underwood, Cavendish, Dowling, Fogelman, & Lawson, 1996). Each of the topics is sub-divided into collections of tasks that are sequenced in terms of performance at different *levels*. When students achieve high mastery at one level, the ILS automatically raises them to the next level. The random nature of the presentation ensures that task performance correctly represents level. The worksheets vary in quality, but many are generally attractive in their presentation and creative in the way they probe understanding, particularly with the use of 2-D representations of appropriate teaching materials in mathematics (e.g., Multi-base Arithmetic Blocks, Place Value Charts, fraction and decimal diagrams). SuccessMaker provides online student resources: *Help* (provides answers), *Tutorial* (directs how to do a task), *Toolbox* (calculators, rulers, tape measures, etc.), *Reference* (provides definitions), and *Audio* (reads text to students through earphones). The worksheets can be printed to provide off-computer activity.

The random nature of the worksheet presentation means that *SuccessMaker* does not provide sequences of activities that can address student misconceptions. There is also a tendency for questions to be closed and to base performance on speed (with time delays resulting in the ILS defaulting to incorrect). The use of the Help and Tutorial icons automatically grades performance as incorrect. Because of their focus on rising through the levels as rapidly as possible, many students in this study tended to avoid using these aids.

The very nature of an ILS marginalises the teacher's role and removes students' initiative and autonomy (Bottino & Furinghetti, 1996). Furthermore, the one-student-at-a-time structure of the ILS discourages cooperative learning by groups of students. This is contrary to current views that learning with computers should be cooperative (Sivin-Kachala, Bialo, & Langford, 1997), particularly with respect to higher cognitive functioning (Carnine, 1993; Riel, 1994), investigations and the construction of links (Wiburg, 1995). There also appears to be insufficient task variety in *SuccessMaker* to prevent repetition; thus many students in this study tended to become bored. Some tasks have novel presentation and solution formats which many students found difficult to interpret. However, *SuccessMaker* does provide feedback to students on the correctness of their responses (desirable for effective learning according to Sivin-Kachala et al.).

Learning and *SuccessMaker*. In a re-analysis of studies into the effectiveness of *SuccessMaker*, Becker (1992) found very little evidence of this ILS improving student learning. He argued that the only significant improvements were found in studies supported by the manufacturers and that these had flaws. A more recent study by Underwood et al. (1996) found some statistically significant improvements from the use of the core mathematics course in primary and secondary classrooms, although the sample sizes from the primary classrooms were too low to meet Becker's criteria for significance.

In a study comparing *SuccessMaker* progress in 23 schools across 6 months with changes in mathematics knowledge as measured by a standardised test, McRobbie, Baturo and Cooper (2000) found no significant improvement as measured by the test even though the ILS reported significant gains. This finding was supported by case studies of students' progress where data from interviews indicated that children with rapid ILS improvement had acquired little mathematical knowledge. As Baturo, Cooper and McRobbie (1999) argued, the worksheet nature of the ILS makes it susceptible to the same pedagogical flaws as were found by Erlwanger (1975) in the Individually Prescribed Instructional (IPI) packages that proliferated in the US in the 70s. Nevertheless, *SuccessMaker* was reasonably popular in many schools in Queensland, Australia. Therefore, Baturo, Cooper, Kidman and McRobbie (2000) explored the factors that appear to influence teachers' endorsement of the ILS. This study found that *SuccessMaker* was endorsed in cases where there was strong supervision, follow-up of students' difficulties, integration with other teaching, external rewards and some novelty with respect to computers. The chance for endorsement appeared to diminish if teachers did not support it philosophically, if rosters were inflexible and if more exciting computer options were available. All of the teachers who endorsed *SuccessMaker* did so because they believed it had contributed to improved levels of mathematical and affective performance in their classrooms.

This paper reports on a follow up study to the Baturo et al. (2000) study in which we investigated:

1. whether the schools' levels of endorsement had changed over a period of three years; and
2. issues which influenced the schools' patterns of endorsement and use.

Method

Data for this study came from two sources. The first set of data was from interviews conducted with administrators, computer coordinators, teachers, teacher aides and technical staff from the 23 low socioeconomic schools involved in the first year of the project (1997). These interviews focused on (1) logistics, management and use of the ILS, (2) teachers' beliefs about teaching and learning and the ILS's role in teaching and learning, (3) perceptions of students' likes, dislikes and preferences with respect to the ILS, and (4) the schools' levels of endorsement of the ILS (see Baturo et al., 2000). The second set of data came from a questionnaire that focused on the logistics, management, use and long term endorsement of the ILS. The questionnaire was sent in 2000 to each of the schools involved in the study; 9 primary and 8 secondary schools responded (1 school had closed and 5 schools did not respond to the questionnaire). In order to clarify their responses to the questionnaire, follow-up telephone interviews were conducted with some of the teachers at five of the schools.

Eleven of the 17 schools that responded had over 20 percent students from indigenous backgrounds. Within Australia, it has been noted that many students from indigenous backgrounds tend to experience significant difficulties in learning mathematics taught in schools (Zevenbergen, Atweh, Kanes, & Cooper, 1996). According to Kepert (1993), many indigenous students have cultural backgrounds that will not immediately allow them to access the mathematics taught in schools. This viewpoint has been supported by Frensh, Frensh, Matthews, Stephen, & Howard (1994) who claimed that too often mathematics is taught in ways that do not take into account the various learning styles of indigenous students.

The data from the interviews and the questionnaires were first tabulated in order to ascertain changes in levels of endorsement of *SuccessMaker*. Each school was classified in one of three categories: Full Endorsement, Partial Endorsement and Non Endorsement. If a school unconditionally endorsed *SuccessMaker*, it was classified in the Full Endorsement category. However, if its endorsement of *SuccessMaker* came with some reservations or conditions, then it was classified in the Partial Endorsement category. Those schools who did not endorse *SuccessMaker* were classified in the Non Endorsement category. After changes in levels of endorsement had been ascertained, the data were further analysed in order to identify factors that had impacted on the levels and the quality of the schools' endorsement of *SuccessMaker*.

Results

During the three-year period in which *SuccessMaker* operated within the schools, levels of endorsement for the ILS decreased. Of the 6 schools that originally fully endorsed *SuccessMaker* in 1997, only 3 were fully endorsing it in 2000, 2 were only partially endorsing it, whilst 1 of the schools no longer endorsed it. With respect to Partial Endorsement, of the 7 schools in this category in 1997, 2 had withdrawn their endorsement by 2000. The four schools who had not endorsed it in 1998 had not upgraded their level of endorsement.

Table 1

Levels of endorsement for SuccessMaker (1997 and 2000)

	Endorsement		
		Final (2000)	
Initial (1997)	*Full*	*Partial*	*None*
Full	3	2	1
Partial	0	5	2
None	0	0	4

The changes to the levels of endorsement were based on both pedagogical and logistical factors. For example, the school that changed from Full to Non Endorsement offered the following reasons: (1) extremely negative feedback about *SuccessMaker* from the students, (2) a realisation on the teachers' part that *SuccessMaker* was not an effective learning tool, and (3) the logistical problem of fitting sessions with *SuccessMaker* into the timetable. The two schools which changed from Partial to Non Endorsement indicated that their reasons for reducing their level of endorsement for *SuccessMaker* as being: (1) the transfer of the teacher who originally implemented *SuccessMaker* from the school; and (2) the feelings of other staff within the school that *SuccessMaker* was not an effective learning tool. The logistical problems of training staff, how to access reports and worksheets from *SuccessMaker*, and of timetabling the three regular 15 minute sessions per week (as recommended by the publishers of *SuccessMaker*) in the computer laboratories together with perceptions about the limited educational value of *SuccessMaker* were the reasons offered by the two schools that changed from Full to Partial Endorsement.

An analysis of the data revealed that all 10 schools which either partially or fully endorsed the use of *SuccessMaker* in 2000 made extensive use of external rewards to keep the students engaged on task. Data from the interview and follow-up telephone interviews seemed to indicate that the use of external rewards played a very important role in maintaining student activity on *SuccessMaker*. In contrast to this, the 7 schools who did not endorse SuccessMaker did not utilise external rewards.

When the data was subjected to more detailed analysis, it was noted that the reservations or conditions that underlay Partial Endorsement of *SuccessMaker* had undergone important qualitative changes during the three years. In 1997, partial endorsement was primarily based on price and logistical issues (such as timetabling computer time). By 2000, Partial Endorsement no longer was primarily based on these issues. Instead, it was based more on educational issues such as:

1. the limited roles *SuccessMaker* could play in a school's mathematic education curriculum program,
2. the limited set of clientele for which *SuccessMaker* activities were deemed appropriate, and
3. the need for supervision.

Limited roles for *Successmaker* in mathematics Curriculum program

All of the schools that partially endorsed the use of *SuccessMaker* in 2000 clearly indicated that they did not perceive that it had general application across the whole mathematics curriculum program. This sentiment about *SuccessMaker* was probably best encapsulated by the comment from one of the high schools, namely, that *SuccessMaker* "can be a valuable component of a learning program".

These schools identified specific niches or roles for *SuccessMaker* within their mathematics curriculum program. For example, most of these schools felt that *SuccessMaker* provided good reinforcement learning activities and was excellent for "drilling the basics in...getting kids ready for high school". Some of the schools also felt that *SuccessMaker* provided effective individualised sets of learning activities for remedial mathematics students.

One of the primary school teachers indicated that she felt that *SuccessMaker* provided worthwhile learning activities in a few specific problematic topics such as division. She thus endorsed the use of the ILS for assisting in the teaching of these specific topics. She, however, felt that its learning activities in many topics were not pedagogically sound so she did not endorse the use of *SuccessMaker* in these latter topics.

Limited set of clientele

The teachers from the schools partially endorsing *SuccessMaker* firmly believed that it is only suited for targeted groups of students such as those in need of remediation and/or those students "who have literacy and numeracy deficits". They were very negative about its suitability for use with the general student population. This notion that *SuccessMaker* should only be used for remediation and learning deficit students probably is best epitomised by a teacher from one of these schools. When she was asked during a follow-up telephone interview during 2000 whether she would endorse *SuccessMaker* for use with her own son, she quite unequivocally stated "no" because she felt that her son was a good learner and thus did not need *SuccessMaker*. However, she was most happy to endorse it, in her classroom, for those students who had learning difficulties.

Need for supervision

All of the 7 schools that partially endorsed the use of *SuccessMaker* in 2000 strongly emphasised the need for supervision. As one of the high schools noted, *SuccessMaker* could be a very effective learning tool:

> *providing they (the teachers) are careful about how they use it. It is not to be used as a child minding device.*

An analysis of these 7 schools' responses to the questionnaire revealed that they perceived that three types of supervision were needed in order for *SuccessMaker* to operate effectively:

1. supervision of the operation of *SuccessMaker* system
2. supervision of the students' behaviour whilst interacting with *SuccessMaker*
3. supervision of the mathematical learning.

Supervision of the operation of SuccessMaker

All of these 7 schools had experienced on many occasions problems with the operation of *SuccessMaker*. For example, the ILS had a history of crashing, especially if some of the graphic capabilities of *SuccessMaker* were being used. Other types of operational problems that occurred included difficulties in extracting students' scores and in extracting hard copies of worksheets. Thus, as one school said, students could not be left to work on *SuccessMaker* without someone (a teacher or a teacher aide) being there "to drive the wheels".

Supervision of student behaviour

Although each of the 7 schools partially endorsing *SuccessMaker* in 2000 indicated that a teacher and/or a teacher aide was necessary in order to ensure that the ILS was functioning properly, their major reason for stressing the need for supervision of the ILS sessions by a teacher or a teacher's aide was to ensure that the students stayed focused on the task. As one of the primary schools stated:

> *the program needs human resource to supervise students to see they are on task and not just random selection.*

Supervision of mathematical learning

Most of the schools partially endorsing *SuccessMaker* expressed skepticism about its publisher's claims that it provided necessary interventions when students were experiencing difficulties with the topics being covered. These schools indicated that they had found it necessary to act upon reports of problems immediately they were identified either by *SuccessMaker's* assessment mechanisms or by the teacher/teacher's aide. Their actions generally took the form of small group tutorials away from the computer where, if necessary, recourse was made to concrete teaching materials.

One teacher in fact created a very systematised mechanism for this process. She got each child to record his or her score on assessment tasks and to print out the worksheets they were having difficulty in completing. She used this information to plan specific mathematics lessons in order to meet the needs of these students. She thus effectively integrated *SuccessMaker* mathematical activities into her other mathematical learning activities.

Discussion and conclusions

Over a period of three years, both quantitative and qualitative changes have been made to the schools' endorsement of *SuccessMaker*. In 1997, 13 out of the 17 schools fully or partially endorsed *SuccessMaker*. By 2000, this number had fallen to 10 schools. However, the most important changes educationally were the qualitative changes that had occurred to the nature of partial endorsement. Whereas in 1997, the reservations/conditions that underlay Partial Endorsement were primarily based on price or logistical issues, by 2000 the reservation/conditions were primarily based on educational issues such as *SuccessMaker's* curriculum limitations, the limited clientele for which *SuccessMaker* is appropriate, and the need for supervision.

However, it should be noted that the educational philosophy underlying the practices employed in those schools partially endorsing *SuccessMaker* in 2000 seemed to have been based on a deficit model of education. This is reflected in the schools' comments about *how* and *with whom* they believed that *SuccessMaker* could be successfully utilised. It is also reflected in the extensive use of external rewards utilised in these schools to keep students on task whilst using the ILS. The deficit model of education assumes that underachieving students lack essential skills or orientations which allegedly hinder their academic achievement and that the major task of education for these types of students is to "fill-in" these deficits. In recent years, serious doubts have been expressed about the limitations of the deficit model. Liedke (1995), Gonzales (1993) and Hamovitch (1994), for example, have found that educational interventions based on the deficit model are not successful because they are often insensitive to the societal and cultural backgrounds of many underachieving students (particularly those from indigenous backgrounds). Because of this, many educators are arguing for educational curriculum and teaching methods that promote and build on students' existing repertoires of knowledge and incorporate their home cultures and history. Thus, it could be argued that the use of *SuccessMaker* to overcome the deficits of underachieving students may not in the end result in significant, long term educational gains by students at these schools.

It also should be noted that the schools who still endorsed *SuccessMaker* in 2000 had instrumentalist viewpoints about the nature and discourse of mathematics. That is, they viewed mathematics as a static corpus of isolated facts, rules and procedures which students needed to learn. Because *SuccessMaker*'s random presentation of mathematical content, its structuring of mathematics closely matched that of the teachers in these schools.

Therefore, although there seemed to be a movement towards a focus on student learning amongst the teachers in the schools who partially endorsed *SuccessMaker* in 2000, methods of teaching and ways of utilising the ILS still were very transmission/absorption in nature (just as they were in 1997). Their methods of teaching mathematics were not being modified to focus on collaborative, socioconstructivist principles such as those being promoted by most current mathematics education curricula and reform documents.

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Videopapers: Investigating new multimedia genres to foster the interweaving of research and teaching

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This paper reports an ongoing investigation on how mathematics teachers discuss and relate to a new genre of research publication that we call "videopapers." Videopapers are multimedia documents that link and synchronize digitized films with subtitles, text with interpretations and transcriptions, and images or software tools associated to the content of the videopaper. We interviewed pairs of high school mathematics teachers as they encountered and discussed a videopaper based on a PME paper presented two years ago. We reflect on the different roles of the classroom video and the paper and on the process of data analysis enabled by the videopaper.

The Study

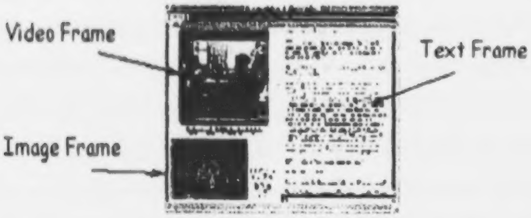
The capacity of educational research to influence teaching and learning is often questioned and dismissed (Kaestle, 1993; Sullivan et al., 2000). Educational research tends to be seen unfavorably in comparison to research in the hard sciences. In contrast to fields like medicine or engineering, the evidence generated by educational research seems to many to be inconsequential and inconclusive. This comparison is itself a product of cultural assumptions and expectations. It reflects the notion that valid research should generate evidence out of comparing different teaching and curricular approaches to improve students' learning; accordingly, a research report would have to compare alternatives and show that one is most beneficial. Teachers would then be able to make informed decisions about their practices. This is a view endorsed and pursued by many educational researchers (Campbell & Stanley, 1963; Clifford, 1973). Others approach the role of educational research differently by stressing the need to examine and question different ways of thinking about the nature of teaching and learning. From the latter point of view, a research report is expected to articulate new perspectives on teaching and learning. By questioning assumptions and articulating new possibilities teachers might conceive of their roles differently, or see what students do in a new light. It is not so much a matter of controlled comparisons but about being generative and help to re-think what is usually taken for granted. Case studies (Stake, 1995), ethnographies (Bolster, 1983), and narratives (Carter, 1993) are some of the publishing styles preferred in these lines of work.

In this paper we report an investigation on how teachers read and talk about case studies based on the analysis of videotaped classroom episodes. The relationship between teachers and research-oriented case studies is the subject of a small literature. Bolster (1983) and Stake (1995) propose that case studies are relevant to teachers because educational events are not governed by formal rules but by intentions and on-the-spot perceptions, which can only be conveyed by complex and detailed ethnographies. On the other hand, Kennedy (1997, 1999) argues that teachers engage with different pieces of research literature not on the basis of the paper's "genre" (quantitative comparison, case study, narrative, etc.) but to the extent to which they elaborate on issues that matter to them (e.g. classroom management, minority students, etc.). Still others stress the centrality of teachers' participation in authoring these case studies. Cochran-Smith & Lytle (1993) discuss key ideas underlying the "teacher-research" movement, one of whose main tenets is that the relevance of research is associated with teachers' shifting their professional identities, becoming actors in the work of research, and seeing teaching as intimately related to investigating questions on their own practices.

A focus of our study is to understand how the introduction of filmed classroom episodes as part of the case might generate new ways in which teachers perceive and interact with the case. The actual use of video is likely to have a deep impact on how a case is "read" or "viewed." One common observation is that educators who are shown a classroom videotape tend to immediately judge teacher's actions as being more or less good or bad, and frequently contrast his or her actions with how it "should" have been done; it is a tendency to shift the filmed episodes to a background in order to highlight one's own preconceived notions of a correct or ideal course of events. This attitude differs from one in which the observer focuses on the filmed interactions, strives to trace the inner origins of the utterances, and attempts to use the evidence to imagine how things look like to the participants in the film. We will refer to this last attitude by using the term "data analysis."

The emerging technologies of digital video are creating many new possibilities to facilitate data analysis. Digital video is simple to use in a manner that is richly connected to other forms of information (e.g. text, images, subtitles, software simulations, etc.). These possibilities are beginning to be exploited both for the development of research (Hall, 2000; Stigler & Hiebert, 1997), and for teachers' professional development (Lampert & Ball, 1998). Sabelli and Dede (1998) offer as an example research by Jacobs et al. (1997), which involves the "collaborative annotation of video-based case studies of educational practice that include ancillary information such as student products and teacher reflections" (p.9).

Together with other colleagues, we are working to develop a new genre of research publishing that we call "videopapers." Videopapers, as we conceive of them, are multimedia documents that include a text frame, a video frame, and an image frame:



Videopapers can be seen with a web browser, such as Netscape. All the components are linked and synchronized. For example, buttons can be inserted in the text that will play a pre-selected interval on the video; images can be made to appear at particular times in the image frame; and the video can trigger the display of certain pages on the text frame. In order to produce a videopaper we have developed a software tool called "VideoPaper Builder"[i] that allows authors to interconnect and synchronize the different components without having to be a programmer or even technically savvy.

In order to conduct the present study we chose a paper that had been presented at PME in Israel (Solomon & Nemirovsky, 1999), developed a videopaper out of it, and interviewed high school mathematics teachers. We conducted and filmed interviews with pairs of teachers. The teachers were asked to read the paper in advance. Then they were introduced and given access to the videopaper during the interview. The paper is based on a 16-minute classroom conversation in a high school math class taught by Solomon. The two main themes of the paper are the nature of open-ended problems and the sources for the "sense of direction" emerging in a classroom conversation. The authors argue that what makes a mathematics problem open-ended is not so much its textual definition but the classroom culture within which it is discussed and figured out. They also contend that the sense of direction of a mathematical conversation does not follow pre-planned paths and it is co-developed by the teacher and some of the students. The videopaper includes the text of the paper, the 16-minute digitized film with subtitles, and synchronized images displaying the content of the overhead transparencies being projected in front of the class.

Our approach to the analysis of the videotaped interviews shares a number of commonalities with Interaction Analysis as described by Jordan and Henderson (1995), and the interpretive approach described by Packer and Mergendoller (1989). Rather than approaching the filmed interviews with a predetermined coding scheme, we allowed the analysis to "emerge from our deepening understanding" of the events unfolding on the video-taped record (Jordan & Henderson, 1995. p. 43). We treated the participants' utterances and actions as processes accomplished over time and in interaction with others and we focused our attention on the details and meanings of these actions and utterances (Packer & Mergendoller, 1989). Our data analysis took place in a group of four researchers with varying interests and expertise who continually challenged each other's observations and required of each other the grounding of interpretations in observable events on the video record (Jordan & Henderson, 1995. p. 45).

For the purpose of presenting our analysis we will excerpt from two interviews. One interview was with June and Ron, who are beginning teachers, and another with Cara and Cher who have almost 30 years of teaching experience. All of them teach mathematics in high schools located in the Boston area.

The Interviews

We organize the selected excerpts from interview transcripts in two parts: 1) Reading the paper/watching the video, and 2) Data analysis.

➢ Reading the paper/watching the video

Before the interview, teachers had read the paper, which included a transcription of the classroom conversation integrated with commentaries and analysis. A transcription captures some aspects of a classroom interaction but it necessarily leaves out many others. Transcribing is selecting aspects deemed to be important and making them suitable for the print medium. A large amount of "filtering" happens in transcribing classroom interactions. The reader is expected to reconstruct the events and therefore she has to assume whatever had been filtered out. In order to "picture" what went on, it is essential for the transcript's reader to project his own assumptions. Although not everything falls under the lens of the camera and the microphone picks up only part of what is being said, a video record undergoes less filtering than a written transcription. Film preserves the original tones of voice, gestures, facial expressions, etc. Another difference is that the video introduces its own time: if an utterance took 3 seconds, one has to spend 3 seconds to hear it, whereas the reading of its transcription is not constrained by the original duration. There were many examples of how these differences played out in teachers' conversation. The one that we have chosen took place in the interview with Cara and Cher. As they read the paper, they "pictured" a certain classroom interaction that resembled, in many ways, the kind of interactions they were used to in their own classrooms. Their implicit picture positioned Solomon standing at the front of the classroom next to the overhead projector. Right after viewing the videotape, Cara and Cher commented with surprise on the fact that Solomon was seated and reflected on their own perceptions on "being seated:"

Cara: I still take a negative connotation to being seated during a class. I mean, it's ingrained in me, you have to be out their performing. So that was my first observation of the film, was, gee he was seated. And it was a good thing. I mean, I took it as a supportive, he's part of the, not part of the discussion. He's letting them do it, and he's just writing down their observations.

(...)

Cara: I mean, the whole effect that he was seated.

Cher: That makes a big difference.

Cara: That physically changes the whole climate of the classroom. At least in a math classroom. I don't think it's so much in a history classroom, or an English classroom, where they're reading and writing papers and all. Maybe I'm wrong. But think from your math classrooms, when you came in, didn't you more or less expect the teacher to be up front, doing something? If you think back? (...) That role is very slowly changing in math classes. (...) because I think it speaks volumes. Rather than being Moses on the mountain, handing down the things.

The transition from reading the paper to watching the video prompted Cara and Cher to encounter their own tacit assumptions. In addition, they felt that reading the paper prior to watching the video was important in another regard: getting a sense of what to look at in the video and approaching the video with *their own* questions in mind. For example Cara felt puzzled by how open-endedness had been characterized in the paper. She commented on this as she was watching the video for the first time:

Mr. Solomon (in video): There's not necessarily one correct answer.

Cara: See, that part I didn't understand.

Later she clarified her comment:

Cara: the one thing that disturbed me when I read this (paper), and was anxious to see it in the video, is this, the concept of an open-ended question. And this is something we were kicking around a lot.

This role of "paper-reading" as a source of questions and issues with which to watch the video was made explicit at a later part of the interview:

Teresa: Was it beneficial for you to read the paper physically before you came in and watched the video?

Cara: Yes. (Cher: Definitely.) Yeah. Because I knew where it was going to head to. You know what I mean? I would have been pausing it and thinking things out myself, you know. Trying to keep at the same level as the student. And if I hadn't been prepared, I wouldn't have known what to be watching.

Cher: Well, yeah, and you could watch this, since you, since I knew what the, where it was going. Watch this, and focus on the watching instead of looking at what was going on and watching at the same time. So I think it was good to have, I mean for myself.

➢ Data analysis

June came to the interview with a deep interest in figuring out why Solomon's class had solved a problem of number sequences using successive differences. This solution was for her an unusual one, and she wondered whether it would have ever taken place in her own class. This concern of hers led her and Ron to work on data analysis in order to trace the origins of this idea within the filmed classroom conversation. Their data analysis was grounded in their non-linear examination of the video, with important references back to the paper, and took a form that we call "narrative account." Their narrative account for the origins of the "differences of the differences" idea integrated rich perceptions of the film, which gave them a sense of what kind of persons the students were and of their subjective experiences:

June: {referring to one of the students, Jamal} He wants to see the numbers up there. Maybe he's a visual guy. Maybe he's used to saying I'm seeing something here and my gut tells me that there's something about this, and throw it out, and let's see if somebody else in the class picks up with a fresh brain. The fact that he says put the numbers up there, I think, (...) he has a gut feeling about that.

June and Ron's development of a narrative account prompted them to search for evidence regarding "who had said what" and "when" by going back and forth in the video and in the paper. Their emerging narrative became expressed in a typical linguistic form: "so and so said this and then someone else....":

June: And then Nadia just keeps plugging away, and then she realizes that she did something wrong with her subtraction, and she says ok, wait I might have a different number. I forget where {in the paper} that is. {Flipping through the hard copy of the paper.} And she's like working away there, while other people are discussing stuff. Ok. yeah. {Finds reference} Nadia says. "I didn't stop at finding differences." She's the one who went to the differences between the differences. I think. And, then she got 12, and then 16 ... and then Jesse says "maybe it's 18." And then Nadia says ok wait, and she goes back. And she's plugging away at the numbers while everyone else is talking, and then she comes back. And then she says 125. So, she obviously made the correction and went back out, and did the reverse of taking the differences by adding on 6 to get the next one, and then 24 to get the next one and then 125. Cause that's not recorded in the conversation, nor in the video.

Both the sequence of the paper's pages as well as the video's timeline were important tools to organize ideas and to get the sense of "before and after" relating the different events that took place during the classroom conversation. Note that while their narrative account incorporates many elements visible in the film, it also reflects others that are not recorded but that were likely to have happened, such as when June said "obviously" and stated an inference about Nadia's approach that was recorded neither in the video nor in the transcript. The following excerpt shows Ron's development of a narrative account and June's repeated contributions and occasional surprises.

Ron: Maria said she did the differences.
Teresa: where are you {to Ron who was controlling the movie player on the computer screen}?
Ron: She says when she went home, she did the differences. She did multiplication. Alright. So that's where the word differences comes up first.

June: ohhh {surprised}

Ron: Then Margaret says there's no sequence, the only common ... but then for some reason, where does the differences {of differences} come out?

June: Jamal again.

Ron: like who says 1, 8, 27, 64 ... I mean, 7 ...

June: Jamal, here {points to quote in transcript}. He says ...

Ron: the differences between the differences between ...

June: {Flipping through paper} He comes up with 7 and Molly quickly says the other two numbers.

Ron: It says {pointing to quote in text} "she probably had already taken the first differences."

June: Oh right, she reacted, that's right.

Ron: But Jamal was the first that brought the differences in the discussion.

June: and I think that he sensed that there were other people that, since Molly saw it, he said yeah, yeah, he wanted {gestures forwards to imaginary board} to see them {the numbers} up there. And I think that kids sometimes do do that. They'll say, I know there's something in there.

Note in the last remark by June her analysis of Jamal's actions that had introduced the differences of the differences approach, albeit in an unintended way. According to June, Jamal wanted to have his number sequence written on the overhead transparency because he felt that "there is something there" although he was uncertain about what that "something" was; and she immediately connected this observation about Jamal with a personal statement: "kids sometimes do *do* that." This excerpt shows how a narrative account expresses at once: grounded evidence (e.g. "here, he says"), interpretations ("he sensed that there were other people..."), and the background of life experience (""kids sometimes do *do* that.").

Conclusions

Paper and video are both important in different ways. The classroom videotape makes possible to get a "feel" for what teachers and students say or experience, to encounter one's own assumptions about the classroom interactions, and to formulate questions with a great degree of ownership. The paper introduces a particular interpretation, it helps to develop a sense for "what to be watching", and its transcriptions highlight what, among the massive amount of information available in the video, appears significant to the author.

Digital video embedded in videopapers facilitates data analysis by enabling a non-linear search of utterances and events and the development of complex narrative accounts encompassing grounded evidence, interpretations, and the teachers' background of life experience. This suggests that videopapers may be particularly suitable for teachers' engagement with data analysis.

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[1] The VideoPaper Builder for Macintosh can be downloaded from www.terc.edu/brp/vpb/vpb.html

USING PROBLEM SOLVING TO IDENTIFY MATHEMATICALLY GIFTED STUDENTS

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Despite growing awareness of the characteristic behaviours of mathematically gifted students, the identification method of choice in most New Zealand schools continues to be results on standardised tests. This study looked at a range of measures for identifying mathematically gifted students in elementary school. Of the six measures identified by the literature as being useful, student's responses to a series of difficult mathematical problems proved most useful in identifying giftedness. Two cases are discussed in which standardised test results differed from ability to solve problems.

Introduction

As the general description of a gifted individual has evolved over the years, so too has the description of the mathematically gifted. In the past, the mathematically gifted were usually identified by means of their results on achievement tests. While this practise continues to a large extent today, a number of researchers have shown that high scores on achievement tests do not necessarily imply mathematical giftedness (e.g., Span and Overtoom-Corsmit, 1986). Researchers and educators alike are looking more and more at the behaviours of students that might indicate mathematical giftedness (e.g., House, 1987). Many of these characteristic behaviours are evident when the mathematically gifted student is engaged in problem solving.

Over the last thirty years there has been an increasing awareness of the importance of problem solving in the teaching and learning of mathematics and an increasing understanding of the particular characteristics of mathematically gifted students that make them adept problem solvers. Krutetskii (1976) studied 192 students aged between six and sixteen, 34 of whom were judged as mathematically gifted. The students were asked to work on a variety of non-standard mathematical problems. He found that mathematically gifted students differed from less able students in their problem solving behaviour in that they were able to: view the content of a problem analytically and synthetically, generalise problem content and solution method, exhibit curtailment when solving similar problems, be flexible, readjust solution techniques where necessary, look for simple, direct, elegant solutions, investigate aspects of problems before trying to solve them, remember and generalise curtailed structures of problems and their solutions, and tire less when doing mathematics than when doing other subjects.

Recent research on problem solving and problem posing has highlighted the potential within these related areas for assessment of mathematical ability. Lowrie and Whitland (2000) studied problem posing in Grade Three students and found valuable information on the students' mathematical understanding and ability was gained by studying the problems they posed. Cifarelli (2000) describes in depth the problem solving of a college student, focussing on her methods to gain a solution as much as on the solution itself. The study revealed much about the mathematical understanding and ability of the student.

The measures that are most commonly recommended for the identification of gifted students in New Zealand are standardised tests, teacher nomination, parent nomination, self nomination and peer nomination (Dale, 1993). This study looked at these five measures and their usefulness in identifying mathematically gifted students. It also added a sixth measure, the student's problem solving ability, because of the strong links that have been identified between problem solving ability and mathematical giftedness.

This study

Sixty-six students participated in this study. Fifty-six of the students were from two mixed ability year six classes (age 10) and ten of the students, of a similar age, were in a class for gifted students. The gifted students attended normal state schools for four days each week and the private gifted school for one day a week.

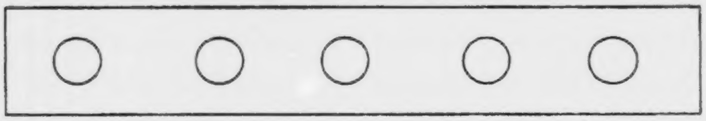
Seven of these students were already judged to be mathematically gifted and provided a benchmark for the study.

Information on each of the six measures used for identification of mathematical giftedness in this study was collected in the following manner.

1. Standardised test: Progressive Achievement Test in Mathematics (P.A.T.) (NZCER, 1998) age percentiles were obtained for each student. Students routinely sit this test in the first month of the new school year.
2. Teacher nomination of which of their students were mathematically talented.
3. Self-appraisal: students were asked to rank their own ability in mathematics.
4. Peer nomination: peers were asked whom they thought were the best mathematicians in their class.
5. Parent nomination: parents were asked whether or not they thought their child was mathematically talented.
6. Problem solving: each student sat two problem solving tests involving a total of six problems. The problems were based on those used in a study on the problem solving attributes of the mathematically gifted by Span and Overtoom-Corsmit (1986). The six problems used are shown below.

1. *On a summer day a grandmother walked along the beach with her granddaughter. They saw bike tracks in the sand. They tried to work out the order in which the bike tracks were made. In which order do you think they might have been made? (A diagram followed showing a series of four intersecting tracks with some of the intersections covered by footprints)*
2. *If I add a father's age to that of his son's the total is 50 years. The father is 28 years older than the son is. How old is the father and how old is the son?*
3. *Sarah went to the shops and bought 4 magazines; Metro, the Listener, More and the New Zealand Woman's Weekly. In how many different sequences can she read her magazines?*
4. *I have two barrels of water. One barrel contains twice as much water as the other barrel. I pour 20 litres of water out of each barrel. Now one barrel contains three times as much water as the other. How much water was there in each of the two barrels to start with?*
5. *Tim's neighbours have just moved to another town. New neighbours will arrive next week. Tim has discovered that two of the new neighbours are children. He wonders what the chances are that at least one of the children will be a boy. What do you think?*
6. *In an office in town people are called by blinking lights. Each employee has a personal combination of one or more lights. There are exactly as many*

combinations as there are workers. There are 5 different coloured lights in a straight line as shown below. How many workers are there?



Each problem was scored using a 7-point rubric. The problems chosen were expected to be difficult for the more able mathematicians in the classes to solve. This proved to be the case, with only 17 of the 66 students who attempted the problems scoring 50% or higher and only two students scoring 90% or more on the problems.

Despite the care taken in choosing the six problems, inevitably some of them proved to be more useful than others in differentiating between the more and less able problem solvers. Problems that a majority of students solved correctly, or came close to solving, were seen as too easy. Problems that very few of the students could solve, or could come close to solving, differentiated between the very able and the less able problem solvers. Problems three, five and six proved to be the most difficult problems. Two of these are permutation problems and the other is a probability problem. Of the three problems which were less successful in differentiating between the more and less able problem solvers, two could be solved using a guess and check method. Problem one proved relatively easy for students to solve for no very obvious reason.

Of the five measures tested, self-appraisal, peer nomination and parental nomination proved to be unreliable as a means for assessing mathematical giftedness and are not discussed further here.

There were strong links between teacher nomination and high standardised test results. Of the 23 students who were nominated as mathematically gifted by their teachers, 19 had test results of 90% or higher. This was largely to be expected as these test results are used in many schools to identify the mathematically gifted.

The problem solving results proved to be most interesting. Looking at the students' answers and their solution methods told us much about their mathematical ability. To illustrate this point I have chosen two students who have similar test results but very dissimilar problem solving results.

Tof and Dan scored very highly on their standardised tests, gaining 99% and 98% respectively. From their similar high test results we might expect them both to achieve well in the problem solving tests. This was not the case. While Tof gained the highest problem solving score of 95%, Dan only achieved 26%. A close look at the methods these two students used to obtain solutions to the problems yields still more interesting information.

For the second problem Dan tried to find an appropriate algorithm to help him solve this problem. He wrote:

50
- 28 Son 22
22 Father 38
+38
50

His addition was incorrect for the second algorithm and he reached an incorrect solution. He was not sure how to solve this problem and seems to be 'grabbing at numbers'.

In contrast, Tof wrote:

50
-28 22 ÷2=11=Son
22 Dad = 11+28=39

This is a very clear and appropriate way of solving the problem. Tof understood the nature of the problem and used appropriate strategies to reach a solution. His mathematical understanding is evident in his response.

A similar difference between these two boys' mathematical ability was evident in their solutions to problem six. Dan used a listing strategy but unfortunately ignored the fact that the coloured lights could not change their position. He labelled the lights r, y, o, b, g. His first five solutions were:

r y o b g
y o b g r
o b g r y
b g r y o
g r y o b

He then went on to provide combinations using four lights, two lights and one light. His answer was '25'.

In comparison, Tof recognised the mathematical nature of this problem and set about solving it appropriately. He wrote:

$2 \times 2 \times 2 \times 2 \times 2$
$2^3 = 8 \times 2^2 = 4$
$8 \times 4 = 32 - 1 = 31$ workers

He was the only student to get a correct answer on this particularly difficult problem. He recognised the nature of the problem and used appropriate mathematical tools to reach a solution. His solutions have much in common with Krutetskii's (1976) descriptions of the work of mathematically gifted children.

The wide difference in Tof and Dan's mathematical abilities was very apparent in their responses to the problems, yet their test results were almost identical. Teachers relying on these test results to identify giftedness would rate these two children as similarly capable and similarly entitled to any extension programme. Their different abilities, as evidenced by their problem solving solutions, would suggest otherwise.

If standardised test results were used as the primary assessment of giftedness, the top 10 students from the 66 students studied would be as shown on Table 1. Note the range of problem results, from a high of 95% percent to a low of 26%. All of the mixed class students with high test results were nominated as mathematically gifted by their teachers. In contrast, the gifted class teacher (myself) did not nominate Dan, even though he had a test result of 99%. Having worked with gifted children for some years now I relied more on my own judgement of Dan than on his test result when deciding whether or not to nominate him as mathematically gifted. His low problem solving results supported this judgement.

Table 1. Problem solving, standardised test percentile and teacher nomination results for children scoring in top ten (age percentile) of the standardised test

Student and class (Mixed or gifted)	Tof (G)	Rav (G)	Dac (G)	Has (G)	Bob (G)	Dar (G)	Gem (M)	Jup (G)	Dan (M)	Stn (M)
Problem total as %	95	93	76	67	50	48	69	67	26	52
P.A.T. age percentile	99	99	99	99	99	99	98	98	98	96
Teacher thinks talented (Yes)	Y	Y	Y	Y	Y		Y	Y	Y	Y

Analysis of the problem solving scores for students with relatively low standardised test result yielded interesting information also. Thp and Rog, who both scored 60% on the standardised test, scored very differently on the problem solving. Thp scored a very low 14% and Rog a much higher 52%. Their solutions showed a marked degree of difference in their mathematical understanding.

Discussion

While information accrues on the characteristics of mathematically gifted students there remains no easy method for their assessment. Teachers faced with a barrage of behaviours to identify and quantify tend to defer to what is straightforward and seemingly defensible in the form of standardised tests. This study sounds a strong note of caution for this practise. Using a score of 90% or more as an identifier of giftedness failed to identify six excellent problem solvers and identified eleven students with low problem solving results. Standardised tests may not be the most appropriate means for assessing giftedness for a variety of reasons. Firstly, the tests may require students to do little more then complete algorithms. This may test a student's memory and ability to repeat a learned procedure, rather than a student's mathematical ability. Secondly, the more able students may experience a ceiling effect on sitting a grade level test. They may be capable of high results in tests one or more years ahead of their grade. Grade level tests may mask this. Thirdly, the test used for this study was a multi-choice test.

Whether students guessed the correct answer or whether they understood the questions and used appropriate means to solve them is unclear in such a test. The problem-solving test gave a real insight into the mathematical ability of the students who sat it.

Of the problems used in this assessment, three stood out in discrimination of the top children. Two of these questions involved permutations, including the magazine question above, and one involved probability, both of which are known to be challenging concepts.

It is apparent that standardised tests alone are a poor test of giftedness. A short additional assessment is needed to identify those children in need of extension. Problems similar to those used in this study may provide the appropriate assessment.

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STUDENTS' RESPONSES TO A NEW GENERATION ILS ALGEBRA TUTOR

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This study reports on the quality of discourse between pairs of students and a new generation ILS and uses this discourse to explain some of the successes of such programs. Split image video recordings were analysed to provide a rich set of data. The quality of cognitive scaffolding provided by the ILS and the social discourse that developed between pairs appeared to foster the learning of algebra by some students. The nature of the discourse between the students and the technology and among students differed. For some students the cognitive and social scaffolding was inadequate and resulted in limited time on task. The study has impactions for the role of teachers in classrooms where such technology is the primary stimulus material.

Integrated Learning Systems (ILS) have used computing technology to harness a student model that records and updates histories of individual students, an expert system that models ideal student actions, a pedagogical model that makes teaching decisions, and a graphical interface with input and output devices for interaction to implement traditional methods of teaching and learning, in particular drill and practice and similar behaviourist approaches to teaching and learning (McArthur, Lewis & Bishay, 2000). Such programs generally have well defined goals, such as factual knowledge and/or procedural skills that can be measured on standard tests. The computer-student interactions are usually controlled by the computer: it provides a stimulus, the student responds, it analyses the response and then provides appropriate feedback and further stimuli. Usually the software breaks down the content to be taught into small units, assesses progress, and then moves on to the next unit or provides remedial instruction (Maddux, Johnson, &Willis, 1997).

ILS programs tend to be *clean* software (Papert, 1993) in that they try to mimic quality instruction by reducing mathematics to formulas describing procedures to manipulate symbols. Because of this, the quality of scaffolding or temporary support provided to students until they can perform the intellectual tasks on their own (Ertmer & Cennamo, 1995) is very important. The software may be ineffective if it does not provide cognitive support for students as they move through their *zone of proximal development* (Vygotsky, 1987). As well, Salampasis (2000) noted that the additional cognitive load of studying within an electronic medium while learning new work might be beyond some students. He argued that the task of both learning the content and learning how to navigate and learn within a new medium may exceed the working memory capacity of these students and result in disorientation, the inability to cognitively process the material being learnt, and impeded learning (Cooper, 1998).

The ILS. *The Learning Equation* (ITP Nelson, 1998) is an ILS designed to be the major mathematics teaching resource for the first four years of high school. It is a complete course that extended from Years 7 to 10. It is a multimedia environment composed of voice and textual explanations, practice questions (where text cues guided students who make mistakes), summary activities, and self-tests. The software uses a cyclic approach with each of its topic units covered in each of the year levels. Generally, each unit comprises four phases. The first is an application or mathematical modelling situation where the key concept was related to an applied problem. The second is a guided explanation of the concepts and procedures with reference to a problem. That is, the students are lead through the logic behind the concepts and procedures by a series of prompts and explanations. The third phase provides practice questions, word problems and terminology activities to consolidate and extend the knowledge introduced in the initial phases. In this phase, the students can select to see the answer, see a model solution or try a problem of similar structure. The final phase provides a self-test where students are given a selection of the types of questions studied in the lesson unit. The students can see their own responses and view correct solutions with detailed working shown.

Previous studies on *The Learning Equation* indicated improved student performance on standard tests (Bracewell, Breuleux, Laferriere, Benoit & Abdous, 1998; Norton, Cooper, & McRobbie, 2000; Pfaus, 1998). This occurred for both able and less able students (Pfaus, 1998). Students were found to be more actively engaged in activities (Pfaus, 1998) and more involved in discussion (Norton et al., 2000) than in traditional classes. The importance of peer interactions in facilitating student learning when working with technology has been noted previously (Tao, 2000). The students indicated they liked being able to work at their own pace (Pfaus, 1998). Thus, it is apparent that most of the research carried out on the effectiveness of *The Learning Equation* package has endorsed its potential to enhance mathematics teaching and learning and affect.

Norton et al., (2000) put forward two explanations for the improved student cognitive outcomes from use of *The Learning Equation*. The first was that the quality of instruction and scaffolding provided by the ILS's virtual environment might have effectively mimicked quality instruction. The second was that the classroom discourse was such that socially constructed understanding in the constructivist tradition may have been facilitated.

The aim of the study reported in this paper was to follow up Norton et al., (2000), to examine the interactions of students with the technology and with each other as they worked on the software. The focus was to determine how different students responded to the cognitive scaffolding and the social environment of *The Learning Equation*. It was hoped that examination of these variables might shed light on the apparent success of the software.

Method

Subjects and Contexts. The subjects of this study were 28 Year 9 students (about 14 years of age) in a secondary school of 650 students located in a middle class suburb in the metropolitan area of Brisbane, the capital of Queensland. Of these, a pair of capable boys, a mixed ability pair of girls and a pair of girls who were regarded as average students were purposefully selected using pre-test results for detailed study to illustrate a range of student responses to the software. The class was taught by an experienced mathematics teachers who was regarded as a quality teacher by his peers but whose normal teaching conformed to that described as the "school mathematics tradition" (Gregg, 1995, p. 443).

Data collection methods. The methods used were observation, collection of artefacts, interviews and tests. The student pairs were observed over six lessons and split-screen videotape data (combining feed from the computer with a video) was recorded for two lessons. This enabled the face reactions of the students and their discussions to be superimposed alongside *The Learning Equation* software screen, showing interactions between the students and the technology. The student pairs were interviewed following the observations. All students were administered pre- and post-tests with respect to algebra achievement. The results of these were reported in Norton et al., (2000). As the focus of this study was to examine students' response to the *The Learning Equation* software, the data from these tests are not provided in this paper.

Analysis. A hermeneutic interpretive and naturalistic approach to data analyses was adopted (Denzin & Lincoln, 1994). Information was analysed for commonalities cumulatively across the life of the study.

Results and Analysis

In *The Learning Equation* class, students sat and worked in pairs at small desks upon which the networked computers were located. Typically, the lessons started with an overview of the tasks to be undertaken that day. On most occasions, the early part of the lesson was also used to model key procedures that students had been required to complete for homework. Interactions between the three pairs and the teacher were very limited. For most of each lesson, students worked almost exclusively on the computers. The classroom was quite noisy.

The three pairs of students recorded divergent responses to *The Learning Equation*. The capable pair (Malcolm and Brendan) and the mixed ability pair of girls (Michelle and Sarah) achieved good marks in both the *operational* and *structural* components (Sfard, 1991) of the post-test after working on the ILS. The performance of Malcolm, Brendan and Sarah was consistent with their previous grades in mathematics, while Michelle's performance was an improvement upon her previous grades. The third pair (Candice and Lota) performed poorly after working on *The Learning Equation*. Their results in mathematics in previous tests had been average.

When working on *The learning Equation,* Malcolm and Brendan took turns with the mouse and keyboard to input responses. They initially kept a record of procedures "so you know where you are if you make a mistake, you can go back and do that bit"; however, as the study progressed they ceased this behaviour. They increasingly used mental arithmetic; that is, most "minor" computations, which might involve two or three steps, were carried out without resorting to pencil and paper algorithms. When working with larger numbers the pair resorted to using a hand held scientific calculator, often reading the problem aloud to each other.

If, they made an error, the pair would read and re-read the given information (as evidenced by the movement of the mouse indicator over the screen) and then discuss the possible options. Most often these discussions were very brief and justification was not attempted; it was rare for a discussion to last more than 10 seconds. They would then try an option and see if the input resulted in affirmation from *The Learning Equation* that they were correct. They appeared to operate in small bites of information before trying it out and moving to the next trial. Their comments were typically of the form, "well why don't we try dividing both sides?" After several failed attempts, the pair would cease study and randomly input until the software provided hints and finally the correct answer. Sometimes, they were systematic in using a process of elimination in order to get the correct answer. They systematically tried all probable solutions without an apparent preferred sequence until the software accepted their response. Sometimes the pair discussed why the answer was correct but this was rare. They engaged in reflection only occasionally.

Malcolm and Brendan exhibited clear enjoyment when their work was rewarded with a tick. The immediate feedback seemed to motivate them. Their cognitive time on task was very high, with little off task chat noted on the videotapes. What snippets of off task chat there were seemed to act as mental breaks and did not usually last more than a few seconds. They repeated the self-checks up to three times, trying to improve upon their score. The boys explained, "We want to get that one question we always get wrong right, we kept doing it to get a past." They rarely asked for the assistance of the teacher explaining "he takes too long to get here." Instead they either worked it out themselves, used the cognitive scaffolding provided by the software, or used the feedback facility to cheat the system; they became autonomous learners. They often did not follow the prescribed sequence of activities recommended by the software, but moved about according to their own preferences.

The following is a synthesis of their comments on *The Learning Equation* environment. The paragraph is a joint construct since the boys shared their evaluation of the program.

> *We would have done better by working with the computers because if your get an error you can go back and re do it. In a normal class, you can not really do that because of the limited number of questions of a particular type. But we work harder in a normal class because a teacher can supervise you all the time. When you work in a pair, it is good because you can sometimes help each other work it out. The problem questions lets you see how the maths is related to the real world better than a textbook. We liked the pictures and how each question is explained. And you get sample solutions and not just the answer like you do with a textbook. We would like to work with the computers again. There is no mathematics teacher we have had that we would prefer over the computer. The teacher should just tell us what units have to be done and then let us do it.*

In summary, the boys appreciated: (a) the cooperative working in pairs; (b) the practice examples, the variety of activities and the cognitive scaffolding provided by the ILS; and the medium of delivery (the computer).

Like the boys, Michelle and Sarah shared turns, reduced their quantity of note taking, enjoyed success, and gradually became largely independent of teacher assistance. However, they discussed and argued more than the boys before submitting a response. They also rationalised to each other more frequently. Comments such as the following were frequent while they worked, "How did you get that? … Well, I just divided both sides by x squared … Why does that make sense to you?" When one did not understand, they would try to explain it to each other and, unlike the boys, did not quickly resort to a form of guessing strategy. Comments like, "no you don't divide you take because the other way is add," were common when they made an error. This seemed to represent a genuine attempt to understand the underlying structures. In summary, their discussions were overtly focused on algebra structure and procedures; they particularly provided a rich discourse in algebra procedures.

Unlike the boys, Michelle and Sarah generally followed the suggested sequence of the program, including most of the examples. The time discussing each question they did not understand meant that they completed fewer questions than the boys had. When the girls had the option of selecting from "try again", "see the answer" or "see a complete solution", they most often selected "see the answer" as a first option and tried to explain the result to each other. After several attempts without understanding, they would select "see a complete solution". When competing the self-checks, the girls operated as a pair and shared the workload so that it was really a learning activity rather than a testing activity.

The following comments represent a summary of their assessment of *The Learning Equation.*

> *The program is good, as we went along we got better and worked better together. With the computer you don't get as bored because you can read the instructions rather than just listening. But when you have a teacher, they can show you to do this step and then this step and I like that kind of teaching and help. We work harder on the computer because we do it together and that helps us to work hard. We also talk a lot more, helping each other, in a normal class we focus more on our individual work. The teacher tells us to "shut up" if we talk. But sometimes on this thing, I get really frustrated because I don't do well on the self-check. I work so hard and just die when I get poor result on the self-check (Michelle). However, if the teacher is really good at explaining then I would prefer to work in a normal class.*

Overall the impression was that, like the pair of boys, the girls had formed an efficient symbiotic relationship helping each other with their learning. They placed greater emphasis on the importance of cooperative work and discussing than the boys, and they had some complaints about the cognitive scaffolding provided by the software. They felt that there was not enough information to enable them to understand the procedures. Like the boys, they appreciated the variety of work and stimulus media; but, unlike the boys, they felt that the program encouraged them to work harder than in a traditional class.

The third pair, Candice and Lota, had a poor attitude towards *The Learning Equation*. Initially, they were non-committed, but they gradually grew hostile toward the software and trying to learn from it. This process started in the first lesson when they had trouble logging on. Three lessons later they were still having problems. It seemed a case of "Murphy's law"; technical "hitches" seemed to plague them. The girls quickly developed anger toward the technology and resented using it. They complained that the structure supplied "was not detailed enough and did not make it clear what had to be done". This was particularly so when procedures required more than one step, a frequent occurrence for the ILS. They resisted using pencil and paper even when encouraged to do so.

Candice and Lota appeared to require higher teacher input and supervision than the other two pairs, both in terms of behaviour and mathematics, in order to work with the ILS. The teacher continually exhorted them to "work". They would not use their calculators unless told to get them out, they would not begin work until told which activities to work on, and they would continue repeating old work unless told to move on to new section. In order to progress through the mathematics activities, these girls frequently typed in random letters and the answer was provided by the technology, sometimes as a first option. Often the girls did not process the data nor did they read the explanations provided, but simply progressed to the next task. The girls constantly complained that *The Learning Equation* was "too complicated". Arithmetic problems, particularly dividing and multiplying by fractions, constantly thwarted the girls and made it difficult for them to complete the algebra. The girls' response to frustration was to put their hand up for teacher help. Often this was not forth coming, so the girls went off task, discussed social issues, and distracted other pairs of students. Their attitude was summed up simply, "we hate it, it does not explain like a teacher ... it is too much for our brains and it aggravates us!". Both girls wanted to return to a classroom where a teacher taught the mathematics. They believed they would get more encouragement and better explanations from a teacher that from *The Learning Equation*.

Analysis

The cognitive time on task for the successful pairs of girls and boys was high and their knowledge of the operations and structures of algebra improved. Both pairs commented that the cognitive scaffolding provided by *The Learning Equation* was a positive factor. They liked particularly that every question was explained, and that explanations "showed you this step then this step". They also believed that the paired interaction with the program was beneficial. As was stated, "you can sometimes help each other work it out", and "we work harder because we do it together".

The boys used a form of "immersion" in order to learn to do the problems. This included guessing and doing the self-checks repeatedly until they achieved mastery. However, while the boys progressed in their structural knowledge of algebra (as tested by word problems), their discussions did not reflect a structural orientation. Michelle and Sarah differed from Malcolm and Brendan in that the social aspect of the cooperative learning process was much more important and pronounced in their behaviour. As well, their discussions showed that they tried to understand the structures of problems.

For Malcolm, Brenda, Michelle and Sarah, the cognitive and social scaffolding provided by *The Learning Equation* was sufficient to foster a quality-learning environment. However, for Candice and Lota, the combination of mastering a new instructional medium and new mathematical content provided a cognitive load that was beyond them; they had cognitive overload. They found the cognitive scaffolding of the ILS inadequate; as a pair, they could not provide each other with the support they each needed. These factors together with the limited intervention of the teacher in both the cognitive and social domain resulted in them spending limited time on task and little movement within their *zone of proximal development*. Both girls exhibited a hostile attitude towards the ILS, technology in general, and mathematics; and they learnt very little algebra from *The Learning Equation*.

Implications

The study provides evidence that when well-structured ILS programs are used with pairs, the cognitive scaffolding of the ILS can combine with the social discourse that develops between students to promote learning (in this case, learning of algebra operations and structures). However, this success is not uniform. Different pairs may adopt quite different discourses with the software and each other, and this may be gender based. Some students, particularly those with lesser mathematical background and a greater need for teacher intervention and direction, may well show limited cognitive progress and exhibit negative attitudes to the technology and mathematics. For such students, the role of the teacher remains critical.

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INSIGHTS INTO CHILDREN'S RULER CONCEPTS – GRADE-2-STUDENTS` CONCEPTIONS AND KNOWLEDGE OF LENGTH MEASUREMENT AND PATHS OF DEVELOPMENT

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The paper is concerned with the qualitative investigation of pre- and postconceptual knowledge of grade-2-students with respect to their ruler concepts. The analysis of children`s drawings of rulers, their supporting explanations and their ruler techniques indicates that they have already constructed ruler concepts with different ways of understanding of the key measuring aspects prior to formal instruction. Their ruler concepts frequently exceed an instrumental understanding of linear measurement. However, some children interpret measuring in a familiar arithmetical framework simply as counting even after measurement has been formally introduced.

THEORETICAL BACKGROUND

We owe the understanding of the development of children`s length concepts primarily to the work of Piaget, who is widely recognized as the pioneer of research on measurement (PIAGET et al. 1974). He showed that the concept of length measurement depends on the comprehension of the construction and co-ordination of linear units. The process of linear measurement requires the ability of integrating space and number concepts in an idea of iterative, dividable and countable units. This idea is based on knowledge and conceptions of units and numbers and their integration into the process of linear measuring (BOULTON-LEWIS et al. 1996, HIEBERT 1984, NUNES et al. 1993).

Length concepts are an important part of elementary mathematics education. The introduction of length is the first formal measurement process to be taught and provides the basis for more sophisticated measurement concepts such as area or volume. Traditionally, the introduction of magnitudes such as length follows an instruction sequence (German: „didaktische Stufenfolge"). This instruction implies that children develop and use informal units, prior to the use of standard units and conventional measurement instruments. The underlying goal of this approach is that children on the one hand recognize the relationship between the one-dimensionality of length and the single countable objects and on the other hand realize the necessity of standard units (OSBORNE & WILSON 1992; RADATZ et al. 1998).

However, the iterative use of informal units requires above all the competence of counting and the accurate joining of objects. Students may not pay attention to the length of the objects nor consider them as units (HIEBERT 1984). The neglect of these aspects during formal instruction may explain why fourth and fifth graders frequently demonstrate insufficient knowledge and misconceptions of length units (CARPENTER et al. 1988; GRUND 1992). An increasing number of publications in the last decade questions the traditional sequential introduction of length (AINLEY 1991, BOULTON-LEWIS et al. 1996, BRAGG & OUTHRED 2000, NUNES et al. 1993). Recent studies indicate that students have already had experiences with culturally developed tools of length measurement in everyday life, e.g. with a ruler or a folding-rule. Even if classroom instruction is based on the use of informal units, students of all grades tend to compare visually or use their everyday measuring skills rather than using informal units (BOULTON-LEWIS et al. 1996, NUNES et al. 1993). A ruler has not only a concrete relation to the reference context of measuring but also a theoretical relation to the representation of the measurement process. It involves mathematical figures and presents an iconic illustration of structural connections between conventional units and numbers (STEINBRING 1993). BRAGG & OUTHRED (2000) as well as HIEBERT (1984) emphasize that many students are able to use a ruler for measuring and have mechanical knowledge about reading and using the ruler scale. But most of them obviously have only a poor understanding of the measurement process. They don't know more „than rules about rulers" (BRAGG & OUTHRED 2000, 97).

According to the „moderate-constructivist" view, children do not simply accept everyday practices or phenomenona in a passive way but rather construct subjectivly significant concepts in an active-ideosyncratic way in interaction with their environment (GERSTENMAIER & MANDL 1995). Hence, by using a ruler children construct individual understanding of length measuring and develop ideosyncratic concepts regarding the construction and co-ordination of marks, spaces and numbers. Therefore, students` ruler concepts imply on the one hand their measuring and drawing skills and on the other hand their understanding of the concepts underlying these procedures, i.e. their knowledge and conception of the relationship between the measurement of length and the numberline represented on a ruler scale.

In this paper the following questions with respect to the conceptual ruler knowledge of children will be investigated: In how far do grade-2-students focus on the represented key aspects of linear measuring (the structural connections between linear units and numbers) when they use a ruler? In how far do they interpret these symbols as a representation of measuring system?

METHODOLOGY

The qualitative longitudinal study underlying this paper is concerned with the investigation of the development of length measurement concepts of grade-2-students. 12 children from an urban school in Münster were selected as case study children according to their performance in a pretest about their knowledge of length and their teacher`s assessment of their mathematical abilities and performances. This process led to the selection of two girls and two boys in the following three categories: low, average and high achievers. These children were interviewed shortly before the formal sequential introduction of the magnitude „length" (pre-interview), a week as well as six months after this unit (1st and 2nd post-interview). The interview tasks were designed to include a variety of components of length measurement in different contexts. Instead of paper-pencil-tests practical tasks were chosen because of their correspondence to measuring in a real life context and to facilitate the use of conventional measurement tools and objects. The half-standardized interviews were evaluated following the „interpretative paradigm" (BECK & MAIER 1993). The interview episodes referred to in this paper address iconic and verbal ruler representations and the techniques of ruler use.

Episodes	Questions	Measuring tools
Ruler pictures	*This is meant to be a ruler* (the children receive a sheet of paper with the drawing of a rectangle – 16,6 cm x 2,6 cm). *Please fill in, what is missing in order to make it look like a typical ruler.*	None
Verbal explanations	*You have drawn numbers and marks. Can you tell me why you used those numbers and marks?*	
Drawing	*a) Please draw a line with a ruler which is 10 cm (25 cm) long.* *b) Please draw a line with a ruler which is 2 mm (12 mm) long.*	They could choose between a plastic ruler (20 cm) or a wooden ruler (30 cm); on both rulers „0“ is marked 0,5 cm from the edge.
Measuring	*a) Which of these drawn lines is as long as this wooden path?* (The children are given 4 paper stripes with lines of 20, 23, 24, 25 cm and on the table in front of them is a wooden path made of two sticks, 20 cm and 4 cm long, like in this picture:) *b) How long is the distance between these two sticks?* (15 cm)	

Table 1: Questions to the ruler concepts

RESULTS AND DISCUSSION

The focus of analysis in this paper is on the ruler pictures, because they provide iconic evidences of the interpretations of the structural relations of measuring. When children draw an object or a phenomenon they consciously turn their attention to their rather vague mental images, organize their knowledge about the use and construction of the object and focus on the significant subjective structural characteristics (BIESTER 1991). Both, the ruler pictures, which are supported by verbal explanations, and the ruler techniques open a window in the students' world of ruler concept.

In this paper, the paths of development of ruler pictures are described and analyzed.

Ruler pictures: All the interviewed students were able to construct ruler pictures and to emphasize significant subjective key characteristics of the ruler scale. Their ruler pictures can be assigned to one of four types of drawings which result from the construction and co-ordination of numbers and marks (NÜHRENBÖRGER, in press):

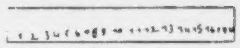
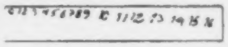
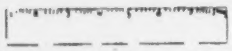
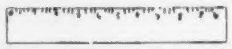
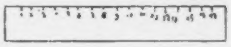
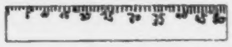
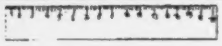
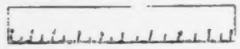
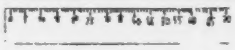
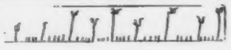
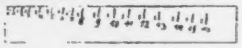
Number-Ruler		**Number-Intermarks-Ruler**	
			
Starting point „1“	Starting point „0“	St.pt. „1“ and 5 intermarks	St.pt. „0“ and at first 4, then 3 intermarks
Number-Mark-Ruler		**Unit-Ruler**	
			
St.pt. „1“ and counting in steps of one	St.pt. „1“ and counting in steps of five	St.pt. „edge“ and a fivefold subdivision	St.pt. „0“ and a tenfold subdivision
			
St.pt. „0“ and counting in steps of five with a fivefold subdivision	St.pt. „0“ and counting in steps of five with a tenfold subdivision	St.pt. „0“ and a double subdivision with irregular intervals	St.pt. „0“ and an irregular subdivision and irregular intervals

Table 2: Examples of types of ruler pictures with different chief characteristics

A **Number-Ruler** is characterized by numbers following the counting sequence from left to right and starting with the first counting number „1“ (if children started with „0“ they showed a deeper relation to the measurement scale). Children who drew these pictures consider numbers to be a subjective dominant and significant aspect of a ruler. They interpreted the measurement scale on the basis of their arithmetical and counting skills; that means that they connected the linearly ordered numbers of a ruler with their conceptions of numbers. This seems obvious because in general, numbers are mentally internalised in an intuitive way as a numberline from left to right. Children drew pictures of Number-Ruler merely in the pre-interview.

Most of the students drew marks and numbers which indicates that they have understood the marks to be a key aspect of the measurement scale. The **Number-Intermarks-Ruler** only has marks between the numbers without any visible subdivision into equal units. The marks rather seem to be an insignificant element which decorates the ruler. However, marks had a characteristic signification for the children so that some subjectively important marks were counted and noted in the familiar arithmetical context. The intervals between the numbers were constantly equal because the marks were counted and noted rhythmically. But it may be possible that the students have perceived a „conventional concept of equal intervals“ (PETITTO 1990, 71). They recognized that rulers usually have equal intervals, but could not explain their structural significance in connection with conventional units.

If children drew marks together with numbers as a visible iterated unit they drew a **Number-Mark-Ruler**. On the one hand this ruler picture shows again an arithmetical interpretation of the ruler as a numberline. But on the other hand the role of zero and especially the spatial distribution of numbers and marks indicate an already promising anchor of the interpretation of centimetre as a dividable unit. Some students counted a certain quantity of marks (or intervals) and consequently wrote numbers in an appropriate counting sequence. Few of the pictures show individual „sense constructions” regarding the unit subdivision presented on the scale.

Students` drawings of **Unit-Rulers** contained a connection between number and space concept which led to the idea of an iterated and subdivided unit. The students showed different facets of understanding of units, both for equal intervals and for rules of subdivisions. While one student drew only a double subdivision with irregular intervals, others drew a ruler picture containing a tenfold subdivision and equal intervals. The starting point of the ruler scale was particularly important. A ruler beginning with „0“ or starting right at the edge shows that the drawer recognizes that „1“ represents the first line segment. In contrast, a ruler with the starting point „1“ indicates an orientation on the sequence of number words. But they do not automatically exclude a deeper understanding of measurement, e.g. one student called his ruler picture with starting point „1“ a „one-minus-subtraction-ruler“.

Paths of development of the ruler pictures: The relative small sample of 12 students already demonstrates the diversity of children`s ruler concepts with respect to their drawings prior to formal instruction (see Table 3). It is remarkable, that four students drew an Unit-Ruler, one of them even with a tenfold subdivision.

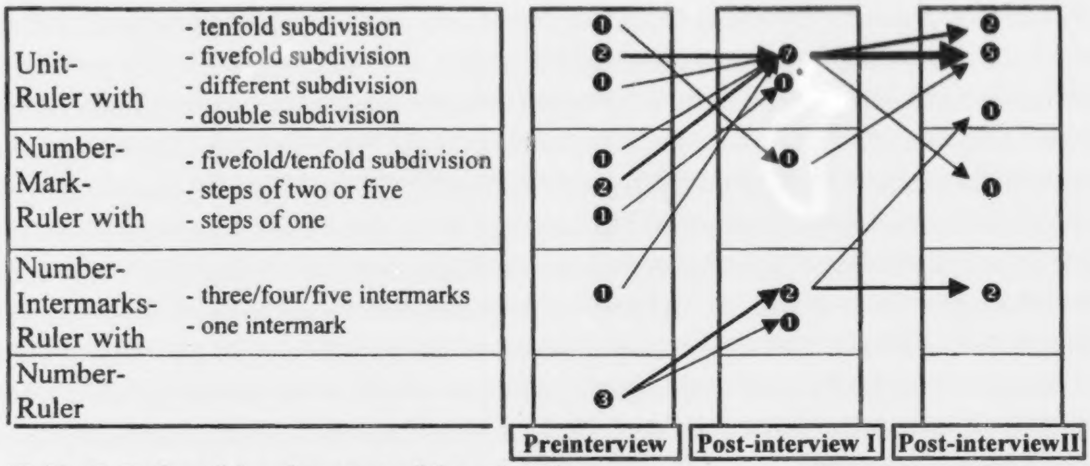

Unit-Ruler with
- tenfold subdivision
- fivefold subdivision
- different subdivision
- double subdivision
Number-Mark-Ruler with
- fivefold/tenfold subdivision
- steps of two or five
- steps of one
Number-Intermarks-Ruler with
- three/four/five intermarks
- one intermark
Number-Ruler
❶
❷
❶
❶
❷
❶
❶
❸
❼
❶
❶
❷
❶
❷
❺
❶
❶
❷
Preinterview
Post-interview I
Post-interviewII

Table 3: Paths of development of the ruler pictures ❸: Number of children
*One child was absent in the 2nd post-interview

The development of the ruler pictures indicates a growing orientation towards the structural construction of the measurement scale. From the pre-interview to the first post-interview two different paths of development can be recognized:

- from the Number-Ruler to the Number-Intermarks-Ruler and
- from the other three ruler types to the Unit-Ruler with a fivefold subdivision.

It is remarkable that even directly after the measurement unit none of the students was able to draw a correct ruler and that three ruler pictures still referred to an arithmetical number concept. The analysis of the second post-interview shows that the majority of children developed further structural understanding of rulers in the absence of formal instruction. For example, two students show a development path from a fivefold subdivision Unit-Ruler to a tenfold one, while two other students still drew the less sophisticated Number-Intermarks-Ruler.

Verbal explanations: The explanations of the students about their ruler pictures contained key words according to the numbers and marks which generally describe a measurement or arithmetic context. Only a few students did not refer to any content.

Explanation of numbers		***Examples***
Reference to measurement	- The number of units	*„One can measure how many centimetres."*
	- Information about length	*„One can measure how long something is."*
	- Information about ruler length	*„In order to know how long the ruler is."*
	- Visual help for measuring	*„There is no need to look at the marks."*
Reference to arithmetic	- For counting	*„In order to count."*
	- For calculating	*„In order to calculate."*
No reference	- Conventional	
	- No idea	*„I don`t know."*

Explanation of marks			***Examples***
Reference to measurement	- units	different	*„These little marks are millimetre, and the long ones are centimetres."*
		only one known	*„These long marks are centimetres. The little ones are 1, 2, 3, ... 8 centimetres."*
	- „Distance-holder"		*„The four intermarks keep the distance."*
	- Visual help for measuring		*„In order to measure exactly"*
Reference to arithmetic	- For counting		*„In order to count."*
	- For calculating		*„In order to calculate."*
No reference	- Conventional		*„On every ruler there are little marks.*
	- No idea		*„I don't know."*

Table 4: Different explanations to the measurement scale with some examples

The interpretation of numbers and marks as units such as metre or centimetre indicated an anchor of measurement understanding. Few students already knew at the beginning of grade-2 that a ruler represents two different, but connected units. Other students saw an interval between two numbers (or „two long marks") as a basic iterable unit called centimetre without knowing that there is also another subdivided unit. Those children were merely able to use a unit for repeated counting. However, few of them were able to construct an explanation of the gaps between the marks that demonstrates „measurement sense" - they interpreted the interval as a „distance-holder" or as a unit which occasionally is also called centimetre. Some children used terms such as metre or centimetre without any unit concepts. They either interpreted marks as „number stations" or they were unable to explain them. In a sense these terms played the role of a „number companion" which refer to length.

Few students were not at all able to express insigths („measurement sense") with respect to the construction of the formal scale. They only explained the symbols on a ruler in a number and counting context or in a verifying-conventional way.

Measuring and drawing with the ruler: Nearly all students were able to use a ruler correctly for drawing centimetres. They aligned the lines with zero or back from the number to zero. Most the case study children were even able to construct a millimetre concept although millimetres are only taught in grade-3. While few students aligned millimetres correctly and utilized the references between one centimetre and ten millimetres, others interpreted the measurement scale in an idiosyncratic way – they explained e.g. a line of two millimetres as following: „*Up to ten, because one millimetre is up to five*" or „*up to the second longer mark*" or „*up to the longer mark just before the number two*" or „*up to two*". These creative „sense constructions" indicated subjective interpretations of the relations between numbers and marks on the formal scale. But they showed on the other hand that some students have a poor understanding of subdivision - they only knew the „number-unit" centimetre.

In contrast to the skills of drawing centimetres with a ruler, many students, even some who drew a ruler with starting point „0", had problems with measuring an object because they were not sure about ruler alignment: „*Should we align a ruler with the scale or with the other side - with the edge, with zero or with one?*"

The procedure of ruler alignment was not a stable one; many students varying their procedure with regard to the situation: Although they were able to identify the length of an object as the difference between its starting and end point on the ruler, they sometimes did not pay attention to how they aligned the object with „0". Dominating for them was obviously the result of the measurement solution, because they tried to place the ruler next to an object in a way that the end of the object was marked by a number and not by an intermark. It seems that they followed an inaccurate mechanical procedure without taking into consideration what the intervals and the numbers on the ruler scale represent.

CONCLUSION

The analysis of the ruler concepts suggests that children at the beginning of grade-2 already have a subjective understanding of the measurement scale represented on a ruler. Their ruler pictures in particular demonstrate that students are able to imagine a ruler and to perceive key aspects of measuring. Their everyday experiences with measurement not only promote an intuitive approach to the technical use of rulers, but also influence the development of individual ruler concepts. These concepts vary among children and change in different paths within a schoolyear.

However, the diverse representations of characteristic features of rulers, does not necessarily indicate an elaborate understanding of the relations between linear units and the measurement scale. Although students are able to use a ruler in measurement situations, they often do not possess structured insights into the construction and co-ordination of units. They connect instead their imagination of measuring with their number concept and interpret some of the measuring aspects in the context of their arithmetical understanding of numbers and counting. Their use of a ruler is based on rules, which merely refer to the application and reading of numbers. Therefore children do not have problems with drawing and (partly) measuring. This differentiated perspective on ruler concepts supports and elaborates on earlier findings of HIEBERT (1984) and BRAGG & OUTHRED (2000).

The current research results suggest that the traditional formal introduction of length with non-conventional units might not help students to develop an elaborated concept of linear measuring, because it does not allow them to make connections to their previous everyday measurement experiences with conventional tools. Instead, dealing with (mainly three-dimensional) informal units obscures the linear nature of the unit of measure and the children interpret measuring with informal units in a familiar arithmetical framework simply as counting. Hence, students cannot visualize the measuring process as a composition of linear units. „Neither zero nor the iteration of line segments can be made explicit when informal units themselves are counted, thus reducing the possibility that students are able to make the important link with the underlying linear unit concept" (BRAGG & OUTHRED 2000, 103). NUNES et al. (1993, 53) emphasized that „relying on conventional units, which have already been chosen and built into an instrument, does not make measurement more difficult." This study has shown that grade-2-students have idiosyncratic anchor imagination of centimetre and millimetre. Their everyday knowledge about the ruler scale and their skills of the ruler use should serve as a starting point for formal instruction and guide the development of an elaborated length concept. A sophisticated understanding of measurement requires both, „hands-on" measuring activities with rulers *and* especially discussions and reflections about the key aspects of measuring repeatedly stimulated and encouraged by the teacher. The comparison and analysis of individual ruler pictures of the children in the classroom for example can help the teacher to identify their „pre-conceptions" and also foster such a discussion and reflection.

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RESEARCH REPORTS

Connecting Partitioning and Iterating: A Path to Improper Fractions

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In this paper I illustrate how an approach to establishing fractions based on a child's multiplicative relations among whole numbers provided one child (Joe) with both affordances and constraints for constructing meaning for both common and improper fractions. Joe established "one third" of a composite number as that number which, when multiplied by three, will give the composite number. In the computer microworld TIMA: Sticks, Joe modeled this multiplicative approach by finding a stick that he could repeat three times to equal the given "number-stick." The Sticks environment provided a venue for Joe to construct unit fractions as "repeatable connected numbers." In order to construct non-unit (common) fractions and (eventually) improper fractions, Joe had to connect this multiplicative approach to fractions with his part-of-whole view of a fraction.

Understanding fractions greater than the whole (improper fractions) has been a stumbling block for many children (Olive, 1999, Tzur, 1999). A major obstacle to their understanding stems from the over-emphasis in curricular materials on representing fractional quantities as the number of parts shaded of a partitioned figure. Interpreting 3/5 solely as "three parts shaded out of five parts total" provides no means for making sense of 7/5. A junior in college once complained to me "How can you have seven fifths of something?" when faced with the following problem (referring to a long thin rectangle on the test page): *This bar is 7/5 of another bar. Draw the other bar.* She was blocked by her part-in-whole interpretation of a fraction. She could not imagine the given bar as seven of five equal parts of another bar.

The research reported in this paper is part of an on-going retrospective analysis of videotaped data from a three-year constructivist teaching experiment with 12 children (Steffe & Olive, 1990; Steffe, 1998). The major hypothesis to be tested in the teaching experiment was that children could (and should) reorganize their whole number knowledge in order to build schemes for working with fractional quantities and numbers (the rational numbers of arithmetic) in meaningful ways. This reorganization hypothesis (Olive, 1999) contrasted with the prevailing assumption that whole number knowledge is a "barrier" or "interferes" with rational number knowledge (Behr et al., 1984; Streefland, 1993; Lamon, 1999).

Description of the Teaching Experiment

A team of teacher/researchers, led by Steffe and Olive, worked with the children, primarily in pairs, outside of the classroom once a week for approximately 20 weeks a year for three years. All teaching episodes were videotaped. Computer microworlds called *Tools for Interactive Mathematical Activity* (TIMA) (Biddlecomb, 1994; Olive & Steffe, 1994; Olive, 2000) were specifically designed for the teaching experiment. The TIMA provide young children with possibilities for enacting their mathematical operations with whole numbers and fractions. They also provide the teacher/researcher with opportunities to provoke perturbations in the children's mathematical schemes and observe the children's mathematical thinking in action.

We began working with Joe during his third grade year. Joe is an African-American student who was often in trouble with his classroom teacher. Our emphasis during the first year was on investigating the children's whole number multiplicative operations. We determined that, by the end of the first year, Joe had constructed an *explicitly nested number sequence* (Steffe & Cobb, 1988; Steffe, 1992). A hallmark of the explicitly nested number sequence (ENS) is the construction of *composite units* (units of units) that are necessary for multiplicative operations (Steffe, 1992). In a series of activities that we designed on the computer using the TIMA: Toys microworld, Joe was able to find how many (hidden) cookies would result from baking three trays of cookies, each tray having 7 hidden cookies, using his multiplication facts. He was also able to determine, however, that when he had 7 trays of cookies (a total of 49 hidden cookies), he would need to bake three more trays in order to have a total of 70 cookies. He was able to work with the difference of 70 and 49 as three units of seven because he knew that 70 was ten sevens.

The Construction of Fractions as Repeated Connected Numbers

For the first half of the second year of the project Joe worked without a student partner with a teacher/researcher, Azita (a doctoral student working with the project). The focus during this period was on using the children's whole number structures and operations in situations that we would regard as involving *continuous quantities*. Most of the activities took place in the computer microworld TIMA: Sticks. The manipulatives in this microworld were horizontal line segments that we called "sticks." Sticks were created by the child dragging the computer mouse across the screen. Thus the extent of a stick was the result of a deliberate motion on the part of the child. Sticks could be copied, joined together to form longer sticks (with vertical marks indicating each of the sticks that were joined), marked with vertical tick marks, partitioned into equal parts using a numeric indicator for the number of parts, and broken into separate, smaller sticks determined by the marks or parts created in the stick. Pieces could also be cut off of a stick using a mouse click. Any stick could be measured with reference to any other stick that had been designated as a measuring unit. The measure would appear as a ratio fraction or whole number. Sticks on the computer screen could be hidden from view by rectangular regions called "covers."

Through the activities with TIMA: Sticks Joe was able to use his composite units to construct what Steffe and Wiegel (1994) have called a *connected number*. Connected numbers are constructed by a child mentally projecting his/her concept of a whole number (e.g. eight) into an unmarked line segment to establish an "interiorized continuous but segmented unit" as a situation of the child's number sequence (p.121). This connected number 8 (say) would contain within it an implicit nesting of the number sequence from 1 to 8, and the sense of those 8 units being united into one composite unit. Connected numbers carry with them a notion of number as indicating relative size (length). Children were able to instantiate connected numbers in TIMA: Sticks by creating what we refer to as *number-sticks* --that is sticks created from joined repetitions (iterations) of an established unit stick. Number-sticks were referred to by their number-name. That is, a stick created from 8 repetitions of a unit stick would be called an 8-stick.

The following protocol illustrates how Joe (J) and Azita (A) used activities with number-sticks to establish fractions as *repeated connected numbers*. For instance, one third of a 24-stick is that stick, which repeated three times, will make a 24-stick. This approach to fractions of composite wholes provided Joe with the opportunity of using his whole number multiplication scheme (*a composite units-coordinating scheme*) to find the appropriate stick. This teaching episode took place in the fall of Joe's fourth grade year, the second year of the Project. A set of number-sticks had been constructed at the top of the playground area of the computer screen, separated by a long thin cover that stretched across the full width of the screen (see Figure 1). Sticks created by repeating or joining copies of sticks from this set of number-sticks were also named as number-sticks (e.g. a stick created from 4 repetitions of the 6-stick was a 24-stick)

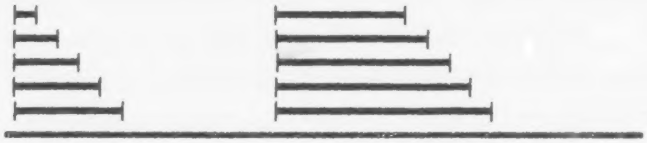

Figure 1: A set of Number-Sticks in TIMA: Sticks

Protocol I

While Joe has his eyes closed, Azita makes a 24-stick below the cover using repetitions of a copy of one of the sticks above the cover and erases the marks from her 24-stick.

A: *The stick that I used was one third of the length of the stick I have right here* (pointing to the unmarked 24-stick).

Joe measures the stick (24 appears in the number box). He then smiles to himself and counts down the set of the number sticks ending on the 8-stick. He copies this stick and repeats it 3 times to make a stick the same length as the 24-stick.

A: *That is right!*

J: *You said one third, so what will be…three times eight is 24.*

Azita suggests doing more problems with the 24-stick. Joe wants to use a 21-stick but Azita asks him to do one more with the 24-stick.

A: *Think of a stick you could use to make the 24-stick and tell me what fractional part of the 24-stick it would be, and I will try to tell you what size stick it is and how many times I should use it.*

J: *Close your eyes.*

Joe trashes the 3-part stick and looks at the set of number sticks.

J: *O.K. I didn't have to do nothing.*

J: *It's umm... It's one sixth.*

A: *The stick that you used is one sixth of the 24 stick?* (Joe nods his head.)

A: *So, I want something, I want a stick that when I repeat it six times would give me…*

J: *No!*

A: *Would give me the 24..*

J: *One fourth!* (at the same time as Azita is speaking).

A: *Oh! You used the one-fourth stick?* (Joe nods yes.)

A: *You used one-fourth, so I want a stick that when I repeat it 4 times will give me the 24, and I think that is the 6-stick! What do you think?* (Joe nods yes.)

Azita copies the 6-stick and repeats it 4 times to make a stick the same as the 24-stick.

Joe knew to use the 8-stick for 1/3 of the 24-stick because "three times eight is 24." I regard this as a modification of Joe's composite units-coordinating scheme (a multiplication scheme) whereby Joe could coordinate the elements of one composite unit (3) with the elements of another composite unit (8) to produce the target number 24. Repeating the 8-stick three times to match the 24-stick was an enactment of this coordination. Joe's interpretation of "one third" as something that when multiplied by 3 gave the total number, was a novel application of his units-coordinating scheme. Joe's strategy supports the major hypothesis of the project: that fractions could emerge from such modifications of the children's whole number schemes (Steffe and Olive, 1990; Olive, 1999; Olive, 2000).

Joe's realization that he did not have to do anything in order to pose the problem for Azita (as the unmarked 24-stick was still visible on the screen) indicates that he was able to *imagine* himself acting within the microworld. According to von Glasersfeld (1981), such imagined actions are critical, as it is through re-presenting mentally their actions that children construct their numerical operations. Joe hesitated in naming the fraction when posing his problem for Azita. I suggest that he was trying to hold both the imagined stick in his head and the number of times he would have to use it. Joe ended up using the stick-size to generate a name for the fraction rather than the number of times he would have to use that stick. Joe realized his mistake as soon as Azita voiced her interpretation of 1/6. Azita's explicit use of fractional language to describe her imitation of Joe's actions in TIMA: Sticks provoked a reflection by Joe on what he had imagined, and provided scaffolding for making sense of the fractional terms. He corrected his error rather than going with it and accepting Azita's actions. This indicates that Joe could generate his result prior to action through both numerical calculation and visualized action. It is in this sense that Joe was constructing meaning for unit fractions as *repeated connected numbers*.

Connecting Partitioning, Segmenting and Numerical Operations

While this approach to unit fractions of composite wholes was productive for Joe, it had limitations that stemmed from its attachment to known multiplication facts. In the continuation of the teaching episode in Protocol I, Joe successfully identified the 4-stick as 1/7 of the 28-stick, but when asked by Azita what fraction of the 28-stick two 4-sticks joined together would be, Joe responded with "One fourteenth…because you add one seventh and another seventh it makes 14." Streefland (1993) refers to such errors as N-distractors (miss-application of whole-number arithmetic to a fraction situation). Unit fractions were results of numerical operations at this point for Joe; they were not yet quantities that could be united.

Joe had difficulty with making equal portions of a stick in sharing situations. His visual estimates for sharing a stick into three or four equal portions were not very accurate. Rather than making a mental partition of the stick into 4 equal parts, and indicating this partition by placing three marks on a stick, he would draw an estimate for 1/4 of the stick and repeat this estimate 4 times to see if it matched the stick. In the teaching episode following the one above, we attempted to provoke Joe's partitioning operations by asking him to make fractions of a number-stick for which there was no available number-stick (e.g. make a stick that is 1/4 of a 27-stick). Our hypothesis was that, without a known multiplication fact to solve the problem numerically, Joe would need to mentally partition the 27-stick in order to make a reasonable estimate for 1/4 of the stick.

Joe had created the 27-stick by repeating an unmarked 9-stick three times. The screen display consisted of a set of number sticks as in Figure 1 (above). the 3-part 27-stick and an unmarked copy of this 27-stick. Joe's first estimate was approximately half the length of a 9-stick. He repeated this estimate four times to create a stick approximately the same as 2/3 of the 27-stick. His next estimate was approximately the same as a 9-stick. He only repeated this stick three times and then trashed it. He adjusted these estimates, gradually getting closer to a visual approximation for 1/4 of the 27-stick. He did not use his whole number multiplication facts to help with choosing an initial estimate (e.g. a 7-stick might work as 4 times 7 is 28 -- a fact he had used in the previous episode to find 1/7 of 28). There was little indication that Joe was mentally partitioning the target stick (unmarked 27-stick) to gauge his estimates for 1/4 of the stick. He appeared to be attempting to segment the unmarked stick with his estimate and adjusting his estimate until the segmentation worked. In the continuation of this teaching episode, however, Joe did use his multiplicative relations to make an estimate for a stick that would be 1/7 of a mystery stick that had been drawn freehand on the screen. He saw that a 2-stick, repeated 7 times was approximately 2/3 of the unknown stick. He made a second guess that the 3-stick would be 1/7 of the mystery stick because “It's about 27. 21, I mean.” Azita continued using the 2-stick Joe had originally chosen, creating 10 repetitions to make a stick that was about one unit stick short of the target stick. Joe confirmed that he was thinking of the mystery stick as a 21-stick when he then exclaimed “21! That's what I said!” and that the 3-stick would be 1/7 of the 21-stick because "If you use 3 seven times you might get 21."

This last task may have provoked a connection of Joe’s partitioning operations with his segmenting approach to unit fractions. He could imagine seven of something being equal to the unknown stick. This relation had to be imagined as a visual partition of the unknown stick, because he had no whole number to operate with. Thus, Joe’s *mental partitioning operations* were activated. Joe was then able to make a connection between his numerical strategy for finding a unit-fraction (find the missing factor) and his *equi-partitioning scheme* by estimating a numerical value for the length of the unknown stick.

Confirmation for this possible connection between Joe's numerical strategies and his partitioning operations came in the next teaching episode. Joe accurately chose a stick that was 1/5 of a target stick where no numerical values were known. Four sticks and the unknown target stick were arranged on the screen as shown in Figure 2.

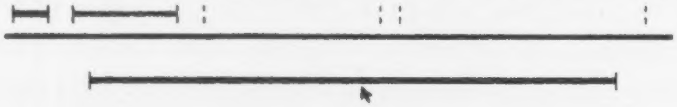

Figure 2: Which stick is 1/5 of the dark blue stick?

Joe copied the second of the four upper sticks below the lower target stick and repeated it 5 times to make a stick that matched the target. In order to have quickly chosen the correct stick, I hypothesize that he made a mental partition of the target stick. Later in this episode, Joe found 1/3 of the target stick (the third upper stick). Azita asked him to make a stick twice as long as this 1/3-stick. Joe did so by repeating the 1/3-stick. When asked what fraction of the target stick this repeated 1/3-stick would be, Joe at first compared it to the longest stick in the top row of sticks. He said this last stick was a bit shorter than his stick. When pressed for a fraction name he called his stick "two thirds" of the target stick. He also called three repetitions of the 1/3-stick a "whole stick" and "three thirds." He had established the inverse relation between a part and a whole that paralleled his numerical approach: What number multiplied by 3 will give me the whole? Further along in this same episode Joe was able to link a part-in-whole view of 5/11 with his multiplicative view of a stick that was "five times as long as the 1/11-stick." He justified calling his five repetitions of the 1/11-stick "Five elevenths!" "Because its 5 and its part of 11." He was able to project the stick that was the same as 5 parts out of the 11 (a disembedded stick) back into the 11/11-stick.. He was now able to combine unit fractions into composite parts of a whole through iteration. Joe was beginning to connect partitioning of a unit whole with the generation of both unit and non-unit (common) fractions. We refer to Joe's fractional scheme at this point in the teaching experiment as a *partitive unit fractional scheme*.

From Common Fractions to Improper Fractions through Iteration of Unit Fractions

The last teaching episode described above occurred just before the winter break in December of Joe's fourth grade year. In the next teaching episode (that took place after the winter break in February) Joe extended his fractional scheme into a *partitive fractional scheme* for common fractions. He was then able to generate improper fractions such as 6/5 and 9/7 through iteration of unit fractions. This was the beginning of Joe's *Iterative Unit Fractional Scheme*.

Protocol II begins after Joe had created 3/5 and 4/5 of a 5-part stick by drawing sticks that were the same length as 3 and 4 parts of the stick respectively. The 5-part unit stick, 3/5-stick and a unit stick with one mark 4/5 the way along the stick.

Protocol II

A: *That's really neat! Now I'm really hungry. I want you to make me another one. I want you to make me 6/5 of that candy* (meaning the unit stick).

J: *Can't!*

A: *Why not?*

J: *You only got 5 of them.*

A: *Five what?*

J: *Fifths.*

A: *You only got 5 fifths. So is there any way of making one, do you think?*

J: *Make a bigger stick.*

A: *Make a bigger stick. How much bigger do you think it should be?*

J: *One more fifth.*

A: *O.K. Do you want to show me?*

Joe pulls the end part out of the original stick that has a mark at the 4/5 position only, and joins this one piece to the original stick to make a stick one fifth larger than the original.

Joe was using his partitive fractional scheme to make 3/5 and 4/5 of the unit candy stick. What is important about the above protocol is how Joe was able to interpret Azita's request for 6/5 of the candy as being one more fifth than the whole bar. This was a possibility for Joe because his partitive fractional scheme included unit fractions as repeatable units. Thus 6/5 was six of one fifth, which was one more fifth than the whole stick. Joe was not constrained by a *part-in-whole* restricted view of fractions as were some of my college students! In Protocol III (later in the same teaching episode), Joe confirms that he can now create fractions greater than the whole through iteration of a unit fraction. Joe had successfully estimated 1/7 of a candy stick and had used his estimate to mark off all seven 7ths on the original candy stick. He had then pulled out 4/7 of the candy stick to give to Dr. Steffe.

Protocol III

A: *Can you make a stick that is...say 9 times as long as the one...seventh stick* (points to the first part of the 7/7-stick)?

Joe cuts off the first part of the 4/7-stick. He accidentally cuts this part again so he joins it back together and erases the resulting mark. He then repeats this 1/7-stick 9 times to make a 9-part stick.

A: *How long is that stick?* (Joe thinks for 3 seconds.)

J: *Nine sevenths.*

A: *Why?* (Joe thinks for 15 seconds.)

A: *You are right. It is 9/7 but why do you think it is 9/7?*

J: *Because it was...you were making these the sevenths* (pointing to the parts of the 9-stick) *so each of these would be one seventh.*

A: *That's really nice!*

In this last episode, Joe was able to work with a fraction as both a part of a whole (the 3/5 and the 4/7) and a unit part *out of a whole*. In repeating a 1/7-stick 9 times to make a stick 9 times as long as 1/7 of the original whole stick, Joe was able to go beyond the whole, and name the resulting stick as nine sevenths "because ...you were making the sevenths, so each of these would be one seventh". In this sense Joe had taken the 1/7-stick as an *iterable one* to generate a composite unit of 9, each one of the 9 units being 1/7 of his 7/7-stick. These operations suggest that Joe had not only constructed a *partitive fractional scheme* (for generating common fractions) but was on his way to developing an *iterative unit fractional scheme* for generating *improper fractions.*

Conclusions

The sequence of episodes with Joe, described in this paper, indicate how powerful a child's thinking about fractions can become when it builds from his numerical schemes and operations, assisted by appropriate language scaffolding by a sensitive teacher. They also indicate the role that both partitioning and iterating operations play in constructing fractions greater than the whole. The affordances created through Joe's use of TIMA: Sticks were also important factors in Joe's construction of improper fractions. Being able to make a stick that is "9 times as long as the 1/7-stick" through repetitions of a 1/7-stick provided Joe with an instantiation of his iterable unit fraction.

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MEASURES IN CABRI AS A BRIDGE BETWEEN PERCEPTION AND THEORY

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Abstract

In the context of teaching geometry at secondary school level, we study students' measurement activity in solving open problems in the Cabri environment. Different uses of measures are observed in the protocols' analysis, which points out that the students' approaches are double-sided, namely perceptual and/or theoretical. A tool like measures in Cabri (in the same way as dragging) can foster the passage from perception to theory and back again. The shifts back and forth are productive in the construction of a proof, after the conjecturing phase.

Introduction

At the level of Mathematics Education research, the issue of the use of new technologies has been addressed from different points of view. Studies on how the introduction of new technologies change learning in the classroom and how they can be integrated in the classroom practice have been carried out (e.g. Artigue, 1997; Arzarello et al., 1998b; Laborde, 1998; Healy, 2000; Hoyles & Healy, 1999; Mariotti & Bartolini Bussi, 1998; Sutherland & Balacheff, 1999).

The ongoing research project we are involved in, is concerned with students' cognitive behaviour when using a dynamic geometry environment, Cabri-Géomètre (Baulac et al,1988), in the context of the proving process[1] in geometry (Arzarello et al., 1998a). The classroom experiments involve secondary school students, who are asked to solve open problems (Arsac et al, 1988) in Cabri, with the aim of proving their conjectures.

The problem we study is the transition from conjectures to proofs (see e.g. Boero et al, 1996), as part of a long term activity: students' transitions from the perceptual level to the theoretical one and backwards. As perceptual level, we mean the activities in which the students use perception (e.g. to see if a quadrilateral is a square, only by eye). As theoretical level, we mean activities like producing a conjecture in a conditional form and validating it with a proof. The relationships between perception and theory prove a key element in mathematics, particularly in geometry (see for example Arsac, 1992).

The different tools provided by Cabri (e.g. dragging, constructions, measures) are useful mediators (Mariotti, to appear) in helping students in the transition mentioned above. In a previous study, we investigated in particular the role of dragging (Arzarello et al., 1998b; Olivero, 1999) within this framework[2].

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[1] By proving process we mean the process of exploring, conjecturing and proving in an open problem.

In this paper we analyse the role of measures in Cabri in the above transition. Our hypothesis is that measures in Cabri constitute a bridge between the perceptual level and the theoretical one and may be a support for managing productively the interaction between the two levels.

The role of measures in Cabri: two transitions.

Our interest in the role of measuring in the proving process comes from the fact that students activity involves the use of measures in different phases of the process and with different purposes. Certainly, the need for measures comes from the perceptual level, when the students have the intuition that, for example, one side equals the double of another one. However, when they read the measures on the screen[3], they are no longer at a purely perceptual level: they are working at a higher level, because they are looking for an answer: *Is my intuition true or false?*

Measures work as a tool which can provide different kinds of answer, because they foster perception but at the same time lead towards the theory:

a. a dichotomic answer of qualitative kind (yes/no, e.g. knowing if two quantities are equal or not, or if one is bigger than another one);

b. a quantitative information (a number, which tells the measure of a quantity with respect to a measurement unit)[4];

c. a relational answer, that links the measures of two or more quantities (e.g. if one side equals the double of another one).

Therefore it is important that measurements are introduced correctly in the school curriculum, in order to avoid obstacles, misunderstandings or conflicts[5] in the learning process. The role of the teacher proves fundamental in relation to the different uses of measures described above, in order to teach the students:

a. to manage the qualitative answer;

b. to use the quantitative information as a number within an interval of uncertainty[6];

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[2] A variety of dragging modalities was identified, each one showing a particular aim to be achieved in the proving process. The different modalities used by students were considered as revealing different cognitive activities.

[3] Similar considerations may concern the use of measures on paper or using other tools.

[4] The quantitative side of the answer linked to the use of measures usually makes students feel safe and certain about results; in particular, weak students usually rely on measures.

[5] In another work (Olivero & Robutti, 2001) we point out some cognitive conflicts connected to the use of measures in Cabri.

[6] An example of good management of the interval of uncertainty is in Olivero & Robutti (submitted).

c. to understand the different status of the numbers obtained by measuring a quantity and the relationships between the quantity and other ones.

The students will then be conscious that measures are useful in the proving process in order to discover, to make conjectures and, finally, to prove, but that they are not sufficient to construct a proof. Measures (and dragging too) are useful when you want *to know* if something is true; however, they are no longer useful if you want *to prove* that something is true: at this point measures need to be substituted by relations of logical consequence. Measures in Cabri (in the same way as dragging) are mediation tools between the perceptual level of students' mathematical activity and the theoretical one.

Students' measurement activity reveals two typical cognitive shifts, which were also revealed by the use of dragging (see Figure 1):

1. the transition from the perceptual level toward the theoretical level;
2. the transition from the theoretical level toward the perceptual level.

If the pupils trust measures and 'believe' measures are absolutely exact, they stay at a perceptual level. While if they use the information provided by the feedback of measures to formulate a conjecture in a conditional form ('if ... then') and to see the figure as a generic example (Balacheff, 1999), they pass to the theoretical level (Laborde, 1998).

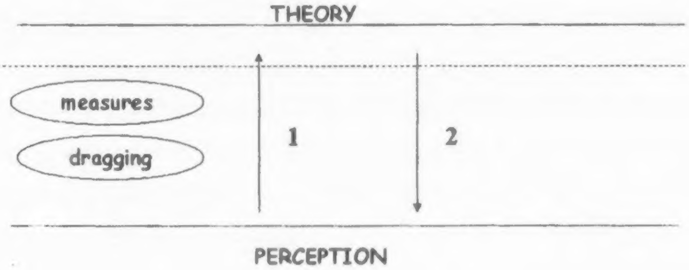


Figure 1

Classroom observations have shown that the modalities of using measurements in the first transition, are for example the followings:

- when the students do not have any precise ideas about the configuration, they explore the situation randomly: they take measurements of some elements of the configuration ("mesures exploratoire", Vadcard, 1996), in the same way they as use wandering dragging (Arzarello et al., 1998b);
- when they do a guided exploration of the configuration, measurements are used to put in order a set of different cases, in order to explore them, in the same way as guided dragging (Olivero, 1999), or together with it.
- as a means of checking the validity of a perception: students see some features of the figure, but they are not sure of their perception, so they use measures on the

figure in order to validate the perception ("mesures probatoire", Vadcard, 1996), and they remain at a spatio-graphical level (Laborde, 1998).

As far as the second transition, the students use measures for example with these modalities:

- after formulating a conjecture, sometimes measurements are used to check the conjecture within Cabri, in order to refute or to accept it[7].
- after constructing a proof, the students go back to Cabri in order to understand the proof and to get a better explanation: new experiments are made in Cabri and measures are used, normally, on the static figure (see, for example, Olivero & Robutti, 2001).

In the following we will present some exploratory examples which illustrate the previous ideas.

The two transitions: some exploratory examples.[8]

All these examples are taken from different Italian Secondary School classrooms (15-17 years old students) which took part in our research project. The classroom sessions consisted of both group work and classroom discussions (Bartolini Bussi, 1998). Two observers took fieldnotes and video-recorded one group for each session and the discussion.

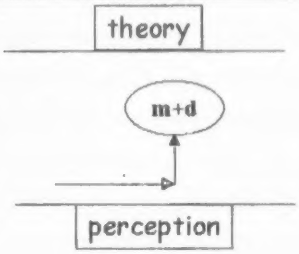
EX1[9]. Students M & N, group work..

12.M: diametrically opposite points (D, E)

13.N: They are on the same line (D, B, E)

14.M (while N is dragging)*: They are on the line, right? OO' and DE are parallel. Put measures. I think it is 1/3, ...2/3, ...the double.*

The students are in the process of formulating a conjecture, so they are moving from perception to theory. At the beginning *(#12-13)* they formulate an observation coming from a perception. Then they use both dragging and measurements to go on with the exploration. The information M sees in the process is a relational answer: the relationship between two quantities is discovered *(#14)*.



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[7] This use is very similar to the dragging test modality (Arzarello et al., 1998b), by which students check the exactness of a construction.

[8] We do not present the statements of each problem because they are not important for the analysis we carry out. However, here is an example of the kind of open problems students worked with:

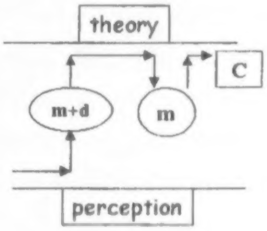
"Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, K, L, M of pairs of consecutive bisectors.

- Drag ABCD, considering different configurations. What happens to the quadrilateral HKLM? What kind of figure does it become?
- Can HKLM become a point? Which hypothesis on ABCD do you need in order to have a point? Write down your conjectures and prove them."

[9] The figures represent the two transitions described above. m=measures, d=dragging, C=conjecture

EX2. Students R & M, teacher T; group work.

90.T: You need to find a different property from the one you used for the construction.
91.R: Let's try to put measures in a particular case.
92.M: Draw a square.
93.They draw a square and then drag one of the sides.
94.R: One side gets bigger and the other one gets smaller in the same proportion.
95.T: How would you say this in mathematics?
96.M: When one gets bigger and the other one gets smaller, they are inversely proportional.
97.T: Not really. They are inversely proportional when the product of two variables is constant. Is this the case?
98.They do some calculations on the measures of the sides of the quadrilateral in different configurations.
99.R: No, the product is not constant.
100. M: Let's have a look at the sum.
101. R: The sum is constant.
102. M: So the sum of the two opposite sides is equal. The conjecture is: if a quadrilateral is circumscribed to a circle, the sum of the opposite sides is equal.

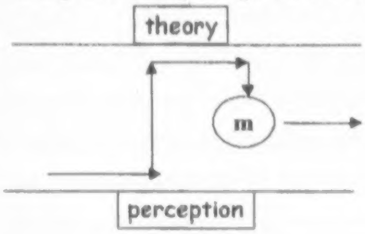


The students do not have any idea in mind. They know they have to find a property but they do not know what kind of property it can be. So they decide to start from a particular case. They put measures as a means to get ideas from. They start dragging *(#93)*. While dragging they look at how measures change on the sides of the quadrilateral and compare them. At first a relational information is seen *(#94)*. The teacher stimulates the students to move towards the theory *(#95)*. Some work at a theoretical level is done *(#96-97)*. In *#98* they go back again to the Cabri measurements and do some calculations. Relationships between the measurements of the sides are observed and then transformed in a conjecture, at a theoretical level.

EX3. Students D & G, group work.

25.*D: They do not coincide, because…look* (she points at the 4 axes)
26.G: And this is a square, I bet it!
27.D: Wait, what does the perpendicular bisector cut…It cuts the side in a half, doesn't it? …so this should be the midpoint…
28.G: Ah…but I want to know if it is a square
29.D: Use measurements!
30.G takes measurements of all the sides.
31.D: Maybe not…look, they are all different!

The starting point is the perceptual level: D observes a characteristic of the figure *(#25)*. G goes on with another perception *(#26)*. At this point D moves to a theoretical level, recollecting a property of the perpendicular bisector *(#27)*. G goes back to the perceptive level and wants to check her perception of the figure being a square *(#28)*. The means D chooses to validate the perception is the use of measurements. The conclusion D gathers from the measures is a dichotomic qualitative information: the sides are all different *(#31)*. At this point they are at an intermediate level between perception and theory.



EX4. Students L & S; group work.

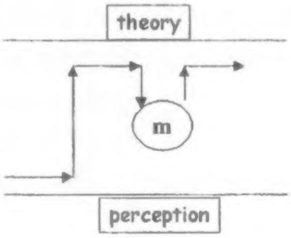
110. L: UR is parallel to AB because you can see it by eye. But let's check it.

111. L marks a point on UR and a point on AB, draws the segment between these two points, marks the internal alternate angles, takes measurements.

112. S: they are equal!

113. L: yes, they are parallel.

This episode starts with a perception *(#110)*; L explicitly says he is observing the figure *by eye*. However he wants to check this observation. There is a jump towards the theory when the students recollect a property of parallel lines *(#111)*. Then there is a move back to the use of measurements. At first a qualitative information is gathered *(#112)*. Then another move towards the theory is made *(#113)*.



EX5. Teacher T, student F, classroom discussion.

85.T: So your conjecture is...

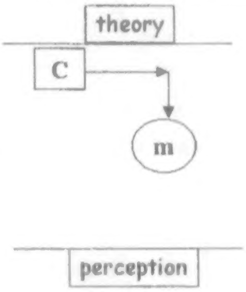
86.F: if a quadrilateral can be circumscribed to a circle the bisectors all meet at a same point.

87.T: when can a quadrilateral be circumscribed to a circle?

88.F: When the opposite sides...when the sum of the opposite sides is equal.

89.F calculates the sum of the opposite sides using the measurements on the Cabri figure.

The students have a conjecture *(#85-86)*, so they are at a theoretical level. The teacher provokes them to explain the conjecture by characterising the circumscribed quadrilaterals *(#87)*. In order to do that, after mentioning the defining property for circumscribed quadrilaterals *(#88)*, they use measurements *(#89)* in order to show that this property is true. The function of measurements is to provide certainty.



Conclusions and issues for further research

From the cognitive point of view, the previous analysis shows that in the proving process there is a richness of back and forth shifts between the perceptual level and the theoretical one. This productive interaction is supported by the use of measures in

Cabri, both as a tool on its own and as a tool in addition to others, e.g. dragging. The cognitive analysis

A cognitive perspective on the use of technology seems to be particularly important if one wants to deeply understand how new technologies affect the teaching and learning process. Knowing in which way a cognitive behaviour changes when working with a software, for example, might be essential for developing suitable classroom activities in which the potential of technology is exploited together with the mathematics involved. The research results concerning dragging and measures we presented might be successfully exploited in the teaching practice, for example by showing students the different possibilities they have and 'teaching' them how to use these tools in a productive way.

And, last but not least, measures (not only in Cabri, but in general) should constitute a key topic in the school curricula as far as their epistemological role is concerned, as it is pointed out by many ongoing curricolar projects that are taking place in US (AAVV, 1998), Belgium, New Zealand, and, recently, Italy.

Some possible issues for further research may be: developing a deeper second order analysis (Arzarello & Bartolini Bussi, 1998) concerning the cognitive interpretation of measurements activities, in relation to the whole proving process; analysing in details the role of the teacher in managing the introduction of technologies in the classroom, with particular respect to the different tools they embed; studying this approach from the embodied cognition perspective (Arzarello, 2000; Nunez, 2000); analysing the role of other mediators, as for example language (Arzarello, 2000; Radford, 1999), in relation to the Cabri tools.

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PUTTING THEORY INTO PRACTICE

Growth of appreciating theory by student teachers

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Abstract

This report presents the raw data from an ongoing study into the manner in which student teachers connect theory with actual practice. In this case, practice is primarily used to indicate the digitised teaching practices recorded in the Multimedia Interactive Learning Environment (MILE), a computer-based environment that provides an investigative representation of teaching in an actual classroom setting. A tool has been developed that incorporates 15 signals of expected theory implementation by student teachers. A prototype of the tool will be presented, as well as the considerations that underlie its design. The student teachers' independent group work and subsequent semi-structured interviews with some of the student teachers have been taped (audio recordings). Similarly, the teacher educator's lessons have been videoed. Furthermore, data has been obtained from the analysis of portfolio documents from the student teachers. A selection of raw empirical data is described on the basis of two observations.

Theory and practice in primary school mathematics teacher training

Primary school teachers in the Netherlands are trained at a PABO (Primary School Teacher Training College), which offer four-year programmes at the higher professional education level. Educating primary school mathematics teachers essentially involves introducing (pedagogical) content knowledge and pedagogics to pre-service teachers, and providing them with the opportunity to conduct interrelated fieldwork in primary schools.

Since the 1970s, mathematics has played a leading role in both the developments in primary education and at PABO teacher training colleges.

In the 1970s and 1980s, a programme for training primary mathematics teachers was developed (Goffree, 1979, 1982, 1983, 1984). This programme was characterised by the attention it focused on the development of student teachers who are educated to acquire– as an integrated whole –the following abilities: solve problems at their own level, learn the didactics of mathematics education and practical teaching techniques (Goffree & Oonk, 1999).

From 1990 – 1995, a team of 10 mathematics teacher educators developed national standards for primary school mathematics teacher training. In 1995, they published their work in the form of a handbook for their fellow teacher educators. In this handbook, three pillars of teacher training are identified: construction, reflectivity and narrative knowing (Goffree & Dolk, 1995). In other words, student teachers are taught to acquire personal (practical) knowledge primarily through reflection on practical situations, which knowledge generally has a narrative character. The idea behind this is that student teachers can integrate the larger theoretical ideas with their practical knowledge by reflecting on (theory-heavy) practical situations. As such, MILE can be viewed as an extension of this new vision of teacher training education.

MILE

The goal of the Multimedia Interactive Learning Environment or MILE (Dolk, Faes, Goffree, Hermsen & Oonk, 1996) is to give student teachers a possibility to investigate teaching practice (primary school mathematics) in a specific way. The developers of MILE were inspired by the Michigan MATH project (Lampert & Ball, 1998). Seven theoretical orientations guided the development of MILE (Goffree & Oonk, 2001).

MILE is comprised of – currently 70 – recorded lessons, discussions with teachers, supervisors and other K 1 – 6 materials. From the archive, it is possible to study each lesson in its entirety or in short fragments. Key word searches of the fragment (clip) descriptions and lesson dialogue (transcripts) can be done using the search engine. Every fragment reproduces a teaching instance and, in MILE, is provided with a short description that provides further elucidation. MILE is intended as an extensive collection of situations, from which the student teachers can acquire practical knowledge (Elbaz, 1983; Verloop, 1991) as a narrative way of knowing (Gudmundsdottir, 1995). Theory can be integrated into practical knowledge by reflecting on 'theory-heavy' practical situations.

Experience has shown that student teachers are often not only focused on the actual teaching of mathematics when watching the fragments, but also on general didactic and educational issues. MILE thus offers the possibility to use the school subject mathematics as an arena for theoretical reflections that connect with larger pedagogical ideas.

Research subject

The research, about which is reported in this paper, attempts to answer the question: How do prospective teachers make connections between theory and actual practice when a digital representation of actual practice (MILE) is made available to them?

Considerations in the design of the research tool

How can one demonstrate that theory plays a role in the student teachers' study of practical situations? Schön (1983) has demonstrated that 'theory in action' is primarily implicit in nature. Only the astute observer (expert) is in a position to notice signals from suchlike theory in action. We decided to generate a list of possible signals to support our observations of student teachers at work. The signals were developed on the basis of theoretical orientation and discussion, which incorporate

the individual practical knowledge (and wisdom) of the researcher and the results from an explorative study conducted earlier (Oonk, 1999).
The tool assumed a background role during the research and evolved from a scoring list to a frame of reference. After several observations, the tool proved to be too extensive as a scoring list. The idea to use the tool to analyse the data collected came about after it had been used to only indicate the direction observations were to follow. The tool, however, appeared far too crude for this purpose and further refinements are necessary.

The tool

This prototype is a refined version of the first. Each signal is coupled with an example (paradigm of a theory in action) with references to sources of the theory cited.

1. While observing practical situations, student teachers can refer to the theory that comes to mind.
   Example: student teacher points to a teacher who interprets the product of 2 x 5 and, in doing so, employs the rectangle model (Treffers & De Moor, 1990, p. 75).
2. Theory is used to explain (as a means to understand) what occurred in the practical situation observed.
   Example: student teacher explains the method employed by the pupil who is using MAB (base ten) material as a working model (Gravemeijer, 1994, p. 57).
3. The student reflects the intention of the teacher or pupil(s) with the help of theory.
   Example: student teacher points out the 'mirroring technique' applied by the teacher as a means to the pupil reflect his own actions (Van Eerde, 1996, p. 143).
4. The student teacher substantiates an idea arising from observing a practical situation.
   Example: student teacher explains the process used by the teacher concerning the transition from context to model, based on an idea about the teacher's opinion of contexts (Treffers et. al., 1989, p. 16).
5. The theory generates new practical questions.
   Example: student teacher wonders at which level (phase) of learning multiplication the pupils are (Goffree, 1994, p. 280).
6. Theory generates new questions about the student teachers' individual notions, ideas and opinions.
   Example: in referring to the theory of the next zône of development, the student teacher wonders whether she is approaching her pupils (during fieldwork) at the appropriate level (Lowyck & Verloop, 1995, p. 154; Van Hiele, 1973, p. 101).
7. The student teacher can theoretically underscore her personal beliefs about an actual practice situation.
   Example: student teacher explains her opinion about a positive working environment that according to her is created by the teacher and based on classroom environment theory (Lowyck & Verloop, 1995, p. 62 Lampert & Ball, 1998, p. 123).

8. The student teacher estimates the practical knowledge of the teacher and identifies its theoretical elements.
   Example: student teacher describes the practical knowledge (of process shortening) that, according to him, motivates the teacher to employ certain actions (Gravemeijer, 1994, p. 58).
9. Student teacher reaches certain conclusions from his observations based on theoretical considerations.
   Example: student teacher reaches the conclusion that group work and beginning with repeated counting better fit the foreknowledge and experience of the children (Simons, 1999, p. 579; Van den Heuvel-Panhuizen et. al. 1998, p. 60).
10. Making connections between practical situations in MILE and own fieldwork experiences with the help of theory.
    Example: student teacher establishes similarities between approaching a pupil in MILE and a pupil in his/her own practical training group (Goffree, 1994, p. 211).
11. (Re)considering points of view and actions on the basis of theory.
    Example: student teacher revises her opinion about a pupil's approach to multiplication, basing it on a fellow student's reflections on the theory behind the strategy employed (Van den Heuvel-Panhuizen et. al., 2000, p. 47).
12. Constructive analysis (= adapting given teaching material) that is underpinned with theory.
    Example: student teacher adjusts a given course by incorporating contexts that provoke 'didactic conflicts' (Van den Brink, 1989, p. 203).
13. The student teacher shows his appreciation of theory.
    Example: student teacher expresses her appreciation of theory when she is able to explain the solution strategy employed by a pupil (Lampert & Ball, 1998, p. 70).
14. Realising the usefulness of theory as a tool for reflecting on actual practice ('reflection on action').
    Example: in a logbook, student teacher describes his modified views on theory in favour of RME (Schön, 1983, p. 278; Lowyck & Verloop, 1995, p. 137).
15. Developing a personal theory to underpin his interpretation (creation) of a practical situation.
    Example: student teacher develops his/her own theory about open and closed questions (Boekaerts & Simons 1993, p. 208; Von Glasersfeld, 1995, p. 192).

Methodology

The study involved two classes (each with 25 student teachers) at a primary school teacher training college during the testing phase of the MILE course, 'The Foundation'. Ten 2-hour meetings were held. For the methodology and the development of the tool, the position was taken to conduct this section of the study in such a manner that we could expect to optimise the results in terms of the signals of theory use (Glaser & Strauss, 1967). The method of triangulation was then selected (Maso & Smaling, 1998), 4 pairs of student teachers were observed and interviewed, and a participating study of the group work with two student teachers was conducted.

In addition, the teacher educator was observed during the 10 meetings involving the entire group in order to inventory the incentives offered to stimulate the use of theory by student teachers (their theories in action). The data was recorded on audio and video tape. Furthermore, data was obtained from the analysis of portfolio documents from the student teachers.

The observations described below offer an impression of the results to date. A more detailed report will be presented at PME-25.

THEORY IN ACTION (1)

Discussing Fadoua's mistake

The MILE fragment shows a pupil, Fadoua, and her teacher, Minke, at the instruction table during a grade 2 independent working session. In a diagnostic discussion, Minke wants to attempt to identify the way of thinking behind the mistake (18-6=11) Fadoua made in her written work. It appears that Fadoua counts backwards starting from 18 ('initial error', a well known standard error) and whilst counting backwards also skips two numbers (12 and 14).

The theory that makes this practical situation more comprehensible is a result of research into subtraction strategies employed by young children, in particular the method of counting backwards. Initial errors, counting mistakes and counting too far are well-known problem areas. To avoid problems in the transition from manipulative to mental calculations when learning to shorten procedures, structural models divisible in “fives structures” can be employed to learn to subtract to twenty (Gravemeijer, 1994; Van den Heuvel-Panhuizen et. al., 1998).

After watching and analysing the video, the student teachers, Denise and Marieke, discuss the most appropriate way to assist Fadoua. Denise initially suggests solving the sum using 18 blocks (units). Marieke rejects this, however, since she believes it doesn't solve Fadoua's counting problem. She rejects the second suggestion – using the number line – for the same reasons. Marieke ultimately agrees with Denise when she suggests using the reckon rack.

Denise’s foremost argument is that the fives structure of the reckon rack can help Fadoua either by directly subtracting 6 or by splitting to yield 8-6 or 18-6. ‘And that doesn't involve counting anymore,’ she says.

After analysing another MILE fragment (involving a talk between the two grade 2 teachers, about the transfer), Marieke reaches the conclusion that Fadoua has most likely mastered splitting the numbers to ten. Based on their interpretation of the teaching method for learning to use the reckon rack (doing, seeing, working it out in your head), they list all the points for helping Fadoua.

In the previously outlined discussion between student teachers Denise and Marieke we see theory in action when they compare, face and consider, on the basis of theoretical perspectives, which material or model is (or is not) appropriate and why. A similar process occurs when they design an explanatory approach for Fadoua, partially on the basis of theoretical considerations (the reckon rack teaching method).

THEORY IN ACTION (2)

Considerations on the use of tangible material (manipulatives)

In the workshop part of the second session, the student teachers are given the task of characterising three MILE situations, A, B and C. Using that analysis, they are to attempt to select two of the three situations that bear the closest interrelationship. In fragment A, teacher Minke is teaching her pupils to memorise the five-times table. On the magnetic board, we also see two 2 by 5 and 3 by 5 tile squares Minke used before this part of the lesson. She now asks who knows a multiplication fact by heart. Kimberley knows that 5x10=50 (Minke alters this to 10x5) and Vincent knows that 8x5=40.

In fragment B, teacher Willie has the class count the five-times table by making leaps of 5 on the number line. Dwaen knows that 9x5 is just ahead of 10x5.

In fragment C, we see 20 transparent cylinders, each containing 5 balls, on the edge of the blackboard. Minke asks which multiplication table sum matches this. Clayton's answer is 20x5.

The theoretical background in all three situations concerns the method of teaching multiplication tables and, more specifically, that of memorising the products. One possible factor could be the grid (rectangle) model in situation A, which is represented by the tile squares, which, although visible, are not in use at that point. Situation B is about repeated leaps on the number line, whereas, in the case of 9x5, the 'one less than' the anchor product 10x5 strategy comes into it. In situation C, theory makes an appearance in the form of visualising the multiplication structure based on the context. The structure elicits the use of strategies (Treffers & De Moor, 1990; Goffree, 1994).

To her group partner Loes, student teacher Linda speaks out in favour of the tie-in between situations A and C. The tiles and the tennis balls are 'tangible' (manipulatives), the number line is not (it is very likely that this terminology was borrowed from the educator during the previous plenary meeting). Using the tiles and the tennis balls, the children can assemble groups themselves. You can see the tennis balls as 20 groups of 5, but also as 10 groups of 10. The number line 'always remains a unit'. According to Loes, you can do all that on the number line too, by adding dashes. In her view, the tiles, the number line and the tennis balls are 'three equal aids'. When Linda speaks out for the relationship between situation A and B and the difference with situation C, she demonstrates theoretical notions about (in)tangibility and about grouping and structuring. She has a vague idea of the number line being a more abstract, 'intangible' construct, as opposed to Loes, who seems to see the number line as a tool or working model.

THEORY IN ACTION: SUMMARY

We see theory coming into action in situation (1) as the student teachers (re)assess their viewpoints with regard to the question which is the material or model of choice for coming to grips with a particular problem. That also happens if they substantiate their construction (help for a pupil) by using theory.

In situation (2) signals that theory is being applied become more evident when a student teacher (Linda) compares practical situations and justifies the relationship between two situations by basing her reasoning on theoretical considerations.

Conclusions

Intended to portray the connection between theory and actual practice made by student teachers, the research tool evolved into a data analysis instrument. As the study progressed, overlap was eliminated and new signal characteristics were added. Further refinement of the characteristics is considered necessary.

The interim results of this ongoing study reveal that student teachers use theory as a means to understand and explain practical situations. The frame of reference of second-year student teachers (comprised of personal experiences as pupil and trainee, supplemented with theoretical information about education and training from lectures at the PABO teacher training college) appears somewhat diffuse and fragmented. It remains difficult to draw a boarder line that separates practical wisdom from theory.

The mental ability to articulate observations of and reflections on practical situations in theoretical terms remains largely undeveloped. The current culture reigning at the teacher training colleges also seems to hamper the development of this. As a result, there is a real danger that student teachers will hang on to their personal (subjective) theories undesirable for professional development.

The student teachers themselves believe that working with MILE enables them to apply and further explore the knowledge that they already have. A number of them demonstrated a budding appreciation for theory. Continued study should reveal whether the signals they display actually point to integrated knowledge of theory and practice, or, in other words, practical knowledge.

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PROBLEM SOLVING IN THE MATHEMATICS CLASSROOM: A SOCIO-CONSTRUCTIVIST ACCOUNT OF THE ROLE OF STUDENTS' EMOTIONS

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Abstract. *The present study aims at documenting from a socio-constructivist perspective the relations between students' mathematics-related beliefs, their emotions, and their problem-solving behavior in the mathematics classroom. To investigate these relations we did a multiple-case study of 6 students (age 14). The use of questionnaires as well as more qualitative methods like thinking aloud and interviews, allowed us to trace students' problem-solving behavior in its affective, motivational, and cognitive dimension. Results indicate that students' beliefs as well as their task-specific perceptions determine the interpretation and appraisal processes underlying emotional experiences. These emotional experiences tend to be closely related to cognitive and metacognitive experiences and determine the problem-solving process in specific ways.*

Theoretical framework

Recent theories on cognition and learning stress the situatedness of every learning activity and point to the close interaction between cognitive, conative and affective factors in students' learning and problem solving. A socio-constructivist account of learning and problem solving (e.g., Cobb & Bowers, 1999) takes emotions and feelings to be as much a constitutive and integral part of problem solving as cognitive and metacognitive strategies. Moreover, within such a perspective emotions are not primarily characterized as typical neurological processes which, together with their corresponding expressions and feelings, can be studied independent of the individual and context. On the contrary, emotions are perceived as being fundamentally grounded in and defined by the broader social-historical context that constitutes the individual as well as by the immediate social-context wherein the problem solving activity is situated. For students this context is in the first place the instructional context. One can claim then that every emotion is situated in its instructional context by virtue of four characteristics. First, emotions are based on students' cognitive interpretations and appraisals of specific situations. Second, students construct interpretations and appraisals based on the knowledge they have and the beliefs they hold, and thus they vary by factors such as age, personal history and home culture. Third, emotions are contextualized because individuals create unique appraisals of "similar" events in different situations. Fourth, emotions are unstable because situations and also the person-in-the-situation continuously develop. One can conclude then that emotions clearly have a rationale with respect to the local social order (Cobb, Yackel, & Wood, 1989).

In the field of mathematics education, McLeod (1992) - although not specifically situating his research within a socio-constructivist perspective - already acknowledged the relevance of studying the interactions between students' beliefs, their emotions, and the specific context of the mathematics classroom when he argues

> "Since beliefs provide an important part of the context within which emotional responses to mathematics develop, we need to establish stronger connections between research in beliefs and research on emotions in the context of mathematics classrooms." (p. 581)

Rarely scholars have addressed in their research this relation between students' mathematics-related beliefs and emotions experienced *in* the classroom. More generally, the emotional reactions of students have never been major factors in research on affect in mathematics education (for the exception see e.g., DeBellis & Goldin, 1993). This lack of attention to emotion is probably due to the fact that research on affective issues has generally looked for factors that are stable and can be measured by questionnaire. Nevertheless, in the last decade several researchers in the field of mathematics education advocating a situated perspective have begun to study students' learning and problem solving including analyses of motivational and affective processes, next to cognitive processes (e.g., Isoda & Nakagoshi, 2000; Lester, Garofalo, & Kroll, 1989; Prawatt & Anderson, 1994). They take a more socio-constructivist approach when investigating students' emotions and focus on the dynamic interplay between student and context, using a variety of research methods (e.g., interviews, video-stimulated recall interviews, on-line heart rate measures) that should enable them to represent the student's perspective on problem solving, rather than the researcher's perspective. After all, research from a socio-constructivist perspective that focuses on the individual has to document how students engage in classroom practices and dynamically reorganize their ways of participating in them (Op 't Eynde, De Corte, & Verschaffel, in press). This approach stresses intentionality and emotionality, next to intellectuality, and takes activity and meaning as its basic currency. It implies a shift for researchers from an observer's perspective to an actor's perspective (Cobb & Bowers, 1999). What matters is not so much the classroom environment, practices, and (emotional) experiences as observed by the researcher, but the meaning students (and teachers) give to it and upon which they act.

The results from the studies mentioned above point to several important relations between the classroom context, students' beliefs, their emotions and their problem-solving behavior. To further investigate these relationships and to advance the search for appropriate research instruments, we developed in our center a research project to study the role of emotions in students' mathematical problem solving, taking a socio-constructivist perspective and focusing on individual students as the unit of analysis. The pilot study discussed here was set up to find out if the research methods and instruments used would enable us to grasp some aspects of the dynamic interplay between the student and the class context, and in the mean while would learn us something more about the relations between students' mathematics-related beliefs, their emotions and their problem-solving behavior.

Research design

This pilot study took place in the second year of junior high school (age 14) in three classes from different schools. The classes had basically the same curriculum for mathematics but differed in the general level of secondary education they followed.

All students of these classes were administered a self-developed questionnaire on mathematics-related beliefs [MRBQ] (Op 't Eynde, De Corte, & Verschaffel, in preparation). Two months later (after presenting the questionnaire to the classes) we made a selection of six students, one high and one low achiever out of each class as evaluated by the teacher. They were asked to solve a complex realistic mathematical problem during a regular mathematics lesson and had to fill in the first part of the "On-line Motivation Questionnaire [OMQ]" (Boekaerts, 1987) after they had skimmed the task and before they actually started to work on it. The problem consisted of a one page long story about the Balkan war between the Serbs and the people from Kosovo. A group of Kosovarian refugees tried to go to Albania through the mountains. In the mountains a women gives birth to a baby that appears to be ill and urgently needs specialized medical care. There are two possibilities, one with a delta plain of the Red Cross, another on food and by car. The students had to calculate the fastest way, given the different speeds of the respective means of transport and the distances they have to travel. Basically the problem consisted of four sub-problems that had to be solved successfully to find the correct solution, i.e. the fastest way.

Every student was asked to think aloud during the whole problem-solving process that was also videotaped. Immediately after finishing, the student accompanied the researcher to a room adjacent to the classroom where a "Video Based Stimulated Recall Interview" [VBSRI] took place (Prawatt & Anderson, 1994). The interview procedure consisted of three phases. In the first phase the student and the researcher watched the videotape and the student was asked to recall what he did, thought and felt while he was solving the problem, especially during those episodes that he was not thinking aloud. In the second phase the interviewer asked questions for clarification, more specifically 'what and how questions' relating to what he saw on the screen, what the student told him, what the student wrote down on the OMQ, and what he wrote on the answer form. Finally, in the third phase, the researcher tried to unravel the subjective rationale for the student's problem-solving behavior. He looked for the interpretations the student gave to certain situations. The "why questions" that were asked, made underlying beliefs more visible and as such clarified the relation between beliefs, emotions and problem-solving behavior.

A qualitative vertical analysis of the different data resulted in six rich narratives of the way students handled and experienced the problems. These narratives were then content analyzed. After these vertical analyses of each student's problem-solving process, a more horizontal approach was taken to look for recurrent patterns and/or fundamental differences that might deepen our understanding of what happens during problem solving and more specifically the role of emotions in this process.

Results

The results indicate that, in general, there is an individually changing flow of emotional experiences that follows from students' interpretations and appraisals of the different events that occur during problem solving in class. We found that solving a problem in class, even the same problem, usually consists of an individually different chain of events for each student. For instance, whereas some students were confronted with a lot of obstacles when solving the problem, others encountered less difficulties they had to deal with; but all of them experience a number of different emotions in the course of solving the problem.

Frank, for example, after already having experienced some difficulties solving subtask 1 and becoming frustrated in the process of doing it, ended up relieved because he was finally able to solve subtask 1. Then, he continued with subtask 2[1]:

Ðakovica is another 14 km.

at 20 km/h

that is,...wait,...

"Wait,.... that 20 km/h.... I seem to have forgotten how I had to do it, then I took a quick look, and then..."

Frank takes his calculator

"Actually, I did not really need the calculator there. I wasn't thinking properly, and then I panic, and then I immediately want to go to my calculator, and if then I stop and think for a moment, I probably know again what I have to do"

INT: "If you are searching for the solution, you have tried something, you end up at the 20 km/h, you grasp your calculator, and you don't know what to do exactly, how do you feel then?"

Frank: "A little bit, I don't know how to put it, you don't feel well because you need to go to the calculator. I always want to do as much as possible without it."

"I did not know immediately how I had to go from 20 km to 14 km."

Franks moves his hand back to the calculator but he redraws it

INT: "At such a point, yes - no, yes - no, ..., how do you feel then?"

Frank: "Then I get something like... come on what is this all about!!.... Usually I focus then on some point where there is nothing, and then I go through everything again, 20 km in one hour, how do I get to the 14 km."

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[1] *Thinking aloud data* are written in bold;
Students' comments in the VBSRI are placed between " ";
Questions of the interviewer are preceded by INT;
Data from observation by the researcher are written in italics.

INT: "How did you manage to find it then?"

Frank: "I just kept searching how I could get from the 20 km to 14 km. First, I thought to take some bigger numbers and that is why I was thinking of using the calculator. I don't know how I came to it, but at a certain point it worked. I was not really thinking, and then I thought, hey this is not possible, and then I start to think again and then I managed."

20 km in 1 hour

no, no, no; yes

20 km divided by 10

1 hour divided by 10

2 km times 7 is 14

6 minutes times 7 is 42 minutes

Clearly, Frank experienced different emotions when solving subtask 2 as, for example, panic, frustration, and angry. Not only do students encounter individually different obstacles when solving the same problem, but even when they are confronted with the same or comparable events these are in some cases interpreted and appraised differently according to the person (his knowledge and beliefs) and the class context.

For example, our data show that negative emotions (e.g., frustration) usually were experienced at moments that students were not able to solve the problem as fluently as they anticipated. Experiencing the inadequacy of a cognitive strategy used, is apparently as much an emotional as a (meta)cognitive process. However, the nature and the intensity of the emotion experienced, when confronted with a comparable cognitive block, can differ significantly between students. Confronted with a difficulty in an early stage of the problem-solving process, one of the students became hopeless, stating that "If I'm already not able to solve this, than I surely will not be able to solve the rest of the problem". Another student also got stuck at the same point, became a bit annoyed, but experienced this as a challenge, and tried to find a way around it. Despite the fact that they both indicated to be highly motivated and confident to solve this problem (based on the results of the OMQ), they interpreted and appraised the first difficulty they encounter in an entirely different way. Differences in their more general mathematics-related beliefs (as measured by the MRBQ), more specifically their general competence beliefs, possibly grounded in the different social contexts they function in, can account for this (see also infra).

These examples also reveal another aspect of the role of emotions in mathematical problem solving. In most of our cases, an emotional experience always triggered students to redirect their behavior looking for alternative cognitive strategies or heuristics to find a way out of the problem. However, big differences were observed in the effectiveness and/or efficiency of the cognitive strategies used.

Frank, after panicking following a first attempt, keeps focusing on subtask 2. At a certain point he thinks about working with bigger numbers (a well-known heuristic strategy), but in fact he stays with the given numbers and looks for a way to bridge the gap between 20 km and 14 km, with good result.

Steve uses another strategy getting stuck with subtask 2, crying out "Come on pal". He decides to continue with the next subtask, hoping that this will help him to find out, how he has to solve subtask 2.

None of the students in the pilot study really used coping or emotional regulation strategies to control their behavior. Some thought of it, but did not use it. Ellen, for example, persists in going on although she has not made any progress for five minutes. Asked if she would act the same way at home, she answered.

Ellen: "No, then I would take a brake and relax for a moment, so that I can try it again afterwards."

Int: "You don't do it here, why not?"

Ellen: "Because I have to keep on working. You are not allowed. You don't do that in class. You are not just going to stop and leave, telling just leave me alone for a while guys, I will continue in a few minutes"

This example also illustrates how students' behavior is determined by the beliefs they have about the practices that are or are not allowed in the mathematics class context. More generally, students' descriptions and explanations of their emotional experiences in the video based stimulated recall interviews usually refer to underlying belief-systems. Combining these data with the results on the mathematics-related beliefs questionnaire and the on-line motivation questionnaire, we found that specifically students' general and task-specific competence and value beliefs appeared to determine the interpretation and appraisal processes underlying the emotional experiences. For example, the experience of a cognitive block as challenging rather than frightening or demotivating in many cases seemed to depend a lot on students' beliefs about their competence (see supra). Of course, one might assume that their specific beliefs about the nature of solving these kinds of problems and the class teacher's acceptance of getting stuck in the process also color their interpretations and appraisals. However, these very specific beliefs nor their social correlates, i.e. the classroom norms and practices, were the focus of attention in this study.

Conclusion and discussion

This pilot study already strongly suggests that emotions are very much part of problem solving in our mathematics classrooms. Especially negative emotions as, for example, frustration and anger were frequently experienced by the six participating students. These emotions almost seem to be an integral part of problem solving. Indeed the absence of an obvious method to immediately solve a presented task as a major defining characteristic of a problem (Mayer & Wittrock, 1996) implies that those who really want to reach the goal state, i.e. find the solution, will find themselves frustrated at some points in the process. Teaching students how to solve mathematical problems then necessarily implies that we have to learn them how to deal effectively with those feelings of frustration or sometimes anger.

The main aim of this pilot study was to test the general research approach and the quality of the methodology and instruments in function of the main study. It shows how the use of a variety of research methods and instruments in a complementary way can enable researchers to trace students' ongoing interpretations and appraisals of events constituting their problem-solving processes. As such the analysis of students' emotional experiences inevitably also includes an analysis of cognitive and conative processes, on the one hand, and of characteristics of the (subjective) task context (the events), on the other hand. It allows us to grasp the student's, actor's, perspective and the meanings underlying his activities when solving a problem, clarifying the dynamics that constitute his problem solving in class. In this way, this kind of research is a good example of how one can study the individual from a socio-constructivist perspective and what can be learned from it. Of course, a deeper and more complete understanding would have been obtained if the focus on the individual could have been complemented with an analysis of classroom interactions. This appeared to be very difficult to realize in one research project, given the available recourses and the constraints implied in doing research in the classroom. However, as argued above, the absence of an *explicit* analysis of the classroom context does not contradict the socio-constructivist nature of this study.

Although there is a lot to gain from these kind of in-depth studies of students' behavior, one has to stay aware of the restrictions implied in the methods and instruments used. For instance, bringing a researcher, video- and audio equipment into the classroom always interferes in some way with normal classroom life. There is sometimes a very clear influence on students' experiences in class, as can be illustrated by the following clarification given by a student during the video stimulated recall interview "I was nervous in the beginning, because there was a camera". However, from this study we have also learned that after a few minutes students' behavior in the presence of the camera becomes gradually more normal, i.e. similar to their behavior when the camera is not present. Teachers' observations in the classrooms confirm this. Nevertheless, researchers have to be aware that the research setting, even when it is situated in the classroom and stays as close as possible to authentic classroom activities, does influence students' behavior. By explicitly allowing students to deal with these aspects in the interviews researchers might be able to trace some of these unintended influences and take them into account when interpreting the results.

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SOME COGNITIVE ASPECTS OF THE RELATIONSHIP BETWEEN ARGUMENTATION AND PROOF IN MATHEMATICS.

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Abstract

The purpose of this paper is to analyze some aspects of the relationships between argumentation and proof. Our assumption is that argumentation and proof can be compared from two points of view: content and structure. Toulmin's model (Toulmin, 1958) can be a tool to compare the two structures. The paper shows how Toulmin's model can be used to detect some structural analogies and changes between argumentation and proof during the solution of geometric problems needing the production of conjectures and related proofs.

Introduction

We will consider the solving process of geometric problems in which students interact with dynamic environments that are represented by Cabri-Geometry software. We consider a situation in which the student produces an argumentation during the production of the conjecture and then constructs a proof of this statement. The purpose of this paper is to analyse the relationships between argumentation and proof. Our research aim is to analyse similarities and differences between the structures of the two processes. In this paper we will consider situations in which the cognitive unity (Boero, Garuti, Mariotti, 1996) works.

In general, in dealing with problems asking for a conjecture, the solution is not immediate. Then the production of an argumentation during the construction of a conjecture is expected. We gave some open-ended problems to 12th-grade students in Italy and in France. The students worked in pairs on a computer running the Cabri-Geometry software. In order to favour an argumentation activity between the students, we decided to gather them in pairs. Cabri-Geometry was chosen because our hypothesis was that the software could help the students to identify the geometrical proprieties which are beneath the figure construction and which are necessary to the production of proof.

1. Cognitive unity between argumentation and proof

The relationships between the production of a conjecture and the construction of proof has been an objet of study from a cognitive perspective. Actually, research studies showed the possibility that some kinds of continuity exists between the two processes. In particular, continuity can take the following shape:

"*During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of previously produced arguments according to a logical chain*" (Boero, Garuti, Mariotti, 1996).

This phenomenon is referred to by the authors as **cognitive unity**[1].

During the solving process, which leads to a theorem, we may suppose that an argumentation activity is developed in order to produce a conjecture. When the statement expressing the conjecture is made valid in a mathematical theory, we can say that a proof is produced. This proof is a particular argumentation based on a mathematical theory. We want to compare the argumentation process in producing a conjecture and proof.

2. From the relationships between conjecture and valid statement to the relationships between argumentation and proof.

The relationship between argumentation and proof is strictly connected to the relationship between conjecture and valid statement We might say that argumentation is to a conjecture what mathematical proof is to a valid statement (Balacheff, 1999).

Really, a conjecture could be provided without any argumentation. A conjecture can be a "**fact**", derived directly from a drawing, from an intuition and the like. In this case there is not an explicit argumentation justifying this fact. But, we are interested to the following kind of conjecture:

Let us define *a **conjecture** a statement strictly connected with an argumentation and a set of conceptions* (Balacheff, 1994) *where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it.*

The conjecture can be transformed into a valid statement if a proof justifying it, is produced.

Let us define a ***valid statement** as a statement which is provided with a proof referring to a mathematical theory. The statement is valid because a mathematical theory allow the construction of a proof that justifies it.*

We are interested to compare the processes used to construct the conjecture and its validation: argumentation and proof.

The analysis of the solution process from the perspective of cognitive unity needs tools that allow the comparison between an argumentation process and a proof. Our purpose is to find these tools.

3. Cognitive unity in content and in structure

The previous research studies about cognitive unity considered the conditions for its existence (see Boero & al., 1996; Garuti & al., 1998). We are rather interested in its working. Our assumption is that the argumentative process in producing a conjecture and a related proof can be compared from two points of view: content and structure.

The "cognitive unity" considered by Boero & al. (1996) concerns the content. It is possible to observe whether there are analogies or differences between argumentation content and proof content. During the production of several theorems, there are many similar content elements in the argumentation and proof, therefore we can say that it is frequent to find cognitive unity (Pedemonte, 1998).

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[1] The word « argument » refers to a reason given to support or disprove something. In this paper the word « argumentation » refers to a discursive activity (cf. Grize, 1996) based on arguments.

We think that it is interesting to analyse and compare the two processes also from the structural point of view. We wish to become able to detect analogies and differences between argumentation structure and proof structure.

It is possible to compare the argumentation and proof according to relevant structural aspects like deduction, abduction and induction. In a deductive argumentation, the statement is deduced from the data by means of a principle (which permit the inference) allowing its assertion from the data. In an abductive argumentation the statement is deducted before the data is identified (Arzarello, 1998). In this case a principle allows the assertion of a statement even if all the data are not available. In an inductive argumentation the statement is deduced as a generic case after research from specific cases.

Only the deductive argumentation can be easily and directly transposed into a deductive proof. In order to transform an abductive argumentation into a proof the structure needs to be reversed. The inductive argumentation has a structure far away from the structure of a deductive proof; in this case, a link between argumentation and proof can be found only when the argumentation contains the "generic case".

According to the previous analysis we can expect that even in the case of "cognitive unity" (which concerns content) the transition from argumentation to proof may demand relevant (and sometimes difficult to perform) changes concerning structures – in particular those from abductive or inductive argumentation to deductive proof.

4. The Duval's answer

According to Duval (1991), deductive thinking does not work like argumentation: there is a "gap" between the two processes even if they use very similar linguistic forms and proposition's connectives. The structure of a proof may be described by a ternary diagram: data, claim, and inference rules (axioms, theorems, or definitions). Within proofs, the steps are connected by a "recycling process" (Duval, 1992–1993): the conclusion of a step serves as input condition to the next step. On the contrary, in argumentation, inferences are based on the contents of the statement. In other words the connection between two propositions is an intrinsic connection (Duval, 1992–1993): the statement is considered and re-interpreted from different points of view. For these reasons (according to Duval) the distance between proof and argumentation is not only logic but also cognitive: in a proof, the epistemic value[2] depends on the theoretical status whereas in argumentation it depends on the content. Then it is easy to observe the cognitive distance between the two processes.

"*... Pour passer d'un mode de fonctionnement à l'autre* (argumentation and proof), *une décentration à l'égard du contenu d'une part, et une prise de conscience de l'existence d'une autre valeur épistémique d'autre part, sont donc nécessaires. On peut donc parler aussi de "distance cognitive" entre le fonctionnement d'un raisonnement valide et celui d'une argumentation*". (Duval, 1992-1993).

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[2] The epistemic value is the degree of certitude or convinction associated with a proposition (Duval,1991).

According to Duval, the distance between these two processes can explain why most of the students don't understand the necessity of a mathematical proof: if there is an argumentation that justifies the statement the proof can be unnecessary.

Some doubts are currently expressed about the nature and the educational relevance of the gap between argumentation and proof, as described by Duval (in particular, see Douek, 1999). We share these doubts. We think that there are some very similar elements between argumentation and proof. In particular, assuming that a proof is a particular argumentation, both argumentation and proof structures can be described by a ternary diagram.

This is the reason why we need a tool to compare the structure of the two processes.

5. How to analyse or compare the structure of the argumentation process and the proof process ?

We have built up a theoretical framework to analyse argumentation structure and proof structure. Toulmin propose a model for the argumentation structure (1958). We use this model as a tool to compare the structures relating to the two processes: argumentation and proof.

In any argumentation the first step is expressed by a standpoint (an assertion, an opinion or the like). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting it. It is important to provide the justification or warrant for using the data concerned as support for the data-claim relationships. The warrant can be expressed as a principle, a rule, and the like. The warrant acts as a bridge between the data and the claim. This is the base structure of argumentation, but auxiliary steps may be necessary to describe an argumentation. Toulmin describes three of them: the qualifier, the rebuttal and the backing. The force of the warrant would be weakened if there were exceptions to the rule, in such a case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. A backing is required if the authority of the warrant is not accepted straight away.

Therefore, Toulmin's model of argumentation contains six related elements as showed in the following figure.

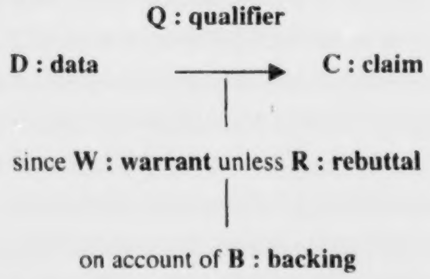
Q : qualifier
D : data
C : claim
since W : warrant unless R : rebuttal
on account of B : backing

Fig.1. The Toulmin's model of argumentation[3]**.**

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[3] Let us illustrate this model with the same example used by Toulmin (1958). *Claim* : Harry is a British subject. *Data* : Harry was born in Bermuda. *Warrant* : A man born in Bermuda will generally be a British subject. *Rebuttal* : No, but it

It is interesting to compare the idea of epistemic value (Duval, 1991) and the idea of the qualifier. The epistemic value of the claim is inherited by the epistemic value of the data. The claim's force is inherited by the data's force. On the contrary, the qualifier is given by the data and also by the warrant's force. The warrant's force is important because the warrant plays a basic role in the argumentation.

If we consider a proof as a particular argumentation, the warrant is an axiom, or a definition, or a theorem, in a specific theory.

Toulmin's model reveals a very powerful tool to compare the process of argumentation and the proof subsequently produced. We can compare the argumentation warrant and the proof warrant. For example if the warrant in an argumentation is based on an intuitive conception, we can see whether in the proof the warrant becomes a theorem of a theory or on the contrary if it remains at the level of conception.

In the following section, we illustrate the use of this model in analysing the resolution process of an open-ended problem.

Interview

The experiment was carried out in four 12th-grade classes in Italy, and in one 12th-grade class in France. The students worked in pairs on a computer running the Cabri-Geometry software. We will transcribe a part of a solution protocol related to the proposed problem. This part is based on the transcriptions of the audio recordings and the written productions of the students. The experiment lasted an hour and a half. The problem proposed was the following:

Problem. ABC is a triangle. Three exteriors squares are constructed on the triangle's sides. The free points of the squares are connected defining three other triangles. Compare the areas of these triangles with the area of triangle ABC (see figure pg. 6).

According to the classification given in the previous section the following types of argumentation can be found in the students' resolutions.

A typical deductive argumentation could be the following. Suppose the student compares the lengths of the base and the height between triangle ABC and one of the external triangles in order to compare the two areas (see figure pg. 6). It's possible to consider the sides of the same square as bases for some triangles and compare the heights considering the small triangles constructed on the heights. The view that the small triangles have two equal angles and an equal side, allows the conclusion that the two triangles are equal under the SAA congruence criterion. Then the large triangles have equal areas.

A typical abductive argumentation could be the following. The student, who wants to compare the two areas, sees that the two bases of the triangles have the same length. It's possible to prove that the heights have the same length in order to prove that the areas are equal. The view that the small triangles constructed on the heights

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generally is. If his parents are foreigners or if he has become a naturalised American, then the rule doesn't apply. *Qualifier* : True : its only presumably so. *Backing* : It's embodied in the following legislation :....

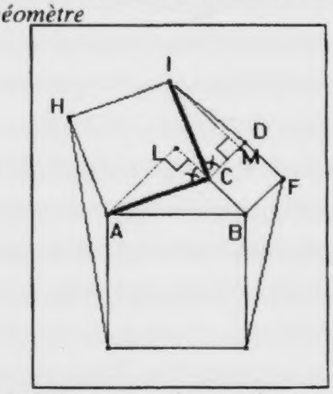
are equal can encourage the search for a theorem to prove this fact. The congruence criterions are remembered and the data to apply one of them is sought out

A typical inductive argumentation could be the following. The student may consider some particular types of the triangles ABC: rectangle, equilateral and the like; or he may consider limit cases, for example, when the points A, B, and C are on the same line. This is an "inductive search" moving from particular to general laws. One of these examples can evolve into a particular example (*exemple générique*: N. Balacheff, 1988) which can lead to the proof.

Example

Using the model described above, we analyze an excerpt of the argumentation and the proof produced by students. Our purpose is to show how the analysis works in order to prove the efficacy of Toulmin's model.

In order to analyse the argumentation, we select the assertions produced by students and we reconstruct the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student's text is in the left column, and our comments and analyses are reported in the right column. The text has been translated from Italian into English. We start the analysis at claim C_7; at this point students are comparing the area of the triangle ABC and the area of the triangle ICD. So far the students spoke about the construction of the heights of the two triangles. They decided to construct the heights in order to compare the areas of the triangles ABC and ICD.

.... Students construct the heights of the triangles ABC et ICD 31. L: I'm prolonging the straight line, yes, the straight line on the segment… what have I done? 32. G: The straight line by the points B and C 33. L: ah it's true ! 34. G: now, we need to do the line perpendicular to this line 35. L: ah there that's it done but you know that it seems they are equal…	*The figure as represented from the students using Cabri-géomètre* 
36. G: almost equal !	C_7: The heights seem to be equal.
37. L: no, more, it seems that they are perpendicular, I have observed this before	C_8: The heights seem to be perpendiculars.
....	*The statements are "facts" where the epistemic value is joined to perception of the figure in Cabri-Geometry.*
44. Students together: hey, these are two triangles ! 45. L: it's true, ALC and ICM these are two triangles…what do they have?	*The Cabri-Geometry drag allows them to see the small triangles. The students realize that the heights are the heights of two equal triangles. The statement is now a fact.*
46. G: we realized… then AC is equal to IC	C_9: The triangles ALC and ICM are equal.

because they are sides of the same square 47. L: wait! 48. G: AC is equal to IC because they are sides the same square, after 49. L: LC… 50. G: it's equal to CM, why ? 51. L: Then… Because it's equal to CM… in my opinion, it's better to prove … no wait this angle is right and this angle is right too.	*The structure of the speech of the students is:* The triangles are equal → to find sides and angles equal congruence criterion *The structure of the argumentative step is an abduction:* $D_9 = ?$ → C_9 W: congruence criterion

The structure of the argumentation is that of an abduction. The students see that the small triangles constructed on the height are equal and they search for a theorem to prove this fact. We can observe that during the proof, students make data explicit in order to affirm that triangles ALC and ICM are equal. The abductive structure of the argumentation is transformed into a deductive structure in the proof. Once obtained, claim C_9 is used to deduce that the heights of the triangles ABC and ICD are equal and consequently that their areas are equal.

The students write the proof:	*The proof structure is a deduction:*
I consider the triangle ABC and the triangle ICD. At once I consider the triangles ALC et ICM and I prove that they are equal triangles for the SAA congruence criterion because we have: • AC = IC because they are two sides of the same square • ALC = IMC because they are right angles (angles constructed as intersection between the sides and the heights) • ACL = ICM because they are complementary of the same right angle (-LCI) In particular IM = AL. Then the triangles ABC and ICD have the same base lengths (as sides of the same square) and the same heights, then they have the same area.	D_9: AC = IC, ALC = IMC, ACL = ICM → C_9: the triangles ALC and ICM are equal W: SAA congruence criterion *If the triangles are equal then it's possible to conclude that the heights are equal, and finally then the areas are equal because the bases are equal.* *The conclusion* C_9 *of the previous step is the date* D_{10} *to apply the inference to the second step.* D_{10}: C_9 → C_{10}: the heights are equal W: inheritance D_{11}: C_{10} → C_{11}: the areas of the triangles ABC et ICD are equal W: formula of area

The protocol is an example of cognitive unity (according to Boero, Garuti and Mariotti, 1996) Indeed, students use the "SAA congruence criterion" both in the argumentation and proof, in order to justify the statements. Words and expressions used in the two processes are often the same ("triangles ALC and ICM are equal", "heights are equal", and the like). But if we look more carefully, we can observe a change between the structures of the two processes: we find an abductive structure in the argumentation (from D_9 to C_9) that is transformed in a deductive structure in the proof. We cannot undervalue the importance of the structure in the comparison between argumentation and proof; it is not unusual that the student tries to transform abduction into a deduction during a resolution process (sometimes successfully, sometimes without getting an acceptable solution).

6. Conclusion

In this paper, we have analyzed some relationships between argumentation and proof; we have used Toulmin's model as a tool in order to compare the structure of the two processes.

In the student's protocols, it is easy to observe cognitive unity regarding the content; but even in this case, as far as structure is concerned, changes are frequently observed. The previous analysis carried out with Toulmin's model clearly reveals the structure of both argumentation and proof facilitating the comparison between them. In particular when students use abduction during argumentation (and this seems to be natural in the production of a conjecture), a structural change is needed and can be detected in students' protocols.

The study reported in this paper is still in progress. Further analysis will be carried out in order to clarify the nature of argumentation (particularly in the conjecturing phase) in order to find other analogies or differences with proof.

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EFFICACY IN PROBLEM POSING AND TEACHING PROBLEM POSING

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In this paper we examine the efficacy beliefs of prospective teachers' in posing problems and teaching problem posing in relation to their ability to construct problems. We used data from 115 questionnaires and 25 interviews to study the structure of perceived efficacy beliefs in problem posing, to examine the relationship between efficacy and actual performance in constructing problems, and efficacy to teach problem posing. The results indicated that students with high efficacy beliefs were more able to construct problems of more advanced complexity than low efficacy students. Significant differences were also found in the level of efficacy beliefs between subjects in terms of their mathematical background, prior involvement in related tasks, and gender.

AFFECT AND PROBLEM SOLVING

Despite the recently intensified interest on the affective domain, Schoenfeld (1992) argued that the arena of beliefs was under-conceptualised and stressed the need for new methodological and exploratory frames. He claimed specifically that "we are still a long way from a unified perspective that allows for a meaningful integration of cognition and affect or, if such unification is not possible, form understanding why it is not" (p. 364). In the following years research on the affective domain has resulted in notable theoretical advances, and there is now an expert consensus that affect is an essential factor in learning interacting with cognition during problem solving activities (De Bellis, 1997). Goldin (1998) proposed a five-component unified model for mathematical learning and problem solving. He considered the affective system as the most important among the five; the other four representations systems proposed were the verbal syntactic, the imagistic, the auditory, the formal notational, and the system of planning, monitoring and executive control.

Some of the questions already cleaned, to some extent, concern the structure and the development of the domain; the construct of affect, however, is still far from being well defined. The affective system includes components such as beliefs, conceptions, views, attitudes, emotions etc. related to mathematics and mathematical learning. If we define *learning*, as the development of one's general and specific "competencies", then affective competencies can be learned and consequently taught in the same sense as cognitive competencies can (Goldin, 1998). Regarding the teaching of affective competencies, the teacher's own belief system has a major role; it functions as a filter influencing knowledge and behaviour. Several components of the affective system have been so far investigated, including self-confidence, self-esteem, self-concept, and self-efficacy. In particular, research has shown that a teacher's sense of efficacy is a reliable indicator of his/her teaching behaviour and effectiveness in bringing about desired learning outcomes. Teacher education programs should, therefore, enhance both the cognitive and the affective domain.

Bandura (1997) defined self-efficacy as one´s conviction that he/she is able to achieve a certain task. By analogy, teaching efficacy can be defined as ones' belief in his/her capability to achieve learning outcomes. Self-efficacy is a context-specific construct in contrast to self-esteem, which is more global. That means that the study of teacher efficacy is more meaningful when carried out in terms of specific teaching tasks rather than in general. Several researchers found that the ability to construct problems and confidence in posing problems are among the most important competencies in mathematics learning, closely related to mathematics achievement. For instance, in models using path analysis the direct influences of efficacy beliefs on students' performance were estimated to range from .349 to .545 (Pajares, 1996). Furthermore, Pajares and Miller (1994) asserted that efficacy in problem solving had a causal effect on students' performance; they found that efficacy beliefs are better predictors of performance in problem solving than beliefs about the usefulness of mathematics, the involvement of students in mathematics, students' gender and experience with mathematics. In general, efficacy to perform a certain task was found to be the most reliable predictor of one´s behaviour in the course of achieving this task (Bandura, 1997; Tschannen-Moran, Woolfock-Hoy & Hoy, 1998).

Efforts to reform mathematics education had a direct impact on the philosophy of mathematics and consequently on instruction. *Knowing mathematics* has been widely identified as "doing mathematics" and *learning mathematics* as equivalent to constructing meaning for oneself and the ability to handle non-routine problems. In this context, problem posing comprises a primary factor that contributes to enhancing students' ability to solve mathematical problems (Leung, 1994). On the same line, a growing consensus that constructivism epistemology could provide the basis to prepare teachers in reform-oriented ways has led to an increased emphasis on developing teachers' ability to construct problems. One of the main responsibilities of primary teachers consists of constructing and/or selecting appropriate, pedagogically rich problem situations and orchestrates classroom activities, which facilitate students' effort to do mathematics on their own.

Problem-posing tasks may take various forms, and thus students can be involved in problem posing through a variety of situations. In this study, preservice teachers were asked to pose problems a) from a mathematical situation (Leung, 1994), b) from a given number sentence (English, 1997), and c) by modifying the structure, the data or the information given in certain problems (Gonzales, 1998). The main purpose of the study was to explore the relationship between the prospective teachers' efficacy beliefs on problem posing in each of the above-mentioned situations and their competence to complete the tasks. In this respect the present study sought for answers to the following questions: a) Is there a significant relation between students' efficacy beliefs in problem posing and their ability to construct problems? b) Is there a significant relation between students' efficacy beliefs in problem posing and their efficacy to teach problem posing? And c) are there significant differences in students' efficacy beliefs about problem posing in terms of gender, prior involvement in problem posing, and mathematical background?

METHODOLOGY

Data collection: The data were gathered through a questionnaire consisting of 31 statements aiming at the clarification of students' involvement in problem posing and problem solving activities, and their efficacy beliefs with respect to these tasks. The questionnaire was administered to 115 preservice teachers during the final stages of their teaching practice. After a first analysis of the responses, 25 students were selected and interviewed. Six of the subjects involved in the interviews were from the low efficacy (LE) group, ten from the average efficacy (AE), and nine were from the high efficacy group (HE). The same tasks were administered to the rest of the students who did not take part in interviews.

The interviews: The interviews were semi-structured and conducted by one of the researchers. During the interviews we used tasks similar to the problems involved in the questionnaire. Each student was asked to construct problems given a mathematical situation, a number sentence, and a problem to modify. The tasks and the initial directions were as follows:

Mathematical situation (Task 1): Construct three problems based on the following story: "Michael, Nicolas, and John drove in succession on their way back from a trip. Michael drove for 80 km more than John. John drove for double the distance Nicolas did. Nicolas drove for 50 km ".

Number sentence (Task 2): "Construct three problems all of which could be solved using the equation 56: 6 = n".

Problem modification (Task 3): Read the following problem and construct up to seven different problems modifying the problem: "The students in a certain school were talking about their favourite singers. One fourth of them voted for singer A, one sixth for singer B, one eighth for singer C, and one twelfth for singer D. What is the student population of the school, if 90 students were undecided?"

The students were at first given time to construct problems on their own. Later on, whenever a student got stuck, the interviewer provided progressively clearer hints about possible ways of performing the tasks. For instance, in the case of Task 1, a common hint was "find a problem in which the answer is not fractional", in the case of Task 3, the students were advised to "insert new information", or "change the unknown", "impose new constraints", etc. If students were proposing an impossible or non-sensible problem, they were considered as failing to achieve the specified task.

RESULTS

Analysis of the questionnaires

Students' responses were factor analysed using Principal Axis factoring with varimax rotation. A five-factor solution, explaining 75% of the variance, was identified as being the most appropriate in isolating distinct scales to identify efficacy beliefs (the loadings of all items were large and statistically significant). The first factor indicated efficacy in Task 1 and Task 2 and explained 21.4% of the variance. The second factor explained 19.71% of the variance and reflected confidence in Task 3, the third factor explained 12.58% of the variance and reflected the subjects' efficacy to *teach problem posing* strategies. The fourth factor explained 11.7% of the variance reflected prior *involvement in problem posing* activities and the fifth factor reflected students' *experience in problem solving* and explained 10.26% of the variance.

In exploring differences among students, we used extracted factors, which reflected efficacy beliefs, as dependent variables and the students' mathematical background (high-school strand[1]), prior involvement in related tasks, and gender as independent variables. Analysis of variance showed that students from the science strand have more desirable efficacy beliefs than students in any of the other two strands (classical and the economics) in all three tasks of problem construction. The same pattern was also found on the teaching efficacy factor. There were no significant differences among the efficacy beliefs of the students from the economics strand and the students from the classical strand.

The subjects expressed significantly higher involvement with problem solving than with problem posing activities ($\underline{\overline{X}}_{ps}= 2.01$, $\underline{\overline{X}}_{pp} =1.66$, $p < .01$). The students' prior involvement with problem posing and problem solving was found to be related to their expressed efficacy beliefs; students with extensive experience in such tasks had a higher level of beliefs in the their ability to construct problems and teach problem posing, than students with limited experience. Significant differences were also found between males and females on the factor efficacy to teach problem posing ($\underline{\overline{X}}_m= 3.11$, $\underline{\overline{X}}_f = 2.71$, $p < .05$). These differences can partially be attributed to the male students' superior efficacy beliefs in their ability to construct problems from a given number sentence over female students ($\underline{\overline{X}}_m = 3.62$, $\underline{\overline{X}}_f = 3.10$, $p < .05$).

Efficacy in Tasks 1, 2 and 3 was strongly correlated to the efficacy to teach problem posing ($r = .58$, $r = .55$, $r = .52$, respectively, $p< .01$). The analysis also showed that "efficacy to construct problems" and "efficacy to teach problem posing" were significantly correlated ($r = .62$, $p < .0$. Finally, the level of students' efficacy beliefs to construct problems was significantly higher than the level of efficacy beliefs to teach problem posing ($\underline{\overline{X}}_{pp} = 3.27$, $\underline{\overline{X}}_{tpp.} = 2.81$, $p < .001$).

Analysis of the interviews

The interviews showed that a) all participants realized the importance of developing problem posing competencies, b) irrespective of efficacy level, they considered problem posing as harder than problem solving, and c) students valued problem posing as the ultimate goal of mathematics learning. "A thorough understanding of problems and problem solving is evidenced when teachers and pupils reach the level of problem posing" (extract from interviews).

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[1] The students come from three high school strands: the science section (emphasis on mathematics and science), the economics section (emphasis and economics), and the classical strand (only core mathematics).

When facing a problem-posing task, 52% of the interviewed students felt uneasy (two of the LE students felt even anxious when assigned such a task). Five of the AE students said that even "hearing the term problem posing makes them feel insecure", and they consider problem posing to be "a very complicated process". The majority of the HE students felt quite comfortable with the task, though one of the students in this group mentioned that he "did not have any real experience and hence he did not like being assigned such a task". The differences among the three efficacy groups of students were more obvious, when the discussion was focused on the specific tasks of problem posing from a mathematical situation, a number sentence, or a given problem.

Concerning efficacy beliefs with respect to teaching problem posing, LE students were less confident than AE and HE students. Specifically, three LE students felt that they were "not well prepared to involve their students in problem posing activities". Explaining their position, they stated that they themselves "faced so many troubles in problem posing" and were "not confident in undertaking such a task". Two others expressed their "reservations..." and "felt more comfortable in teaching problem posing in the lower school grades". The AE subjects were more or less ready to pursue the task, though "they needed more experience with problem posing". On the contrary, five of the HE students stated that they were well prepared to integrate problem posing in their teaching, while two others held the same beliefs as the AE students, i.e., they said that they needed "additional experiences".

A student was rated as successful in a task when he or she was able to construct a good problem with little or no help. Given the initial efficacy statements of the students in each of the 13 assigned tasks, we tested the correlation coefficient between stated efficacy and performance in each of these cases. The correlations were in the range of .724 and .866 and they were significant at the .01 level.

Task 1: To rate the quality of the students' problems we adopted the linguistic and structural criteria established by English (1997) and Silver and Cai (1996). The most important elements in a problem are (a) the type of the question, and (b) the number of important relations involved in the problem structure. The same authors classified problem questions as *conditional, relational, and assignment*; conditional and relational questions are more complicated than the direct assignment questions. Similarly, the number of relations involved in a problem is an indication of the complexity of the problem. Bandura (1997) asserted that efficacy beliefs could be a good predictor of the quality of peoples' work. The average number of conditional and relational problems constructed by each efficacy group was 1.13, .92, and .64 by the high, the average and the low efficacy group, respectively (significant at the .05 level). Finally, HE students constructed eleven assignment questions, eight relational, and two conditional questions. We also observed an increasing trend of complexity from the first constructed problem onwards. The average number of relations per proposed problem was 2.40 for LE subjects, 2.59 relations for AE subjects, and 2.84 for the HE subjects (non-significant difference). Table 1 shows indicative problems constructed by the students in each efficacy group.

Table 1

Examples of problems constructed by the subjects of the three groups

	Low Efficacy	**Average Efficacy**	**High Efficacy**
Ass	How many km did John drive? How many km did John and Michael drive?	How many km did John drive? How many km did the three friends drive altogether?	How many km did the three friends drive altogether? How many km did each of them drive?
Rel	How many more km did Michael drive than John did? How many km did Michael drive more than Nicolas?	Did both Nicolas and John drive more km than Michael did alone? Who drove the more km?	How many more km did Michael drive than Nicolas? How many less km did Michael drive than Nicolas?
Con	none	If Michael drove for 3 hours, John for 4 hours and Nicolas for 2 hours, compare their driving speed. If the average speed of the car were 50 km/h, how long would the journey last?	If the distance of their journey was X km, how many km would each of them had driven in order to reach their destination? If they travelled 800 km and continued driving in the same way, how many more km would John had driven than Nicolas?

Task 2: The students had great difficulties in constructing problems given a number sentence. Most of the students explained their difficulties saying "there is no story to start with…one has to start from the beginning, to create everything in his/her own mind". Three LE students were unable to construct problems eliciting answers other than 9 2/6 (56: 6 = n) despite of being helped by the interviewer. The AE and the HE students could somehow construct a problem, but it was evident that this task was more difficult even for them.

Task 3: The second easier task was constructing a problem by modifying a given one for the majority of students (Task 1 was generally judged as the easiest). About 64% of the subjects initially thought to change the story of the problem, 56% to change the values of the variables or the unknown, 28% to introduce new information and 28% to delete some information. However, only two of the subjects thought on their own to impose new constraints or extend the problem using "what if" strategy.

Another dimension differentiating students in different efficacy groups concerns checking the constructed problems by solving them. The majority of the AE and the HE students checked the solution of the problems they constructed, in contrast to LE students of which only three attempted to check anyone of the posed problems.

DISCUSSION

The findings of this study underline the importance of students' background and involvement with problem solving and particularly with problem posing. The analysis of the questionnaire data provides support to Bandura's (1997) claim that the main source of efficacy beliefs comes from the individuals' experiences with similar and related tasks. As one LE students mentioned, "What I really lack is confidence in myself, that I will succeed in doing a problem ... my prior experience was, so far, to solve a problem than to construct a good problem to assign to students". The high school strand was related to efficacy in problem posing. Since the science students are generally involved in more extensive and rich mathematical experiences than the rest of the students, this factor is not different from one's overall mathematical involvement. Males were in some cases found to hold higher efficacy beliefs than females. One possible explanation could be the masculine "aggressive" attitude to overestimate own capabilities against the feminine moderate attitude, influenced by the well-known role stereotypes.

The significant correlations between the prospective teachers' efficacy beliefs in problem posing and their ability to construct problems indicate that efficacy constitutes a reliable predictor of the subjects performance in problem posing from the types of sources examined in the present study. Furthermore, these beliefs provide a clue about the quality of the results in such a task. For instance, the AE and the HE students were able to construct more problems and of higher complexity, as indicated by the type of the questions raised and the number of relations involved. In addition, these subjects used to test their problems and felt more comfortable with this task of problem posing, in contrast to the LE students who felt anxious, when facing a problem-posing task and seldom did they test the problems they constructed. For instance, one of the LE subjects mentioned, "What I really lack is the confidence in myself, ... that I will succeed in making a problem. My prior experience was so far in solving problems, ... not to make up a good problem to assign the to others".

In conclusion, the above findings suggest that developing efficacy beliefs in problem posing should be an integral part in any preservice teacher education program. Efficacy beliefs constitute "an important component of motivation and behaviour" (Pajares, 1996, p. 341) and consequently are important for integrating problem posing and problem solving in classroom instruction. The correlation found among the efficacy in problem posing and the students' beliefs about teaching this activity suggests a possible focus for further research.

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INVESTIGATING PUPILS' IMAGES OF MATHEMATICIANS

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Abstract

This paper describes a research project that used a variation of the Draw-A-Scientist-Test (DAST) to investigate and compare the images of mathematicians held by lower secondary pupils (ages 12-13) in five countries. We report that with small cultural differences certain stereotypical images of mathematicians, and some surprising and disturbing images, were common to pupils in all of these countries. Further, we found that for pupils at this age, mathematicians are for all practical purposes invisible to pupils, and so the images they adopt to fill this void arise either from the media, or, in some cases, from negative experiences in mathematics classes.

1. Introduction

For the past decade there has been increased discussion and research about images of mathematics and mathematicians (Furinghetti, 1993; Henrion, 1997; Lim & Ernest, 1998; Rock & Shaw, 2000) as well as what has been referred to as the "mathematics image problem" (Howson & Kahane, 1990; Malkevitch, 1989; 1997). Research has shown that images of a particular field can affect who goes into that field (see Henrion, 1997), and where students may see mathematics as unattractive, and as populated with persons very different from themselves, they may be less likely to seriously consider entering it (NSF, 1998).

Lim and Ernest (1999) point out that it is only through ascertaining how popular or unpopular mathematics is, that measures can be created to change and improve its public image. And if, as Jaworski (1994, p. 218) seems to imply, learning mathematics is related to *being* a mathematician, what she calls "being mathematical within a mathematical community," then where pupils have images of mathematicians that are inaccurate it may hinder their study of mathematics.

The decision to investigate pupils' images of mathematicians came after seeing images produced from a class assignment given to her pupils by a colleague in

New York City, to: *Draw your perception of a mathematician*, one of which is reproduced in **Figure 1:**

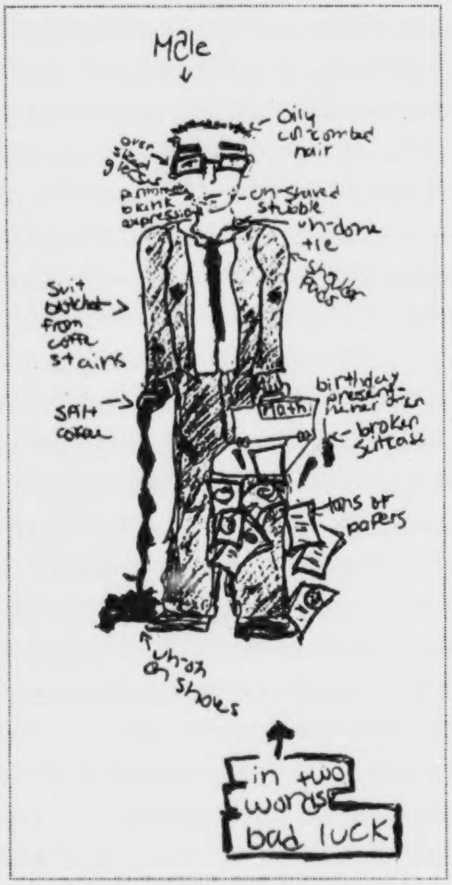
Male
Oily un-combed hair
over sized glasses
blank expression
un-shaved stubble
un-done tie
Shoulder Pads
Suit blotched from coffe stains
Spilt coffee
birthday present- never open
Math
broken suitcase
tons of papers
un-tied on shoes
in two words bad luck

Figure 1 Female 7th Grade (UK Year 8) Pupil

We then learned of similar studies involving pupils' drawings of scientists, the *Draw-A-Scientist Test* (DAST), (see e.g., Chambers, 1983; Finson, Beaver & Cramond, 1995; Huber & Burton, 1995; Matthews and Davies, 1999; Barman, 1999), which showed that the images pupils held of scientists were increasingly stereotypical and increasingly male, by the time pupils leave the middle grades.

2. The Aims and Framework of the Research

We came to feel that images held by pupils aged 12-13 (Year 8 in the U. K. and Europe; 7th grade in the U. S.) might afford us some view of pupils at a sensitive age—one in which studies show negative attitudes begin to form (see Aiken, 1970; also Lucas, 1981) and we hoped to understand whether any of these images were held in common in the different countries involved in our survey. The 476 pupils involved in the study were from schools in the following countries: USA (n= 201), United Kingdom (n= 99), Finland (n= 94), Sweden (n= 49), Romania (n= 33).

3. Research Methodology and Methods

The study was primarily qualitative and interpretive in design, with purposeful sampling since in some cases we had to rely on colleagues in other countries to administer the survey tool.

Pupils were given a one page two-sided questionnaire that asked them to "*draw a mathematician at work*." There were also two open-ended prompts: the first asked pupils to explain and enumerate the circumstances under which one would need to hire a mathematician; the second prompt asked pupils to explain their drawings. We hoped with the first question, to ascertain what it is that pupils think mathematicians actually do. With the second prompt, we hoped pupils would give more information about the gender of their mathematician and perhaps reveal something further about their beliefs. The combination of pupils drawing and writing with interviews of some of the American pupils provided a triangulation of the data.

5. A Sampling of the Results

Pupils' written explanations for why a mathematician would be hired showed that they believe that mathematicians do applications similar to those they have seen in their own mathematics classes, including arithmetic computation, area and perimeter, and measurement. They also believe that a mathematician's work involves accounting, doing taxes and bills, and banking; work which they contend includes doing *hard sums* or *hard problems*; yet in interviews pupils could supply no specifics about what such problems entail. We came to conclude that for pupils of this age, mathematicians are invisible.

The drawings from the surveys showed many similarities among the different cultures. Upon examination, we identified seven sub themes among them:

Mathematics as coercion, in which pupils drew mathematicians as teachers who use intimidation, violence, or threats of violence on their pupils. This was a completely unexpected theme that emerged;

The foolish mathematician, in which mathematicians were depicted as lacking common sense, fashion sense, or computational abilities;

The overwrought mathematician, in which mathematicians were depicted as looking wild and being *overstrained*, to quote a pupil from Sweden;

The mathematician who can't teach, in which a classroom is drawn which the mathematician cannot control, or in which he doesn't know the material;

Disparagement of mathematicians who are depicted by pupils as being too clever or in some other way contemptible;

The Einstein effect; and

The mathematician with special powers, which may include wizardry and special potions.

We detail two of these sub themes: *mathematics as coercion,* and *the mathematician with special powers,* because they afford us a surprising view of pupils' images from their classroom experiences. And while some of the drawings could easily fall under more than one of the sub themes, the hope and intent was to highlight international commonalties amongst them.

5.1.1 Mathematics as Coercion

This sub theme can be seen in two drawings from Finland, and in drawings from Sweden, the United States and the United Kingdom. In each, the pupil has drawn a large authority figure intimidating someone smaller, sometimes with violence or threats of it. It is interesting that there was no drawing of this type from Romania, even as pupils' mathematics classes are very demanding.

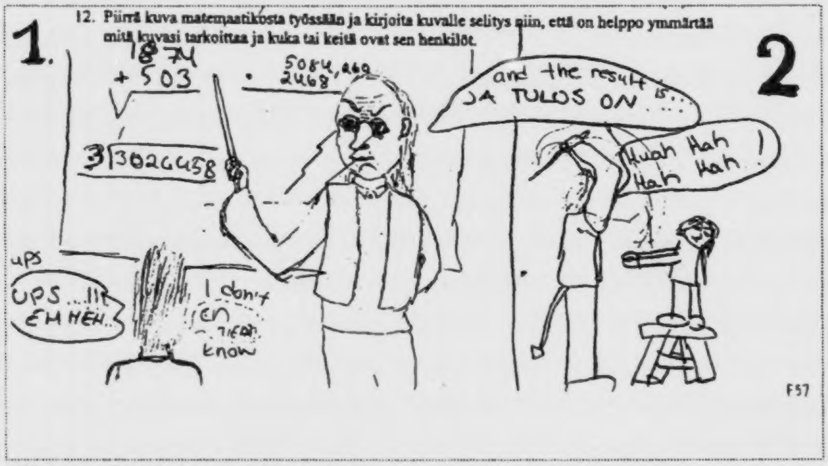
12. Piirrä kuva matemaatikosta työssään ja kirjoita kuvalle selitys niin, että on helppo ymmärtää mitä kuvasi tarkoittaa ja kuka tai keitä ovat sen henkilöt.
1.
1874
+ 503
5084,469
• 2468
3|3026458
ups
UPS ...!!! EH HEH..
I don't
EN TIEDÄ
know
2
and the result is...
JA TULOS ON ......
Huah Hah Hah Hah !
F57

Figure 2 Finland—Female pupil

In the drawing in **Figure 2** from Finland, a Svengali-like figure prompts a trembling pupil in the first panel, then, in the second panel, a devil's tail peeks out from his coat as laughing maniacally, he beats the pupil for not knowing the answer to a simple arithmetic problem. The difference in their stature is accented by the pupil's having to stand on a stool.

In two drawings, from Finland and Sweden (see **Figure 3**), there are rifles pointed at smaller figures. In **Figure 3**, a pupil is also being asked to do simple arithmetic. The pupil wrote about his drawing: *He is a strong mathematician. If you answer wrong he* [will] *KILL you.*

It is jarring to see images of guns and violence from countries whose societies are not known for this type of behaviour, within schools or without. Similar drawings from the United States, which however has a regrettable history of violence in its schools, nevertheless contained intimidation but in a different form, with no such threats of violence coming from a teacher.

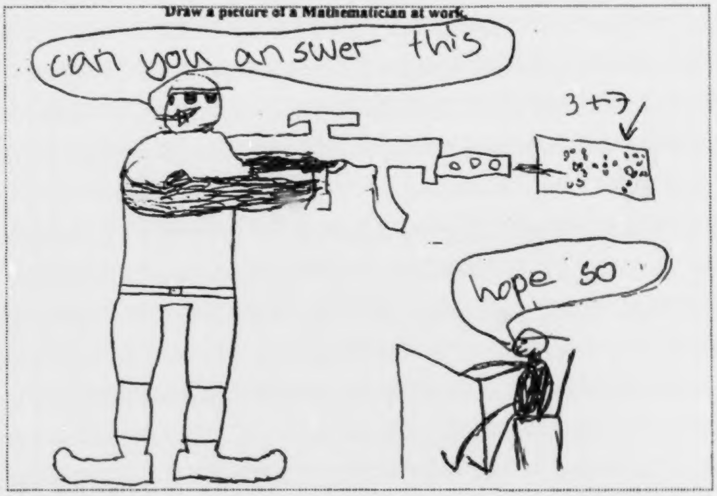
Draw a picture of a Mathematician at work.
can you answer this
3+7
hope so

Figure 3 Sweden—male pupil

The drawings from the U.S. and U.K. also contain large authority-figures. In the American drawing the girl who drew it wrote: *A white Caucasian male saying complicated things to a class of small children (only 1 child represented).* And in the drawing from the U.K., a menacing-looking the teacher is drawn on a stage ordering punishment, evincing what Nolan and Francis (1992, p.46) call a teacher centred conception of teaching in which the teacher "occupies the centre stage of the educational drama."

In these drawings again, the differences in statures between the authority-figures and the pupils is notable. And it is worth noting, too, that pupils have chosen to draw *small children* although the pupils creating these drawings are no longer small children, but in their early teens. It is possible that for the pupils creating this type of drawing, the experiences that have produced such images come from a time when they were much younger and felt more keenly their own lack of power. But these images are now carried into the present, with the result that the image of mathematics represented in each of these drawings is that of a bewildering and intimidating subject, placing pupils in a situation over which they have no control; of being excluded from the world the teacher inhabits—the teacher on a *stage* is one example of this remove—of sitting powerlessly in a class while a large adult says *complicated things.*

Davis & Hersh (1981, p. 282) have illuminated the origin of this perception of powerlessness in the minds of students:

> Mathematical presentations, whether in books or in the classrooms, are often perceived as authoritarian…Ideally, mathematical instruction says, 'Come, let us reason together.' But what comes from the mouth of the lecturer is often, 'Look, I tell you this is the way it is.' This is proof by coercion.

The theme of power is a large one in children's literature, often including secret and supernatural powers. It may also be a large factor in what is being referred to as "the Harry Potter phenomenon" (Jacobs, 2000), on both sides of the Atlantic, for as one reviewer of the fourth Rowling book (Acocella, 2000, p.77) observed: "The subject of the Harry Potter series is power, an important matter for children, since they have so little of it." In interviews, Rowling (see, e.g., Fraser, 2000, pp. 5-6; 8) has spoken about her experience of feeling intimidated in mathematics class at school.

The large size of the teachers in these drawings would seem to indicate pupils' perceptions of having more often had to deal with conformity and authority than sense-making in the classroom.

5.1.2 The Mathematician With Special Powers

The drawings in this theme contained references to special powers, from a Superman-like *S* on the chest of a mathematician drawn by a Romanian pupil to a figure creating a *maths potion*, drawn by a pupil in the U.K., to a series of wizards who appeared less than benign.

The very idea of a *maths potion* or super power implies that something extraordinary is necessary in order to do mathematics. And it is also related to the general invisibility of the mathematical process, for with the process hidden, mathematical facility looks more like a power than an ability, which anyone has the possibility to learn.

And although these are not the same wizards as in Harry Potter—the books had not caught on at the time of this survey as they have since—comments on the meaning of the magic in relation to the books are still, we believe, relevant. For as Jacobs (2000) indicates, anything that is "sufficiently inscrutable" might as well be the product of wizardry.

6. Implications for Pedagogy and Conclusions

Along with our finding that for pupils of this age, mathematicians are essentially invisible, is the conclusion that pupils appear to rely on stereotypical images from the media to provide images of mathematicians when asked.

We could not have anticipated how much pupils' drawings would provide a window onto their experiences in their mathematics classes. We believe that the drawings created by the pupils contain valuable insights with significant implications for teachers, their training, and their practise.

Pupils appeared to use experiences of having been intimidated in mathematics classes (*You should know this!)* and their criticisms of teachers for doing this, at times to depict mathematicians in their drawings in a vengeful manner, something with which they were aided by images of mathematicians in the media. How pupils can be made to feel in a classroom by teachers, appears centrally in many of the drawings, from feeling intimidated at not knowing something, to being dazzled by a teacher's polish and ability, to exploiting a teacher's inability to control a classroom, and I think seeing all these possibilities portrayed in the drawings can spur rich discussions and significantly raise teachers' consciousnesses.

The projection of supernatural powers onto mathematicians appeared in drawings by pupils in each country. When Arthur C. Clarke (in Jacobs, 2000) observed that, "Any smoothly functioning technology gives the appearance of magic," he could as easily been commenting on what pupils perceive as a "smoothly functioning" ease many teachers exhibit with mathematics, a facility that to many pupils also looks like magic.

The dominant image of a mathematician that emerged from this study is that of a white, middle aged, balding or wild-haired man. This points to both a gender and a racial gap in pupils' images of mathematicians, which is consistent with those findings of the DAST.

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Following students' development in a traditional university analysis course

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This paper presents a framework to follow students' development through a formal mathematics course. Built on learning episodes of two groups of three students attending a traditional university course, it highlights different cognitive demands learners will have according to their personal learning strategy.

Introduction

In this article we trace the development of students through the learning of limit concepts in sequences, series, continuity and differentiation, using and refining the routes of learning identified as *formal* and *natural* (Pinto 1998, Pinto & Tall, 1999). In this study we focus on three levels of development in each route. Formal thinkers attempt to base their work on the definitions, but the handling of three nested quantifiers imposes great cognitive strain. Some fail to cope with the complexity of constructing meaning, often focusing on manipulating the symbols and inequalities rather than the logic. Natural thinkers reconstruct new knowledge from their concept image. Unsuccessful learners in this route attempt to interpret the definitions in a personal way that fits their imagery, rejecting the formal theory or leading to conflict. Students who get past this initial hurdle cope with the definitions and deductions in their own personalised ways. Formal learners essentially construct the theory by deduction, coping with the great cognitive strain as best they can, producing a deductive formal theory. Natural learners—working from their concept imagery—reconstruct it taking account of more general ideas met in the course. They must then develop the formal theory from their reconstructed imagery, producing a formal theory integrating both imagery and deduction. This summary of data leads to a concise framework of development (table 1).

	Formal learning	**Natural learning**
1 Initial obstacles	Routine (based on concept definition) (a) disjoint from images, partial procedures (b) attempt to link to images, weak links	Informal (based on concept image) (a) formalism rejected, maintaining images (b) formalism embedded in informal knowledge, with some conflict
2 Theory Building	Formal construction Constructing formal theory	Formal reconstruction (with some conflict) (a) Thought experiments, reconstructing images (b) Deductions reconstructing formal theory
3 Formal theory	Formal (deductive)	Formal (integrated)

Table 1: levels of development in natural and formal routes to learning formal mathematics

Methodology

This study builds on a cross-sectional analysis of three pairs of students selected to best approximate prototypical strategies of learning from eleven students who were studied in seven interviews at three-week intervals. At this stage of theory development we move from a study of a single limit concept (the convergence of a sequence) to take into account successive encounters with sequences, series, continuity, differentiation, and a final reflective interview with each student. Our purpose is to study the ways in which different students interpret the material presented to them by the lecturer.

The 'formal' route of learning

Three levels of development are identified for the 'formal' route, which we describe as:

1: **Initial Obstacles** (working with the concept definition).
  1a: routine approach, disjoint from images, focusing on rules and procedures, partially achieved.
  1b: routine approach, some link with images, causing conflict, unable to coordinate processes.
2: **Theory building** (with some evidence of routinising reflectively).
3: **Formal deductive knowledge.**

We consider three students, Rolf, Robin and Ross, who focus on the formal structure of the course. Rolf's constructions define level 1a; he withdraws from the course after the first ten weeks. Robin starts at level 1b, and then consolidates his work at level 2. Ross distrusts his limited visual imagery and concentrates on the formal theory at levels 2 and 3. On occasion he uses visual imagery unsatisfactorily and this part of his activity is also considered.

Rolf attempts to build from the formal definitions and proofs from the outset. DEFINITIONS are rote-learned inaccurately. However, in handling questions involving direct calculation, he is able to provide ARGUMENTS that seem to be based on the formal theory. For instance, he explains the constant sequence 1, 1, 1, ... is convergent, saying:

> "Take the sequence 1, 1, 1, ... the *N* is just any greater than zero. Big *N* is just 1 basically."
> (Rolf, first interview)

He relies on the authority of the lecturer to decide what is important:

> "...I have been told that a formal proof is a proof involving epsilons. That's why I think that."
> (Rolf, second interview)

He focuses on rules, procedures and his IMAGES arise from his routine activities rather than from pictures. He is confused rather than in conflict (level 1a):

> "*I understand the reasoning, but I mean, I can do it.* And the reason I can do it is maybe because I understand it err I don't know. I don't know if I understand everything or I don't, because I don't know. If I am not understanding something that I don't know, then I don't know basically, you see what I mean. I mean, as far as I feel, I understand everything about this, [...] but I haven't a clue. You see, I don't know."
> [Okay, you are saying you can work it out.]
> "Yes, *I can work it out.*" (Rolf, third interview)

At the beginning of his second term, Rolf transferred to another university.

Robin begins in the same way as Rolf, routinising mechanically, but unable to coordinate processes (level 1a). DEFINITIONS and ARGUMENTS may be categorised in the same manner as Rolf. However, Robin attempts to link IMAGES built from the formal theory to previous experiences, which generates conflict (level 1b).

> "I mean, the making ... I could understand it said: by finding the value of N ... it actually ... is the proof that the ... sequence is tending to zero. That was a bit awkward, I couldn't get that to start with. I could understand what the definition was saying, but, to me, it didn't seem to be proving ... that it went to zero. ... I'm not ... not fully happy with this. There's still things I've got to get sorted out. But ... I get ... I have made some progress since I started."
>
> (Robin, first interview)

We understand that time pressure and weakness in his elementary mathematical background hampered his attempts to construct meaning for the formal theory.

Robin sometimes evokes visual images; he explained that imagery was helpful for him to get a sense of what was being proved.

> "... I was saying that in Analysis II, we did a lot ... from the theorems, he (the lecturer) would say: here is a graph for the theorem; well, he is showing what is going on, and that, to see it pictorially, helps to understand incredibly, really." (Robin, seventh interview)

However, he is unable to translate such analogical representations into formal language, and continued to focus on the formal route while failing to cope with the full complexity of the definitions.

Ross—the most successful formal learner—starts by constructing formal knowledge through routinising reflectively on the theory presented in the lectures (level 2). DEFINITIONS are formal, correct with occasional slips. ARGUMENTS are meaningful, based on the formal theory. IMAGES are generally built from the formal theory, usually compartmentalised from previous experiences. For instance, contrasting old and new experiences does not cause him conflict as in the limit of a constant sequence.

> "Umm 1, 1, 1, ... umm yes, because it's a set of ones, so it would tend to 1, because ... it's not changing. But ... now again when she [the lecturer] first said that like 1, 1, 1, ... tends to 1, then I thought it was bit strange, because you tend to think of the sequence going up and then gradually getting closer and closer to a value so that if you go sufficiently far out then it's reached a value, whereas 1, 1, 1, ..., I mean, you can take the first term it's already at the limit umm but I mean ... this ... definition works for that, so ... so it must tend to value ... so..."
>
> (Ross, first interview)

Ross forms strong conceptual links between successive notions of limit. For example, he perceives the link between series and the previous formal theory of sequences:

> "Umm ... well, the way, I mean, in the order of the things we've done umm it makes sense the feature of relating the series back to sequences, which was what we started off doing so ... if you can convert a series into a sequence ... then you just deal with a sequence ... ... I suppose ... umm by ... relating a convergent series with convergence of sequences then ... you are building up all of this umm area of maths from a very small ... ... each new step that you do up series you will be relating to something you have already done so ..." (Ross, third interview)

This aspect is a characteristic of Ross's performance during the course.

Ross maintains his strategy of practising routines and familiarising himself with the formal theory (level 2). There are times when he is unable to solve problems, such as writing a proof that a real function defined on **Z** is continuous, even though he could write down a formal definition of continuity for a function $f: \mathbf{R} \to \mathbf{R}$. His DEFINITIONS are always formal, correct or nearly correct. He always tries to relate ARGUMENTS to the formal theory. He realises the weakness of his own visual imagery and seeks the safety of formal definitions and deductions. On the few occasions where he evokes visual IMAGES they are often linked to previous experiences, with conflicts or inconsistencies. For instance when discussing the function $f(x) = x\sin(1/x)$ he argues:

> "...I think there is sort of ... the oscillations are getting closer and closer together and ... so I suppose a tiny change in x produces a huge change in the gradient and so at nought itself there is no way to find the gradient ... I suppose ... it's going up and down (??)"
>
> (Ross, sixth interview)

He seems to be talking about $f'(x)$ rather than $\frac{f(x)-f(0)}{x}$ as x nears zero.

In this last example we see a student we have classified as a formal learner using visual links. We do not claim that formal learners *never* use imagery. Taking the formal route means relying on formal definitions and proof to generate the theory. Meaning arises from the formalism itself rather than from any underlying pictorial images.

Following the 'natural' route of learning

The three levels proposed for the natural learner are as follows:

1: **Initial obstacles** (based on the concept image).
  1a: formal theory is rejected, not assimilated,
  1b: formal theory is embedded within the old.
2: **Theory building** (in conflict with formal theory).
  2a: thought experiments reconstructing images,
  2b: Deductions reconstructing formal theory.
3: **Formal theory** (integrated with imagery).

Cliff, Colin and Chris are natural learners developing through the course at different levels. Cliff, the least successful, reveals some characteristics of level 1, Colin's conflict characterises level 2a and the most successful, Chris, works at levels 2 and 3.

Cliff interprets new knowledge in terms of old, at levels 1a and 1b, ignoring the new experience if it does not fit his previous experiences, or rote-learning, to respond as he perceives being required by the course assessment. ARGUMENTS are image-based, or eventually pragmatic, and IMAGES are not reconstructed to fit the formal theory. For instance, when asked to consider the convergence of a constant sequence, he says it does not tend to a limit because

> "1, 1, 1, 1, 1 ... no, it's just 1, 1, 1, 1 continuously." (Cliff, first interview).

As the course progresses, the formal theory is never assimilated (level 1a). DEFINITIONS are not reproduced formally and are given in terms of imagery:

> "Err ... continuous means what ... I've got my definition of continuous function as ... I can draw the graph and you don't lift the pencil if it's continuous..." (Cliff, fifth interview)

or restricted to particular prototypes:

> "Umm ... well, I think it's any function ... any function can be differentiated ... any normal function."
> [What is a normal function for you?]
> "Umm ... like $y = 3x$ or $y = 3x + x^2$." (Cliff, sixth interview)

This instance shows Cliff's evoked images of functions as prototypes restricted not to continuous functions, but to polynomial ones (Vinner, 1983; Vinner & Dreyfus, 1989). In the seventh interview, he is back to the same strategies as in the very beginning (level 1b), rote-learning definitions in a way that is not suitable for formal deduction.

In a middle trajectory, Colin starts in the same way as Cliff and interprets new knowledge in terms of old. Sensing results and theorems are true, he does not understand *why* he is being asked to prove results that he believes to be obvious.

> "It seemed to be a silly question that ... if a_n tends to 1 then if you question when a_n is greater than $\frac{3}{4}$... this is a bound, it seems ... I don't know why... ..." (Colin, second interview)

As the course progresses, he gradually perceives his IMAGES in conflict with the formal theory (level 2a); allowing him to attempt to begin attempting to tackle proof:

> "... now I am sort of getting into it (into the course) ... what I am supposed to be doing ... how I am supposed to be proving things ... so it's got a bit easier." (Colin fifth interview)

DEFINITIONS remain distorted or eventually rote-learned and ARGUMENTS are image based such as:

> "... $\sin x$ is differentiable because I can always find the gradient of $\sin x$."
> (Colin, sixth interview)

Conflict between old and new images makes him move a little:

> "How do I prove this ... umm well, you just show that the umm you can work out the gradient for all points on the curve."
>
> [Okay, but how do you prove that ...]
>
> "How do I prove it ... umm ...
>
> $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$
>
> umm ... so like that ... (Colin, sixth interview)

In this sense the natural learner's difficulty with formalism corresponds to the formal learner's confusions with the underlying concept image. However, Robin's formal trajectory discussed earlier contrasts with Colin's conflicting 'natural' route. Robin does not know *what* he is proving so he builds a theory with weak conceptual links. Colin does not know *why* he is being asked to prove statements but he senses their truth.

Colin's old images deeply interfere with any reconstruction and he appears to be in a state of conflict until the end of the course. For example, after presenting a proof that 0.999...(recurring) is 1, using partial sums and the definition of convergence of a series, he comments:

> "... ... It's sort of ... I understand it should be 1 ... and that the limit of the sequence is actually 1 just ... 1 down as notation. It just it's a bit hard to let go of 0.9999 recurring ..."
> (Colin, seventh interview)

Chris—the most successful natural learner—constantly seeks reconstruction of old knowledge to build new. (level 2) and finally develops an integrated formalism (level 3) in which he can write both the quantified form of the definition of limit and also verbalise his own image for it:

> A sequence has a limit if and only if as the
> sequence progresses, eventually, all values of the
> sequence gather around a certain value.

(Chris, seventh interview)

His DEFINITIONS are always correct, or nearly correct. ARGUMENTS are based on thought experiments, most of the time articulating images reconstructed with the new theory. For example, a classical question resolved in classroom by handling inequalities:

> If $a_n \to 1$, prove that there exists $N \in N$ such that $a_n > \frac{3}{4}$ for all $n > N$.

has its solution verbalised by Chris as an experiment

> "I chose epsilon as 0.1 ... and showed that a_n lies between 0.9 and 1.1; so it must be greater than $\frac{3}{4}$." (Chris, second interview)

In general Chris is able to present his arguments formally. However, as he builds on his imagery (level 2a), he sometimes formulates image-based thought-experiments, such as initially restricting his concept of divergence to plus or minus infinity he also visualises continuity as "drawing the graph without taking the pen off the paper". We hypothesise that solving conflicts between old and new ideas is central to the natural route (see also Vinner, 1991). For instance, the fact that a constant sequence converges according to the formal theory surprises Chris, because it conflicts with his old images:

> "(Laughter) I don't know really. It definitely it will ... it will always be one ... so I am not really sure (laughter) ... ... umm ... it's strange, because when something tends to a limit, you think of it as never reaching it ... so if it's ... 1 ... then by definition it has a limit but ... you don't really think of it as a limit (laughter) but just as a constant value." (Chris, first interview)

As in the case of Cliff, such images may have been evoked by the use of the word "tends to" in the formulation of the question (see Schwarzenberger & Tall, 1978; Cornu, 1991; Monaghan, 1991). Data shows Chris adding new facets as additional information, as a 'natural' strategy of coping with new experiences, closely related to *how* we function in our everyday world. In his last interview, he resolves his earlier conflicts with constant sequences:

> "Umm ... it's that ... where the values are the same, it doesn't deviate at all."
>
> [Doesn't deviate.]
>
> "Doesn't deviate on the line ... and then that must be the limit." (Chris, seventh interview)

Discussion: following students' development

Tables 2 and 3 summarise the development of the students over the twenty weeks of the course. (The shaded entries represent transitional movement into the given level.)

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Rolf (a) Robin (a& b)	Rolf (a) Robin (b)	[Rolf withdrew] Robin (b)		
2. Formal Construction				Robin	Robin
	Ross		Ross		
3. Formal (deductive)		Ross		Ross	
					Ross

Table 2: Students following an essentially formal route

	Sequences	Series	Continuity	Derivative	Final Interview
1. Initial obstacles	Cliff (a) Colin (b)	Cliff (a) Colin (b)	Cliff (a)	Cliff (a)	Cliff (a)
2. Formal Reconstruction			Colin (a)	Colin (b)	Colin (b)
		Chris (a&b)	Chris (a&b)	Chris (a&b)	
3. Formal (deductive)	Chris				
					Chris

Table 3: Students following an essentially natural route

Rolf, Robin and Ross are classified as 'formal learners' who build their schemas by routinising and familiarising themselves with the formal constructs. They attempt to handle quantified statements, constructing representations which are propositional (see Eysenck & Keane, 1997). Rolf cannot cope with the formal definition and only reproduces partial procedures (level 1a) before withdrawing from the course. Robin finds great difficulty initially, with his underlying images in conflict in sequences, series and continuity, (level 1b) but persists in his formal route and manages to start making formal constructions (level 2) in dealing with derivatives. Ross works mainly at level 2, memorising the definition and working with the formal proof. He shows an understanding that the topic 'series' has essentially already been formalised using ideas from the previous section ('sequences'). This is acknowledged by classifying his responses as being in transition to a full formal theory. In the final interview he is classified at level 3.

Cliff, Colin and Chris are classified as 'natural learners' because they use their own imagery as a starting point to build their theories. These learners support their construction with analogical representations (Eysenck & Keane, 1997), which they build using their old imagery. Cliff clings to his informal imagery, making no headway with the formal theory (level 1a) throughout the course. Colin makes an attempt at the formal theory building it first on imagery (level 1b) then modifying his imagery (level 2a) as he moves on to continuity, finally beginning to build formalism on his modified images (level 2b). Chris operates at level 2 and moves to level 3.

Drawing conclusions

This research unfolds aspects of individual construction of knowledge that differ from other proposed frameworks. For instance, the strategy of encapsulation of process into object (Dubinsky *et al*, 1988) does not appear to provide a model to explain the cognitive strategies of the 'natural' route of learning. Rather than constructing a concept image from a defined object, abstracting from 'actions on objects', the successful natural learner understands the defined object by reconstructing it from the concept image.

At the other end of the spectrum, we note that there are concept images not developed through formal deduction (Tall & Vinner, 1981) which nevertheless may seem coherent with the formal models. These may or not be capable of translation into formal language. The concern is that such concept images may give the learner an apparent grasp of the theory, though they may not be sufficient to guarantee long-term success.

In the teaching of advanced mathematics, this research shows there does not seem to be a single methodology or 'formula' that works for all students. Even first class students encounter times where they struggle badly. Learners have different cognitive demands, according to their own strategies of learning. Perhaps teachers may address the needs of both routes to learning. However, there is still a distinct possibility that natural learners may be confused by formal instructions just as formal learners may be confused by references to imagery, so that it may require more subtle treatments for different kinds of student.

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MATHEMATICS SUBJECT KNOWLEDGE REVISITED.

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Our research over the last few years about teachers' mathematics subject knowledge, has led to a model for thinking about subject knowledge which distinguishes between knowledge needed to pass examinations and that needed to help others to come to know. This paper explores this model in depth and uses interviews with pre and in-service teachers responding to questions on graphs and percentages, to exemplify the model.

Introduction

Teachers hold many professional knowledges, knowledge about pupils, systems and structures; about styles of teaching and learning; about management, resources and assessment as well as knowledge about the subject. Research offers definitions of professional knowledge and the different forms of knowledge that a teacher holds, (Brown & McIntyre 1993, Cooper & McIntyre 1996, Desforges & McNamara 1979, Ernest 1989, Marks 1990, Calderhead & Shorrock 1997, Banks et al 1999).

> *It is clear that learning to teach involves more than a mastery of a limited set of competencies. It is a complex process. It is also a lengthy process, extending for most teachers well after their initial training. (Calderhead & Shorrock 1997, p.194)*

This paper considers mathematics teachers' subject knowledge and describes a model for different aspects of a teacher's knowledge about mathematics. Subject knowledge is an aspect of teachers' professional knowledge known to be problematic, but one which 'has provoked more controversy than study' (Grossman et al 1989).

> *While one can infer from studies of teacher thinking that teachers have knowledge of their students, of their curriculum, of the learning process that is used to make decisions, it remains unclear what teachers know about their subject matter. (Wilson et al., 1987 p.108)*

Research in the particular area of subject knowledge and pedagogical content knowledge (Shulman, 1986, Wilson et al. 1987, Tamir 1988, Aubrey 1997) explores the transformation of subject matter knowledge for the classroom; teachers' knowledge about explanations, tasks and activities, about styles of teaching and learning. But it does not include explicit detail of how such subject knowledge is held in an intellectual way by teachers, other than is shown by activities or explanations given. The mathematics is rarely explicit. Whilst Shulman and others have categorised the different components of subject knowledge and discussed its transformation through classroom events, our research data was used to investigate the ways in which teachers' subject knowledge in mathematics is held and transformed.

We agree with Buchmann (1984) that teachers need a rich and deep understanding of their subject in order to respond to all aspects of pupils' needs: 'Content knowledge of this kind encourages the mobility of teacher conceptions and yields knowledge in the form of multiple and fluid conceptions' (ibid. p.46). Evidence from our earlier research led to a hypothesis that teachers' subject knowledge in mathematics is held in two forms either as *learner-knowledge* or as *teacher-knowledge* in mathematics; the former is the knowledge needed to pass examinations; the latter is the knowledge needed to plan for others to come to learn the mathematics. Auditing the former might be necessary (currently demanded on Initial Teacher Education (ITE) courses, DfEE 1998) but is not sufficient for the developing teacher. Such audits offer limited list-like perspectives of knowledge to be held by teachers, a view shared by others.

> *The shared assumption underlying [such] research is that a teacher's knowledge of the subject matter can be treated as a list-like collection of individual propositions readily sampled and measured by standardised tests ...[these] researchers ask what a teacher knows and not how that knowledge is organised, justified or validated ... [such research] has failed to provide insight into the character of the knowledge held by students and teachers and the ways in which that knowledge is developed, enriched and used in classrooms. (Wilson et al 1987p. 107)*

It is the very 'character of the knowledge' in mathematics that we sought to provide insight by interviewing experienced and pre-service teachers.

A model for describing subject knowledge

The nature of the initial research has been reported elsewhere (Prestage and Perks 1999a). Analysis of teachers' responses to discussions of subject matter gave categories of phases in which aspects of their mathematics knowledge was held. This offered ways to explain the differences in the types of responses and how transitions might be described, leading to a model. Whilst the roots of the model lie in a variety of research, an explanation of its components is more easily given via our ITE students.

Graduate mathematician students to our secondary pre-service course arrive with personal subject knowledge (*learner-knowledge*) that enables them to answer mathematical questions. When asked to calculate the division of one fraction by another or to differentiate a function, all respond correctly. When asked why the answers are correct they do not know. They can do mathematics but they do not necessarily hold 'multiple and fluid conceptions'. ITE students also bring with them their personal beliefs about 'being a teacher'. Their view of teaching is to replicate the learner-knowledge they hold for others to learn, a view sometimes held by experienced teachers (Prestage 1999). Their subject knowledge is often ill-connected and they have to work on this when planning for teaching (Perks & Prestage 1994).

During ITE courses, students gain different knowledge and understandings of other professional traditions - some national like the National Curriculum and examination systems and some local traditions from particular schools such as schemes and textbooks. Learner-knowledge and professional traditions merge in the first instance to create classroom events for others to engage with learning mathematics (figure 1).

This combination is also evident from the experienced teachers. High on the list when justifying decisions about the curriculum were text books and other departmental resources, experiences of learning mathematics, ideas related to teaching practices (Baturo & Nason 1996; Ball 1988,1990) and from the new legislative curriculum.

There comes a moment for many teachers (often early in their professional education) when they realise that giving their learner-knowledge directly to pupils does not work:

> *if you present a problem to the class and there is a need for them to know something about a particular shape and the area of it, and therefore they would set the agenda ... things are encompassed in a problem and the pupils are setting the agenda.*

Reflection upon these classroom events, with the integration of learner-knowledge and professional traditions, leads to the beginnings of practical wisdom that enables teachers to adapt activities from the professional traditions to suit their particular circumstances (figure 2).

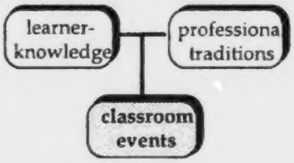

Figure 1

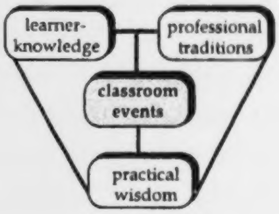

Figure 2

Pre-service teachers developed practical wisdom during their teaching experiences as shown in their evaluation of lessons. Many of the experienced teachers talked in the interviews about the consequences of diverse classroom interactions, of altering the teaching decisions in order to respond pupils' needs. Classroom experiences give rise to practical wisdom based on perceived learner needs and new explanations and contexts are found to support learner-knowledge. However, the learner-knowledge is not necessarily questioned, the teacher may still offer the rule "change the divide sign to a multiply and turn the second fraction upside down" but previous lessons may have included "how many quarters in one whole, how many in two etc." and other starting points to support the learning of the algorithm. Existing research makes assumptions that teachers have full access to subject matter knowledge and that it is transformed by activities developed for teaching. We argue that for both experienced and novice teachers much of this subject matter knowledge remains as learner-knowledge and is not transformed into teacher-knowledge, as Aubrey (1997 found):

> *There is however, little evidence to suggest that the development of project teachers' subject matter through teaching occurred. The capacity to transform personal understanding, thus, depends on what teachers bring to the classroom. Whilst knowledge of learning and teaching and classrooms increases with experience, knowledge of subject content does not. (pp.159-160)*

In certain topics there was evidence from the research data that some teachers had thought more about the subject matter, beyond reacting to pupils. For one primary co-ordinator curriculum decisions were made in response to the pupils in her Y2 class 'depending on where the conversation goes'. There was evidence of her deliberate decisions about progression through the mathematics. This aspect of deliberate reflection towards teacher-knowledge was also partially evident in other interviews and also emerges occasionally in the data from the student teachers (Perks 1997).

We believe that 'good' teachers need to reflect upon classroom events not simply to consider their success or failure for the students but to reconsider their own personal understandings of mathematics, to reflect upon the 'why' not only of teaching but also of mathematics. We would argue that it is in this way that they come to own a better personal knowledge of mathematics (teacher-knowledge), that learner-knowledge (the only explicit content knowledge) requires transforming through deliberate reflection.

This analytic process requires a synthesis of the reflection on the three elements of figure 2. The model is completed in the form of a tetrahedron, where the struts represent the reflective/analytic process.

This research explores each of these vertices to provide exemplification.

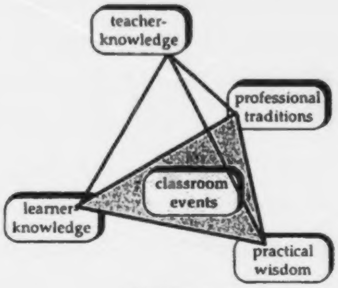

Figure 3

The current project: methods and analysis

Four experienced teachers (ET) and four pre-service teachers (NT) (at the end of their course) were interviewed to gather data about subject knowledge. Two different areas of mathematics were used, one from number (percentages), as previous research (Prestage 1999) had shown more evidence of teacher-knowledge in the area of number and one from data handling, to compare data from another project (Soares & Prestage 2000. The teachers were asked to complete several questions (Appendix) and to talk about the solutions in construction as well as their understanding of the pedagogical issues. The interviews were recorded.

Analysis of the data looked for evidence for describing the vertices of the tetrahedron (Figure 3). Whilst certain prompts during the interviews led to particular descriptions of subject knowledge, aspects of all the phrases of the tetrahedron were interwoven through the data.

Learner-knowledge: this was the data taken as they solved each question with some interim prompts such as "Why are you doing that?"; analysed by strategy.

Professional traditions: aspects of this were found in response to 'why' a particular strategy had been chosen; evidence of the impact of their own schooling, texts, schemes and government policies.

Practical wisdom: this was evidenced in response to 'how would you teach this in the classroom' and from any pupil difficulties and errors described.

We have taken teacher-knowledge to be an amalgam of all of these categories. We looked for evidence of wider connections in the mathematics, reflections upon the integration of the three categories, and when the interviewee was aware of the consequences of choices made for teaching with a definite purpose for doing the mathematics beyond getting the right answer.

Findings

Leaner-Knowledge

This was evident throughout the interviews as the teachers described their solutions. There was more variety in methods offered for the percentages questions than the graphs questions. The approach to the solutions for the graph questions focused on scale, type of graph, axes and care in plotting. There was some uncertainty on the difference between discrete and continuous data. For percentages the methods varied from mental calculations to use of calculator but the interchange of, e.g 73% to 0.73 to 73/100 was common but there was surprising variety of solutions in a small group. There were few errors in calculation which were not corrected immediately, but for the graphs two of the novice teachers tried to use time on wrong axis.

Professional Traditions.

For percentages when asked why they had done the calculation in that way, the teachers gave three types of replies:

1. using algorithms which depend on what they had done at school - the fraction and '10% is divide by 10';
2. mental methods which they were aware of having adapted from what they had done at school, or methods thy felt they had devised to make it more efficient to do it mentally;
3. the use of the calculator.

For ET1 who was using the National Numeracy Strategy (NNS, DfEE 1999) in year 7, the influence of the NNS was clear. She mentioned the examples and the ways they had make her think about methods. She felt that she had adapted her own learner-knowledge in working with this new professional tradition (our language).

For graphs, the novice teachers had done similar work at school or at university and the error about the axis for time was attributed to a learned response from school "time is the dependent variable - so it's across". For the experienced teachers the influence was more likely to be what they had taught - "the textbook always uses bar charts for shoe sizes" - "we have just been using spike graphs in our new scheme".

Practical Wisdom

For the work on graphs any description of classroom activities tended to focus on children's errors. The teachers described how to help pupils to choose the correct graph, which way to use the paper, the problem of scale, where to draw the axes, labels such as the axes and title and accuracy in general. The experienced teachers were very similar in the speed and manner in which they described what they would work on. The only noticeable difference in response lay between the two sets of novice teachers. The good novices were as good at identifying errors and activities as the experienced teachers, whereas the others were slower and had a much more limited range of possible activities which they described. The major teaching approach relied on telling the students what to do before they made errors. There was talk of such things as "discussion of scale", but this tended to mean question and answer. It was suggested that pupils needed to work on different data sets. Only ET1 and NT1&2 talked about activities such as offering tables and ready-drawn graphs with errors for the pupils to identify or having several graphs on cards for the pupils to choose the 'best' and then justify their decisions.

Percentages saw a difference between the two good novices and the rest of the interviewees. *Only NT1&2 felt that teaching pupils to change a percentage to a* fraction and then do 'fraction of' was unnecessary, describing the algorithm as "usually unhelpful, it confuses rather than helps". All of the others felt that they had to teach this algorithm - their justification lying in the professional traditions, "it's in the school scheme" or their own learner-knowledge "that's the way I learned it and it's the way to understand it". For other written methods, some teachers described using 10% as the base for teaching another routine and the others used 1%. These written methods were described by means of examples.

All but two interviewees, ET 2&3, mentioned mental work. Listening to children's methods and to develop activities was given importance. ET1 highlighted the stress placed on this in the NNS. For the novices, this was an expected inclusion as the place of mental methods had been a major strand of the course they were just completing.

Only ET3 and NT4 failed to mention calculator work, but only ET1 and NT1&2 talked in any detail. NT1&2 offered justification for decimals, "the calculator and the way percentage keys work". They talked about the importance of linking percentages to decimals directly, almost as pattern recognition: 76% = 0.76, 123% = 1.23

Discussion

The purpose of research was to find teachers' mathematical descriptions from to exemplify the phases in figure 3. Evidence for triad at the base of the tetrahedron is easy to identify from teachers talking about mathematics. However, much teacher-knowledge appears implicit rather than explicit in the data and has to be inferred. More questions or a longer time frame would have been useful. This raises the challenge of identifying what would help teachers to articulate their subject knowledge in this way, when they have not been expected to articulate it in this form. Whereas, for us as tutors on a pre-service course, we must be able to in order to help our students come to refine their subject knowledge for better mathematics teaching.

Teacher-Knowledge

To identify aspects of teacher-knowledge, we were looking for evidence of the teachers knowing the connections between the areas of mathematics and having a purpose for the activity. From the data, it appears that the implicit teacher-knowledge for percentages is more developed than that for graphs. More variety of connections and approaches were given. All of the methods of calculation would be recognised by others, whereas some of the data questions might be unfamiliar even as learner-knowledge to some.

The answers from ET1 showed her using a variety of methods, including instant recall. She used connections to other methods - "10% of works the same way as divide by 10 so we shift the numbers one place", as did NT1 "10% is like dividing by 10 or multiplying by 0.1, so 10% of means a shift of the numbers one place". There is a strong connection to other aspects of place value and a shifting image. NT4 also appears to be connecting yet when asked why he used "10% of is divide by 10" he replied "That's what you do".

For the graphs questions, there is less variety; there is evidence of different learner-knowledge related to the mathematics of the data. The teacher-knowledge related to grouped data and choice of appropriate diagram is not very explicit. There is, however, some sense of purpose for the tasks ET 1 and NT 1&2, other than practising drawing, to develop pupils' decision making skills, in choosing appropriate diagrams.

Discussion

Shulman argued that pedagogical content knowledge was the missing paradigm for teaching and we agree with its importance. We believe that designing activities for the classroom is necessary but not sufficient to develop the type of subject knowledge, the *teacher-knowledge*, we feel essential for the development of strong mathematics teaching. Nor is it acceptable to demand more learner-knowledge. More of this kind of knowledge is not sufficient for the development of teacher-knowledge. The analytic process has been long recognised and is essential for the intellectual professional.

> *Ideally we may wish to have teachers who are not only competent actors in the classroom but also who are practitioners capable of understanding what they are doing, why they are doing it and how they might change their practice to suit changing curricula, contexts or circumstances. This produces a tension between the need for teachers to understand teaching and the need to be able to perform teaching. (Calderhead & Shorrock 1997, p.195)*

Teacher-knowledge in mathematics allows teachers to not only answer the questions correctly but also help to build a variety of connections and routes through knowledge, that provides answers to 'why' something is so (Prestage, 1999). It is our contention that only when such *teacher-knowledge* is informing classroom practice that the real needs of learners and the challenges of mathematics are addressed. There are threads of teacher-knowledge emerging from the data, but the interviews did not sufficiently challenge the teachers to articulate this aspect. There is still the need to delve more deeply if the model is to be strongly exemplified.

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Appendix

Graph questions.

1. Shoe survey - a lesson on variation

Shoe size	1	2	3	4	5	6	7
Number of pupils	1	0	3	7	8	6	2

2. Stretching elastic

Number of marbles	1	2	3	4	5	6
Length of elastic (cm)	9	10	11	13	14	15

3. Falling Spinners

Some pupils made a paper spinner. They put a paper clip on the bottom and timed how long the spinner took to fall. *Their teacher asked them to investigate whether the number of paper clips affected the time it takes spinners to fall*. They added more paper clips, one at a time, timing each fall.

Number of paper clips	2	4	6	8	10
Time for spinner to fall (s)	4.5	3.0	2.0	1.5	1.5

Percentage questions.

1. 35% of £40 2. Increase £80 by 10% 3. 73% of £90

Atomistic and holistic approaches to the early primary mathematics curriculum for addition

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Abstract

A study of teaching and learning in early primary British mathematics classrooms (children aged 4 -6 years) has identified contrasting ways in which the curriculum is approached. Focusing on the teaching of addition, one of the first formal arithmetical concepts taught in school, the researcher identified that children taught in through an atomistic approach to the curriculum, taught sequentially with reference to psychological research on children's conceptual development of addition, made less progress than those taught through a holistic approach, where the focus was on counting and the number system. The mathematical structure of number, the opportunities given to children to mathematize, and alternate views on cognition are used to consider the reasons for these findings.

Introduction

In the twentieth century the curriculum in British primary schools, and especially the mathematics curriculum, has been heavily influenced by Piagetian constructivism (Walkerdine 1984). Sociocultural (Lerman 1996) and Social Practice theories (Lave et al. 1991) while influencing views of learning in primary education (e.g. Wood 1998) have had little observable effect on the mathematics curriculum. Activities for young primary children emphasise practical activity, with a range of structure apparatus (Dienes 1964) in order to develop relational understanding (Skemp 1979). Lacking confidence in their ability to provide for the developmental needs of individual children, many teachers have tended to rely on structured published mathematics schemes (Walkerdine 1984; Desforges and Cockburn 1987). These schemes tend to be based on research evidence on the order in which concepts and strategies are developed, with the assumption that teaching should follow the same order; despite evidence to the contrary (Denvir and Brown 1986a, b).

The development of addition strategies has been summarised by Nunes and Bryant (1996) drawing on work by Carpenter, Romberg and Moser (1992) Gelman and Gallistel (1978) and others. The first strategy, and the most naive, involves counting out each set (e.g. 1,2,3, then 1,2,3,4,5) combining the sets and counting all of the new, larger set (1,2,3,4,5,6,7,8), and is referred to as counting-all (or the sum procedure). Subsequently children realise that they do not need to recount the first set of objects, but can 'count-on' from this set (3...4, 5, 6, 7, 8). Baroody and Ginsberg (1986) showed that this counting-on strategy might be 'invented' by children, while Fuson and Fuson (1992) developed ways to teach it. At the same time children begin to learn some of the number facts, especially the double facts (e.g. 5 + 5 are always 10) (Gray 1997), and then to use these facts to derive others (if 5 + 5 are 10 then 5 + 6 are 11) (Thompson 1997). Further developments in addition strategies require an understanding of place value to solve multidigit addition (Nunes and Bryant 1996).

Context

The research is part of a wider study which examined the teaching and early learning of addition in early primary classrooms in Britain. The relationship between teaching and learning was examined at the level of classroom interaction, in the carrying out of mathematical tasks. The mathematics lessons of two classes in each of two schools were observed over a period of six months, involving four teachers and 112 children aged 4 to 6 years. The mathematical focus of the study was the learning of addition, one of the first formal mathematical concepts taught in school. This formed a basis for exploring the factors involved in the teaching of mathematics to young children, and their learning.

The methodology was qualitative, with participant observation the main method of data collection. Detailed fieldnotes were taken of all mathematics lessons observed; short unstructured interviews with teachers were carried out before and after the lessons. The children's understanding of number concepts and addition was assessed at both the beginning and the end of the observation period through a series of informal activities and games. The data was analysed using a grounded theory approach, which produced patterns of recurring variables. Analysis of these variables, grounded in the theoretical framework of the researcher, provided analytical pictures of teaching and learning, from which the findings emerged. One set of findings related to the way that the curriculum was structured and presented to the children, and it is this aspect that will be considered here.

Approaches to the mathematics curriculum

The study identified two contrasting approaches to the primary mathematics curriculum evident in the two schools studied.

Analysis of the addition curriculum at Ashburne School identified key characteristics of the way that the mathematics was broken down and taught to the children, which I have defined as **'atomistic'**. Atomism is defined as 'any doctrine or theory which propounds or implies the existence of irreducible constituent units' (SOED, Brown 1993). The most significant characteristic of the atomistic approach to the teaching of addition was that the curriculum was broken down into elements which were taught in a set order, which related to the findings on children's development of arithmetic strategies. There was an assumption that if research has found that children develop increasingly complex addition strategies, then the teaching of these strategies in a set order would result in learning. Analysis of the lessons observed showed that other significant characteristics were found to be related to this developmental approach. An Atomistic Approach showed:

- an initial emphasis on small numbers [small numbers],
- teaching addition in isolation from subtraction [isolation];
- emphasis on procedures rather than patterns and relationships [procedural];
- the use of physical and predominantly cardinal representations of number [cardinal];
- progression from one 'stage of development' to the next [developmental];
- the repetition of similar activities in order to reinforce the procedure [repetition].

These characteristics can be seen in the following transcript from one of the classes studied at Ashburne School. The lesson was designed to encourage the children (aged 6 years), who currently used 'count-all' to solve addition, to use a 'counting-on' strategy. It was one of a series of similar lessons in which the children used the number line to model addition [repetition; isolation]. The children had made a 1 to 20 number track [small numbers] which they used to play a game. The children took turns to throw a die and put that number of counters on the track [physical and predominantly cardinal]. On second and subsequent turns they were required to use the language of adding-on.

> Barry threw five and placed five counters, one each on the squares marked 1 to 5. The next time it was Barry's turn he threw another five.
>
> Teacher M. So, count out five to one side first of all, and then count-on from that number and put the cubes down one at a time.
>
> Barry 5, 6, 7, 8, 9, 10.
>
> (Teacher M:3)

This game formed a link between the use of discrete objects e.g. counters to model addition and the use of the number track, between cardinal and ordinal number. The children were encouraged to count-on (i.e. to say 6, 7, 8, 9, 10 rather than 1, 2, 3, 4, 5 as they added on five) and the teacher reinforced the language of counting-on: "So, 5 count-on 5 is 10" [procedural]. Over time the children learnt the procedure and could use the number track and counters to model addition, but in subsequent lessons and without these aids were seen to be no further on in their use of counting-on to solve addition.

The characteristics were not only seen in the lessons observed; they were also evident in the school's planning documents for mathematics and the mathematics scheme used in the school, showing that it was a curriculum, rather than individual teachers', approach.

Analysis of the addition curriculum at the second school, St David's, identified very different characteristics in the way that the mathematics was broken down and taught to the children, which I have defined as '**holistic**'. The most significant characteristic of the holistic approach to the addition curriculum was an emphasis on the number system, with calculation seen as one element of the relationship between numbers within the overall system. Further analysis showed other characteristics were related to this approach. A Holistic Approach showed:

- an emphasis on large numbers as well as small ones [large numbers],
- an emphasis on patterns in number and pattern spotting [pattern];
- discussion of the inverse relationship between addition and subtraction [inverse];
- emphasis on relationships rather than procedures [relationship];
- emphasis on ordinal representations of number and the development of mental imagery [ordinal];
- a eclectic curriculum in which the children are immersed in a wide range of activities, with little apparent sequence [immersion].

In both classes observed at St David's there was a heavy emphasis on counting, especially in the whole class introduction to the mathematics lessons. This was not the counting of objects, nor counting along the number track to encourage counting-on which I had seen in Ashburne School, but oral repetition of the number words in units and in larger steps, counting in twos, fives, tens etc., forwards and backwards. While I had initially not seen this as directly related to the teaching of addition, analysis showed that it was teaching essential skills towards the development of mental addition strategies where numbers are added by counting-on in units or larger steps. The children were developing mental and oral, ordinal representations of number which would help them count-on [ordinal]. The counting was not restricted to small numbers. During discussion of a hundred square [ordinal], Teacher D. highlighted the multiples of ten and she and the children (aged 5 and 6) counted in tens to one thousand.

Debbie ... count with me and we'll go all the way down[1] to one hundred and then see if we can keep going, 10, 20, 30, ... 100, 110, 120, 130, ... 190, 200 ...

Some children are still counting with Debbie on two hundred, others saying a hundred and twenty, all picked up again at 210. They continued counting to 800.

Debbie I am absolutely amazed. Give yourselves a big clap. Can anyone tell me what happens after 970, 980, 990 ?

Zeb 991, 992

Will A thousand

Rita it goes on and on (Teacher D:7)

Many of the children were not confident in keeping track of the hundreds as they counted and they relied on their teacher to provide the next hundred number, but they were confident at the pattern of tens. They were praised and felt good about counting with such high numbers [large numbers]. The counting sparked further discussion which showed the children's wider understanding of the number system including negative numbers [relationships].

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[1] I believe that the teacher talked of counting 'down' rather than the more usual counting 'up' to 100, since the number square showed the tens increasing down the right hand column.

These relationships can also be seen in an earlier lesson, centred around the number seven. Before the lesson Teacher D. talked to me about what she was going to do.

> "Some days we just look at a number and see how many different ways we can make sums that have that answer. Some of the children will just be able to do simple addition or subtraction, some might get into patterns [patterns]. I hope that all the children will be able at least to try. I think it is important for them to look at sums this way ... there can be lots of different sums with the same answer, you can get there by doing different things" [relationships].
>
> (Teacher D:4 - pre-lesson discussion)

D. wrote a large number 7 in the centre of the board.

D. Who can think of a way to make seven?

Rob Three and four.

[At each suggestion D. wrote the calculation in symbols on left-hand side of the board (3 + 4). Subtractions were recorded on the right-hand side of the board.]

Nozumo Four and three.

Ruth Seven and zero.

Lucia Twelve take away three ... er ... five.

John Twenty take twelve.

D. Nearly

John Twenty take thirteen.

Nozumo Two add five.

Ellen Six and one.

Zeb Five and two.

John Twenty-one take fourteen; twenty-two take fifteen.

Ruth Twenty-three take away sixteen.

Ellen Twenty-four minus seventeen...

Beaty one add one add one add one add one add one add one, and 10 take away 3.

John Seven take seven add seven; eleven take four, twenty-seven take twenty... (Teacher D. 4)

I observed that the children were using pattern [patterns] to generate more examples, applying their knowledge of pattern to addition. Nozumo (one of the new reception children, aged 5) seemed aware of commutativity, providing 4 + 3 after Rob gave 3 + 4, and later 1 + 6 when 6 + 1 was already on the board [relationship]. John uses inverse (7+7-7) [inverse] and equal addition. The children were working mentally and at their own level. Nozumo, Zeb and Rob (aged 5) were able to concentrate on simple one step addition, while experiencing the subtraction and more complex multi-step calculations offered by the others [immersion].

The advantages of a holistic approach to the curriculum

I have identified differences between the two school's approaches to the curriculum which I have categorized as Atomistic and Holistic. While these differences were found between the two schools, by analysing the data for counter examples, I found that the approaches were also influenced by the individual understanding of the teachers and their beliefs about mathematics and learning, though this paper does not allow for further discussion of this. In this section I want to look at why a holistic approach may be preferable to an atomistic approach to teaching addition, drawing on three areas of theory.

The first of these relates to the structure of the number system itself and, in particular, the additive composition of number, which Nunes and Bryant (1996) define as " any number *n* can be decomposed into two others that come before it in the ordinal list of numbers, in such a way that these two add up exactly to *n*" (p. 46). A special, and important, case is that of place value, since larger numbers can be decomposed into their constituent multiples of ten and units. Nunes and Bryant found that an understanding of this characteristic of number is important not only for addition of large numbers, but is also essential for children to learn to count-on. It is the basis for understanding that a number can be added to another number directly, without them having to be reduced to their unitary elements and counted singly. Nunes and Bryant emphasize the importance not only of counting, but also of "understanding the relative value of counting units and their additive composition" (p. 51). It therefore follows that teaching children to count-on using a number track is likely to be ineffective if they do not understand this additive structure of number, and demonstrates why direct teaching of small areas of arithmetic do not necessarily result in the learning of those particular areas (Denvir & Brown 1986). The concept of additive composition of number provides insight into the advantages of a holistic curriculum. Such a curriculum allows an understanding of the relationships between numbers and patterns within numbers, which characterize the additive composition of number, and are essential for addition strategies more complex than counting-all. This is more than an argument that place value should be taught in advance of addition (an atomistic approach). It is to argue that addition must be seen as an integral part of the number structure rather than a mathematical topic in its own right.

A second reason why a holistic approach to the curriculum may be preferable to an atomistic one relates to the nature of mathematics and learning. A holistic curriculum gives children greater opportunities to mathematize, to act in a mathematical way. Patterns and relationships are an essential part of mathematics which enable the development of conceptual structures (Skemp 1971; Hiebert 1986). Procedural learning, identified as a characteristic of an atomistic curriculum, can result in limited understanding: children are able to carry out the procedure but not understand why. When the procedure has been forgotten, or the necessary materials are not to hand, the learner has no way to reconstruct it. I have shown how breaking the curriculum down into very small pieces which are learnt in isolation (atomism), can result in such procedural learning. Teaching children the 'bigger picture' first, an understanding of the way that the number system fits together (holism), offers them an overall structure in which constituent sub-topics within arithmetic, such as addition, can be defined.

For the third and final explanation of the advantages of a holistic curriculum I want to appeal to the wider human experience of learning. Chomsky (Chomsky 1980) proposed a language acquisition device (LAD) to explain how children learn to speak. He reasoned that, since language was extremely complex, young children must have some special way to make sense of it. They appear to learn not only by copying the language that they hear, but by constructing their own logical rules. For example, young children will often generalize regular forms of the past tense to include goed (went), seed (saw) or buyed (bought). Chomsky therefore argued for a specialized area of the brain, specific to language learning and complete with a LAD. Bruner argued that such a LAD was in fact culturally influenced, offering instead a language acquisition support system (LASS, Bruner 1986).

I believe that language acquisition is not a unique part of learning. All children's learning in the world outside school is as an experience of immersion, in language and in culture. It is in the nature of children to make sense of the world around them. It would seem logical to assume that to teach mathematics through immersion into the number system, is to take advantage of the way that children learn. To break it down into 'bite sized pieces' is analogous to teaching children to speak using only nouns first, or allowing them only to relate socially to one other human being because more than one may confuse them. While the curriculum in British primary school mathematics classrooms has been based on a constructivist perspective on learning, founded on an active, practical approach to learning and a developmental (and therefore atomistic) view of progression, to view learning as resulting from 'immersion' is to see learning from a sociocultural perspective. Such a perspective appears to accord more clearly with the findings of this research.

Summary and Implications

In this article I have identified differences in the way that the mathematics curriculum is presented to young children found in a study of two schools and which I have described as atomistic or holistic. An atomistic approach to the curriculum breaks the addition curriculum into its developmental stages and teachers target teaching to the next stage, according to the teacher's perception of the children's needs. Activities are designed to address these stages, to teach counting-all, counting-on, and place value, starting from cardinal representations of number. This was the predominant curriculum experience at Ashburne School. A holistic approach to the curriculum sees addition as part of the relationships within the number system. Teachers offer children a range of activities which develop facility with number including counting and locating, and patterns in number as well as more formal addition tasks. This was the predominant way of presenting the curriculum at St David's.

I have shown how these findings are based on the data collected in this study, and finally given some explanations for why a holistic approach may be preferable to an atomistic approach to teaching young children addition. I have considered three ways to explain why a holistic approach to the curriculum may be preferable to am atomistic approach. These have been based on an understanding of the additive composition of the number system, an understanding of mathematical reasoning, and a personal hypothesis about children's learning.

This implies that children will learn early addition more easily if they experience a holistic curriculum. More complex addition strategies will not be developed through direct teaching of these strategies but through an understanding of the complexities of the number system. Children, offered a broad range of activities which explore the number system in its complexity, are able to develop understanding and skills which are specific to addition. However, in order to teach in a holistic way teachers may have to change their views of mathematics, addition and how children learn. Offering teachers a holistic curriculum may not prevent them from reverting to an atomistic approach if this is all that they understand.

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FACTUAL, CONTEXTUAL AND SYMBOLIC GENERALIZATIONS IN ALGEBRA[1]

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Abstract: *The purpose of this paper is to investigate, from a socio-cultural semiotic perspective, novice students' pre-symbolic and symbolic types of generalization of patterns. The investigation is carried out in terms of the semiotic (linguistic and non linguistic) means of objectification that Grade 8 students display in the attainment of the goals of generalizing mathematical tasks. The results suggest that while rhythm and movement, as well as differentiated ostensive gestures (e.g. 'grotesque' and 'refined' pointing), play a central role in pre-symbolic generalizations, symbolic-algebraic generalizations require a desubjectification process ensuring the desembodiment of spatial-temporal embodied mathematical experience. In order to deepen our understanding of the cognitive and semiotic requirements underlying pre- and symbolic generalizations, in the last part of the paper, I discuss the desubjecification process in terms of the relation between the object of knowledge and the through-sign-knowing-subject.*

1. INTRODUCTION AND FRAMEWORK

In a certain sense, generalization is one of the more natural human semiotic processes. As John Mason remarked many years ago, in one of the sessions of the 20th PME Conference Algebra Working Group, if we were to communicate without being able to make generalizations, we would be restricted to be pointing to objects around us. Any word, in fact, is the result of a generalization: it applies to a range of objects (not necessarily present) and can be used in a variety of situations. Semiosis, as it is intended here (that is, as the use of words and other signs in human activity), allows one to go beyond pointing. Within semiosis (and only within semiosis), can objects be objectified in a process that goes from the use of signs (marks, names and the like) as pointers of attention to more and more complex presentation and re-presentation systems involving new signs, meanings and layers of generalization.

In this paper, I want to pursue my investigation of the students' processes of generalizing by looking into the way students deploy and mobilize signs (words, letters, etc.) to accomplish mathematical generalizations. In (Radford 1999) I focused on the way novice students, interacting with their teacher, underwent a process of dynamic and differentiated understanding allowing them to achieve the elaboration of a *meaning* for the general term of a pattern. In (Radford 2000) the analysis was brought further and several algebraic generalizing strategies were examined. This analysis was carried out in terms of the various meanings with which signs were endowed by the students and the semiotic role that students ascribed to signs as a way to convey relations between the particular and the general. One of the reported results was the identification of the nature of the signs that the students tend to use in the elaboration of the first algebraic formulas: it turned out that these signs appear genetically related to the arithmetical concrete actions and to the objectification of these actions in speech. More specifically, novice students often use algebraic symbols as *marks* or *abbreviations* of key words belonging to a *discursive* non-symbolic semiotic layer. Thus, the students' symbolic expression $n \times 2+2$ mirrors the utterance "The term times two plus two" previously produced

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during the students' discursive activity (Radford 1999, p. 95). Following Peirce's terminology, I suggested that the students' first algebraic signs were *indexical* in nature, inasmuch as they stand for their objects in such a way that, like pointers, they appear as *indicating* the place of the objects to which they refer.

Given the strong genetic connection between algebraic generalizations and generalizations achieved in previous discursive layers of mathematical activity, it seems then, that the investigation of the semiotic modes of functioning of the latter needs to be pursued further if we want to envisage some pedagogical actions to promote new meanings for algebraic signs in the classroom. In this line of thought, the purpose of this article is to offer an exploratory investigation of pre-symbolic types of generalization in patterns and to contrast them with the algebraic symbolic ones.

2. METHODOLOGY

To do so, I will interweave theoretical reflections that draw from Bakhtin's theory of speech (Bakhtin 1986) and Voloshinov's philosophy of language (Voloshinov 1973) with relevant passages coming from my classroom-based research (more details about the methodology can be found in Radford in press). I will present excerpts of the discussions held by one of the Grade 8 students' small-groups (the students will be identified as Josh, Anik and Judith) and I will make oblique reference to the work of other small-groups. The data mentioned here involve students in their very first contact with symbolic algebra and relate to a classroom mathematical activity designed in collaboration with the teachers to immerse students into the social practice of algebraic generalization. I shall focus here on one of the activities based on the classic triangle toothpick pattern (see *Table 1*). The activity included several tasks, among them the following: (a) to find the number of toothpicks required to make figure number 5 and figure number 25 (b) to explain how to find the number of toothpicks required to make any given figure and (c) to write a mathematical formula to calculate the number of toothpicks required to make figure number 'n'.

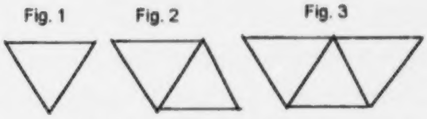

Table 1 . Toothpick pattern

3. RESULTS AND DISCUSSION

3.1 Factual Generalizations

In this episode, after finding out the number of toothpicks in figure 5, the students turned to the next question that asked to find the number of toothpicks in figure 25. Josh notices the following pattern:

1. Josh: It's always the next. Look! (*and pointing to the figures he says*) 1 + 2, 2+ 3 [...]
2. Anik: So, 25 plus 26...

This led them to write the answer as 25+26=51.

As evidenced by this passage, the students did not have much trouble calculating the number of toothpicks in the concrete figure 25. What is more important, they did so *not* by counting the number of toothpicks, figure after figure up to figure 25, but by a process of *generalization*. The generalization thus achieved is what I want to call a *factual generalization*, that is, a generalization of numerical actions in the form of an *operational scheme* (in a neo-

Piagetian sense) that remains bound to the numerical level, nevertheless allowing the students to virtually tackle any *particular* case successfully. Their objectification takes the form of a process of *perceptual semiosis*, i.e. a process relying on a use of signs dialectically entangled with the way that concrete objects become perceived by the individuals. In this process, the mathematical structure of the pattern is revealed and ostensively asserted by linguistic key terms in the students' utterances. This is what Josh does in using the term 'the next' –a term reflecting the perceived ordered position of objects in the space. Another key term is the presence of adverbs like "always" (line 1). As noticed elsewhere (Radford, in press), these adverbs underpin the *generative functions of language*, that is, the functions that make it possible to *describe* procedures and actions that potentially can be carried out reiteratively.

The semiotic means to objectify factual generalizations are varied. In another small-group, one of the students sums up her group discussion by saying: "O.K. Anyways, Figure 1 is plus 2. Figure 2 is plus 3. Figure 3 is plus 4. Figure 4 is plus 5", and she points to the figures on the paper as she utters the sentence.

Here, the objectification is accomplished in a different manner. In this case, we do not find adverbs and spatial-positional terms. Actually, to obtain a similar generalizing effect, the students rely on the *rhythm* of the utterance, the *movement* during the course of the undertaken numerical actions and the ostensive correspondence between pronounced words and written signs. Rhythm and movement here play the role of the adverb "always".

Although rhythm and movement are also present in Josh's utterance ("Look! 1+2, 2+3") we would say that, in the second group, rhythm and movement create a cadence that, to some extent, dispenses the students from using other explicit semiotic linguistic means of objectification and also provide room for a type of social understanding based on a great deal of implicit agreements and mutual comprehension. We saw how the students understood Josh's brief utterance and agreed upon the numeric actions to be performed. In the second group, the students understood that "Figure 1 is plus 2" means "the total number of toothpicks in Figure 1 is equal to 1 plus 2", etc. However, factual generalizations, without having or being objectified by more specific linguistic terms or specialized symbols, cannot gain a more general status – they remain context-bound.

To further objectify the factual generalization through language is not an easy task, as we shall see in the next subsection.

3.2 Contextual Generalizations

The next task of the mathematical activity required the students to write an *explanation* of how to calculate the number of toothpicks for *any* given although non-specific figure. The characteristics of the required explanation in the mathematical activity introduced two new elements– a social-communicative one and a mathematical one.

The social-communicative element: In this task, the social aspect of understanding was shifted. Indeed, the explanation presupposes an addressee who is tacitly thought of as being absent from the actual scene in which the students' small-group activity unfolds. Implicit and mutual agreements of face-to-face interaction had to be replaced by objective elements of social understanding demanding a deeper degree of clarity in the communication.

The mathematical element: In addition to the social-communicative element, a new abstract object has been introduced into the discourse: the question, in fact, asks for *any* although non-specific figure. The two new aforementioned elements led the students to move into another layer of discourse (for some specific difficulties that students usually encounter in understanding this level of generality see Radford 1999). As in the previous episode, we shall present here excerpts of the students' dialogue. After a relatively long period of discussions and arguments, the students arrived at the following formulation:

1. Anik: Yes. Yes. OK. You add the figure plus the next figure ... No. Plus the ... [...]
2. Anik: (*she writes as she says*) You add the first figure ...
3. Josh: (interrupting and completing Anik's utterance says) ... [to] the second figure.

Here, particular cases have been displaced and put in abeyance. Rhythm and ostensive gestures have also been excluded. What then are the semiotic mechanisms of objectification that the students display? And what are the epistemological and conceptual consequences?
First, notice the insertion of the addressee through the personal pronoun "You" (see line 1). Second, the addressee becomes interwoven with the new mathematical object: the addressee will indeed perform an *action* ("You *add* ...") not on concrete numbers but on *abstract objects* ("You add *the figure* ...). Abstract objects hence not only become abstract objects *per se* but become related to the actions required by the task and to the subject performing the actions. Third, the emergent abstract objects are objectified here by expressions like 'the figure', 'the next figure'. Such terms indicate a contrast with their surrounding; they have this kind of semiotic power to fix the students' attention (in the sense explained in our Framework: see Section 1). Through them, the students provide themselves with the capacity to achieve a fixity of reference much in the same way as 'deictic' or 'demonstrative terms' like 'that' and 'this' do in speech (see a clear example in the protocol analysis given in Radford 2000, p. 86).
We see then, that in terms of objectification, instead of a grotesque pointing, the abstract object appears as being objectified through a refined term pointing to a non-materially present concrete object through a discursive move that makes the structure of relevant events visible thereby creating a new perceptual field.

As a consequence of this linguistic objectifying process based on a refined but still ostensive way of functioning, the abstract objects are contextually conceptualized in reference to the particularities of the concrete mathematical objects. The latter stamp characteristics such as the spatial position of the sequence and a temporal sequencing action on the former, as clearly indicated in the utterance "You add the figure *and the next* figure", an expression that reveals indeed *tense* and *spatial aspects of contiguity*. The abstract objects are hence abstract while bearing at the same time contextual and situated features that reveal their very genetic origin. Their genesis also relates them to the individual who performs the actions on them. Because of the specific mode of objectification, subject and object bear an almost invisible but extremely powerful contextual dimension that allows the subject to perspectively see the emergent mathematical object.

All in all, without using letters and capitalizing on factual generalizations (which function as a guiding structure), the students hence succeeded in objectifying an operational scheme that acts upon abstract —although contextually situated— objects and indicates mathematical operations with them, ensuring thus the attainment of a new level of generality. These objects, belonging to a non-symbolic language, are not genuine mathematical objects in the traditional sense of the word. However, these objects abound in classroom discourses, where they become part of the ontogenetic process of construction of the latter. This is one of the reasons to pay careful attention to their genesis and their functioning.

Let us call these types of generalizations, performed on conceptual spatial-temporal situated objects, *contextual generalizations*.

Contextual generalizations differ from algebraic generalizations on two important related counts. First, algebraic generalizations involve objects that do not have spatial-temporal characteristics. Algebraic objects are unsituated and atemporal. Second, in algebraic generalizations the individual does not have access to a perspectival view of the objects. As Bertrand Russell noticed, in the world of mathematics (and of pure physics), space and time are seen impartially "as God might be supposed to view it". And to emphasize the non-subjective character of space and time in mathematical descriptions, he then added that, in such descriptions, "there is not, as in perception, a region which is specially warm and intimate and bright, surrounded in all directions by gradually growing darkness." (Russell 1976, p. 108).

How then will the students proceed to the des-embodiment of their spatial-temporal embodied situated experience? How are they to produce the voiceless symbolic algebraic expressions? This is an extremely complex problem impossible to exhaust in the few remaining pages. I will, however, focus on one of the elements of the des-embodiment of spatial-temporal embodied experience, namely, a cognitive/semiotic dimension involving what I want to term the subject's *desubjectification* process –a process that stresses changes in the relation between the object of knowledge and the through-sign-knowing-subject.

3.3 Symbolic generalization

3.3.1 Bypassing the 'positioning problem'

In the next passage, the students did not symbolize the factual generalization based on the pattern "the figure plus the next figure" that we discussed in subsection 3.2. Actually, they worked out a different algebraic symbolization, as shown in the following excerpt:

1. Josh: It would be n + n...
2. Annie: n + ... OK. Wait a minute! ... n ...
3. Judith: Yes. n plus ... yeah it's n .... [...] plus n plus 1.
4. Annie: Yes! n + n+1! (*that is,* $(n+n)+1$*, as it will become clear later*) [...]
5. Judith: Yes. Because, look! Look! ...
6. Annie: Your first figure is 'n' right? Plus you have n because it's the same number...
7. Judith: Because, look! Look! 4 + 4 = 8 + 1.
8. Josh: n plus n plus 1.
9. Annie: Bracket plus 1. (*they write* '$(n+n)+1$')
10. Judith: OK. Let's try it. Example... [*Josh says:* $4+4=8$ *and Judith adds:* $4+4=8+1=9$].

There is an aspect of the desubjectification process in which the students have succeeded so far, namely, the insertion of a speech genre based on the *impersonal voice*. This is evidenced by the students' utterances produced in lines 8 and 9. Thus, "Your first figure" in line 6 becomes "n" in lines 8 and 9. Furthermore, in contrast to the subjective utterance in line 6, lines 8 and 9 no longer make any allusion to an individual owning or acting on the figures. And with this, the traces of subjectivity start fading in a process where personal voices (e.g. "*I* add", "*you* put") and the general deictic objects (e.g. "*this figure*"), underpinning the previous mathematical experience, have to shift to the background thereby providing room for the emergence of objective scientific and mathematical discourse.

But there were other aspects of the desubjectification process that proved to be more difficult to confront. To understand this, we have to raise the following question:

> *Why did the students not symbolize the generalizing strategy based on 'the figure plus the next figure' that they objectified before?*

As we shall see later, when we turn to the teacher's intervention, the change in strategy is related to the students' difficulty in symbolizing 'the next figure', something that requires finding a way to forge a symbolic link between the figure and the next figure and their corresponding ranks. This problem, previously referred to as 'the positioning problem' (Radford in press), results from the dramatic change in the mode of denotation that the disembodied algebraic language brings with it, caused by the exclusion of linguistic terms conveying spatial characteristics (e.g. 'the next') and their links with the now vanishing acting individual ("I", "You", etc.). As such, the 'positioning problem' is part of the desubjectification process that the mastering of the algebraic language requires and its presence here is a token of the difficulties that the students encountered engaging in this desubjectification process.

3.3.2 The teacher's intervention

When the teacher came to see the students' work, she noticed the discrepancy between the students' explanation (written in the previous task) and their current algebraic expression. She decided to further immerse the students into the objectifying process by commenting that the symbolic expression did not say the same thing as their explanation in natural language so she asked if they could provide a formula that would say the same thing. Josh continued:

1. Josh: That would be like n + a or something else, n + n or something else.
2. Anik: Well [no] because "a" could be any figure [...] You can't add your 9 plus your ... like ... [...] You know, whatever you want it has to be your next [figure].

When the students reached an impasse, the teacher intervened again: Teacher: "If the figure I have here is 'n', which one comes next?" Then Josh, thinking of the letter in the alphabet that comes after n, says: "o".

The teacher's utterance shows how her attempt to help the students overcome the 'positioning problem' is underpinned by the spatial-temporal dimension of the general objects alluded to in the previous section (e.g. the figures are dynamically conceived of as coming one after the other). It is an open research question whether or not the mathematical meanings required to understand the denoting actions underlying the 'positioning problem' need to be imported (at least to some extent and probably within some variants) from previous non-symbolic contextual semiotic activities, as the teacher did here. If meaning is not seen as living in self-contained systems, the answer would be yes. At any rate, the teacher's intervention helped to refine Josh's understanding and to align it with the one required in the social practice of algebra. Finally, after reworking the case of figure 5, the students noticed that 6, that is, the number of the figure that 'comes next', can be written as 5+1, which was then reinterpreted as 'n+1'. In an attempt to recapitulate the discussion, the teacher asked:

1. Teacher: This would be ...? (*referring to the expression '(n+1)' that the students had previously written on their page*)
2. Anik: It's the next [figure]!
3. Teacher: (*approvingly*) Ah!
4. Anik: OK! There, now. I understand what it is I'm doing.
5. Judith: OK.

6. Anik: You put your 'n', 'n' is your figure, right?
7. Judith: Yes.
8. Anik: OK. So, what we can do is n equals the figure ... [...] n + 1 equals the next figure, right?
9. Judith: Right. (*Anik writes the answer* $(n+1)+n$ *).*

The teacher's intervention made it possible to overcome the positioning problem. It does not mean, however, that the students definitely secured new modes of denotation. The logic of signification behind the algebraic language requires a deeper engagement in the process of desubjectification. If subjective voices are no longer on the surface of the new students' mathematical discourse genre (see line 8 in the last dialogue), they are not far from it either (see line 6). Furthermore, the relation between the acting subject and the object upon which s/he acts is still perspectival in nature. It has not reached God's unperspectival view –to borrow Russell's metaphor. This is why students insist so tenaciously that brackets have to be written, as in line 9 here and in line 9 subsection 3.3.1. This is why the expression reached here, that is, $(n+1)+n$ and the expression $(n+n)+1$, reached in the beginning of section 3.3 are seen as different by the students –the reason being that they refer to two different actions.

The relation between $(n+1)+n$ and $(n+n)+1$ leads us to the relation between the signified object and its signifier. This was exactly the question that Frege asked in his article *On Sense and Denotation* (Frege 1971). Within Frege's semiotics, the alluded symbolic expressions are denoting the same mathematical object and the difference between signifiers account for differences in the modes of denotation and their respective senses. And he took sense as one of the ingredients of meaning, actually the only one related to the truly logical or mathematical aspect of the object to which the symbolic expressions refer. What the students' dialogues suggest is that to reach desubjectification and to end up with the objective kernel of the algebraic generalization, meaning has to be disembodied and become thus pure mathematical sense.

4. SYNTHESIS AND CONCLUDING REMARKS

In this article, we identified three types of generalizations related to geometric-numeric patterns. These generalizations appear as operational schemes relying on different semiotic means of objectification. While factual generalizations remain bound to a numerical level and their objectification is based on a process of *perceptual semiosis* stressing the patterned effect through adverbs of generative actions (e.g. 'always') or through articulated semiotic devices like *rhythm* and *movement*, contextual generalizations take, as their arguments, general non-specific numeric objects. These proto-mathematical objects displayed on a still non fully mathematized layer of discourse, are objectified through linguistic, non symbolic terms e.g. 'the figure', 'the next figure'. In doing so, in the course of a discursive practice, the students achieve a fixation of attention and extract from the undifferentiated horizon of objects certain elements that make apparent new objects that are beyond direct perception (indeed, the term 'the figure' *is not* 'figure 1' or 'figure 2' or figure 3, i.e. any of the figures *shown* on the activity page). Yet, the students' type of denotation is one that conveys the embodiment of the mathematical experience. It provides the students with a perspectival view of the emergent general objects. As a result, the proto-mathematical objects bear a very important characteristic: they remain *contextual* objects because of their spatial-temporal mode of being. They are abstract deictic objects.

The semiotic means of objectification underpinning these types of generalization shed some light on a question that has been tormenting me for the last couple of years. The question is related to the novice students' meaning of signs in algebraic generalization of patterns. As I noticed in the Introduction, and the phenomenon was again visible in the episodes seen in this paper, the students' signs in their first algebraic expressions bear the characteristics of associative indexes whose primary function is that of an abbreviation. The analysis presented here suggests that in this semiotic operation, the students succeed in accomplishing the devoicing of subjectivity. The suspension of subjectivity (related to objectivity) was recognized by Kant in his *Critique of Pure Reason* as one of the two conditions for knowledge. The second one that he contemplated concerned the exclusion of time (that Kant related to logical necessity). It is in this regard that the major cognitive and epistemological problems appear. Indeed, as we saw, the phantom of the students' actions still haunts the algebraic symbols. The difficulty of the effacement of the individuals in the action that they produce was noticed by Piaget (1979) during the course of his investigation of children's sensorimotor stage and talked about the individuals' *décentration* of their actions. One of the reasons for the persistence of the action as a link between subject and object may be that, as Vygotsky (1997) suggested, actions appear as a formidable source of meaning in the emergence of the child's semiotic activity.

The question of the individuals' actions and their semiotic objectification (discussed from other theoretical perspectives and in different contexts by Arzarello 2000 and Núñez 2000) appears as an important element in contemporary understandings of the ontogenesis of algebraic language. The analysis offered in this paper evidenced some tensions caused by a shifting in the relation between the knowing subject and the object of knowledge imposed by the cultural requirements of algebraic and scientific languages. As we saw, natural language accounted for close dialectical forms of relationship between subject and object. In algebraic language, the relationship between subject and object is shattered. The dual reference subject/object becomes lost and it is no longer possible to talk about e.g. "your first figure". The students now have to refer to the objects in a different way. Deprived of indexical and deictic spatial-temporal terms, the new objects have to be denoted in a layer of discourse where they bear a different kind of existence and where the subject denoting them has to become (to use a term from Lacanian theory of discourse) *decentred* (see e.g. Bracher *et al.* 1994). The epistemological and didactic understanding of the decentration of the subject urges us to reflect on and envision new dialogical and semiotic forms of action in the activities that we propose to students during their insertion into the phylogenetically constituted social practice of algebra.

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UNDERSTANDING DATA COLLECTION

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Recent changes in the research agenda, fuelled by curricula changes, have focused on considering what 'statistical thinking' really means. To assist educators in both curriculum design and assessment more needs to be known about students' statistical understanding. This paper takes up the theme by considering students' responses to two open-ended tasks, based on scenarios involving data collection. The first task requires a suitable data collection method to be suggested, while the second task suggests the method but asks for implementation details. In both, a justification for the answer is elicited. A developmental sequence of nine levels was identified and the responses to the two data collection questions were analysed. The SOLO Taxonomy was used as the theoretical framework to assist this process.

Introduction

As more researchers focus on students' statistical understanding, some aspects of statistics are being more thoroughly researched than others. Just as the understanding of simple probability has been identified as critical to the statistical process, so too is a basic understanding of data collection. Too often students are given data to work with or told how to collect data, rather than experiences which involve data collection decisions. Shaughnessy (1997) advocates encouraging teachers to give students a chance to show what they *can* do statistically. This should include making decisions about data collection and more research into students' understanding of data collection is needed.

The SOLO Taxonomy (Biggs & Collis, 1982) is being increasingly used as a framework in both probability and data handling. SOLO levels have been used to classify student responses concerning; data representation (Reading, 1999; Chick & Watson, 1998), data reduction (Reading & Pegg, 1996), data interpretation (Reading, 1998) and uncertainty (Moritz, Watson & Collis, 1996). This paper explores students' responses to questions concerning the understanding of data collection, using the SOLO Taxonomy as the theoretical framework.

The SOLO Taxonomy

Detailed descriptions of the SOLO Taxonomy can be found elsewhere (see for example, Biggs & Collis, 1991). The model, which allows a deep analysis during categorisation of students' responses, consists of five modes of functioning, with levels of achievement identifiable within each of these modes. The two modes relevant to the research being reported are the ikonic mode (making use of imaging and imagination) and the concrete symbolic mode (operating with second order symbol systems such as written language). The three relevant levels identified within each of these modes are: *unistructural* - with focus on one aspect, *multistructural* - with focus on several unrelated aspects and *relational* - with focus on several aspects in which inter-relationships are identified. A cycle of growth forms as the three levels recur within the modes, with the relational level response in one cycle similar to, but not as concise as, the unistructural response in the next. Different cycles of levels are identified by the nature of the element on which the cycle is based.

Research Design

One hundred and eighty secondary students, selected randomly over gender, mathematical ability and academic years were tested on a range of statistical questions. This paper reports on the responses to a two part question which presented short scenarios to students and then asked about some aspect of the related data collection. The question was open-ended and students were asked to explain the reasons for their answers. Part I of the question sought students' ideas on method of collecting data, while Part II was more specific, the method was given and implementation details were sought. For a more detailed discussion of the analysis of responses to these questions see Reading (1966).

Analysis of Responses to Part I

The question as presented to students is shown in Figure 1. Based on the depth to which the response indicated the ability of the student to understand the collection of the data, three major groupings of the levels were identified.

PART I Question

Radio stations have their own way of working out the most popular song on the radio and they often produce Top 40 charts. Imagine that you have been asked to do this independently of the radio station and answer the following questions :

(i) Describe the best way to find out what the most popular songs are on the local radio station.

(ii) Why did you decide to find out this way?

Figure 1

First Group (No Method Suggested)

Observed responses, which were coded into two broad levels (1 and 2), attempt to rationalize the requirements of the question but show no real concern about actual data collection.

Level 1 These responses do not fully address the question, suggesting the use of data that have already been collected rather than collecting data to address the problem. For example a Year 7 student wrote:

(i) *You could ring up the radio station, ask someone that works there, go in and ask them.*

(ii) *It was the first thing that came into mind. It would be easier than worrying about it.*

Level 2 These responses indicate that all aspects of the question have been considered, but a suitable explanation as to why the answer was chosen was not given. For example a Year 12 student gave a reason to collect data rather than for the method chosen:

(i) *Watch programs such as Rage or Video Smash Hits then maybe listen to the radio to see if it is right, or just ring up the station and ask.*
(ii) *So that I know what to expect if they ever play a song on the radio.*

Second Group (Concern with Physical Aspects of Data Collection)

Responses in the second group (coded as Levels 3, 4 and 5) are concerned with rationalizing the method of data collection. These responses attempt to describe suitable data collection, but are mainly concerned with physical aspects collection such as, the time or cost involved. There is no evidence of concern for the quality of the resulting sample.

Level 3 These responses indicate that, in attempting to justify the suggested method of data collection, focus was directed back to the question and not to any specific aspect of the collection of the data. For example a Year 7 student wrote:
(i) *Have a piece of paper sent to all houses, get them to write their favourite songs on them and return them to the radio station.*
(ii) *I decided this way because I think it would be a good idea.*

Level 4 These responses give reasons, with an explanation for the method chosen, which focus on physical aspects of the collection process. There is no real concern for the accuracy of the resulting sample. For example a year 9 student wrote:
(i) *Have a phone in census or a questionnaire that is put through the public for their forty favourite songs.*
(ii) *I decided to find out this way as you can get a larger amount of information in a relatively short amount of time.*

Level 5 These responses indicate that concern for the physical aspects of the data collection have been rationalized. However, the only concern for the quality of the resulting sample is that data have been collected in such a way that the sample is fair or accurate with no indication as to how this is to be achieved. For example a Year 8 student wrote:
(i) *By finding what music is bought as singles most at the music stores.*
(ii) *Because it is the most accurate way of finding out this.*
and a Year 12 student wrote:
(i) *Have people ring in and vote for their favourite song. The song that is most popular will be no. 1, the second most popular no. 2 and so on.*
(ii) *To give everybody an equal chance of giving their opinion of their favourite song.*

Third Group (Concern with Quality or Accuracy of Resulting Data)

The final group of responses (coded as Levels 6 and 7) indicate concern for the quality or accuracy of the data in the resulting sample.

Level 6 These responses indicate the need for sample selection to be arranged so as to produce a range of data in the sample, based on one variable. For example a year 12 student used 'time' as the variable:

(i) *Do a random survey on the radio turning it on at different times of the day at different intervals noting the songs that are being played.*
(ii) *Because it gives you a less bias opinion and view on the popularity of certain songs. You can get a wider census area, making the results more realistic.*

Level 7 These responses indicate that selection of the sample has been based on more than one variable in an attempt to improve the range of the responses which are collected. For example a Year 12 student used 'age' and 'background':
(i) *Collect group of people of varying ages and background who listen to the radio and ask them their favourite songs.*
(ii) *Not biased to any group of people and asks people who are interested in music because they listen to the radio.*

The results, arranged by academic year in Table 1, illustrate a number of interesting points. First, there are only three students (5%) from Years 11 and 12 whose responses fall within the first group (Levels 1 and 2), whereas in Years 7 and 8 there are nine students (15%), in this group. Second, there are only three students, all in Year 12, whose responses were coded as Level 7. Last, there was an overall bulge at Levels 3 and 4. This bulge is consistent in all years, except Year 12 where the bulge shifts to Levels 4 and 5.

Table 1 - Response Level by Academic Year for Part I

Level			**Year**				**Total**
	7	**8**	**9**	**10**	**11**	**12**	
1	5	1	1	0	2	0	**9**
2	1	2	3	5	0	1	**12**
3	10	9	9	6	10	5	**49**
4	10	14	9	12	12	9	**66**
5	2	3	4	5	5	7	**25**
6	2	1	4	1	1	5	**16**
7	2	0	0	0	0	3	**3**
Total	**30**	**30**	**30**	**30**	**30**	**30**	**180**

These results suggest that, when dealing with data collection, the level of response improves progressively with academic year, although, the bulge at Levels 3 and 4 suggests that many students are more concerned with sample selection based on physical aspects rather than quality of the data.

Analysis of Responses to Part II

Answering this question (Figure 2) meant that students did not need to be concerned about the method of collection, the actual details of the sample were required.

A similar hierarchy of levels of response was observed for Part II so examples of responses for Levels 1 to 7 are not included. However, two other Levels 0 and 8, were observed.

Part II Question

There are often surveys of the community to see what T.V. programs they like to watch. The editor of the school magazine is interested in writing an article about the viewing habits of the students at A.H.S. and asked you to find out the information.
(i) You are only able to ask 30 students from the school. Which students would you select to ask ? (Don't use names)
(ii) Why would you select these students ?

Figure 2

Level 0 These responses (in the First Group before Level 1) indicate that no attempt at all has been made to answer the question.

Level 8 This response (in the Third Group after Level 7) indicates that the selection of the sample, based on a number of variables, also takes into account the composition of the population from which it is drawn. This is a more thorough attempt to make the sample representative. This unusually good response from a very insightful Year 8 student, who achieved at Level 6 in Part I was:
(i) *I would try to select a broad spectrum of the populous, taking into account age and social groupings. I would keep the divisions proportionate to what they are in the school environment e.g. there are 80 people in one social group and 20 in another therefore i would take 4 people randomly from group 1 and 1 person from group 2.*
(ii) *I would use this method to be sure of getting the full range of viewing habits within the school, but so as not to overestimate the statistical effects of minority groups.*

Three interesting points arise from the results, arranged by academic year in Table 2. First, there are no students from the two senior years whose responses fall within the first group (Levels 0, 1 and 2), whereas in Years 7 and 8 there are a number of students, eight (13%).

Table 2 - Response Level by Academic Year for Part II

Level			**Year**				**Total**
	7	**8**	**9**	**10**	**11**	**12**	
0	1	0	0	0	0	0	**1**
1	1	0	0	0	0	0	**1**
2	3	3	0	3	0	0	**9**
3	1	0	0	0	1	0	**2**
4	12	7	8	3	1	3	**34**
5	7	9	10	5	7	11	**49**
6	2	7	10	13	13	6	**51**
7	3	3	2	6	8	10	**32**
8	0	1	0	0	0	0	**1**
Total	**30**	**30**	**30**	**30**	**30**	**30**	**180**

Second, there are only seven (12%) students from Year 7 and 8 whose responses were coded as Level 7 or 8 compared to eighteen (30%) Year 11 and 12 students. Last, there is an overall bulge which varies from year to year, ranging from Levels 4 to 5 for Year 7 through Levels 4 to 6 for Year 9 to Levels 5 to 7 for Years 11 and 12.

These observations suggest an improvement in the quality of responses with increasing academic year and the ability to show more concern with the accuracy of the sample when the method of collection is specified.

Comparison of Part I and Part II

The framework developed appears to be adequate for explaining students' understanding of the basic concepts of data collection. An upwards shift in the level of response with increasing academic year is more pronounced in Part II than Part I. There is an association between the level of the coding and the part of the question being answered ($\chi^2 = 109.5$, 6 d.f., is very significant, $p < 0.001$). Far less responses are coded into the Levels 0 to 3, and far more responses in the Levels 6 to 8, for Part II of the question than for Part I. Even the large bulge of responses at Levels 3 and 4 for Part I has shifted up to Levels 4, 5 and 6 for Part II.

This suggests that, given a sampling scenario, students are able to discuss the practicalities of the collection of data, but find it difficult to rationalize this sufficiently to consider the consequences of the sample choice on the data produced. However, once students have been prompted with some information about the sample and are able to concentrate on which members of the population to choose, consideration is given to the aspects of selection which affect the sample selected and hence the quality of the data collected.

The significantly higher level attained in Part II suggests that, with prompting as to the physical details of the sample, students were able to demonstrate a deeper understanding taking into consideration the variables which might possibly affect the resulting data.

SOLO Taxonomy Framework

The levels established for the classification of the responses, along with the structure of the SOLO Taxonomy, were used to create a framework which could be used in future to code student responses to data collection questions. The first group of three levels exhibit ikonic features while the second and third groups represent two different cycles in the concrete-symbolic (CS) mode.

Ikonic responses suggest that the required task could not be linked with any sort of symbolic representation. Level 1 responses were coded as a mixture of unistructural (U) and multistructural (M) responses. As the task had not been addressed, it is difficult to determine how many visual cues from the question are in focus or what personal beliefs and experiences have been drawn upon, without further investigation. Level 2 responses correspond to the relational level within this mode.

The responses in the second and third groups have been able to link the concepts in the question to concrete experience. The answers include reasons linked directly to the practical aspects of data collection or to concerns about the accuracy of the sample. These responses are in the CS mode with two cycles of U, M and R levels.

The first cycle involves consideration of physical aspects of the data collection. The elements in the first cycle are the practicalities which need to be taken into consideration when data are to be collected. Typical considerations are the number in the sample and the type of data to be collected as influenced by things, such as, the cost and time involved. A relational response in the first cycle is not achieved until the student is able to consider all physical considerations as a functioning set, and hence come to the realization that more needs to be considered. The U, M and R levels in this cycle correspond to the Levels 3, 4 and 5 identified earlier.

The second cycle involves appreciating that the method of selection of the sample influences the quality of the responses. Reasons given in responses now indicate that some attention has been focused on ensuring that the data collected presents a suitable range of opinions. The elements in the second cycle are the various variables that may be used in the selection process to ensure an accurate sample. A relational response in the second cycle is not achieved until a variety of variables have been considered as concern is centred on making the sample as representative of the population as possible. In this cycle, the U, M and R levels correspond to the Levels 6, 7 and 8, as outlined earlier.

The main feature which distinguishes the concrete-symbolic mode responses from the ikonic mode responses is evidence of the recognition that data need to be collected to address an issue. Ikonic mode responses either suggest using information collected by others or consider a personal judgement sufficient. CS mode responses discuss one or more aspects of the data collection process. Within this mode, the first cycle responses are only concerned with physical aspects of data collection, while second cycle responses consider the influence of variables related to the method of selection on the quality of the data.

Conclusion

Three major findings have evolved as a result of this study. First, students are better able to consider variables influencing the selection of a sample when the physical aspects of the data collection process have already been resolved for them.

Next, the three broad groupings identified, namely, *No Method*, *Concern with Physical Aspects* and *Concern with Accuracy* assist in determining the stage a student has reached in understanding data collection. *No Method* responses are addressing the issue but not by collecting data, while the other two groups deal with data collection, *Concern with Physical Aspects* responses in a less statistically sophisticated fashion than the *Concern with Accuracy* responses. These groupings offer educators a means to better follow student thinking when planning lesson sequences within the curriculum and assessing specific student outcomes.

Last, the groups of levels identified can be categorized as cycles of U-M-R levels, based on the SOLO taxonomy. The *No Method* group is a U-M-R cycle in the ikonic mode where the elements of focus are the facts in the question. The other two groups represent two U-M-R cycles in the CS mode. The elements of focus in the first cycle, the *Concern with Physical Aspects* group, are the physical aspects of collection while the focus elemens in the second cycle, the *Concern with Accuracy* group, are the various variables which could affect the accuracy of the sample. Educators could make use of the suggested framework when considering the quality of responses to gain a greater awareness of what students really know, understand and can do.

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PREREQUISITES FOR THE UNDERSTANDING OF PROOFS IN THE GEOMETRY CLASSROOM

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Geometry is a field, which is a good starting point to teach and learn mathematical argumentation, to explore mathematical concepts, to fill the gap between every day life and mathematics, and to value mathematics as a part of human culture. Accordingly, geometrical competence has to be regarded as an important prerequisite of understanding mathematics. Our research aims at identifying aspects of geometrical competence. Based on empirical data from upper secondary school, we argue that high level geometrical competence is specifically influenced by spatial ability, declarative knowledge, and methodological knowledge.

1 Theoretical framework

The role of geometry in the German mathematics classrooms has changed considerably in the last decades. Despite the fact that geometry is part of the regular curriculum, it is often regarded by teachers as a less important topic. While this is well known for primary schools there is little research concerning secondary schools. In contrast, mathematicians as well as mathematics educators agree that geometry should be an important part of mathematics education (Lehrer & Kazan, 1998).

1.1 Understanding of proof and geometrical competence

Proofs and mathematical argumentations play an important role in the geometry classroom. Particularly in recent years, many researchers contributed to the description of the role of proofs for the development of mathematical competence. Authors like Hanna and Jahnke (1993), Hersh (1993), Moore (1994), Hoyles, (1997), Harel and Sowder (1998) have pointed out that in both, mathematical research and school instruction, proving spans a broad range of formal and informal arguments and that being able to understand or generate such proofs is an essential component of mathematical competence. In constructivist-oriented mathematics instruction, the critical exchange of arguments and elements of proof is accorded new significance.

Some empirical surveys of North American high school students (Senk, 1985; Usiskin, 1987) and pre-service teachers (Martin & Harel, 1989) have revealed wide gaps in respondents' understanding of proofs. Healy and Hoyles (1998) made a significant contribution to the field with their recent systematic investigation of students' understanding of proofs, ability to construct proofs, and views on the role of proof. Their empirical study was conducted in various types of schools spread across England and Wales. Almost 2,500 tenth grade students, nearly all of them from the top mathematics set, participated in the study. The results show that even these high-attaining students had great difficulties in generating proofs. The students were far from proficient in constructing mathematical proofs, and were more likely to rely on empirical verification. However, most of them were well aware that once a statement has been proved it holds for all cases within its domain of validity. Moreover, they were frequently able to recognise a correct proof, though their choices were influenced by factors other than correctness, such as perceived teacher preference. Students considered that their teachers would be more likely to accept formally-presented proofs, though they were personally more likely to construct proofs which they deemed to have an explanatory character. In all domains, students with higher levels of mathematical competence outperformed less able students.

Recent research in the field of cognitive psychology has focused on the cognitive processes and specific knowledge structures needed to solve geometry problems. Geometrical reasoning has been investigated in detail by setting various types of test items, observing students and experts working on the items by means of think-aloud protocols, and computer simulation of thought processes. Koedinger and Anderson (1990) emphasize that when experts construct geometrical proofs, they do not merely retrieve definitions, axioms and theorems from the memory and combine these to make logical deductions. On the contrary, they skip details of the proving process, and outline their argumentation in broad terms, taking a constructivist approach. They use visual models, in which they are able to "see" properties and connections, and "pragmatic reasoning schemas" such as set patterns for individual steps in the proving process. As stated by Koedinger (1998), this indicates that geometrical competence is not merely a question of talent, but of specific skills and knowledge.

Geometrical competence does require specific knowledge; it is based on general psychological mechanisms that are central to other domains of mathematics as well as to thinking and problem solving in general. Where geometrical knowledge is concerned – as shown by the cognitive psychological research cited above – a distinction must be drawn between declarative knowledge and methodological knowledge. Moreover, metacognition can be identified as a general mechanism. Various components of general intelligence are relevant; according to Clements and Battista (1992), spatial reasoning is of particular importance for geometrical competence. They suggest that geometrical competence is largely dependent on spatial visualisation skills, but that spatial ability can also be enhanced by exposure to geometry.

1.2 Research Questions

The present study integrates the lines of research described above. Geometrical competence and its cognitive prerequisites are investigated by reference to TIMSS items, with a particular focus on respondents' understanding of proof. The research questions to be addressed in this paper are as follows:

- Is geometrical competence dependent on students' declarative knowledge, methodological knowledge, metacognitive competences and spatial abilities? What is the relative importance of each of these factors in explaining interindividual differences in geometrical competence?

- Can Healy and Hoyles' (1998) findings on the connection between students' ability to construct proofs and their views of the role of proof be replicated in another instructional culture, namely the German mathematics classroom? Is is possible to identify prerequisites for the correct understanding of proof?

2. Design of the Study

In the present study, geometrical competence was assessed using nine TIMSS items from the so-called *advanced mathematics* domain in the upper secondary level. The sample consisted of 81 students from German schools (48 female), 59 of them attending a regular mathematics course and 22 an advanced course, who tackled selected TIMSS items as well as additional tests (metacognitive assessment, declarative knowledge, understanding of proof, spatial reasoning), and were videotaped as they worked on the geometry items using the think-aloud method. Based on analyses of the entire set of TIMSS items, the nine items were allocated to two proficiency levels (Klieme, 2000). The lower level items were answered correctly by more than half of the students in the international TIMSS population, the three higher level items by one third or less of the students.

The first prerequisite of geometrical competence to be measured independent of the TIMSS items was declarative geometrical knowledge. Linking up with earlier work on the conceptual knowledge required for mathematical problem-solving tasks (Klieme, 1989; Reiss, 1999), we chose a central concept of school geometry, namely "congruence", for the evaluation of students' declarative knowledge. Students were asked to give a definition, an example, a visual or graphic portrayal of the word "congruent", and to name a mathematical theorem in which the concept features. The students' open-ended answers were coded according to a specially developed category system; one point could be earned for each of the four aspects.

Methodological knowledge was assessed using an item from Healy and Hoyles' (1998) proof questionnaire. The item dealt with the question of whether a given triangle could be proved to be isosceles. Students were presented with a correct formal proof, a correct narrative proof and two incorrect arguments. They were then asked to assess the correctness and generality of each of the four arguments.

As a measure of general intellectual abilities, particularly of spatial ability, an instrument which is well-known in Germany – Stumpf and Fay's so-called *Schlauchfiguren-Test* – was administered. *Schlauchfiguren* presents different views of complex tubular figures, which have to be judged with respect to the specific point of view. This kind of task has been shown to predict mathematical problem-solving competence (Klieme, 1989). Validiation studies have shown that the test calls for both spatial ability and deductive reasoning. It is therefore a suitable instrument to capture those aspects of general intellectual ability which are cognitive prerequisites of geometrical competence.

Scale	Number of items	Theoretical maximum	Mean	Standard deviation	Reliability (Cronbach's α)
Geometrical competence	9	14	6,90	2,60	.53
Level I/II	6	8	5,67	1,77	.49
Level III/IV	3	6	1,23	1,49	.35
Methodological knowledge	8	8	3,52	2,14	.68
Declarative knowledge	4	4	1,49	0,95	.47
Spatial reasoning	21	21	13,17	3,95	.76

Table 1: Distributional parameters and reliability of scales

Table 1 provides an overview of the various scales, the number of items in each, and the most important distributional parameters. The estimated reliability (Cronbach's α) for our sample is also shown. Because of the limited test time, we were only able to administer short tests, particularly for conceptual knowledge and the two levels of geometrical competence, the scales for which consisted of only three to six items. The estimated reliability for these scales is correspondingly low. If extrapolated to tests of the standard 20-item length, however, an acceptable α of between .71 and .84 emerges in all cases. This suggests that the constructs behind the indicators represent dimensions of ability which may be regarded as reliable. If significant correlations are found in spite of the technical limitations and the associated lack of reliability of our instruments, it can be assumed that these are valid findings with relevant effect sizes.

3. Results

3.1 Descriptive findings

In the following we will report on certain aspects of our findings. Figure 1 shows the percentage of students providing correct solutions for each of the nine geometry items administered in our study, along with the corresponding results for the international TIMSS sample and the German national TIMSS sample.

The results show a remarkably high level of correspondence across the three samples, both in the average achievement level and in the performance in each of the nine questions. Averaged out across the nine items, 53% of the students in our sample provided correct solutions, compared to 51% in the international sample and 47% in the German sample. Across the nine items, the correlations between the performance in our sample on the one hand and in the representative German and international TIMSS samples on the other amount to .97 and .89 respectively; both of these correlations are highly significant. The relative strengths and weaknesses of the German students are thus also reflected in our small sample.

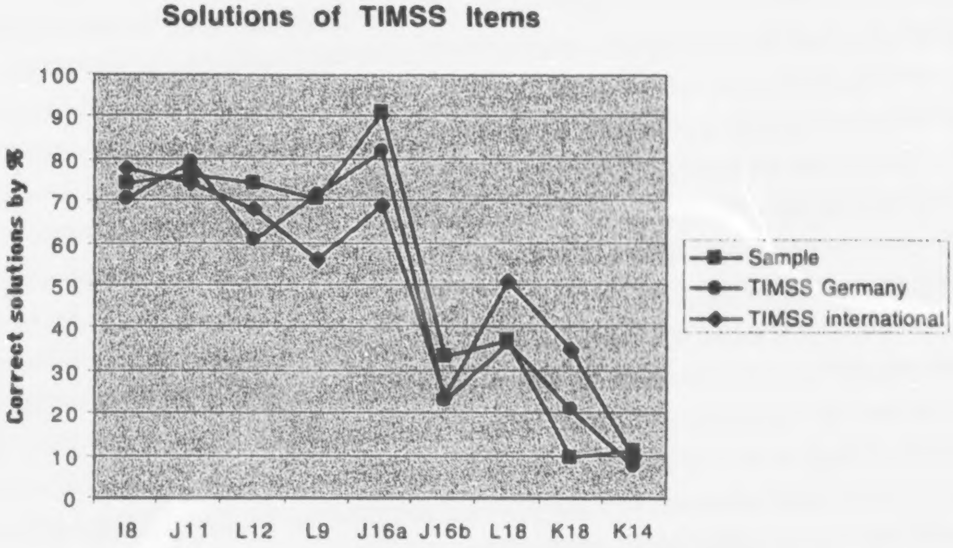

Figure 1: Solutions of TIMSS Items

In view of the observation that very few students (20% and 35% of the representative German and international TIMSS samples respectively) were able to construct correct Euclidean geometry proofs, we also expected the levels of performance to be rather unsatisfactory in students' understanding of proof and their views of the role of proof (taken from Healy and Hoyles, 1998). Interestingly, our students also found it much easier to judge given proofs than to construct their own proofs. This confirms the results of Healy and Hoyles (1998).

Proof / feature	Relative frequency (in percent)	Corrected item total-correlation
Correct formal proof		
/ correct	57	.49
/ general	57	.45
Correct narrative proof		
/ correct	42	.45
/ general	30	.38
Empirical argument		
/ incorrect	46	.13
/ not generalizable	60	.26
Formal, circular argument		
/ incorrect	33	.39
/ not generalizable	27	.40

Table 2: Components of methodological knowledge (understanding proof)

As shown in Table 2, 57% of our respondents recognised the correct formal proof (using congruence) to be correct, and the same proportion of participants correctly appreciated its generality. A similar proportion of respondents recognised a purely exemplary, empirical argumentation to be incorrect: 46% said that the argument was incorrect, and 60% recognised that it was not generalisable. However, the low item-total correlations of these two answers (see right-hand column of Table 2) showed that even students with a low general understanding of proof were aware the purely empirical argument was incorrect and not generalisable. Our findings on the respondents' declarative knowledge, are also less than satisfactory from the standpoint of mathematics education. When asked to describe the concept of "congruence", 82% of respondents were able to illustrate the concept in a sketch, most of them drawing congruent triangles. Less than half of the respondents were able to give an example of congruence, however. Only about one in ten of the students mastered the mathematically central components of the concept, i.e., were able to provide a definition of the concept and name a mathematical theorem in which it features (e.g., a theorem of triangular congruence).

3.2 Explaining Geometrical Competence

We will now explore the relations between the scales for geometrical competence, methodological knowledge and declarative knowledge. Table 3 shows the intercorrelations, calculated as rank-correlation coefficients (Kendall's tau), on which this discussion is based. In addition to the mathematical dimensions of competence and knowledge, the two general psychological predictors – metacognition and spatial reasoning – are also included in the table.

Scale	(2)	(3)	(4)	(5)	(6)	(7)
(1) Geometrical Competence	.76**	.62***	.20*	.24**	.24**	.33***
(2) Geometry, level I/II		.27**	.10	.18*	.21*	.23**
(3) Geometry, level III/IV			.22**	.23**	.21*	.37***
(4) Methodological knowledge				-.01	.05	.12
(5) Declarative knowledge					-.02	.09
(6) Metacognition						.09
(7) Spatial reasoning						

Table 3: Intercorrelations of scales (Kendall's tau – b) *) p< .05 **) p< .01 ***) p< .001

The most important finding is that all four predictors exhibit significant correlations with geometrical competence. This lends support to our basic hypothesis that geometrical competence is dependent on methodological knowledge, declarative knowledge, metacognition, and spatial reasoning. The correlation matrix does not actually allow such causal interpretations to be made; but interpreting the results in the light of other research on geometrical knowledge (Reiss & Abel, 1999; Reiss & Thomas, to appear) makes it plausible to assume that scales (4) to (7) tap the *prerequisites*, and scales (1) to (3), the *results* of development of geometrical competence.

As expected, stronger correlations with the predictors emerge at the higher levels of geometrical competence (items on TIMSS proficiency levels III and IV) than at the lower levels of geometrical competence (levels I and II). Understanding of proof is a vital ingredient at the higher levels of competence, but is irrelevant to performance in the easier TIMSS geometry items. This confirms our assumption that the TIMSS proficiency levels really do reflect different standards of (geometrical) competence.

4. Discussion

In our study, students were presented not only with TIMSS geometry items, but with a number of additional test components. This enabled us to assess various types of mathematics-related skills and general psychological competencies that could possibly be prerequisites of geometrical competence. Where the geometry items are concerned, the performance of the students in our sample was well in line with the profile of results obtained for the national and international TIMSS samples. It is thus possible to assume that our findings can be generalised to these study populations.

Our findings provide evidence for the validity of the TIMSS advanced mathematics tests. In particular, we were able to demonstrate that items from the higher levels on the TIMSS proficiency scale really do make more complex demands on the problem solvers, calling for a broader base of declarative knowledge (e.g., comprehension of geometrical concepts such as congruence) and methodological knowledge (e.g., an understanding of proofs, their generality, etc.).

Investigation of individual items has shown that the demands made by each item vary greatly. In some cases, only spatial and deductive reasoning ability is required, in other cases conceptual and/or methodological knowledge is also essential. In other words, TIMSS items make different demands, and cover many different facets of geometrical (or general) mathematical competence. The test is nonetheless one-dimensional, as model testing indicated. This means that the various facets of mathematical competence are not independent of one another, but rather that they are highly correlated. In the German school system at least, high levels of geometrical competence are accompanied by high levels of overall mathematical competence, a good understanding of proof, and differentiated conceptual knowledge. In other words, geometrical competence does not develop in isolation.

These findings, revealing the students' inadequate understanding of proof, can be regarded as an important indication of where the problem areas in mathematics instruction lie. Indeed, this was the basic approach taken by Healy and Hoyles (1998). In the context of a theory of situated cognition, however, the discrepancy between abstract knowledge about the correct construction of proofs and (at least partly) erroneous personal preferences is easy to understand and can be positively evaluated: students bear the context in mind when evaluating differing formulations of mathematical arguments. This is precisely the sort of approach encouraged in modern, reform-oriented conceptions of mathematics instruction. After all, students should not only experience mathematics as a set of fixed rules. On the contrary, they should be able to construct appropriate mathematical arguments both in school and in applied contexts. Our findings indicate that the topic of "proof in mathematics instruction" is particularly well suited as an introduction to mathematical argumentation – precisely because of this juxtaposition of views and preferences. "The goal is to help students refine their own conception of what constitutes justification in mathematics from a conception that is largely dominated by surface perceptions, symbol manipulations, and proof rituals, to a conception that is based on intuition, internal conviction, and necessity" (Harel & Sowder 1998, p. 237).

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VERBALIZATION AS A MEDIATOR BETWEEN FIGURAL AND THEORETICAL ASPECTS

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ABSTRACT: The study reported in this paper concerns the dialectic relationship between the figural register and the natural language register when students try to solve plane geometry problems. I will present a theoretical framework and some preliminary results concerning the following problem: how and to what extent does natural language act as a mediator and a control tool between the operational handling *(Duval) of the drawing and the theoretical reference (in our case, Euclidian geometry)?*

1. The research problem and the theoretical framework

In geometry, since we deal with theoretical objects and their representations, we need to state what we mean by *drawing*, *figure* and *geometric object*. I considered the definitions given by Parzysz (1988) and by Laborde and Capponi (1994).

Parzysz suggests that "*the FIGURE is the geometrical object which is described by the text defining it* " and "*The figure is most often REPRESENTED*" (Parzysz pg. 80). Parzysz calls *drawing* the illustration of a figure.

Referring to Parzysz' elaboration, Laborde and Capponi propose the following definition: *"Drawing can be considered as a signifier of a theoretical reference (an object of a geometric theory, like Euclidean Geometry or Projective Geometry). A geometric figure involves the joining of a given reference to all of its drawings: it can be defined as the set of all couples which have the reference as the first term, while the second term belongs to the universe of all possible drawings of the reference".*

Referring to the abovementioned elaborations, from now on I will consider *geometric object* the *object of a geometric theory related to a definition.* The *description* will be the *verbal presentation of the geometric object* (i.e. the text of the definition). By *drawing* I then mean *one of the different graphical expressions of the definition itself.*

I can now define the *figure* (F) as the set of couples made up by the geometrical object (O) and one among the drawings (d_i) that are material representations of that geometrical object (O):

$$F = \{(O, d_1), (O, d_2), (O, d_3), \quad \dots (O, d_i)\}.$$

In this way the theoretical aspect is linked to the graphic one and a kind of bridge is established between them.

The differences (and relations) between drawing, figure and geometrical object play a very important role in handling the drawing when trying to solve a plane geometry problem. Therefore, we adopted the *operational handling of a drawing* considered in Duval's theory (Duval, 1994): operational handling of drawing (*appréhension opératoire* [1]) involves an immediate perception of the drawing and its different variations (*mereologique,* optical or of position[2]). Our research concentrates mainly on analysing the influence of natural language on the relationship between the operational handling of drawings and the theoretical reference to which it is related. We define *theoretical reference* in a given geometric theory as theorems and definitions of that theory which are related to the figure by the student who is solving the problem. Since solving a plane geometry problem involves reciprocal relationships between drawing and theory, we note a two-way relationship between the handling of the drawing and the choosing of a particular theoretical reference: choosing a particular theoretical reference leads to the operational handling of the drawing and vice versa the operational handling of drawing can suggest how to choose a particular theoretical reference.

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[1] "L'appréhension opératoire est l'appréhension d'une figure en ses différents modifications" .

Three hypotheses can be formulated:

a) The above-described relationship is guided and controlled by natural language.

b) Recognising some useful sub-configurations or some useful geometrical properties may be due not only to the perception of the drawing, but also to the *description* of the considered *geometrical objects*. This might even have a certain influence on the way to consider and analyse the actual drawing.

c) A verbal discourse about the project of resolution (meta-discourse) is necessary in order to give a status of hypothesis or of conclusion to the information which we get by handling the drawing procedures and by the description of geometrical objects.

In this paper I will elaborate a research methodology suitable for testing the above hypotheses and present some preliminary experimental results (intended to support further investigations).

2. Research methodology

Hypotheses a), b) and c) bring in particular to the following questions:

1. Is natural language a tool of mediation and control between operational handling of drawing and theoretical reference in the procedure for solving a plane geometry problem?
2. What role does language play in the meta-discourses elaborated while solving the problem?

These questions immediately pose a problem concerning research methodology: How can we access the students' solving process? Basically, I tried to elaborate a methodology that should be suitable for tackling this problem. Such a problem depends on the fact that language plays two different roles: it is a tool for the researcher (as a revealer of students' processes) and, at the same time, it is a tool for students, because they use it to solve the problem (and our inquiry concerns its role in the solving process). We elaborated a model for analysing protocols, based on the use of language as a revealer, which allowed us to point out the role of language as a problem-solving tool for students. The model is based on the assumption that solving processes are mainly expressed through two registers: the linguistic and the figural. Then, the model distinguishes between two strategies: one developing from the figure drawn after reading the text of the problem, and the other developing from the question posed in the text or from the sub-questions obtained by transforming that question.

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[2] A variation is called "*mereologique*" when it divides the drawing into parts; it is designated as *optical* if it is an enlargement or a reduction of the drawing ; it is called *positional* when the figure background changes position.

We named these strategies *drawing strategy* and *discourse strategy.*

The *drawing strategy* involves handling the drawing (in Duval's sense of operational handling of drawing), or its perceptive apprehension (cf. Duval, 1994) in order to construct a *work environment* by means of a list of information

The *discourse strategy* consists of a structured sequence of questions, starting from the question of the text; or from some key words taken from the text, by talking with a schoolmate, or with a teacher; or from a key configuration isolated in the drawing.

So, the *discourse strategy* is closely linked to the text of the problem but, on the contrary, the *drawing strategy* is not strictly related to the text.

Both strategies may intervene in the same student's solution.

• *Aim of strategies*

If the *discourse strategy* consists of a structured sequence of questions, its aim is a structured sequence of answers. On the contrary, the aim of the *drawing strategy* is collecting information starting from the drawing or by acting on the drawing itself. So, the change of aim in the procedure is the key element that reveals the intention to go on to another solving strategy. The *discourse strategy* usually is a part of a deductive strategy, in which the aim is to prove something. On the contrary, the aim of the *drawing strategy* isn't proving (indeed this strategy is used to create a set of information that constitutes the working environment).

• *Criteria for distinguishing between the two strategies*

We now try to provide some criteria that are useful for recognising a *drawing strategy.* In detail, language makes it possible to recognise this strategy when we can detect:

- words that refer to perception, such as *" you can see that..";*
- words and adverbs indicating space, such *"here, there,.."*; besides demonstrative adjective or pronouns, such *"this (one), that (one), ..."*, accompanied by gestures;
- the present tense recurring frequently
- a descriptive rather than deductive discourse, without any connection linking the information in the list.
- unjustified inferences: they carry the formal shape of ordinary inferences such as *"since we know that..., then it follows necessarily that..."*. But the term "*necessarily*" introduces perceptive evidence and takes the place of *"then, since..."*.

The following is an example containing some of the above-mentioned inferences, made at the drawing level.

123. Taina: "because, since we have OD diagonal, I go on tracing the OD line, then we have the parallel, no, the perpendicular, which is AE, since it is a circle, since we know than OA, OD and OE

are circle radii and that AO is equal to AD and that OE is also equal to AD, then necessarily DE is equal too".

We now try to provide some rules that are useful for recognising a *discourse strategy :*

- the variation in the use of verb forms and tenses. For instance a sentence like *"we should be able to demonstrate that"* points out an attempt to get out of the solving procedure, in order to provide a plan of it.
- the complexity of sentences: coordination between several complete propositions.
- the *final* [3] structure of a sentence as *"to have...it is necessary that..."*. This structure allows us to determine the theoretical reference that guided the answers to the questions.
- the presence of *key words* such as *perpendicular* or *isosceles triangle, height, medians.* These words play a key role for the subject, which refers back to a concept belonging to his/her knowledge system. Therefore, these words are a kind of bridge between the subject's knowledge and the text of the problem or the discussion with some schoolmates. Let's take a look at the word *parallelogram*, for example: it reminds the subject of the quadrilateral figure, then the student will relate it to all the theorems and properties defining it which are part of his/her knowledge system, thus becoming capable of handling the drawing.
- the presence of *Key configurations*, which is recognised and isolated by the subject in the drawing.

3. Experimental situation and early research results

As pointed out above, the object of our research is the role of language in the link between the operational handling of a drawing and the theoretical reference to which it is related. By analysing the students' oral and written texts using the model of *drawing strategy* and *discourse strategy*, we try to distinguish among various behaviours, which we call *action models*, in the students' solving processes. Such models should enable us to define how these working moments are structured and what the switchovers from one to another are. Such models should be made operational by finding ways of relating them to students' behaviour. A short description of the first experiment performed is presented, followed by the early results.

3.1 The experimental situation

We performed a preliminary experiment involving Italian and French Grade X students. They worked in pairs, trying to solve a plane geometry problem involving geometrical objects already studied by the students in the middle school. Audio and video-recordings as well as students' written texts were collected.

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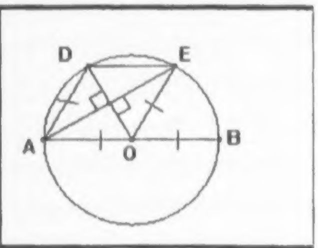
[3] This term indicates that the subject begins to search for the "cause" starting from the "effect" (consequence). The action focuses on a search of the theoretical reference to reach the "effect" that, in the specific case of our experimentation, is the rhombus.

Task

Given a circle C; its centre O; its diameter AB;
D is a point on this circle, so that AD = AO.
The perpendicular to DO through A meets the circle C again in point E.
Prove that OADE is a rhombus.



3.2 Early research results

We identified several *verbal action models* implemented by pupils: some of them will be described (along with the criteria for detecting them).

It is within the context of these models that we are trying to determine the role of language as a problem-solving tool.

Action models in the Drawing strategy

Among the results obtained by analysing the students' *drawing strategy*, there is one action model involving the creation of a list of information and the handling of such data. Let's use a French student's work as an example (Taina worked with Sophie - see later for additional excerpts).

40. Taina: diagonals AE and OD cut each other in their middle point, making a right angle, and AO is equal to EO, EO is equal to AD.

The information in the list is, obviously:

1) Diagonals AE and OD intersect each other in their middle point.
2) They both make a right angle
3) AO is equal to EO
4) EO is equal to AD.

As we can see, the information in the list is not related to each other.

How and where do students get the information (theoretical references, geometrical relations, properties, etc.) for making their own list? We already said that the information in a list can be collected from the drawing, through operational handling or through the perception of it, but it can also be collected through implicit or explicit inferences. Here are two examples: Taina - Sophie and Gaelle - Camille:

- explicit inference:

59. Sophie: Look! AO is a radius of the circle and EO is a radius of the circle too.
60. Taina: then, AO is equal to EO too
61. Sophie: and AO is equal to AD too
62. Taina: so, and AO is equal to AD, so AD is equal to OE

- implicit inference:

36. Gaelle: maybe, look! this one is symmetrical to this one *(OA and DE)* then it is the same.

The above inference is implicit, since it comes out from the drawing interpretation field[4] and not from a transition to a deductive procedure. We can see some revealing signs: the verbs *"to look at"*, related to a perception of the figure, and indicative words *("this one")*.

Experience shows that students try to handle the list whenever it becomes too long to be managed. Such handling involves some operations geared to modify the list. These operations include: putting the information in the correct order, picking up useful information and leaving useless information off the list through inference, adding some information to the list. For instance, the following dialogue is an example of how information is deleted from the list:

26 O: But ...wait, ...look: this one is equal to that one *(Ad = AO)* 27 D: NO, but... look: OD is equal to OE, which is equal to OA, because they are radii of the circle, three radii...then... since this one is equal to that one *(OD = OE)* and OE is equal to AO, then DE is equal to OE	Intervention 26/27 List : C_1 : AD = AO C_2 : OE radius C_3 : AO radius By inference we can get to the information C_4: OE=AO
31 D: Since in the text they say that AD is equal to OA, and that OE is a radius, then it (OE) is equal to AO, because AO is a radius. Then OE is equal to OA, which is equal to AD.	Intervention 31 Handled list (useless information C_2, C_3 kept out of the list through inference) : C_4 : OE=AO C_1: AO=OD

In order to point out the role played by language in the drawing and its handling in the relationship between the operational handling and the theoretical reference, let's see the result of analysing the student's oral and written work:

1) The ordering function of language (language as an *organiser*): there is no order in the information carried by drawing, because it is global and two-dimensional (the operational handling of the drawing doesn't give any ordered information). On the contrary, language is straight and sequential and because of these qualities the information must come out in order.

2) The selective memory function of language (language as a selective memory tool), which makes it possible to select only the useful part of the information given in the drawing. The drawing has everything, but there's even too much! Therefore we need to select the information and thus build a system to keep such data in mind. (For instance, deleting an useless information: see the interventions 26-31)

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[4] The drawing interpretation field was defined by Laborde as the set of spatial drawing's properties which are related to the geometrical properties of the object.
(Laborde and Capponi, 1994, pp. 171 - 172).

3) The role of control in handling the drawing : the presence of language in handling the *drawing* is a tool for controlling the entire operation. (For an example, see the interventions 36-46)

Action models in the Discourse strategy

Based on the results obtained by analysing the students' output related to the *discourse strategy*, two *action models* were identified: one related to the discourse procedure starting from the question of the problem, and the other related to the procedure started by key words. The latter will be described in detail in this report while the former will not be discussed at this time.

The *key words* act like a kind of label and carry out two functions: they let students recall a particular theoretical reference and they can refer to a linking concept, by which it is possible to switch to another theoretical field, leaving the given one (such a situation is not described in this report).

The action model that refers to *key words* can be used to recall the theoretical reference needed for the solution. In this action the word is associated to the concept (theoretical reference belonging to the subject's knowledge system). Usually, the associative operations are started by pronouncing a word or by reading it. The concept allows us to consider particular geometrical objects. In this sense, based on the definition of Parallelogram, we can consider two equal segments which are also parallel segments. This defines a set of information that must be found again in the drawing by handling it: we need to identify two opposite and parallel segments.

Here is an example of how *key words* work:

36. Gaelle: maybe we can prove… well, look at it! This one is symmetrical to the other one (AO and DE), so it is the same.

37. Camille: and then?

38. Gaelle: and then we should be able to prove that (*this is a meta discourse*) it is parallel to that one there (*AO parallel to DE*).

39. Camille: yes, but what we have to say is that this one is the middle point (*the diagonals intersection*).

40. Gaelle: yes.

41.Camille: it is the middle point of this one and of that one (*DO and AE*)…wait! AO is equal to AD… and *what if we could prove that (meta discourse)* triangle DAO is isosceles? Because, you know, it is important with reference to this one (*DO*).

42: Gaelle: yes, because it is the height.

43. Camille: yes, it is the height.

44. Gaelle: Yes, it is also the median ….Yeees!!! it is the median!!!

45. Camille: and this means that it is an isosceles triangle because the height is equal to the median… AE is perpendicular to OD and AH is the height in the triangle ADO (*H intersection of the two diagonals*).

46. Gaelle: then AE cuts OD in the middle.

The sequence of these pieces of information (isosceles triangle, height and medians, as underlined in the text) is the standard sequence of the properties by which an isosceles triangle is described in France. After naming the medians, Gaelle realizes that it is connected to *middle point*; then she relates this word to the theoretical reference, the isosceles triangle, and then she goes on to the discourse procedure.

We notice that the meta-discourse usually reveals the transition to a new strategy (see the above-mentioned interventions no. 38 and no. 41) and it plays a role of control on the solving procedure.

The dialogue reported above is an example showing quite clearly how a change of aim is decisive in transforming the descriptive structure of the discourse into a deductive one, to go on to the *discourse strategy*.

4. Conclusion

In the previous sections I presented a description and a means of interpreting students' works in the specific field of plane geometry problem solving, in particular by concentrating on the role of language in the problem solving process. Up to now, results show that natural language plays a truly important role in plane geometry problem solving because it acts not only as a bridge but also as a guide and a mediator, in the two-way relationships between the handling of the drawing and the theoretical reference which is useful for handling the figure. By using a set of criteria to identify students' main strategies, I was able to partly describe this mediation process and give a list of same functions of the language.

I presented the first steps a long-term research project about language as a didactic tool, aimed at obtaining results to be used in classroom didactic engineering. The first experiment I carried out suggests that further steps in the research project should be made at a micro-analysis level concerning the identification of students' *action models* related to their macro-strategies. *Action models* (once identified) should make it possible to find the appropriate area where the teacher can intervene in students' problem-solving activities.

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HOW DO GRADUATE MATHEMATICS STUDENTS EVALUATE ASSERTIONS WITH A FALSE PREMISE ?

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Abstract

We present an empirical study focusing on diagnosis of graduate mathematical students' logical ability. Patterns of responses on evaluating non-computable implications are identified and related to answers in hypothesis testing (Wason's tasks). "Stable" categories either oriented by logic or by pertinence are strongly linked to success in the final exam. The category of "false logical" answers is the worse. Some issues about didactical consequences are discussed.

Introduction

Mathematical reasoning often requires to use assertions with uncertain or false premises. Such a situation is encountered, for instance, in proving the step in a recurrence, or in proofs needing "case by case reasoning", when the premise is not true for some set of parameters values. In order to master such situations, students must: clearly understand the concept of implication—involving or not involving quantifiers—, master the notion of counterexample, and be fully aware of the distinction between implication and equivalence. More, taking the negation or the contraposition of a given mathematical statement requires to differenciate the implicative link from its components: premise and consequent.

To our knowledge, there were few empirical studies reported on this subject with regards to the litterature on mathematical proof or on natural implication in experimental psychology of reasoning. However, some epistemological or psychological analyses can be found on this point, which all insist on the conflict between everyday reasoning on false premises and mathematical logical reasoning, or which present these situations as paradoxical ones (for instance: Legrand, 1990; Johnson-Laird & Byrne, 1991; Deloustal-Jorrand, 1999; Politzer & Carles, to appear). In a grand tour on why students cannot master mathematical reasoning, Dreyfus (1997) stressed the "shortage of research data" on students' explanations when reasoning in undergraduate mathematics. In fact, studies on reasoning emphasise global proof strategies, organisation and understanding (for instance: Balacheff, 1987; Coe & Rutven, 1994; Duval, 199; Fischbein, 1982). Among researches which focus on details on mathematical logic used in demonstration, Radford (1985) analyses implication by secondary level students; Durand-Guerrier (1996) stresses the role of contingent assessments and the importance of predicats in logic, and not only propositions, for college students.

In mathematical curricula, logic is generally not taught as such, even if textbooks may present it as a tool or as an object at the university level (Deloustal-Jorrand, 1999). The usefulness of logic in compulsory mathematics education is questionned for future non-mathematicians. At high-school or university levels, emphasis is put on mathematical contents (concepts and problem solving): the main idea is that learning to solve problems and proving statements is sufficient for learning to master mathematical reasoning. Our claim is that it is probably unsufficient, because most of the proofs encountered in mathematical cursus, including at the university level, are mainly "going from stated as true to stated as true".

The present study focuses on what happen when such a process is disrupted.

The experimental study

Students and tasks

Students were graduate students (three or four years after *baccalauréat*), candidates from the North Region of France for being mathematical teachers at the secondary level (107 students—59 girls and 48 boys). They answered a test about their global reasoning ability at the beginning of their preparation to the national competitive entry examination. The test duration (4 hours) was sufficient to answer all the 20 questions. It was compulsory, answers were non anonymous, and students were said that the test aimed at informing the teacher on their collective logical competence.

We will here analyse 7 non mathematical or elementary mathematical items, where the task was to assess the value of an implication. In 5 cases the premise of the implication was clearly false; the two other items were the classical Wason's selection task (Wason, 1966) and the Radford's version of it (Radford, 1988). The text of these items is given in annex, with their respective positions within the set of 20 questions. Through the forword, we intended to avoid contract effects, to obtain comments, and to avoid non responses, as far as possible.

Data analysis

We will present the first step of data analysis, based on categories of responses, without taking into account detailed procedures.

Categories of responses

- Three of the questions concerned mathematical or material implications. The implication truthfulness was **non computable**. These items were coded as "*non-computable implication*". Premises were meaningful and even provoking. There were 4 types of responses: true; false; non sense (or variants); no answer.

- Another mathematical implication to be assessed was a **computable one** ("$Hn \Rightarrow Hn+1$"). In a previous part of the question, it was asked to prove the implication *via* a simple computation. It is coded "*computable implication*". The premise was without direct meaning for the students. There were the same types of responses as for non-computable implications.

- Two items were two versions of **Wason**'s selection task, coded as "*Wason* " (W1 and W2). There were 4 types of responses: all correct; W1 correct & W2 non correct; W2 non correct & W1 correct; other or no answer. Two errors were particularly interesting from the point of view of the research of counterexamples: missing the non-Q card (with 7) in W1 and evaluating the procedure 2 as non conclusive in W2. They were coded "non-Q missing".

- The last item was a situation of **social contract** ("*The teacher's sweets*"): "If P then I will do Q", then P is not satisfied and Q is nevertheless performed. Four types of responses were identified: logical true; modifying the context to make P true; contract not respected; other or no answer.

Pattern of responses in evaluating assertions with false premise

The second step of analysis consists in defining patterns of responses on the three similar items (non-computable implication), and expressing their relationship with the other ones, and with the success to the final mathematical competitive exam.

Seven patterns were identified:

- **Logic** perspective, **stable** (LS): 3 "*true*" logical correct answers; 10 students.
- **Logic** perspective, **unstable** (LI): two "*true*" answers; 9 students.
- **Pertinence** perspective, **stable** (PS): three "*non-sense*" answers denying pertinence to the question ; 9 students.
- **Pertinence** perspective, **unstable** (PI): two "*non-sense*" answers; 14 students.
- **Non-conditional** answers (NC): three "*false*" answers (with or without explanation: "*because P is false*"); 22 students.
- **Non-conditional** answers, unstable (NCI): two "*false*" answers; 23 students.
- Answers with **no-dominant** type of answer (SD); 20 students.

Given the global results, the probability to find a dominant pattern (all but SD) if students answered independently to each item would be quite lower than the observed percentage, while the probability to find SD patterns would be higher.

Results

Global results

They are presented, for implications with false premise, in Table1, and for Wason's tests, in Table 2.

Table 1. Number of students given each type of answer depending on the questions

	True	non pertinent	False	other
triangle	15	38	46	8
lights	13	33	51	10
labyrinth	29	24	48	6
$Hn \Rightarrow Hn+1$	57	17	20	13
*the teacher's sweets**	31	37	11	28

* For this item, the "non pertinent question" answer was dominantly "modifying the context in order to make the premise true"; the answer "contract not respected" is put under "assertion false".

The two first items were quite similar. For the "*labyrinth*" item, there were more "correct" answers and less "non pertinent" ones: it could be possible that some of the students missed the premise falsity because they "forgot" the rule of movement.

Table 2. Number of students given each type of answer on Wason's tasks

	Correct	non-Q missing	other errors	no answer
Wason 1	51	40	13	3
Wason 2	77	20	5	5

There were more correct answers with the Radford's version (W2) than with the Wason's selection task (W1). In fact, 43 students (40,2%) gave both correct answers; 34 correctly evaluate the procedures for testing the rule and made errors in the selection task, while only 8 were correct for W1 and made errors for W2.

Relationships between type of questions

Figure 1. Percentage of students with a correct answer to the computable assertion, depending on their patterns of answers to non-computable implications.

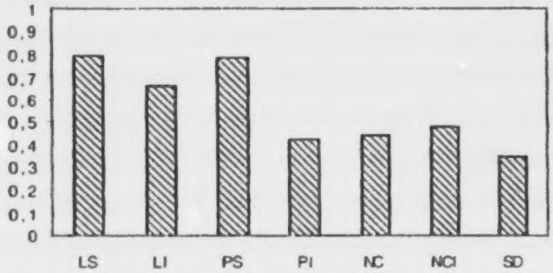
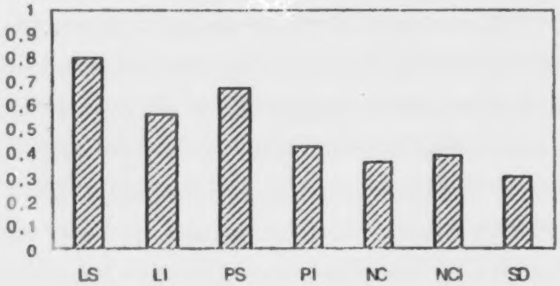


Figure 2. Percentage of students with correct responses to Wason's selection tasks, depending on their patterns of answers to non-computable implications.



Patterns of answers presented similar relationships with the correct answer to the computable implication and to the correct answers to the two Wason's tests. Both stable logic and pertinence perspectives (LS and PS) ensured success, while answers without dominant perspective (SD) led to the worst results.

However, a further analysis showed that 80% of LS students succeeded both the Wason's tests and the computable implication, while the other categories were all under 45% of success (17 % for NC, NCI and SD groups).

Mathematical and material implications versus social contract statements

Figure 3 presents the relationship between the responses to non-computable implications and the logical answer to the social contract item.

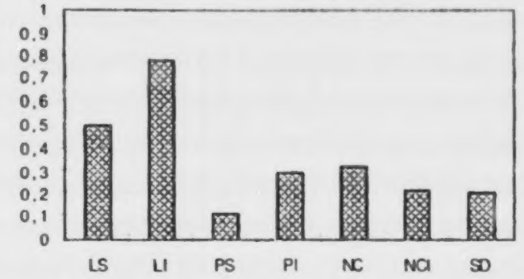


Figure 3. Percentage of students with a correct logical answer to the social contract question, depending on their patterns of answers to non-computable implications.

Logical answers to the social contract question were low (less than 30%), except for students giving logical answers (stable or incomplete) to the mathematical and material implications (globally: 63% of logical responses to the social contract item). The lowest percentage (10%) was observed for students answering with a (stable) pertinence perspective (PS).

Reasoning about false premises and mathematical success

The following table indicates the percentage (and number) of students passing the written test (first step of the competition), and the percentage of students who succeeded in this competition (and entered a teacher training institute: IUFM).

Table 3. Results to the competitive examination

	National level	Regional level	Tested students
Attending the competition	7780	638	107
Passing the written test	28,4% (2212)	28,8% (184)	38,3% (41)
Fully succeeding	12,4% (968)	12,2% (78)	18,7% (20)

With regards to their success in the national competitive examination, the students in our study were above the national and regional levels. This must be taken into account for evaluating the generality of the results.

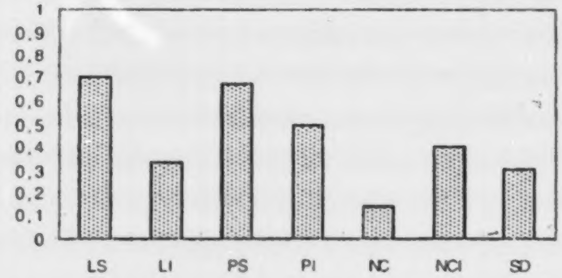


Figure 4. Percentage of students successing the written final exam, depending on their patterns of answers to non-computable implications.

Figure 4 presents the percentage of students passing the written test depending on their patterns of answers to non-computable implications. The best results were observed for students responding from a stable perspective of logic (LS) or pertinence (PS) (70% success); the worst results concerned the students with coherent non-conditional responses (NC: less than 10% success). Students presenting variability in their answers (LI, PI, NCI and SD) were slightly above the national mean (between 30% and 50%).

Discussion and conclusion

Students' answers concerning assertions with false premise and hypothesis testing, and students' success to the final mathematical show interesting relations. Both students centered on logic and on pertinence presented similar success. Only students who focus on logic answer logically to the social contract question. The model: "implication is false when premise is false" appears to be the worse for hypothesis testing and mathematical success.

The results on Wason's tasks have to be underlined: our results not only confort Tweney and Yachanin results (1985) on the fact that scientists rationally assess conditional inferences better that non scientists, but also that graduate mathematics students can do better than the university physicians members tested by Wason.

These diagnosis results are partiel elements of a more complete study (on the 20 questions of the test) which might enable us to analyse the various difficulties —and successes— of mathematical graduate students in logical processes used in mathematics.

The long term purpose of understanding how students process at a fine grained level of their mathematical logic is to identify what could be efficiently taught in this field, while keeping in mind the shared goal expressed by Thurston: the most important in not to prove but to understand mathematics.

The main issue is to improve the logical tool, in agreement with Hanna and Jahnke (1993) who defend the position that "a curriculum which aims to reflect the real role of rigourous proof in mathematics must present it as an indispensable tool in mathematics rather than the very core of that science" (p. 879).

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Annex. The logical test proposed to students consisted on 20 questions (11 mathematical ones, nine requiring no mathematical knowledge). Among them, seven asked to evaluate the value of an assertion or of a rule. The text of these questions is given below, with their number. The test began with the following forword: "Mathematical sentences, assertions, in the following exercices are often expressed in a naive, not formalised form, as it is done in an everyday mathematical text, or even in everyday conversation. It is deliberate, and you have to feel free in your answers, which may (must !) include many comments, and even propose "this question is a stupid one!". In italics the context of the analysed question.

Computable assertion with false premise

I. *Given $\lambda \in \mathbb{R}$, let be $(u_n)_n$ the sequence defined by :* $u_0 = \lambda$, $u_{n+1} = 2u_n + 1$.

Let H_n be the assertion "$u_n \leq \frac{2^n}{3} - 1$".

(a) Is the implication "$H_n \Rightarrow H_{n+1}$" true for some n ? for every n ?

(b) Compute explicitly u_n as a function of λ and n [one may write $u_n = v_n - 1$].

(c) Show that if $\lambda > -\frac{2}{3}$ all tssertions H_n are false.

(d) If $\lambda = 10$, what can be said about the assertion "$\forall n \; H_n \Rightarrow H_{n+1}$" ?

Non computable assertion with false premise: item 1

III. What do you think about the truthfulness of the following assertion:
"Every not flat triangle of the plane, whose mediatrices are not concurring, is an equilateral triangle" ?

Wason's selection task (W1)

IX. Cards are given, with a letter on one face and a number on the other one. One must test the possible rule: "behind a vowel there is an even number". For this purpose, a sample of 4 cards is disposed on the table; what is seen is represented just on the right. What card(s) is/are to be turned on in order to know is the rule is confirmed on this sample?".

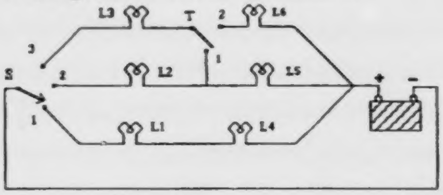
E 4 K 7

Radford's version of Wason's task (W2)

XV. A set of numbered balls are in an urn. Balls are black or white. The following question is "do all white balls in the urn get an even number?"
Four procedures are considered for answering the question:
procedure 1 : balls with an even number are taken out of the urn; then their colour is considered;
procedure 2: balls with an odd number are taken out of the urn; then their colour is considered;
procedure 3: white balls are taken out of the urn; then their numbers are considered;
procedure 4: black balls are taken out of the urn; then their numbers are considered.
For each of these 4 procedures, chose among the two options:
(a) the procedure will certainly enable me to answer the question;
(b) there is a chance that the procedure does not enable me to conclude.

Non computable assertion with false premise (Legrand's Circuit): item 2

XVII. An electric circuit consists on six identical lamps denoted L1, ... L6, and two switches S and T; S may take three positions: S1, S2, or S3, and T may take two positions: T1 or T2.
(a) What may be said about the truthfulness of the following assertion: "if L1 is turned on or if L6 is on, then L3 is on or L4 is on"?
(b) What may be said about the truthfulness of the following assertion: "If L1 is on and if L3 is on, then L2 is on and L5 is not on"?

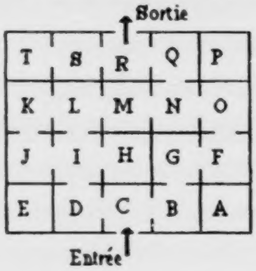


Social contract question

XVIII. In a primary school, the teacher gives a problem to the pupils and says: "search this problem at home; tomorow, if somebody was able to solve it, I will give you sweets". The following day, no pupil has been able to solve the problem. The teacher distributes sweets to the pupils. They protest: "It is not just, we were not able to solve the problem, we get no right to sweets!". Mocking, the teacher answers that she perfectly respected the contract... What are your comments about this story?

Non computable assertion with false premise (Durand-Guerrier's Labyrinth): item 3

XIX. Somebody, called X came through the labyrinth (on the right), from the entrance (entrée) to the issue (sortie), without going twice through the same door. Rooms are called A, B, ..., T.
For each of the seven following sentences, say if it is true, it is false, or if it is impossible to know:
(1) X went through T ;
(2) X went through N ;
(3) X went through M ;
(4) if X went throughO, then X went through F ;
(5) if X went through K, then X went through L ;
(6) if X went through L, then X went through K ;
(7) if X went through S, then X went through T.



INVESTIGATING THE MATHEMATICS SUBJECT MATTER KNOWLEDGE OF PRE-SERVICE ELEMENTARY SCHOOL TEACHERS

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This paper reports recent findings from a project which investigates the mathematics subject knowledge of prospective elementary school teachers, and how this relates to classroom teaching performance. The project was initiated in 1997 in the context of UK government policy to introduce subject content knowledge as an explicit dimension of the 'standards' for the award of Qualified Teacher Status in England. We present some findings about topics that trainees find difficult and show that the extent and security of their subject matter knowledge is related to their teaching competence.

BACKGROUND

Recent changes in the curriculum for Initial Teacher Training (ITT) in England incorporate a stronger focus on trainees'[1] subject matter knowledge (SMK). Notwithstanding the complex relationship between SMK and pedagogical content knowledge (PCK), there is evidence from the UK and beyond which would seem to support this shift of emphasis (Ball, 1990; Kennedy, 1991; Alexander, Rose and Woodhead, 1992; Ofsted, 1994; Simon and Brown, 1996). Government Circular 4/98 (DfEE, 1998) sets out what is considered to be the "knowledge and understanding of mathematics that trainees need in order to underpin effective teaching of mathematics at primary [elementary] level", and charges ITT providers with the audit and remediation of students' SMK.

> Providers must audit trainees' knowledge and understanding of the mathematics content in the National Curriculum Programmes of Study for mathematics at KS1 and KS2[2], and that specified in paragraph 13 of this document. Where gaps in trainees' subject knowledge are identified, providers of ITT must make arrangements to ensure that trainees gain that knowledge during the course … (DfEE, 1998, p. 48)

In this paper, we describe our approach to the audit of the mathematics SMK of 173 primary trainees in 1998-99. This was the first cohort of students following the one-year Postgraduate Certificate in Education (PGCE) course to whom the requirements of Circular 4/98 applied by statute. We had, however, piloted the audit and a draft version of the 'standards' on a voluntary basis the previous year. We present here some findings related to the trainees' knowledge and understanding of proof. Currently, there is evidence for concern in the UK about students' facility with mathematical proof, both at school and at university level (Coe and Ruthven, 1994; London Mathematical Society,

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[1] The official discourse in England refers to students undergoing pre-service preparation for school teaching as 'trainees'. In this paper, we speak of 'students' and 'trainees' synonymously.

[2] In England and Wales, Key Stage 1 (KS1) is the first phase of compulsory primary education, between the ages of five and seven. Similarly, KS2 covers ages seven to 11.

1995; Healy and Hoyles, 1998). One argument suggests that curriculum and assessment reforms in the 1970s and 1980s promoted investigational approaches to school mathematics at the same time as Euclidean point-line geometry went into decline, favouring inductive reason at the expense of deduction. One requirement of Circular 4/98 (detailed later) can be seen as an attempt to address a deficit in the current generation of prospective primary school teachers.

A number of PME papers have considered aspects of elementary teachers' SMK, such as divisibility (Zazkis, 1994), ratio (Klemer and Peled, 1998), place value (McClain and Bowers, 2000), with comment on the relevance of SMK to the professional role of their participants.

OVERVIEW OF GOALS AND METHODS

The project sets out to investigate:

1. those areas of SMK required by Circular 4/98 which prove to be problematic for significant numbers of trainees;
2. whether the expectations of Circular 4/98 are well-founded insofar as secure SMK (or otherwise) is reflected in classroom performance;
3. ways in which trainees' practice in school-based placements is informed by their SMK;
4. the process and effectiveness of SMK remediation through peer tutoring.

Some findings with respect to the first year and the first three goals of the project, incorporating trainees' ability to perceive and express generalisation, can be found in Rowland, Martyn, Barber and Heal (2000). Preliminary findings concerning the fourth goal were reported in Barber, Heal, Martyn and Rowland (1999).

The structure of the primary PGCE under consideration is such that by the middle of January, with fully six months of the course remaining, the main content areas – number concepts and operations, data handling, mathematical processes, shape and space, measures, algebra, probability – have been 'covered' in lectures and workshops, giving the trainees opportunity to recall those topics they have forgotten (for lack of use) since they did mathematics at school. A 90-minute written assessment consisting of 16 test items in mathematics is administered at this point of the course. Each trainee's response to each question includes a self-assessment of their ability to complete it successfully.

The course includes two extended 'practicum' placements in schools in the latter parts of the second and third terms. Given these and other demands of the course, the major SMK remediation opportunity comes between the first and second placements.

Other aspects of the taught course based in the University	Maths SMK audit	Other aspects of the taught course based in the University	School placement 1	SMK peer-tutoring and remediation (with other aspects of the course)	School placement 2
Term 1 (autumn)	Term 2 (spring)			Term 3 (summer)	

Table 1: The chronology of the PGCE course

From this second project cohort, 34 students who were assessed as secure in nearly all of the topics audited became mathematics peer tutors. Following training for this task, they conducted peer tutoring sessions with those students (at most two per peer tutor) who experienced most difficulty with the audit. These peer tutors wrote a feedback sheet on the post-audit progress of each of their tutees. In addition, 12 students acted in a looser 'on demand' support capacity to a self-support group of about five students. Members of these groups self-reported their progress with mathematics SMK.

During school placements, each student works under the joint supervision of a school-based mentor and a university tutor. For the purposes of the project, the two supervisors agreed on assessments of the student's performance in teaching mathematics towards the end of (and in the context of) each placement, against the standards of Circular 4/98.

TRAINEES' MATHEMATICAL THINKING: ASPECTS OF PROOF

One dimension of our research has been to identify what mathematics (within the remit of Circular 4/98) primary trainees find difficult, and the nature of their errors and misconceptions in these areas. Facilities in the four 'easiest' and 'hardest' of the 16 items in the 1998-99 audit are shown in Table 2.

HIGHEST FACILITY		
% secure[1]	Mean score[2]	TOPIC
98[3]	1.96	Inverse operations
95	1.92	Ordering decimals
93	1.90	Divide 4-digit number by 2-digit
92	1.90	Problem solving in a money context

LOWEST FACILITY		
% secure	Mean score	TOPIC
52	1.17	Pythagoras, area
49	1.15	Generalisation
33	0.87	Reasoning and proof
30	0.73	Scale factors, percentage increase

[1] The written response gives a high level of assurance of the knowledge being audited.

[2] A secure response scores 2, which is therefore the maximum possible for the mean.

Table 2: Audit topics with the highest and lowest facility ratings

The table demonstrates some striking similarities with that for the previous 1997-98 cohort (Rowland *et al*, 2000) although facility with the more difficult items is lower. We discuss here aspects of the trainees' understanding of reasoning and proof as evidenced in the audit. At another UK university, Goulding and Suggate (in press) have found, as we did, that proof is a source of particular difficulty for trainees. They add that these difficulties are particularly resistant to remediation within the span of the PGCE course. Circular 4/98 requires that trainees demonstrate:

> That they know and understand [...] methods of proof, including simple deductive proof, proof by exhaustion and disproof by counter-example (DfEE, 1998, p. 62)

The following item was designed to audit this 'standard'.

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[3] All cohort percentages have been rounded to the nearest integer.

> A rectangle is made by fitting together 120 square tiles, each 1 cm^2. For example, it could be 10cm by 12 cm. State whether each of the following three statements is true or false for every such rectangle. Justify each of your claims in an appropriate way:
>
> (a) The perimeter (in cm) of the rectangle is an even number.
>
> (b) The perimeter (in cm) of the rectangle is a multiple of 4.
>
> (c) The rectangle is not a square.

More than one mode of justification is possible for each part, and a proof by exhaustion (listing the 8 possible rectangles) would meet the requirements of all three. We anticipated some deductive arguments for (a), counterexamples for (b) and perhaps contradiction ($\sqrt{120}$ is not an integer) for (c).

Table 2 shows that only one-third of the students made a secure response to the whole question. 30% gave insecure (or blank) answers to all three parts. The percentage of secure[4] responses to the three individual parts were 59, 44 and 52 respectively. In the self-assessment referred to earlier, students indicated a lower level of confidence in their ability to tackle this question than any of the others. Only two percent declared themselves confident to do it, whereas 35% reported that they didn't think they could do it, or didn't understand it, or were too terrified to think about it.

We proceed here with further consideration of part (b) of the question, which fewer than half of the students were able to manage to our satisfaction. The difficulty with counter-examples encountered by the majority of students is consistent with the findings of Zaslavsky and Ron (1998) with top-level 9th and 10th grade school students.

Each student's response to part (b) was categorised and assigned to one of eleven codes (the first column below, Table 3). The second column describes the type of response for each code; the third column gives the percentage of students making that response.

Code 9	No response to part (b)	6%
	NOT SECURE, INCORRECT ANSWER ('true')	
Code 10	States without explanation/justification	6%
Code 11	Refers only to the 10x12 case	10%
Code 14	Because 4 divides 120 (including confuses perimeter with area)	6%
Code 15	Rectangle has 4 sides so 4 divides the perimeter	5%
Code 19	other	5%
	NOT SECURE, CORRECT ANSWER ('false') WITHOUT JUSTIFICATION	
Code 0	States without explanation/justification	3%
Code 2	Hints that a counterexample exists but doesn't give one	12%
Code 4	other	3%
	SECURE: CORRECT ANSWER ('false') WITH JUSTIFICATION	
Code 30	Gives one counterexample	34%
Code 32	Gives two or more counterexamples	10%

Table 3: Percentage responses to the proof item part (b), by code.

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[4] We describe responses as 'secure' rather than 'correct', because some *answers* were correct (e.g. that statement (a) is TRUE) but not adequately justified (e.g. "because 120 is even"). A response that we are calling 'secure' would consist of a correct true/false judgement *and* a valid justification.

Each type of response other than code 9 corresponds to a student judgement concerning the validity of the statement that the perimeter of (every) rectangle is a multiple of 4, and – codes 0 and 10 apart – to a decision concerning an appropriate means of justifying that judgement. A variety of misconceptions seem to underpin invalid arguments, and we consider some of the more interesting ones.

The fact that one third of the students erroneously believe the statement to be true resonates strongly with the findings of Zaslavsky and Ron (1998). Ten per cent of the students seemed to base this conclusion on the fact that it holds true for the 10x12 rectangle given as an example. Perhaps some of these students genuinely believed that the question required them to consider only this particular rectangle, and interpreted "every such rectangle" to mean "every 10x12 rectangle", although this raises interesting questions about when they might consider two such rectangles to be different. Such an interpretation is supported by the response of students such as the one who wrote "The perimeter is 44. 44 is a multiple of 4. 44÷4=11 or 4x11=44." The peer tutor report on another such student read "(she) did not understand what was expected … (she) read the question as though it referred to only one possibility, now sees the need to investigate further". A rather different diagnosis is implied by those students who seemed to be drawing on a 'false conservation' misconception (Lunzer, 1968) i.e that once the area is fixed, so is the perimeter. One wrote "The perimeter has to stay the same otherwise the area will change … the perimeter is always in total 44."

Some students (code 15) argued that (b) must be true because a rectangle has four sides. Again, we can only speculate from their written responses, but these suggest an epistemological orientation which views mathematics as a non-empirical discipline, one in which truth can only be arrived at – or even guessed - by appeal to deductive argument, albeit argument of a spurious kind. There is little or no sense of mathematics as an experimental test-bed, in which they might confidently respond to an unexpected student question "I don't know, let's find out." Likewise, the suggestion that the perimeter is a multiple of 4 because 120 is a multiple of 4 (code 14) seems in some cases to privilege 'argument' over evidence. In others there is a clear case of confusing perimeter with area. Thus, one student wrote "the perimeter is always 120" and another "perimeter = $a \times b$". Confusion between perimeter and area is well-researched and documented e.g. Foxman, Joffe, Mason, Mitchell, Ruddock, and Sexton (1987).

The secure responses all gave counter-examples. Ten per cent chose to give more than one counter-example, even though one is sufficient. A quarter of these described the *general characteristics* of a counter-example, such as "As the addition of the two different side lengths does not have to be an even number (if one length is odd and the other is even, it won't be) the perimeter will not necessarily be divisible by 2(2)=4." Whilst such an analysis exceeds the requirements of the refutation, it seems to point to a desire for explanation – why it is that some perimeters are multiples of 4 and others are not. If a counter-example is deemed to be a kind of proof (that not $\forall x P(x)$), then a single example might typically fall short of one of the purposes of proof – to explain (de Villiers, 1990). Just under a fifth of the 44% who successfully refuted statement (b) with one or more counter-examples actually used the word 'counter-example' in their response, exposing some awareness of the 'syntactic' structure of the discipline (Grossman, Wilson and Shulman, 1989) i.e. the nature of enquiry in the domain of mathematics, and how new knowledge is introduced and warranted.

SUBJECT KNOWLEDGE AND CLASSROOM PERFORMANCE

We move on now to data which have enabled us to build on and update our earlier findings (Rowland *et al.*, 2000) associated with the second of our project goals – investigating the relation between trainees' SMK and their teaching competence. To summarise those findings: with the first project student cohort (N=154), the level of each student's subject knowledge (based on the audit) was categorised as low, medium or high, corresponding to the need for significant remedial support, modest support (or self-remediation), or none. Towards the end of that course, specific assessments of the students' teaching of number[5] were made on the second and final school placement (against the standards set out in Circular 4/98) on a three-point scale weak/capable/strong. These data did not support a null hypothesis that the spread of performance in the teaching of number was the same for the three categories identified in the subject knowledge audit. There was an association between mathematics subject knowledge (as assessed by the audit) and competence in teaching number. Further analysis (Goodman, 1964) pinpointed the source of rejection of the null hypothesis: students obtaining high (or even middle) scores on the audit were more likely to be assessed as strong numeracy teachers than those with low scores; students with low audit scores were more likely than other students to be assessed as weak numeracy teachers. In effect, there is a risk which is uniquely associated with trainees with low audit scores.

For the second cohort considered in this paper, more extensive data from school placements enabled comparison of mathematics subject knowledge with teaching performance (a) on both first and second placements (b) with respect to both 'preactive' (related to planning and self-evaluation) and 'interactive' (related to the management of the lesson in progress) aspects of mathematics teaching (Bennett and Turner-Bisset, 1993). For reasons of space, Tables 4 and 5 below show two of the four 3 by 3 contingency tables, those for Placement 2 (N=164: nine students had withdrawn from the course), together with expected frequencies (in parentheses) based on the null hypothesis that audit performance and teaching performance are independent.

Each table has df=4, and values of χ^2 less than 9.5 support the null hypothesis against the alternative that audit performance and teaching performance are in some way linked ($p<0.05$). The χ^2 values for the preactive and interactive data are 17.8 and 13.6 respectively. In fact, the association between audit score and teaching performance was significant for each of the four analyses. These results confirm our earlier finding and point to the positive effect of strong SMK in both the planning and the 'delivery' of elementary mathematics teaching.

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[5] The restriction to 'number' rather than mathematics was a pragmatic decision determined by the fact that, in that year, the PGCE course was subjected to scrutiny by a government agency, the Office for Standards in Education. The inspectors' brief was to focus on Reading and Number.

		TEACHING PRACTICE PERFORMANCE		
SUBJECT KNOWLEDGE AUDIT		Strong	Capable	Weak
	High	**12 (8.1)**	**18 (14.1)**	**4 (11.8)**
	Middle	**20 (18.5)**	**33 (32.3)**	**25 (27.1)**
	Low	**7 (12.4)**	**17 (21.6)**	**28 (18.1)**

Table 4: Placement 2, preactive

		TEACHING PRACTICE PERFORMANCE		
SUBJECT KNOWLEDGE AUDIT		1 (strong)	2 (capable)	3 (weak)
	A (high)	**13 (8.5)**	**19 (18.2)**	**2 (7.3)**
	B (middle)	**21 (19.5)**	**42 (41.9)**	**15 (16.6)**
	C (low)	**7 (13.0)**	**27 (27.9)**	**18 (11.1)**

Table 5: Placement 2, interactive

CONCLUSION

We have chosen here to highlight the problematic nature of proof as a component of the mathematics SMK of pre-service elementary teachers, adding further weight to the doubts of Goulding and Suggate (in press) that much can be done to remedy trainees' difficulties with proof within initial training, especially given the multiple demands on them in all areas of the curriculum in an intensely pressured course. We would expect that clarity of understanding of the nature of proof and refutation in mathematics would inform the trainees' approach to questioning and enquiry with their students, and we are struck by the robustness under replication of our earlier finding (Rowland *et al*., 2000) that effective classroom teaching of elementary mathematics is associated with secure SMK at a level beyond the elementary curriculum. It may be that, even within the constraints of PGCE courses, greater priority could be given to syntactic dimensions of SMK, although inevitably this would be at the expense of substantive elements. In the light of Goulding and Suggate's comment above, we observe that the second school placement occurred *after* the remediation sessions, yet the association between classroom performance and the audit some five months earlier was maintained. It seems clear that there is need for the development of teachers' SMK as a component of longer-term continuing professional development. With this in mind, it might be more honest and realistic if the attainment of the full range of SMK standards (DfEE, 1998) were re-conceptualised as an ongoing professional process rather than a hurdle to be crossed in initial training.

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STUDENTS' SHIFTING CONCEPTIONS OF THE INFINITE THROUGH COMPUTER EXPLORATIONS OF FRACTALS AND OTHER VISUAL MODELS

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This paper presents some results from a study whose main concern was to investigate students' developing conceptions of infinity and infinite processes as mediated by a Logo-based environment or microworld. The general findings of the study indicate that the environment and its tools shaped these students' understandings of the infinite in rich ways, allowing them to discriminate subtle process-oriented features of infinite processes; it also permitted the students to deal with the complexity of the infinite by assisting them in coordinating the different elements present. The corpus of data is based on case studies of 8 individuals, whose ages ranged form 14 to mid-thirties, interacting with the microworld as pairs of the same age group.

The concept of infinity is central to calculus: infinite processes form the basis for the concept of limit; it is also present in other important areas of mathematics. This concept, however, has always been recognised as difficult and has historically been the origin of paradoxes and confusions. Fischbein et al. (1979) claim that the concept of infinity (and specifically of infinite divisibility) is intuitively contradictory. As explained by these and other authors (e.g. Waldegg, 1988), contradictory situations arise because "finitist" interpretations tend to prevail (such as the idea that the whole is always bigger than the parts), a fact recognised more than 350 years ago by Galileo[1]. Another finding (see also Nuñez, 1993 and Hauchart & Rouche, 1987) is that the context and form of representation are very influential in the type of responses the students give: if a geometric set is bounded, this may become an obstacle for its infinite quantification. It has also been argued (e.g. Waldegg, 1988; Sierpinska & Viwegier, 1989; Sacristán, 1991; Cornu, 1991) that the spontaneous conceptions and intuitions that people have of infinite processes and of infinite (mathematical) objects can become obstacles for the adequate construction of formalised versions of these concepts. However, both Fischbein et al. and Waldegg have found that formal mathematics teaching does not modify students' conceptions and intuitions of infinity. Furthermore, the areas of mathematics where infinity occurs are those that have traditionally been presented to students mainly from an algebraic/symbolic perspective, which has tended to make it difficult to link formal and intuitive knowledge. However, in a context that combines numerical and geometrical contexts through the use of algebraic language, Waldegg claims that some of the obstacles observed in previous cases seemed to have disappeared. This is an important finding which supports the idea that by building connections between different types of representations (in this case through algebraic language) some of the difficulties that arise when working in a single context can be diminished. We thus took it upon ourselves to create situations in which the learning of the infinite infinity could be facilitated by incorporating the use of the computer and the representational systems it provides, even though we are aware that attempts to use the computer for the learning of the concept of limits have pointed to difficulties (e.g. Monaghan, Sun & Tall, 1994).

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[1] Galileo, in his in his *Dialogues Concerning Two New Sciences* (1638), wrote: "difficulties arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited".

Our theoretical approach follows the 'constructionist' paradigm (see Harel & Papert, 1991) adopting the position that the construction of meanings involves the use of representations; that representations are tools for understanding; and that the learning of a concept is facilitated when there are more opportunities of constructing and interacting with, as diverse as possible, external representations of a concept. We thus postulated that at least some of the infinite processes found in mathematics, could become more accessible if studied in an environment that facilitates the construction and articulation of diverse types of representations, including visual ones.

A computer microworld to study thinking in change

Based on this premise, we built a computational set of open tools (diSessa, 1997) — a microworld —, using the Logo programming language, which could simultaneously provide its users with insights into a range of infinity-related ideas, and offer the researcher a window (see Noss & Hoyles, 1996) into the users' thinking about the infinite. The main research issue of the study was *to investigate students' developing conceptions of infinity and infinite processes as mediated by this microworld.*

The microworld provided a means for students to actively construct and explore different types of representations — symbolic, graphical and numerical — of infinite processes (infinite sequences and the construction of fractals, as explained below) via programming activities. In general, the computer setting provided an opportunity to analyse and discuss in conceptual (and concrete) terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and analytic representations.

There were many reasons for choosing Logo, which will not all be outlined here, but in particular Logo is easily accessible for programming activities (recursive programming, specially) by the students. The Logo procedures used in the microworld were generally not given in written form (except for the initial activities). They were usually programmed by the students using suggestions given by the researcher. It is important to emphasise that for all the above activities, "measurement" procedures were used for computing numerical data which could complement the visual models. Most of the time, tables were used for structuring this numerical (and algebraic) data, becoming an additional representation of the sequences under study.

The programming and exploratory activities included:

- *Explorations of infinite sequences, such as* $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, *and* $\{1/n\}$, $\{1/n^{1.1}\}$,..., $\{1/n^2\}$, *and their corresponding series*, through geometric models such as spirals, bar graphs, staircases, and straight lines, and the corresponding Logo procedures, with a complementary analysis of the numerical values. These models were chosen since they constitute a straightforward way of translating arithmetic series into geometric form (e.g. in the 'spiral' type of representation each term of the sequence is translated into a length, visually separated by a turn, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series.) Through the observation of the visual (and numeric) behaviour of the models, students were able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series. The different geometric models for the same sequence provided different perspectives of the same process.

- *Exploration of fractal figures*. These included the Koch curve and snowflake (formed by putting together three Koch 'segments'), and the Sierpinski triangle. The explorations involved the study of their recursive structures (apparent both visually and in the programming code), and dealing with apparent paradoxes at infinity, such as a finite area bounded by an infinite perimeter.

The study

The main phase of the study was carried out with 4 pairs[2] of students of varying ages and backgrounds (in total there were 4 female and 4 male students). Two of these students were as young as fourteen years of age. On the one hand we were interested in introducing younger students to infinity-related mathematical ideas. We were also interested in observing the conceptions of younger students and the ways in which they worked in the environment, in comparison to older students. The pairs were as follows: Pair 1: two 14-year old girls; Pair 2: two high-school boys aged 16 (M) and 17 (J); Pair 3: Two college students (1 male, 1 female) in their twenties in non-mathematical areas; Pair 4: Math teachers in their thirties (1 male, 1 female). Pairs 1 & 3 were not mathematically inclined, as opposed to Pairs 2 & 4. Each pair of students worked for 5 sessions of 3 hours with the microworld. The analysis of the data was carried out through detailed case studies of the interactions of each of these 4 pairs with the microworld.

The role of the researcher was that of a participant observer, suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed, yet allowing students to be in control of the explorations, giving them freedom to explore and express their ideas. Students were informally interviewed throughout the sessions but formal interviews were conducted at the beginning and end of the study.

Students' shifting conceptions of the infinite

One of the advantages of the microworld was that the *behaviour* of the process could be observed, rather than the end result as is usually the case in traditional school mathematics. Observing the behaviour, such as the rate of convergence, played a very important role for giving meaning and finding explanations as to why in a particular instance a process converged or diverged. The exploration of the behaviour was done in several ways which included the observation of the process through its unfolding visual and numerical behaviour, the possibility to compare different sequences and models, and in the case of series, coordinating the behaviour of the series with that of the corresponding sequences.

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[2]. Students worked in pairs with one computer to facilitate the sharing and discussion of ideas (simultaneously providing the researcher with insights into their thinking processes), and give them independence from the instructor. To facilitate the analysis of students' experiences, we worked with only one pair of students at a time, using a clinical interview style.

Students discovered and explored limiting (or divergent) behaviours first through the graphical representations and then carrying out a back and forth process between these representations and numeric values; only in the case of Pairs 2 & 4 was there some degree of a more traditional "mathematical" analysis of the formula. The graphical element played a role in indicating the existence of a limit when there was *visual invariance* through several stages. For example, in the fractal explorations of the Koch snowflake, the apparently invariant visual image conveyed the boundaries of the area, highlighting its independent behaviour from the infinite perimeter that delineates it. At a second level, students would use numeric values, organised into tables, to complement and confirm the observed visual behaviour and give an indication of the value of the limit or divergence of the sequences.

Below, I briefly describe some examples illustrating a few of the conceptions of the infinite shown by students during the study, and how these evolved through the explorations.

- *Intuition that if process is infinite, then it will diverge*: Throughout the study, a common intuition arose, particularly among the less mathematically oriented students (Pairs 1 & 3): the confusion that if a process is infinite then it will diverge. This intuition has been found by other researchers. Nuñez (1993) in particular, explains that the problem arises when there are several competing components (processes) present; that is, when two types of iterations of perhaps different nature (cardinality vs. measure) are confused: the process itself and the divergent process of adding terms to the sequence. Thus, in the case where infinite sums are involved the intuition appears as: "if an infinite number of terms or elements (*cardinality*) is added then the *measure* of the sum must be infinite, it must pass any preset value". For example, in an early part of the study, Pair 1 expected the line model representing the series $\sum \frac{1}{2^n}$ to grow without bounds, since an infinite number of segments were being added. She and her partner were quite surprised to see that the line got "stuck" at a length twice the initial value. Because they were convinced that the line would grow indefinitely, they attempted increasing the scale, but always got a line that eventually "got stuck". By working with the microworld in different ways, as outlined above, these students gradually found ways to make sense of how a process could continue infinitely and not grow to be infinite.

In particular, they focused extensively (as did Pair 3) on the decimal structure of the real numbers, realising that in the decimal expansion of the values under study, the number of digits would increase more and more as the sequence progressed. Thus, the infinite nature of the process was reflected in the decimal structure of the numeric values. On the one hand the infinite process was seen to take place in the "infinitely small"; also, seeing the process from the point of view of the numeric, temporarily disassociated from the geometry, allowed students to cope with the visual boundaries.

However, the misconception discussed here seems to be a deeply rooted one since it would often re-emerge, and was also observed with other students (e.g. Pair 3). It is also interesting that it particularly re-emerged after the explorations of the divergent (i.e. where the value of the infinite sum is infinite) Harmonic series $\sum\frac{1}{n}$, which is a case where the misconception would appear to be true. But generally, as the students gained more experience, this intuition would gradually lose strength and even though the intuition would often briefly re-emerge, it would be more easily dismissed than at the beginning of the study, as more meanings were constructed: e.g. the continuity of the process was found in the decimal expansion, not in the length.

- *Koch curve 'paradoxes': solving an indetermination by coordinating two simultaneous infinite processes.* Above I described how coping with a bounded infinite process requires discerning and coordinating two simultaneous infinite processes: the infinite iterative process of adding or increasing the number of terms, with the behaviour of the process itself (which could be convergent). In the Koch curve explorations we find another example involving the coordination of several simultaneous infinite processes. For some students the idea of an infinite perimeter formed by an infinite number of "zero-length" segments caused anxiety. This was particularly the case of Pair 2. One of the boys in this pair (J) was aware of the problem of having two types of processes involved in the change of the perimeter: the increase in the number of segments, and the decrease in the size of those segments. He realised that *the behaviour of the numerical values* pointed towards the perimeter becoming very large, infinite. But when they considered that the segments *at infinity* measured zero, this seemed to indicate to them that *at infinity* the perimeter would measure zero! In fact, by focusing on the latter process student J would challenge the idea of the divergence of the perimeter: "The segments are getting smaller... The perimeter cannot be infinite...". His partner had a different perspective: he focused more on how the zero-sized segments would affect the *shape* of the figure first concluding that it would become a "smooth" curve with no segments, "an infinite sequence of points". The students were of course dealing with what is formally defined as an *indetermination* (an infinite number of segments of size zero: $\infty \times 0$). The students realised it was necessary to carry out algebraic and numeric explorations to solve the paradoxes. A breakthrough came when student J became interested in how each of the factors (the rate of decrease in the size of each segment vs. the rate of increase of the perimeter in the number of segments) behaved in relationship to each other. He used numerical explorations (structured in a table) to explore the behaviour of the perimeter, verify his hypothesis, and become convinced of the divergence of the perimeter by observing that the perimeter's increase was faster than the segments' convergence to zero. Whereas limit indeterminations are traditionally solved through algebraic manipulation, in this case Jesús overcame the indetermination through analysis of the *behaviour* of each of the elements involved, observing specifically the difference in the *rate* of divergence or convergence of each of the elements and coordinating the two processes involved.

His partner (M), on the other hand, still had a conflict between what his intuitions told him, and his attempt to apply (finite) mathematics and "logical" principles ("a number multiplied by zero is zero" vs. "a number multiplied by infinity is infinite"), and his confusions would resurface during the explorations of the Koch Snowflake perimeter: *"if the number is infinite the perimeter is zero, and what will happen? That all of this will become a point!"* He is considering that at infinity the segments forming the curve would measure zero implying a sort of "collapse" of the curve into a point. His difficulty is related to the epistemological obstacle described by Cornu (1986) of "the passage to the limit" where "that which happens at infinity" seems to be isolated from the dynamic limiting process. Also, this situation is analogous to Zeno's paradoxes[3]: as in that case, there are two components present: the *number* of segments, and the *measure* of the segments. This dilemma highlights the difficulties that emerge when thinking of the infinite with the schema of the finite and which are found historically and by mathematics education researchers (as discussed at the beginning of this paper). For this student it would take a long process of (particularly numeric and algebraic) explorations and reflections to become convinced of the divergence of the perimeter and some of his doubts may not have been clearly resolved. Interestingly, this student had no conflict with bounded area of the snowflake which he attributed to the *shape* of the figure being such that the perimeter simply folds up as it increases, not letting the area grow any further.

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[3] As explained by Struik's (1967, p.43) Zeno challenged the belief that "the sum of an infinite number of quantities can be made as large as we like, even if each quantity is extremely small ($\infty \times \varepsilon = \infty$), and also that the sum of a finite or infinite number of quantities of dimension zero is zero". His arguments highlighted the difficulty of saying that the line is formed by points.

As the examples above show, the perspective adopted, and the context in which the infinite is presented, are likely to have a determinant role on how it is conceived. As David Tall (1980; p.281) points out: "our interpretation of infinity is relative to our schema of interpretation, rather than an absolute form of truth."

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MATHEMATICS TEACHING PRACTICES IN TRANSITION: SOME MEANING CONSTRUCTION ISSUES

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Abstract. *There are a number of questions still to be addressed in coming to understand teacher change in the context of mathematics education reform. In this study, we examine the impact of a change in the organisation of the mathematical content and in the pupils' engagement in activities on the teachers' instructional practices. The results show that such changes have limited effect on both teachers' management and pupils' construction of the mathematical meaning.*

Theoretical issues

Until recently, mathematics classrooms were dominated by instructional practices which view knowledge as a static body of facts and techniques that can be broken down and "passed along" by the teacher through direct teaching (such as lecturing and demonstrating) to the pupils. However, in the last two decades, the findings of research in mathematics education has raised concerns about this approach to teaching mathematics and indicated a need for a reform in instruction in this area. Towards this, a number of approaches to mathematics teaching have been suggested, such as those promoted by the Realistic Mathematics Education model (e.g. Romberg, 1997) and the NCTM's document 'Principles and Standards for School Mathematics' (2000).

These alternative approaches, despite their differences, accept the premise that mathematical knowledge cannot be handed over by the teacher: it must be constructed by the pupil. Here, the teacher is viewed as an informed and reflective decision maker who must provide contexts in which children's own attempts to make sense of new ideas are valued and supported and their current understandings acknowledged. From this perspective, the teacher can no longer be seen as the ultimate source of knowledge and truth; neither can s/he be expected to be in absolute control of the class agenda.

These new forms of instruction often take teachers far beyond their traditional and familiar roles and practices, and they raise some difficult questions: How much of their professional persona can they dare to risk? How can they be sure that what is to be learned is indeed learned? What are the conditions that determine change in teachers and how can these be nurtured? How long are changes sustained? Are changes developed in one mathematical domain carried over into new topics?

For teachers, the shift from familiar instructional practice to a reformed approach is not easily accomplished (Fennema & Neslon, 1997). While they may invoke notions of "good practice", they do so without actually carrying out the practices which are entailed (Desforges & Cockburn, 1987). Research shows that providing teachers with experiences where their own practices are challenged and opportunities to reflect on and rethink about them, has the potential to facilitate new insights and understandings of the teaching process (Aichele & Caste, 1994). Much in the same vein, Yackel (1994) argues that making aspects of their current practice problematic for teachers constitutes a first priority for changing their teaching practice.

Teaching processes cannot be easily divested of particularity without distortion. Furthermore, a thorough analysis of processes that are not stripped of particulars can provide powerful interpretations of classroom events and explanations for common dilemmas. Groves et al. (2000) quote Stigler (1998) who highlights the importance of looking at examples and "say[ing] exactly what it is …that you'd like to see changed". Moreover, they note that the lack of "exemplars of conceptually focused problematic situations" is an important constraint on a more "coherent and conceptually based teaching practice".

Based on the above considerations, we consider the examination of what happens within the mathematics classroom, by focusing on particular instances of the teaching processes, as very important in the study of changes in teaching practices. This is because we can identify specific elements of teachers' practice that change or are resistant to change in various instructional approaches. Of course, it is evident that this study should be carried out from different perspectives - epistemological, sociological, psychological and – in order to provide a rich set of insights into the issues raised when teachers transfer from one approach to another, for example, from a traditional to a reformed one.

In previous studies (e.g. Kaldrimidou et al., 2000) we examined teaching practices in current (traditional) mathematics classrooms from an epistemological point of view. That is, we looked at the way teachers use the epistemological features of the subject matter: definitions, theorems (properties) and solving, proving and validation procedures. The results showed that in the Greek primary and secondary mathematics classrooms, independently of the mathematical topic discussed (algebraic or geometric), the teaching approach used tended to treat the epistemological features of mathematics in a unified manner. This homogeneity was seen as of crucial importance in the pupils' attempt to construct mathematical meaning for themselves. We further noted a dialectic relation between the communicative pattern and the management of the mathematical content within the classroom, arguing that the observed teachers' quick shifts to different pupils (communicative aspect) does not allow the control of the flow of the meaning construction by individual pupils.

Thus, communicative aspects, as they are related to the management of mathematical knowledge, seem to play an important role in pupils' construction of mathematical meaning. In the present study, we look at these two aspects (communication and the management of mathematical knowledge) as they are realised through the teaching practices in a constructive activity-based mathematics classroom, that is, in an alternative to traditional teaching and learning setting.

The study

The data reported here come from a large project[1] which focused on the teaching of mathematics in the nine years of the Greek compulsory educational system (6–15 year olds). The study consisted of two parts: the first was concerned with the current status of mathematics teaching in Greece, while the second focused on experimentation with alternative, pupil-centred mathematics teaching approaches. In particular, with respect to the latter, the research team, in co-operation with a group of well–experienced teachers of both primary and secondary schools, designed and prepared the teaching of eight mathematical topics for five different grades (11-15 years old) within an activity-driven framework which included: (a) the construction of a set of well-structured activities for each topic (unit) aiming at the targeted mathematical ideas, (b) the preparation of the teachers so that they could respond to the demands of the activities and (c) the observation and videotaping of the lessons as well as the provision of feedback to the teachers.

In each case, the activities were sequenced and set up in a such a way as to allow: (a) a gradual but methodical and global approach to the targeted idea in accordance with the mathematical framework, (b) the constructive engagement of the pupils, according to their stage of development and (c) the formulation of a final mathematical result, in accordance with the teaching objectives of the lesson. Teachers were advised to present all mathematical activities as problems to solve, and to challenge and expect the children to: solve them in their own ways; discuss, compare and reflect on different strategies; make sense of other pupils' solutions and strategies, and formulate generalisations.

The research problem addressed here examines the impact of change in the organisation of the content of a lesson and the children's activation (through appropriately designed and mathematically focused activities) on the interplay between the communicative patterns and the management of the mathematical knowledge within the classroom. This serves a double purpose: first, it allows for an investigation into whether teachers' practices using the traditional approach derive from the organisation of the mathematical content and second, it helps in the location of the "deeper" characteristics of their practice (making the assumption that in a new teaching environment, there is an activation of the strongest features of practice considered by the teachers as "good" and effective).

More specifically, in order to locate those elements of the teaching practice which resist and which may be held responsible for the "distortion" of pupils' mathematical meanings in both traditional and reformed approaches, we focus on the way in which the management of the mathematical knowledge and the communicative aspects interact to generate the mathematical meaning, addressing the following research questions: (a) what kind of ideas and meanings regarding mathematical knowledge are encouraged by the teacher? (b) how does the course of classroom interaction hinder or favour the mathematical thinking of the pupils? and (c) how does the communicative pattern elevate or weaken the mathematical meaning constructed by the children?

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[1] The project was financed with resources from the operational plan "Education and Initial Vocational Training" of the 2nd Community Framework Support, European Commission, European Social Fund, Directorate General V.

The data collected consisted of 48 mathematics lessons (from 23 teachers) observed in various primary and secondary classes (11–15 years old) from 13 different schools in three geographical areas of the country. For each teacher, at least two 45 minutes sessions on different topics were observed; these were then videotaped and transcribed. For the present paper, the transcripts of the lessons were analysed by looking at the interplay between the communicative patterns and the management of the mathematical knowledge by the teacher in two phases of the pupils' engagement with the activities and with respect to the epistemological differentiation that is achieved. The two phases concern the management of : (a) the pupils' outcomes at the completion of the unit's activities and (b) the pupils' mathematical thinking at the completion of a unit and the generalisation of the results.

Presentation of the data and discussion

In the following, exemplary episodes from various lessons are used to illustrate the findings. The focus of the analysis is on the interplay between the communicative patterns and the management of the mathematical knowledge as realised by the teacher, with reference to the epistemological elements of the mathematics generated.

1. The pupils' outcomes at the completion of activities: All the episodes below demonstrate the way in which the teachers dealt with the pupils' ideas and solutions after the latter had completed a unit's activity. The focus of the analysis is on the communicative aspects of this manipulation in relation to the mathematical meaning that may be generated by the approach employed.

Episode 1.1. [*The size of the angles of a triangle* (11 years old)]: The activity requires the calculation of the size of the angles of a right-angled and isosceles triangle (only the right angle is marked) and it constitutes part of similar activities. The objective is to help pupils identify those characteristics of a triangle that would allow them to determine the size of its angles.

T(eacher). *Pay attention. This triangle has two characteristics. First, what type of triangle is this Nik, with respect to its angles?*

P(upil). *Right-angled*

T. *Right-angled. With respect to its sides, what type of triangle is this? Tania?*

P. *Isosceles*

T. *Isosceles. Well done Tania. That is, this triangle is right-angled and isosceles. And we know one of the angles, the right angle,. Michael?*

P. *Angles b and c…*

T. *Yes….*

P. *They are each 45^o*

T. *But why?*

P. *Ehhh.. Because ….*

T. *The triangle is ...*

P. *The triangle is right-angled and isosceles.*

The communicative features of the teacher's practice emerging from this episode are: he (a) tends to pose successive questions, often from one pupil to another, 'hunting for the correct' result, (b) breaks down the activities and the pupils' responses with questions, (c) allows very little space for the children to formulate ideas and complete their reasoning and (d) repeats or completes what he thinks important to be phrased overall. As for the mathematical meaning, in dealing with this activity, the teacher accepts a 'leap' in the reasoning concerning the justification of the 45^{o} for the equal angles (i.e. that $a+b=90^{o}$, $180^{o}-90^{o}=90^{o}$ and since $a=b$, each will be $90^{o}\div2=45^{o}$). This prevents the emergence of the general mathematical idea being targeted, that is, the justification of the result on the basis of the properties of a triangle. Only the final result appears to matter, which is related to a simple recognition process of the data (the right-angled and isosceles triangle has its acute angles equal to 45^{o}).

Episode 1.2. [*Percentages* (12 years old)]: Pupils are engaged in an activity asking for the equivalent of 36/45 as a percentage. This is part of an activity in which children learn how to turn fractions and decimals into percentages. One pupil calculates as follows: $100\div=2.22\ldots$ and multiplies the result by 36 ($=79.999\ldots$). But the teacher is looking for an accurate answer.

P. *We have found it, sir. We divided 36 by 45 and multiplied by 100.*

T. *How much did you find?*

P. *0.8 times 100 equals 80.*

T. (tries to generalise) *Thus, when we know a ratio of two quantities ...*

P. *Yes...*

T. *What operation should we perform in order to find the percentage?*

P. *Sir, division!*

T. *Division. What do we divide?*

P. *The numerators by the denominator*

T. *Very nice*

Then the teacher generalises further with x: $x/100=36/45$ and standardises the rule. In this extract, the teacher asks for an accurate solution, rejecting an approximation. In doing so, he limits the development of an interesting process of reasoning and links the result with a rule and then with a formal procedure: we divide the numerator by the denominator. As far as the communicative characteristics are concerned, the same patterns as in episode 1.1 are observed.

2. The pupils' mathematical thinking at the completion of a unit: In this part, the episodes focus on communicative issues in the teachers' treatment of the pupils' mathematical thinking in relation to the pupils' justification which is encouraged.

Episode 2.1. [The sum of the angles of a triangle (11 years old)]: Through a series of activities, pupils are invited to conclude that the sum of the angles of a triangle is 180°. After some construction and measurements, they initially estimate the sum of the three angles of special cases of triangles and then take measurements to confirm their estimations. The activity aims at leading children to a first level generalisation through a phenomenological contradiction (for either long or short sides, the sum remains 180°). The activity should be treated as a whole in order for this contradiction to emerge.

T. *George, what did you find for triangle B. Tell us.*

P. *90°*

T. *90°. Why do you say 90°? All together, eh?*

P. *All together?*

T. *Doesn't the exercise ask you for the sum of the angles?*

P. *Because they are small.*

T. *The angles are small. Right. Tell us Chris.*

P. *130° madam.*

T. *About. Why my boy?*

P. *Madam, estimated by the eye.*

T. *By the eye, right. Charoula?*

P. *180°, madam*

T. *Why 180° Charoula?*

P. *Because, for the right angle, I say 90°, for the other one which is acute, because it is very small, I say 10° and for the other one towards the right angle .. but it is acute, I say 80°.*

T. *About this, I don't know, it might be correct too. ....* Other pupils go on like this suggesting 60°, 120°, 185°, 150°, but the teacher doesn't ask for any more explanations, she simply says:

P. *Did you estimate it by the eye?*

With respect to the communicative patterns that are registered, the teacher: (a) tends to make frequently address successive questions from pupil to pupil, (b) breaks down the activities or the pupils' answers with questions, (c) provides little space for the pupils to complete their reasoning, (d) reduces the exchanges among the children, often repeats statements or completes anything she considers must be said. As for the mathematical meaning of this activity, the property targeted here (theorem: the sum of the angles) was never pushed forward, and the activity slipped into a process (estimation) of finding the result. Thus, the contradiction never emerged and the pupils missed the chance to see the necessity of resolving it by other means (checking by measuring), which would guide them to discover the common property. This also becomes apparent from the pupils' demand to be told the answer. It is clear that the children carry on considering the possibility that there are many different solutions and that it is the teacher who will provide the correct one.

Episode 2.2 [Quadrilaterals (12 years old)]: Following a series of activities, the pupils are invited to determine the characteristics of a rectangle. This last activity aims at encouraging pupils to formulate some definitions.

T. *Which are the characteristics of a parallelogram? Then what does it say? Which are the features of a rectangle?.....* A little later on

P. *The parallelogram has its opposite sides parallel and has 4 sides*

T. *Opposite sides parallel and 4 sides. But this is true for the rectangle too. Stefania?*

P. *In the parallelogram, all the opposite sides are parallel and equal*

T. (Repeats). *Do we have to add anything else? About the angles?*

P. *They are right angles.*

T. *Right angles?*

P. *Acute, obtuse maybe?*

T. *Two acute, two obtuse. Tell us Anna.*

P. *In the rectangle, all the angles are right angles and all the opposite sides ...*

T. *Do we agree?*

P. *Yes* (all together)!

T. *Let's go for the last one, Katerina.*

P. *The square has 4 sides and 4 angles, that is, it is a quadrilateral and its sides are equal and parallel; its opposite sides and its angles are right angles.*

With regard to the communicative patterns, we find the same features recorded in the previous episodes. In addition, the teacher very often repeats and immediately corrects their answers, sometimes explains himself why it was wrong; but he never reasons or asks for reasoning about a correct answer. As for the mathematical meaning, the fact that each shape is treated separately does not allow pupils to appreciate the purpose of creating definitions, that is, to identify and differentiate. Other basic deriving properties, i.e. those which result from the application of a proving reasoning are also missed.

Concluding remarks

In the current era of educational reform, teachers all over the world are being asked to transform their mathematics teaching. This transformation entails more than changing the types of problems and questions posed; it requires changes in teachers' epistemological perspectives, their knowledge of how people learn mathematics and their classroom practices.

In the study described above, examining traditional and reformed mathematics classrooms through the interaction between the communicative patterns and the management of the mathematical knowledge that shapes the generated mathematical meaning, we found that teachers tended to augment reformed instruction with traditional practices and then to modify and change activities so that they resemble past lessons. In particular, we identified some patterns of communication that were repeated and persisted despite a change in the organisation of the lesson's content and the children's activation via constructively designed and mathematically focused activities.

In the exemplary episodes analysed, and in both phases of the pupils' engagement with the activities, three distinctive features were identified: (a) succession of questions and breaking down of the targeted mathematical idea with a simultaneous quick shifting from pupil to pupil in hunting for the "right" answer, (b) reduced opportunities for exchanges among children and insufficient encouragement for reasoning and justification and (c) elevation of the correct answers, repetition of pupils' formulations, acceptance or cancellation of answers and correction of their mistakes by the teacher. All these features have also been identified in the traditional approach to mathematics teaching (see our previous studies) with the exception of the breaking down of an activity and the prevention of pupils from completing their reasoning. These latter two practices emerging in the reformed instructional settings result in a distortion of the characteristics of the content's organisation with consequences on the mathematical meaning.

Thus, we can claim that the dialectic relation between the communicative patterns and the management of the mathematical content within the classroom is confirmed and argue that (a) the communicative aspects recorded are related to the mathematical meaning that emerges in the classroom and (b) the change in the mathematical context and of the classroom's functioning do not affect significantly the way the management of the mathematical meaning is managed. This has important consequences for the development of pupils' mathematical meaning in both traditional and reformed approaches.

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WHEN TABLES BECOME FUNCTION TABLES[1]

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This study explores third-grade students' strategies for dealing with function tables and linear functions as they participate in activities aimed at bringing out the algebraic character of arithmetic. We found that the students typically did not focus upon the invariant relationship across columns when completing tables. We introduced several changes in the table structure to encourage them to focus on the functional relationship implicit in the tables. With a guess-my-rule game and function-mapping notation we brought functions explicitly into discussion. Under such conditions nine-year-old students meaningfully used algebraic notation to describe functions.

Most mathematics educators, ourselves included, tend to view data tables as function tables. But what about the students? Are they learning about functions when they fill out tables? What does it take for third grade students to treat multiplication tables, for example, as function tables? Can they use and understand algebraic notation for representing linear functions? What sorts of activities involving tables might encourage young students to focus on functional relations?

Students begin to deal with (linear) functions and (constant) rates long before they make any sense of an expression like $y = mx + b$. Certain curriculum materials embody these relations without making them explicit in algebraic notation. A multiplication table, for instance, might be thought of as an embodiment of the expression $y = mx$, where x and y are integers along the margins and m corresponds to the number in the expression "times <m> table". The question we raise here is whether children as young as nine years of age can understand functions and algebraic notation for functions.

The studies by Davydov and colleagues (1991/1969) showed that young students were able to use and understand algebraic notation such as $y = 5x+12$. However, in their studies x and y stand for unknowns. We know of no evidence from their work suggesting that students thought of the notation as expressing a multitude of ordered pairs and hence functions; and this is unlikely since problems were invariably constrained in such a way as to require that x and y take on single values. To make sure that students are contemplating *multiple* input and output values, so to speak, it is useful to consider situations where the same function is applied repeatedly. Let us look at a sales context first; then we will move to the issue of how it is embodied in a function table. These are precisely the conditions underlying our work with third graders as we explore how, in keeping with the U.S. National Council of Teachers of Mathematics (2000) Standards, algebraic reasoning and notation can become part of the elementary school curriculum.

In our approach, we treat algebra as a generalized arithmetic of numbers and quantities. Accordingly, we view the transition from arithmetic to algebra as a move from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing relations among variables. This requires providing a series of problems to students, so that they can begin to note

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and articulate the general patterns they see among variables. Tables play a crucial role in this process as they allow one to systematically register diverse outcomes (one per row) and look for patterns in the results.

Our initial steps were inspired by the findings of everyday mathematics studies. When computing the price of a certain amount of items, street sellers usually start from the price of one item, performing successive additions of that price, as many times as the number of items to be sold (Nunes, Schliemann, & Carraher, 1993; Schliemann et al., 1998; Schliemann & Nunes, 1990). If we try to understand their procedure in terms of displacements in a function table, they work down the number column and the price column, performing operations on measures of like nature, summing money with money, items with items. Vergnaud (1983) describes this strategy as a "scalar approach". In contrast, a functional approach presumably relies upon relationships between variables, often variables of different natures. The latter focuses on how one variable changes as a function of the other variable.

But when we look closely at street sellers' strategies we realize that they establish a correspondence of values across measure spaces before proceeding to the next case. The flow of thought proceeds from one measure space to the other, row by row. This is illustrated by the following solution by a coconut seller to determine the price of 10 coconuts at 35 *cruzeiros* each:

> "Three will be one hundred and five; with three more, that will be two hundred and ten. [Pause]. I need four more. That is... [Pause] three hundred and fifteen... I think it is three hundred and fifty." (Nunes, Schliemann, & Carraher, 1993, p. 19).

The street sellers' approach indeed involves a linking of a unique y-value to each value of x. It therefore captures the essential idea of a function and can constitute a meaningful and efficient strategy to solve missing value proportionality problems. In school young children also seem to prefer using scalar solutions (Kaput & West, 1994; Ricco, 1982). Scalar solutions can be a good start for understanding functions. But they are limited in scope and typically do not allow for broader exploration of the relationships between the two variables (Schliemann & Carraher, 1992, Carraher & Schliemann, 2001).

In the classroom study here reported we will look at some specific examples of how third-grade students' emerging understanding of functional relations draws upon and at the same time departs from their previous strategies for dealing with quantities and number relations.

The data come from a broader study aimed at understanding and documenting issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting (see Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 2000, 2001). Our goal was to help children build an understanding of multiplication from an algebraic point of view and as a functional relationship. To reach this goal, we designed activities that aimed at shifting the focus from scalar relations to functional relations and to general algebraic-type notational representation. Through a discussion of children's difficulties and successes, as they participate in these activities, we will explore some of the issues they face in trying to move from their intuitive approaches to a functional approach and from computations to generalizations.

The Study

We worked with a classroom of 18 third-grade students at a public elementary school in the Boston area, serving a diverse multiethnic and racial community. During the school year, we met with them once a week for a period of ninety minutes. The first six meetings were dedicated to additive relations (see Carraher, Brizuela, & Schliemann, 2000). In the seventh week, as the children were working on learning the multiplication tables, we started working on multiplicative relations. Our challenge at this point was to design situations that would allow children to understand multiplication as a functional relationship between two quantities or numbers.

We used what we knew about street sellers and young children's strategies to solve price problems as a point of departure. From our perspective, the organization of data for two related quantities in a table would provide the opportunity for children to use their own scalar strategies but would also allow us to explore with them the implicit functional relationships between two variables. The sequence of tasks we designed was presented and discussed over two weekly meetings (classes 7 and 8). The first two tasks were part of class 7 and the other four were part of class 8.

Task 1: Filling out function tables
We began by asking children to fill out the table in Figure 1. Each child received a work sheet, but we suggested that they could work in pairs and discuss their solutions, helping each other.

Figure 1: The incomplete table

Mary had a table with the prices for boxes of Girl Scout cookies. But it rained and some numbers were wiped out. Let's help Mary fill out her table:

Boxes of cookies	Price
	$ 3.00
2	$ 6.00
3	
	$ 12.00
5	
6	
	$ 21.00
8	
9	
10	$ 30.00

Most of the students in the class first appeared to treat each column, items and price, as if they were separate problems. They would fill out column one by counting by 1's and column 2 by counting by 3's. Their approach leads to correct answers but does not involve them in thinking about the general relationships between price and items. A few children related the task to the multiplication tables they were memorizing and used the latter to fill out the second column in the table.

Task 2: Different ways to go from one number to another
The remainder of this class was dedicated to an activity where the children had to find different ways to operate on a number in order to get to another (e.g., "How do you get from 2 to 8?" and "How do you get from 3 to 15?"). This activity constituted an attempt to have children exploring the multiple relationships between two numbers in a pair. We hoped that this would later help them to focus on determining the relationship in a function table.

The first and most popular solutions were additive solutions such as: To get from 2 to 8 you "add 6 to 2" or "add 2, plus 2, plus 2." As discussions developed, children also used multiplication as alternative ways to get from one number to the other.

Task 3: Focusing on any number (N)

The following week we first presented children with a multiplication table similar to the one they had worked with, except for an added "Nth" row. Our goal here was to encourage children to think about the general relationships depicted in the table. They were asked to answer: What do you think the N means? What is the price if the number of boxes is N?

Again, children easily filled in the blanks by counting by ones in the first column and counting by threes in the second. David, the instructor, asked them to explain how they found the number that corresponded to 4 and one child responded that he added four threes. For the same question regarding the second row, one child explained that it was three times two and another that she had added 4 to 2. For the Nth row, one of the students, Sara, stated: "add 3 up; 11 times 3 equals 33; N probably stands for 11." Other children also considered that N was 11 and that the corresponding value in the second column was 33.

David explained that "N stands for anything." A child volunteered, "It could be any number." After discussion and examples, three children maintained 33 as a response in their worksheets, three left the cell blank, five adopted N+N+N or NNN as their response, and seven adopted the notation 3N or Nx3. One girl wrote on her work sheet the expression Nx3 followed by the equals sign: "Nx3=".

Task 4: Breaking the columns' pattern

After noting the predominance of column-by-column solutions, we decided to introduce breaks in the table sequence (Figure 2), thus hoping to draw children's attention to the functional relationship.

Figure 2: Filling out a table and generalizing to higher values and to N

2. Here is another table. Can you fill in the missing values?

X	Y
1	3
2	5
3	7
4	9
5	
7	
8	
9	
10	
20	
30	
100	
N	

This table was more demanding since it represented a function with an additive term (x→ 2x +1). Children did not spontaneously focus upon the functional relationship and needed external help to complete the table. With help many were able to apply the rule and to complete the table.

Task 5: Developing a notation for the function

The next step was to focus on a general notation for the function. David wrote the rule nx2+1 on the board and worked with the whole class, assigning different values for n and computing the result. The same was done for 3n+2. He replaced n by different numbers, including zero and 1000, and children easily computed the output.

Task 6: Finding the rule from pairs of numbers

For the next activity, pairs of numbers were given and children were asked to find the rule that originated them. For the first trial of this new task, David wrote 3 and 6 as a first pair and 7 and 10 as a second pair. As the discussion below shows, the children found that they had to add 3 to the first number.

> David: Let's work backwards, we'll start from the numbers, and you tell me what the rule is. Can you do that?
> Student: Yes.
> David: All right. I'm going to start, I'm not going to tell you what the rule is.
> Student: You have to do it in pairs?
> David: Well, yes. Hold on. I'm going to start with three [...] Then I'm going to go to there, to six [writes "3→6"]. OK, attention.
> Sara: Can I tell you the rule?
> David: OK, and now, if I start from seven, I'm going to go to 10 [writes "7→10"]..
> Sara: Can I tell you the rule?
> David: Does anybody have the rule figured out? If I start from five, I'm going to go to.
> Sara: Eight.
> David: Yes, I'm going to go to eight. I think somebody knows the rule! Jennifer! What were you thinking? What's the rule?
> Jennifer: Plus three?
> David: Yes! If I start from n, then I have to go to what?
> Students: (Inaudible).
> David: Three?
> Student: You have to add three.
> David: I have to add three to what?
> Student: To the n.
> David: Yes, to the n. So how am I going to write that down?
> Students: N plus three.
> David: Yeah! That's the rule!
> Students: Oh, we need something harder.

The children take the rule "N becomes N+3" as applying to all three of the cases that were presented. In this way, the N stands not necessarily for one particular instance as an unknown, but as a variable in a description of the relationship between the pairs of numbers.

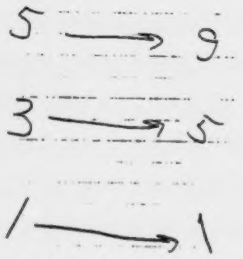
The transition from understanding letters as unknowns, to understanding them as variables is notoriously difficult, even for adolescents. However, in the present activity, with a simple additive function, they follow the idea with little trouble, as Melissa shows:

> Melissa: Yeah, because you have eleven. Well. Lets just say you have ten, then you add three more and which…I mean, you have eleven, then you have three more, equal fourteen. And say if you did it with twelve, that equals fifteen.
> David: Hold on. Twelve becomes fifteen. Yes. That's correct.
> Melissa: And, like the higher we go, the higher the numbers get.
> David: That's right. So, could you do a hundred?
> Melissa: Yeah.
> Students: A hundred and three.
> David: That's great.
> Melissa: And if you do a thousand, it's a thousand and three.

Melissa first offers the cases of 11, then 12, and then attempts to generalize: "the higher we go, the higher the numbers get". This suggests that she is referring to two *sets of numbers*, the numbers chosen ("the higher we go") as well as the numbers that emerge from applying the rule ("the higher the numbers get"). The numbers are connected one to one as ordered pairs; for each number-input there is a respective output number. But she can also mentally scan the diverse cases in an ordered fashion and think about how variations in input are related to variations in output. The numbers co-vary according to a remarkably simple pattern: as input values increase, the output values increase, with the constraint that the latter are in every case precisely three units more than the former.

In response to the children's demand to give them "something harder", David wrote the following number pairs (see Figure 3), one by one, and asked the children to guess the rule he was using.

Figure 3: Input and outputs for n→2n-1



The class discusses the possible rule that generates the output numbers from the input numbers:

> David: [As he writes 3 and 5 in the second row] OK. If I give you a three, you've got to get a five out. You think you still know? You think you know, Michael?
> Michael: Yeah.
> David: If I gave you an n [writes "n→" above the number pairs] then what, OK…
> Michael: For the first one…
> David: For the first one, how do you get from five to nine?
> Michael: Add four.
> David: You add four. And if I add four to three?
> Students: No.
> David: You could've been right. Cause that's one way to get from five to nine (adding 4). However, this rule, it can't be that rule cause it didn't work for the second one (from 3 to 5). Because if I added four, this would become seven, and it became five. Let me give you another example. If I give you a one, do you know what you're gonna get from this?

James: Oh, I know!
David: James. Let's see if he's got it.
James: You have to add two?
David: You add two? So if you add two to five you get how much?
Student: Nine.
David: No, you don't get nine.
Student: Seven.
David: Actually, it's not as hard. If I give you a one, you have to get out a one.
Student: Oh, one times one equals one.
David: One times one would be one. But five times five isn't nine.
Jessica: Sara knows!
David: Sara, give us a, clarify for us.
Sara: Two times that number minus one.
David: Wow! Wow! Sara, come here, write this up here. Write it up here, if you can generalize it.
Sara: [Writes "x 2 – 1" above the second column of numbers, following "n→"].
David: Write the n in front so we remember, n times 2 minus one.
Sara: [Completes the notation "n→n x 2 – 1"].
David: Have you guys got this figured out? Did you see what she did? So you have to use which times table, Sara? This is really something!
Students: Harder, harder.
David: Pardon me?
Students: Harder, harder.

Children who proposed additive rules such as "n→ n+2" may have been thinking only about particular cases. However Sara's rule, "n→n x 2 –1", does not merely describe the relationship between two known values but encompasses each of the cases listed. It is remarkable that she does this in the very lesson in which mapping notation is introduced.

Subsequent discussion in the same lesson showed that only Sara and a small number of her peers were able to generate such "linear function" rules from multiple instances. However, students in the class understood how the rule could account for each of the individual instances. In fact, once a student would propose a rule, other students, including those who did not themselves generate such rules, eagerly volunteered to argue whether the rule worked for the set of instances or just for isolated cases. In this sense they were able to begin to think functionally and to make use of functional notation. Furthermore, when dealing with simpler, additive functions, such as n→ n+3, most students were able to meaningfully generate and use algebraic notation for functions.

Students may not quickly learn to identify linear functions underlying data. Despite this, and perhaps *because* of this, linear functions can begin to be explored as extensions to students' work with multiplication tables. Further, even though not all third grade students will initially identify and represent the functional relationships underlying data tables, they can learn significant things in the resulting discussions and slowly work functional notation into their arsenal of representational tools.

Discussion

In introducing a data table with number of items and prices, we found that the students could correctly fill in the tables, but they did so with a minimal of thought about the invariant relationship between the values in the first and second columns. Several changes were made in the structure of the table and the purposes of the activities to discourage students from working on each column as if it were unrelated to the column next to it.

A guess-my-rule game helped students break away from the isolated column strategies they had been using. One important feature of the game was that there was no discernible downward progression from row to row. This seemed to deter students from viewing the data from a within-measure perspective.

It surprised us that the 9-year-old children were content to look for patterns and functional relations among pure numbers devoid of quantitative reference. They did not need concrete materials to support their reasoning about numerical relations and could even deal with notations of an algebraic nature. In fact, algebraic notation seemed to help them move from computational aspects to generalizations about how two sets of values are interrelated.

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A STUDY OF THE USE OF CLINICAL INTERVIEWING WITH PROSPECTIVE TEACHERS

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The purpose of this paper is to share some of the results of a 15-week study in which pre-service teachers were provided with opportunities to learn to use the clinical interview method with children and other adults. The clinical interview is often regarded as a useful method for understanding children's thinking, and the basic hypothesis of this study was that this technique could be used to help pre-service teachers develop a deeper understanding of the ways in which children learn mathematics and devise better ways of teaching it. Results indicate that the pre-service teachers revised their ideas about the teaching and learning of mathematics, and developed a deeper understanding of the ways in which children build mathematical ideas.

Introduction and Theoretical Framework:

Providing opportunities for teachers at the pre-service and in-service levels to develop insight into their student's thinking is considered to be essential if teachers are to move away from more traditional instructional approaches which emphasize rote memorization and the execution of rules and procedures, and move toward instructional practices which provide students with the opportunity to build concepts and ideas as they are engaged in meaningful mathematical activities (Fennema, Carpenter, Franke, Levi, Jacobs, and Empson, 1996; Ginsburg, 1997; NCTM, 2000; Klein and Tirosh, 2000; Schorr and Ginsburg, 2000). The teacher who has insight into student's thinking can appreciate the sense in students' interpretations and representations of mathematical ideas, and can deal with them constructively. By contrast, the teacher who lacks understanding of student's thinking may understand a concept in a certain way, and genuinely believe that he/she is teaching that concept to his/her students. That teacher often does not realize that a student may not be learning the concept at all, or may be learning an entirely different concept from the one that the teacher has assumed. In either case, there is a wide gap between the thinking of the teacher and the thinking of the student. This can lead to problematic teaching situations, where, for example, the teacher may deal with what is seen as the student's failure to learn by redoubling efforts to teach the concept (as interpreted by the teacher)-- and remains unaware that the student is in fact attempting to learn something else entirely different.

Gaps between teachers' minds and students' minds are widespread, and occur in schools ranging from preschool through university (Schorr and Ginsburg, 2000). This is not surprising, since gaining insight into students' thinking is not easy. Piaget (1976) pointed out that it will take the psychological equivalent of the Copernican revolution for the adult to realize first that the child's thinking does not necessarily revolve around or take a form similar to that of the adult's, and second that children's minds, although often radically different from the adult's, can nevertheless make their own kind of sense.

To better understand people's thinking Piaget, (1952) developed a technique known as "the method of clinical examination" modeled after psychiatrists' methods of diagnosis. Using this method, he pursued the investigation of thinking in the following way: "I engaged my subjects in conversations patterned after psychiatric questioning, with the aim of discovering something about the reasoning process underlying their right, but especially their wrong answers" (Piaget, 1952, p. 244 as quoted in Ginsburg, 1997). The clinical interview, as originally developed by Piaget (1976), is a flexible and deliberately non-standardized method of questioning, which aims at providing insight into children's ways of thinking-- into their personal "constructions"-- which are often different from the adult's. In the clinical interview, the adult poses a specific task to the child, and usually begins with some predetermined questions. However, the adult is free to modify the questions as necessary, depending on the child's apparent understanding of the questions, the child's motivation, and particularly the child's response to the initial question. The interviewer has the freedom to rephrase the questions to ensure that the child understands them, to follow up on interesting remarks, to clarify responses, and even to challenge them so as to establish the child's degree of conviction. The clinical interview method has been used as the basis of a good deal of research on children's understanding of school mathematics for many years, perhaps beginning with Davis & Greenstein (1964), and is now receiving increasing recognition as a major tool for psychological research into cognitive functioning (Ginsburg, 1997).

This research focuses on one method for helping pre-service teachers reduce the gap that exists between children's thinking and teachers' thinking—namely the clinical interview. The central hypothesis is that if used effectively, the clinical interview can be a fundamental method for helping pre-service teachers better understand students thinking, and subsequently devise better ways of teaching mathematics. This research builds upon previous work which shows that it is possible for teachers to learn the clinical interview method and to develop useful forms of it for practical implementation in the classroom (Ginsburg, Jacobs, & Lopez, 1998). That research showed that elementary level teachers from very different types of schools—inner city, suburban, and private--were able to become rather good interviewers and to develop distinctive styles of interviewing appropriate for their classrooms. For example, some teachers developed forms of interviewing individual students; others developed methods for interviewing groups of students; others integrated interviewing into their teaching; and another teacher taught her students to interview each other. Almost all teachers said that the process of learning and implementing clinical interview methods was an extremely valuable experience and indeed changed their whole approach to understanding children and teaching them. The research in this paper also builds upon prior work in a similar course where clinical interviewing was shown to be effective in helping prospective teachers begin to develop new ideas about teaching mathematics (Schorr and Ginsburg, 2000).

Methods and Procedures:

The setting for this study was a course entitled "Methods of Teaching Mathematics for Elementary and Middle School Teachers", a mathematics methods course at an East Coast university in the USA. The course had 23 prospective teachers enrolled, who met with the researcher three hours per week, for 15 weeks. As part of the course, the prospective teachers were provided with opportunities to learn the clinical interview method, interview children, and reflect on the interviews individually and collectively during classroom sessions. They were also provided with opportunities to deepen their own understanding of the mathematics that they were expected to teach, and simultaneously consider the pedagogical implications of teaching mathematics in a thoughtful manner. For example, during weekly class sessions, the prospective teachers would investigate a particular mathematical idea by solving a problem or series of problems related to the idea, generally in a group setting. They would then share their solutions, and compare the strategies chosen, the notations that were invented or selected, and the representations built. They were always challenged to defend and justify their solutions. When appropriate, they would then watch a videotaped interview[1] involving a child or series of children grappling with the same or similar mathematical ideas. During and after viewing the videotaped interview, they would share reflections about the questions posed and the interview techniques used. They would also discuss the child's mathematical thinking, and the pedagogical implications of teaching and learning the mathematical ideas. At times, they would also watch videotaped episodes of children working on similar mathematical ideas[2] and again, discuss the mathematical ideas that emerged, and the implications for teaching. Afterwards, they were encouraged to actually interview a child about the same ideas, and share the results during the next class session. In addition, the prospective teachers were assigned selective readings on the same topics as well as readings taken from state and national standards in mathematics.

All prospective teachers had to interview a child (or in some cases, an adult) every other class session. They had to record what they considered to be significant aspects of the interview (noting the exact words of the interviewer and interviewee by using a tape recorder or videotape) and discuss the overall interview, their reflections on the questions that they asked and responses of the interviewee, and the implications for teaching. Each prospective teacher therefore conducted an interview with at least one child (or adult) every other week for a total of at least seven interviews[3]. Prospective teachers also kept a journal in which they recorded their reflections on the readings, mathematical ideas, or at times, additional reflections on their own or their peers interviews. They submitted their journals on the weeks when they did not submit an interview.

The data for this research includes the written interview logs and journals of the prospective teachers, and researcher's notes.

Results and Discussion:

There are many findings to report regarding changes in the prospective teachers' ideas about the teaching and learning of mathematics and the ways in which children build the mathematical ideas that were investigated. Space limitations permit only a subset of the results to be shared in this paper. These will be limited to three main themes that consistently emerged among the prospective teachers.

Theme 1: The prospective teachers changed their views and beliefs about the teaching of mathematics.

All prospective teachers reported that as a result of interviewing children (and adults), their conceptions about teaching and learning changed. For example, consider one Journal entry submitted by a prospective teacher who shall be referred to as Juanita:

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[1] The tapes and corresponding guide are part of a series entitled "Children's Mathematical Thinking-Videotape Workshops for Educators" developed by Herbert Ginsburg, Rochelle Kaplan, and Rebecca Netley. They are distributed by the Everyday Learning Corporation.

[2] These tapes were often obtained through the Robert B. Davis Institute for Mathematics Education.

[3] Prospective teachers often volunteered to interview additional children and/or adults about an idea that interested them, or when they sought more data about a particular idea.

> Before taking this course, I had a very different idea of what it meant to teach math. I thought that teaching math would be just like it was when I went to school; the teacher did a few problems in the board and we simply had to follow the steps she/he took to come up with the answers. To me this was math, until I took this course. Now I could look back and see that even though I always had good grades, it didn't mean that I knew math. Now I realize that math is more than just finding answers. Math should involve solving problems by reasoning and understanding. After doing my interviews, I was able to see the difference between knowing how to find answers by following an algorithm and finding the answers by using our knowledge and understanding of the subject.

Juanita went on to describe how she would like her classroom to look, and the types of questions and techniques that she might employ. For example, she stated the following:

> Now I realize how important and beneficial it is that we encourage students to think and come up with their own strategies. As teachers, our goal should be to challenge students by coming up with problems…good problems [that] give students the chance to solidify and extend what they know…. As teachers we should encourage students to find the reasoning behind their solutions and also be able to prove them. It's very important that students be able to defend their answers by coming up w/ proofs because this way they will have to have more in-depth understanding of the problem. We have to help our students see that their statements need to be supported by evidence.

Other prospective teachers made similar comments regarding their beliefs about what it means to teach mathematics. One teacher, Maria, noted that she used to believe that using pictorial or concrete representations was not really "doing math", but now she feels differently. As documentation, she recorded these observations as she was interviewing a seventh grade student. The problem she had posed to the student, "Irene" had been similar to one that she and the other prospective teachers had solved and discussed the prior week in class. Maria was interested in seeing the mathematical ideas, strategies, notational systems and representations that would be used by this student. The problem, as recorded in Maria's interview log, was as follows: Juan has 3 pairs of pants and 5 shirts, how many different outfits can Juan wear, please show me your work. Maria recorded the following:

> Now I had a feeling the phrase "please show me your work" would be a problem [for Irene], because I knew it was for me when I was her age. Many teachers do not want to see drawings, they want a mathematical equation. Sometimes, if they don't receive that, the question is marked wrong. I wanted "Irene" to learn that drawing pictures is okay. I remember solving a problem using my head or drawings and having to come up with an equation, So, I would use any old equation that gave me the answer I was looking for, even if it didn't make sense…

Maria noted that Irene worked for a while and then handed her a piece of paper and said "it's probably wrong, but I think it makes sense". Maria included Irene's work in her written log. The work included five circles representing the shirts, and three rectangles representing the pants, along with an answer of 15. Maria continued her interview as follows:

> So I asked [Irene] how did she get that answer and if she could explain it to me. She tells me "I made circles for a the amount of shirts and squares for the amount of pants, then I just paired them up." At this point, she starts to draw lines from a circle to a square, but then she tells me that the lines got too confusing to count so she changes her format. This time she makes the shirts vertically and does something very interesting with the pants [Maria included Irene's written work which again included five circles--each with the word shirt written inside, but this time, each circle had three lines extended, each with a small square at the end and the word pants written inside each square.] She [Irene] tells me that the lines are easier this way (to count) and that it makes 15 outfits. So, I asked her how she figured it out and she tells me "well there's three pairs of pants, and five shirts, so I just paired up every single pair of pants with every shirt."

Maria went on to ask Irene if there could be any other answer, and Irene said that she didn't think so. Maria continued her interview by asking, "why did you say in the beginning that your answer is probably wrong?" Irene responded by saying "because I didn't show my work...". Maria noted that this student did not see "drawing pictures" as doing mathematics. As part of her written log, Maria also discussed the type of reasoning (additive and/or multiplicative) that she felt the student was using. In a subsequent class discussion, Maria used this example to underscore her desire to encourage students to draw pictures. The researcher was able to use this (and other similar examples) to provide an opportunity for Maria and the other pre-service teachers to consider the type of reasoning used by the students, and how students' representations like "drawings" could be linked to spoken words, written words, numbers, symbols, etc..

Theme 2: Prospective teachers noticed that children (and adults) often invent their own strategies when solving problems.

Juanita described an aspect of one interview in which different subjects used different techniques to solve the same problem involving multiplication (one drew figures, the other used different number combinations). This, along with the other interviews and reflections of her peers prompted her to write the following:

> After doing these interviews, I was able to see for myself how everyone has their own invented techniques to solve problems. Everyone sees their own ways as being easier to understand. As teachers, we must keep this in mind because if we teach children using only one method, our method, children may not understand our techniques, to them, our ways may seem complicated, even though we find them easy.

Juanita wrote that children must be able to "make the connections between the symbols and the meaning of the mathematical process". She noted that "as teachers we should allow students to present alternative [methods]...and then lead a discussion about why a particular [method] works or doesn't. This allows the teachers to become more aware of the diversity of different approaches that are used".

Anna, another prospective teacher, decided to interview several adults about a simple mathematical problem to see whether or not they would use a traditional algorithm or an invented strategy. The problem that Anna chose was to find the age of a woman who had a son at the age of fifteen if the son was now nineteen. Anna wrote the following in her log:

> I asked my sister and aunt [and mother] to solve the problem. I was so surprised that we each ended up with a different way of coming up with the same answer. I thought that would happen more with children.

She described their methods, along with the way that she had solved the same problem. For example, she noted that her sister solved it by taking "the tens first, which came out to twenty. She then counted the ones column. She took one from the five and added it to the nine and got ten. She then added the ten to the twenty and got thirty and then added the remaining four to thirty and got the answer thirty four." Anna included these reflections in her log:

> This interview was so interesting to me because it showed me that it isn't just children who come up with different ways of solving even a simple math problem. In the video I only saw children being asked to solve problems and I was somewhat curious how adults would respond. I thought we would be more like-minded since we're older and have gone through a lot of similar math rules, but I was shown wrong…. This has made it even more significant to me that what may be in my mind may not be what's in another's mind, and I have to take that into much consideration!

While the prospective teachers had discussed the use of different strategies in class on numerous occasions, and indeed had even shared the many different ways in which they had solved different problems, Anna still needed to experience this for herself, and the interview provided just such an opportunity. For Anna, this interview continued what the class experience could only begin—help her to realize for herself, that people can approach the same problem quite differently.

Interviews often triggered extensive in-class discussions about why particular methods worked, when and under what circumstances they would work, and how they related to more traditional approaches. This provided an additional opportunity for the prospective teachers to deepen their own understanding of the mathematical ideas.

Theme 3: Getting the right answer does not necessarily mean that the student understands the mathematics.

Perhaps one of the most important "discoveries" that most of the prospective teachers made was that students can get the "right answer" without understanding anything about why their method worked. For example, consider the reflections of another prospective teacher, Oneda. Oneda decided to interview a friend of hers who is an accountant. Her reflections included the following comments:

> I began the interview by asking "Jim" whether he felt completely comfortable with the subject of mathematics. Jim answered the question…with an emphatic "of course…what kind of accountant would I be if I was not comfortable with mathematics." I continued the interview by asking Jim…to add 3/4 plus 1/2. Jim added the fractions and got the answer 1 and 1/4. I asked Jim to explain how he got his answer. Jim mentioned in his explanation that the first thing that he did was to find the least common denominator. I then asked why the least common denominator had to be found, and after thinking for a few seconds he laughed and stated that he did not know why. He seemed to be a little embarrassed and I assured him that many people simply did not know. We then got into a discussion of how he had learned to add and subtract fractions. Jim recalls learning to add

> and subtract fractions by being given the math rule by the teacher followed by numerous worksheets. He basically memorized the rules and then repeated them on similar problems on tests. He did this all through elementary school, high school, and even college with more advanced mathematics. Jim always received good grades.

Her reflections went on to include the following comments:

> The problem is not that one should not learn these algorithms or methods and rules of solving problems. The problem lies in that this is all that is presented to children. This is truly unfair and simply cannot be called teaching mathematics. As witnessed in the videos shown in class, children are capable of much more. They can grasp many concepts that many times adults don't think they can grasp.... Limited understanding of mathematical concepts cannot be blamed on the individual or his or her intelligence level. Rather, the blame must be put on the years of bad teaching that the individual was subjected to. Perhaps even blaming bad teaching is pretty unfair. Many of the teachers that teach in this way are simply teaching in the way that they were taught. It is basically like a bad cycle. I only hope that I can be one of those teachers that helps to break the cycle.

Another prospective teacher, Inez, interviewed a middle school student on ideas relating to fractions as well. Inez noted that while this student could add fractions with like denominators, he had no idea why he needed to have a common denominator. She noted that whenever she asked this student why the denominators had to stay the same, he responded "because it just does" or "[you] just have to". She stated

> Even though he knew the rule about keeping common denominators the same, he did not know why. I noted frustration in his voice when I tried to get him to explain to me the reason behind his answer.

Inez then posed a problem to this student where the denominators were not the same (3/8 + 1/4). She recorded this:

> When adding 3/8 plus 1/4, I could tell he was confused and frustrated for forgetting the rule to solve it. He used the rule of dividing and multiplying fractions, which involved canceling out to simplify. He admitted that he was not sure that he got the correct answer. Even though he got the answer wrong, that it was due to the fact that he was thinking about which rule to use instead of really thinking about what to do. He always got good grades in math...

While the prospective teachers had spent time in class exploring ideas relating to fractions before they conducted their interviews, the interviews provided additional opportunities for them to reconsider their own understanding of the mathematical ideas, and the implications for teaching. This occurred when the interview was shared by the student and/or the researcher, and resulted in additional discussions and explorations.

Conclusions:

The prospective teachers involved in this study felt that their views of teaching and learning had greatly changed as a result of clinical interviewing. They also gained insight into the ways in which mathematical ideas can be built and represented. It would be unwise to assume that by itself, this 15-week experience could be sufficient to dramatically change the prospective teachers' approach to teaching all areas of mathematics. Clearly, the prospective teachers will need longer and deeper experiences in order to make the transformation from perspectives that can be difficult to move beyond (see Simon, Tzur, Heinz, & Kinzel, 2000). However, this study does provide documentation that clinical interviewing can help prospective teachers to consider alternative approaches to the teaching and learning of mathematics and develop an increased awareness of the ways in which people learn mathematics. In addition, the interviews also provided the impetus for challenging and thought provoking course discussions about mathematical ideas—ideas that became more personally relevant to the prospective teachers because they emerged as a consequence of their own interviews with children and adults. Taken together, the results appear to indicate that the clinical interview method can be a valuable tool in helping prospective teachers begin to see mathematics as a "collection of ideas and methods which a student builds up in his or her own head" (Davis, 1984, p. 92), and as a result, devise better ways of teaching it.

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WHAT CONSTITUTES A (GOOD) DEFINITION? THE CASE OF A SQUARE

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The notion of definition is central in the study of mathematics. Several researchers discuss the significant role definitions play in understanding mathematical concepts, in problem solving and in proving. This work deals with mathematics teachers' conceptions of a mathematical definition. Their conceptions of definition were revealed through individual and group activities in which they were asked to consider a number of possible definitions of a square. Data were collected from written questionnaires and recorded observations. The findings point to a number of perspectives underlying teachers' conceptions of an acceptable mathematical definition.

Theoretical Background

Mathematical definitions play a central role in mathematics and in mathematics education. According to Pimm (1993) "The mathematical term *definition* is one of a meta-mathematical marker terms (others include axiom, theorem, proof, lemma, proposition, corollary), terms which serve to indicate the purported status and function of various elements of written mathematics" (p. 261-262 ibid).

The following citation from Wilson (1990) expresses the motivation.for the current study:

> "Although we frequently use definitions, we rarely focus on the nature of definitions. There is little agreement on what constitutes a good definition" (p. 33, ibid).

A definition is a way to create uniformity in the meaning of concepts, it is a tool for communication among human beings, and it is a foundation for proving and problem solving. Pimm (1993) brings the following citation of Ludwig Wittgenstein when speaking about definitions "... in order to communicate, people must agree with one another about the meaning of words" (p. 272, ibid). Borasi (1992) refers to the uniformity aspect of definitions through students' thoughts regarding the use of definitions in geometry: "So we bring unity, to make things uniform..." (p. 14, ibid). Moore (1994) discusses the connections between definitions and proving. According to Moore, there are three possible ways of operating with definitions in doing proofs: (a) using definitions for generating examples; (b) using definitions for justifying steps in a proof; and (c) using definitions for planning an overall structure of a proof.

There is a body of research dealing with the role definitions play in mathematical concept formation and concept understanding. Feldman

(1972) reports on three experiments done to determine the effect of several instructional variables on concept attainment. According to Feldman, providing a rational set of positive and negative instances with a definition was significantly more facilitative in promoting concept learning than a rational set alone. According to Wilson (1990) definitions, examples and non-examples are the building blocks needed to construct mathematical concepts. Klausemeier & Feldman (1975) and Sowder (1980) suggest a model of concept learning, which includes the following steps: recognizing examples, classifying examples and non-examples, and stating a definition of the concept. Vinner (1991) presents the notions of concept image and concept definition as two cells in which the knowledge about the concept is located. He adds, "the ability to construct a formal definition is a possible indication of deep understanding" (p. 79, ibid). Moore (1994), based on Vinner's and others work, suggests a 'concept understanding scheme', which consists of a third aspect - *concept usage* - in addition to concept images and concept definitions.

In spite of the significant roles definitions play in learning and doing mathematics, many students have difficulties in understanding and using definitions. In a study dealing with classification of students' mathematical errors, Movshovitz-Hadar, Zaslavsky, and Inbar (1987) found that many of the errors students perform are related to distortion of definitions. Moore (1994) found that mathematics and mathematics education undergraduate students, who either lack the knowledge of certain mathematical definitions or do not know how definitions may be used, have difficulties in constructing mathematical proofs. Many difficulties students have in constructing meaning of a mathematical concept are related to the compartmentalization between the formal definition of a concept and the (personal) concept image (Tall & Vinner, 1981; Vinner & Dreyfus, 1989; Vinner, 1991).

Several aspects of mathematical definitions are discussed by a number of mathematics educators (Leikin & Winicki-Landmean, 2000; van Dormolen & Zaslavsky, 1999; Pimm, 1993; Borasi, 1987, 1992; Vinner, 1991; Or-Bach, 1991; Leron, 1988). Most of the aspects discussed are considered critical requirements for a mathematical definition. Thus, a mathematical definition must be: *hierarchical* (i.e., based on previously basic or defined terms), *existent* (i.e., having at least one existing instance), *noncircular*, *non-contradicting* (i.e., all conditions of the definitions may co-exist), *unambiguous*, and *independent of the representation used*. In addition, two definitions of the same concept must be *logically equivalent*. There are two aspects on which there is no consensus regarding their ultimate need, that is, it is not commonly agreed whether or not a mathematical definition must be *minimal* (i.e., economical, with no superfluous conditions or information), and *elegant* (this is the most subjective aspect that is hard to articulate). Leron (1988) and Pimm (1993) discuss another relevant feature that distinguishes between definitions: a definition can be either *procedural* or *structural* (according to Leron). In Pimm's terms, it can be either *by genesis* or *by property*. Generally, the discussions on features of mathematical definitions distinguish between mathematical requirements and pedagogical choices.

A number of recent studies, which relate to the features mentioned above, propose ways to facilitate the understanding of definitions in mathematics. Leikin & Winicki-Landman (2000) presented teachers with a number of equivalent definitions of a certain concept (e.g., *absolute value*). Each definition used a different term for the defined concept. The teachers, who were not aware that the definitions were equivalent and that all define the same concept, were asked to investigate the mutual logical relationships between every two definitions. Through these activities they developed an understanding of equivalent definitions and discussed the freedom to choose a definition from a collection of equivalent statements.

In another study Furinghetti and Paola (2000) asked students to consider two alternative non-equivalent definitions of a trapezoid. The authors point to the value of the group discussions focusing on the advantages and disadvantages of each definition. In this study, similar to the work of Leikin and Winicki-Landman, students became aware of the issue of arbitrariness of a definition and the underlying considerations in determining what definition to accept.

De Villies (1998) asked students to define quadrangles. Following their responses, activities and classroom discussions were conducted focusing on the advantages of economical definitions. Through these activities students' tendency to suggest more economical definitions increased.

The current study focuses on ways in which secondary mathematics teachers view a mathematical definition, particularly, the aspects of a mathematical definition they consider critical, from both mathematical and pedagogical points of view.

The Study

The aim of the current study was to investigate mathematics teachers' conceptions of a mathematical definition, through their justifications for accepting or rejecting specific statements as possible definitions of a certain mathematical concept.

In order to focus on the notion of definition rather than on the defined concept, it was important to use a simple and familiar concept. Thus, the square was chosen as the focal concept for the study. The research instrument consisted of a questionnaire with eight equivalent statements, all of which describe a square (see Table 1). For each statement, the participating teachers were asked to determine whether they would accept it as a definition of a square. All along they were prompted to provide justifications for their responses.

In constructing the different statements for the research instrument, attention was given to several features. The statements differed from each other with respect to whether it is minimal or not, whether it is procedural or structural, and the degree of hierarchy in the statement (see below).

As mentioned above, one of the requirements of a definition is heirarchy. However, the kind of heirarchy for a given concept may vary. We distinguish between different levels of heirarchy. For example, we can define an isosceles triangle as *a triangle that has two equal sides*. We can go one step back and define it as *a polygon with three sides, two of which are equal*. The further back we go, the degree of heirarchy decreases. The focal concept of the present study – a square – can be defined based on a rectangle or on a rhombus (this is the 1st and highest level of heirarchy), on a parallelogram (the 2nd level), on a quadrangle (the 3rd level), on a polygon (the 4th level), and so on. Figure 1 illustrates the different levels of heirarchy that are associated with the notion of a square.

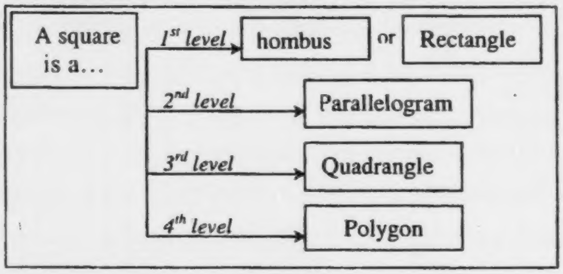

Figure 1

The statements in the research instrument appear in Table 1. The level of hierarchy in statements 2, 3, and 6 is 1, the level of hierarchy in statements 1, 5, and 8 is 3, and the level of hierarchy of statement 4 is lower than 4. However, the level of hierarchy of statement 7 is not decisive, since it relies also on the location of the notion of locus in the hierarchy of geometric concepts.

All except statement 1 are minimal statements in terms of a definition of a square. Statement 4 is the only procedural statement.

Twenty-four secondary mathematics teachers participated in the study. The participants took part in a 90-minute workshop dealing with alternative ways for defining a square. At the beginning, each teacher received a written questionnaire that contained the eight equivalent statements, and was asked to reply to it individually. Then, the teachers were divided into groups of 3-5, and were requested to discuss their answers to the written questionnaire and to try to reach an agreement. The third and last stage was a full classroom discussion, based on reports from the small groups.

Findings

For each statement, there were 4 kinds of responses: (1) accept the statement as a definition of a square; (2) do not accept the statement as a definition of a square; (3) not decisive; (4) no reply. Table 1 presents the distribution of responses to the eight statements in the questionnaire.

The Statement	Accept as Definition of a Square	Do Not Accept as Definition of a Square	Not Decisive	No Reply
1. A square is a quadrangle in which all sides are equal and all angles are 90°.	22 (92%)	1 (4%)	1 (4%)	- -
2. Of all the rectangles with a fixed perimeter, the square is the rectangle with the maximum area.	5 (21%)	15 (62%)	4 (17%)	- -
3. A square is a rhombus with a right angle.	20 (83%)	3 (13%)	1 (4%)	- -
4. A square is an object that can be constructed as follows: Sketch a segment, from both edges erect a perpendicular to the segment, each equal in length to the segment. Sketch the segment connecting the other 2 edges of the perpendiculars. The 4 segments form a quadrangle that is a square.	7 (29%)	14 (58%)	1 (4%)	2 (9%)
5. A square is a quadrangle with diagonals that are equal, perpendicular, and bisect each other.	14 (58%)	7 (29%)	- -	3 (13%)
6. A square is a rectangle with perpendicular diagonals.	18 (75%)	6 (25%)	- -	- -
7. A square is the locus of points for which the sum of their distances from two given perpendicular lines is constant.	10 (42%)	12 (50%)	1 (4%)	1 (4%)
8. A square is a regular quadrangle.	19 (78%)	2 (9%)	1 (4%)	2 (9%)

Table 1: Distribution of Teachers' Responses to the Statements

Although all the given statements are equivalent to a well-known and commonly accepted definition of a square, and all constitute a necessary and sufficient condition for a square, only 5 teachers accepted all 8 statements as possible definitions. Moreover, there was no unanimous agreement among the teachers about acceptance or rejection of any of the given statements. The percent of agreement on acceptance varied from the minimum of 21% for statement 2 to the maximum of 92% for statement 1.

There was also little agreement on the reasons for acceptance or rejection of the statements as possible definitions of a square. The written responses included 65 arguments justifying the acceptance and 54 arguments justifying the rejection of a statement as a possible definition. These arguments were classified into 7 reasons for acceptance and 7 reasons for rejection (Table 2) (it was mere coincidence that in both cases there was the same number of types of arguments). A further analysis grouped the different kinds of arguments according to their underlying perspective: Mathematical, pedagogical, both - mathematical and pedagogical, and embodied cognition. Table 2 presents the distribution of types of arguments that teachers used to support their decisions.

Underlying Perspective	Reasons for Acceptance: The statement is ...	N	Reasons for Rejection: The statement is ...	N
Mathematical	A necessary and sufficient condition for a square	24 (37%)	Not minimal	3 (5.5%)
	Equivalent to a known definition of a square	8 (12%)		
Pedagogical	Simple or clear	15 (23%)	Long or complicated	7 (13%)
	Based on students' previous knowledge	6 (9%)	Not based on students' previous knowledge	22 (41%)
	Familiar	5 (8%)	Not obvious – it requires more work in order to check	5 (9%)
Both - Mathematical & Pedagogical	A procedural description	3 (5%)	A procedural description	3 (5.5%)
Embodied			Based on properties of parts that are not integral parts of a square[*]	8 (15%)
Other	Other	4 (6%)	Other	6 (11%)
	Total	65 (100%)	Total	54 (100%)

Table 2: Arguments for Accepting or Rejecting a Statement as Definition of a Square

Note that about half the arguments (49%) for accepting a statement were based on mathematical arguments, while there was hardly any mathematical support (5.5%) for rejecting a statement. On the other hand, pedagogical considerations played a significant role both in accepting (40%) as well as in rejecting a statement (63%).

Discussion

We begin by pointing to the potential of the activity described in this paper as a vehicle for professional development, in addition to its power of eliciting teachers' conceptions of a mathematical definition and the role definitions play in teaching mathematical concepts. It is not surprising that many teachers drew on pedagogical considerations, which they are

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[*] For example, some teachers, in reflecting on their ways of thinking about a square, referred to the diagonals of a square as non-integral parts of a square (opposed to the sides and angles of a square).

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accustomed to take into account, even though they were asked to respond from their personal perspective, not necessarily as teachers.

We turn to a short discussion of the different perspectives that were identified, and offer some explanatory comments regarding each one.

The mathematical considerations teachers employed for accepting a statement as definition indicate their logical oriented view that there is a degree of arbitrariness in the choice of a definition. For them, an equivalent statement to a well-known definition, or a statement that constitutes a necessary and sufficient condition for a square, qualifies as a definition. Those who rejected a statement for mathematical reasons were convinced that a mathematical definition must be minimal (although, all of them probably teach their students the classical definition of congruent triangles that is not a minimal definition).

The pedagogical considerations that were given by the teachers indicate their expectation that a mathematical definition should be easily comprehended by students. For this reason a definition should be simple, clear, familiar, not complicated, and obvious. In addition, it should be based on students' previous knowledge. The requirement for previous knowledge may be seen as an extension and application of the mathematical hierarchy criterion to the mathematics curriculum, that is, to the order in which students learn (geometrical) concepts.

Procedural definitions seem to cause disagreement. Some teachers accepted the procedural statement from a mathematical standpoint and favored it from a pedagogical point of view, because it portrays the underlying structure of the object and lends itself well to construction of a mental image of the object. However, those who rejected the procedural statement rested mainly on mathematical grounds, and argued that a mathematical definition cannot be procedural. This view reflects the fact that procedural definitions are not very common in high school mathematic textbooks, and it is likely that many of them never came across a procedural definition before.

The last, but probably one of the more interesting considerations, is the embodied one. People are exposed to squares rather frequently in real life contexts, from early childhood. Thus, they probably conceptualize the technical mathematical concept of a square making use of their everyday concept of a square (Núñez, 2000), which appears without its diagonals. Statements 5 & 6 define a square through properties of its diagonals. For a number of teachers this was illegitimate, because the diagonals of a square are not perceived as integral parts of a square.

In this paper we reported findings of one part of a larger study dealing with what constitutes a (good) mathematical definition. Similar findings were obtained for other mathematical concepts in other groups of in-service and prospective mathematics teachers.

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HOW DO MATHEMATICS TEACHERS (INSERVICE AND PRESERVICE) PERCIEVE THE CONCEPT OF PARABOLA?

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The study described in this paper examined mathematics teachers' (inservice and preservice) knowledge regarding the concept of parabola. The participants (33 preservice and 21 inservice) were asked to perform two tasks: in the first one they were given four different verbal definitions of sets of curves, and for each definition they were asked to sketch a curve, which they believed, compatible with the definition, and to describe its properties. In the second task the participants were asked to sketch a Venn-Diagram in order to describe the logical connections between the four sets of curves, which were formed by the four definitions that appeared in the first task. All the definitions concerned the parabola.

The results show that only a few possess a full concept image concerning the parabola and thus a few of them are capable of perceiving the parabola in its algebraic as well as in its geometrical contexts or to identify links between them.

INTRODUCTION

In many countries students in ninth or tenth grade learn how to solve quadratic equations, and become familiar with quadratic functions and their graphic representations. In other words they get acquainted with the parabola as the graph of **quadratic function** i.e., as an **algebraic entity** (though they are not given any formal definition of the parabola). Later on, the parabola appears as a **geometrical entity** in analytic geometry. Although the concept is exposed in its algebraic as well as in its geometrical contexts, teachers often neglect the connections between the two, and do not initiate a discussion about the difference between the concepts in the two contexts.

THEORETICAL BACKGROUND

During the process of learning a certain concept one builds in mind a **concept image** and a **concept definition** (Tall & Vinner, 1981). A concept image is the "*total cognitive structure that is associated with a concept*" and a concept definition is the "*form of words used to specify that concept*". One might hold a concept definition that does not coincide with its mathematical definition or a concept definition that is not necessarily linked to his or her concept image. As a consequence, there is a gap between the mathematical definition of the concept and the way one perceives it. According to Hershkowitz (1990) individuals who possess poor concept images use a few prototypical examples of the concept while considering that concept. They tend to reject as examples figures that do not coincide with those prototypes, because they base their judgment upon visual properties. Individuals who possess somewhat richer concept images base their judgment upon more prototypical examples plus their mathematical properties. They try to apply the properties to the figures they are dealing with, and reject those that seem not to match them. Individuals who possess full concept images hold a wide variety of examples connected to the concept together with their properties. These individuals are able to make correct judgments based upon the analysis of the properties.

Vinner (1991) emphasized the fact that a good learning process is one that integrates concept images and concept definitions, and thus enables to distinguish between examples, counterexamples, and nonexamples of that concept.

Regarding our study, representing the parabola in both contexts induce the creation of two separate concept images and two separate concept definitions, which are different from one another. It is essential to create links between the perception of the parabola as an algebraic entity and its perception as a geometrical entity in order to create a full concept image. Otherwise students may not unify the two into one concept image and never get the complete one.

Teachers should strive to help their students to create those links. Vinner (ibid.) points out the importance of students' experience and the examples of a concept they are requested to deal with. These experiences are crucial for the formation of concept images. Since teachers' instructional foci are constrained by their own mathematical conceptions (Lloyd & Wilson, 1998, Guitiérrez & Jaime,1999), then only if the teachers themselves possess a full concept image of a certain concept, they would be able to convey it to their students. Research (e.g. Stump, 1999) had revealed the existence of a wide gap between the implementation of the recommendations for making changes in mathematics education (NCTM, 1989,1991) and the actual practice. Part of it can be referred to the use of traditional textbooks, but there is no doubt that the implementation of the intended curriculum depends mostly on the teachers' knowledge. This view motivated our study.

This paper describes the results of a study, focused on the ability of preservice and inservice teachers of mathematics to interpret various definitions that are connected with the concept of parabola and on their ability to identify links between them.

RESEARCH DESIGN

Twenty-one inservice mathematics teachers, each having at least ten years of experience in teaching high-school mathematics, and thirty-three preservice teachers who are in their third year of learning towards B.Sc. in mathematics education, participated in the study. The participants were divided into six separate groups (three of the inservice teachers and three of the preservice teachers).

Two main research questions were addressed:

1. What kind of concept image do inservice and preservice teachers of mathematics hold regarding the parabola?
2. What are the differences, if any, between the way in which the preservice and inservice teachers perceive that concept?

In order to answer these questions the participants were asked to perform two tasks. In the first one they were given four different verbal definitions of sets of curves. The first definition was the geometrical definition of the parabola, and thus compatible for every parabola. The others were algebraic definitions of subsets of the parabolas. For each definition the participants were asked to sketch a curve, which they believed, compatible with the definition, and to describe its properties. In the second task the participants were asked to sketch a Venn-Diagram in order to describe the logical connections between the four sets of curves, which were formed by the four definitions that appeared in the first task.

The definitions given in the first task were:

1. **Set no. 1:** λ_1 is an element of set no. 1 iff: Given a line l and a point F not on the line, λ_1 is the locus of points in the plane that their distance from the point F equals their distance from the line l.
2. **Set no. 2:** λ_2 is an element of set no. 2 iff: λ_2 is a graph of a quadratic function of the form: $y = ax^2 + bx + c$, where $a \neq 0$; $a,b,c \in R$.
3. **Set no. 3:** λ_3 is an element of set no. 3 iff: λ_3 is the graph of an implicit function of the form $y^2 = 2px$, where $p \neq 0$, $p \in R$.
4. **Set no. 4:** λ_4 is an element of set no. 4 iff: λ_4 is a graph of a function which its pattern is a product of two non constant linear patterns.

All the participants worked individually on the two tasks. Later on an instructed discussion was held, guided by the researchers. During that session the participants could raise questions, wonders and thoughts in a form of a dialogue or a conversation. Each discussion was tape-recorded and used for further analysis.

RESULTS

The first task

The participants' responses to the first task were first classified and then ranked as poor/non concept images, partial concept images and full concept images, based upon Hershkowitz's (1990) research. Table 1 shows the types of answers obtained, their classification and ranking, and the distribution of the answers in each group (inservice/preservice). The average level of the inservice teachers' performance was slightly higher then that of the preservice teachers' performance. Regarding the first task, it seemed like 38.64% of the preservice teachers possess full concept images as compared to 48.81% of the inservice teachers. Concerning the category that was ranked as "partial concept images" it was found that both groups possess the same concept images.

	Poor/non Concept images	Partial concept images	Full concept images
	Definition no. 1		
	A geometrical shape different from parabola (line, circle, ellipse)	Related to parabola but at the same time related to a function and depended on an axis	A parabola that is not depended on an axis
PS (N=33)	18 (54.54%)	7 (21.21%)	8 (24.24%)
IS (N=21)	0 (0%)	9 (42.85%)	12 (57.14%)
	Definition no. 2		
	—	—	A graph of a quadratic function
PS (N=33)	0 (0%)	0 (0%)	33 (100%)
IS (N=21)	0 (0%)	0 (0%)	21 (100%)
	Definition no. 3		
	A graph of non-quadratic function	Referring only to p>0	Referring to p>0 and p<0
PS (N=33)	8 (24.24%)	23 (69.69%)	2 (6.06%)
IS (N=21)	0 (0%)	14 (66.66%)	7 (33.33%)
	Definition no. 4		
	Two lines	Parabola without any constrains regarding the number of intersection points with x-axis	Parabola with at least one intersection point with x-axis
PS (N=33)	8 (24.24%)	23 (69.69%)	2 (6.06%)
IS (N=21)	0 (0%)	14 (66.66%)	7 (33.33%)
	Averages		
PS (N=33)	30.30%	31.06%	38.64%
IS (N=21)	0%	51.19%	48.81%

Table 1: Types of answers, their classification and ranking (preservice = PS, inservice = IS)

The second task

The most striking result was the fact that a high percent of the participants (39.39% of the preservice teachers and 57.15% of the inservice teachers) did not sketch any diagram. The most common mistake among the inservice teachers who tried to sketch a suitable Venn-Diagram was the misplacement of set no. 4 (23.81% of them had such a difficulty). The most common mistake of the preservice teachers (48.48%) stem from their inability to interpret correctly definition no. 1 or to identify links between that definition and the others. The results are summarized in table 2.

	Venn-Diagram			A sketch which is not a Venn-Diagram	No Sketch
	Correct diagram	Difficulties with the placement of the set of curves, which satisfies definition no. 4	Difficulties with the set of curves, which satisfies definition no. 1		
PS (N=33)	1 (3.03%)	1 (3.03%)	16 (48.48%)	2 (6.06%)	13(39.39%)
IS (N=21)	2 (9.52%)	5 (23.81%)	1 (4.76%)	1 (4.76%)	12(57.15%)

Table 2: Distribution of the results obtained for the second task (preservice = PS, inservice = IS)

ANALYSIS AND DISCUSSION

Table 1 and table 2 show that the inservice teachers' performance was better then that of the preservice teachers' on both tasks and especially on the first one. It has also been found that both groups shared similar difficulties and misconceptions (except for the poor concept images related to the first task). These findings are consistent with other studies focused on preservice and inservice teachers' mathematical knowledge (e.g. Hershkowitz, 1989, Stump, 1999, Guitiérrez & Jaime, 1999).

The first task

The results reveal a strong and clear tendency towards conceiving the parabola as an algebraic entity. It seems that **definition no. 2** is the closest one to the participants' concept images. Hershkowitz (1989) refers to that as "the prototype phenomenon". Hershkowitz (ibid.) had found that when children build their concept image they often use a prototypical example they have in their mind. They base their judgment on that prototypical example and try to impose its properties on other examples of that concept. As was mentioned, the initial examples of the parabola the students are encountered with are graphs of quadratic functions. Such graphs are elements of set no. 2. It seems that this can explain the fact that the inservice as well as the preservice teachers used those examples as prototypes. The way the participants imposed these prototype examples' properties are expressed in their responses to all the other definitions: **Definition no. 3** should have been well known to all participants since it is an integrated part of high school analytic geometry curriculum. Yet, many preservice teachers (24.24%) could not identify the obtained curve or analyze its properties. They attributed the curve function properties. Among the inservice teachers we found a remarkable phenomenon – most of them referred only to the case in which $p>0$. This finding is standing in a contradiction to their responses to definition no. 2. Those responses included lists of properties designating the various possibilities for each of the coefficients that appeared in the pattern ax^2+bx+c. It seems like this can be attributed to the fact that the common algebra books, though deal with the two possibilities, refer only to the possibility of $p>0$ while exploring the properties of the obtained curve. Since the inservice teachers have been using those books for years, it is reasonable to assume that they have developed a prototypical view towards that curve. In their responses to **definition no. 4** most participants in both groups did not mention any constrains regarding the number of intersecting points between the obtained graph and the x-axis. This finding is consistent with former findings mentioned above regarding the perception of the parabola as a prototypical algebraic entity. According to that perception, if two linear patterns are being multiplied the result has to be a quadratic pattern and thus the obtained graph is a parabola, and vice versa. In the responses to **definition no. 1**, many participants (21.21% of the preservice teachers and 42.85% of the inservice teachers) tended to exhibit again their algebraic perception, and associated the parabola the properties of a function. Among the preservice teachers there exists a fairly large group (54.54%) that was unable to identify that definition as a definition of a set of parabolas, and as a consequence they sketched different curves. The main evident difficulty is the participants' inability to identify links between the geometrical and the algebraic representation of the parabola. The mental formation of those links is not obvious since, as was mentioned, each representation bears its own concept definition and concept images. If the prototypical example is an algebraic one, there is a low probability that a full concept image would be acquired without any deliberate intervention. Evidence to the absence of mental links was also obtained from the discussions that were held. The vast majority of the inservice teachers designated that they use both definitions (in accordance to the grade levels). But at the same breath they admit that "*the geometrical definition is embarrassing since it enables us to get a parabola that is not a graph of a function*". Others express their conflict by saying that "*the parabola seems like an algebraic entity that sometimes makes problems*" or "*it is time for us to recognize the fact that we, as teachers, also perceive the parabola as a graph of a quadratic function*". The preservice teachers justified their poor performances by saying "*we didn't learn that subject*" or "*our teachers didn't discussed the subject with us*". Those "excuses" are supporting the research findings, since teachers that do not posses a full concept image cannot convey it to their students.

The second task

Table 2 shows that both groups had great difficulties in sketching a proper Venn-Diagram. 57.15% of the inservice and 39.39% of the preservice teachers did not even sketch any diagram. We believe that coping with that task caused them a certain conflict or embracement, and therefore they have chosen not to expose their lack of knowledge, though the responses were all anonymous.

Sketching a Venn-Diagram requires the ability to identify logical links between concepts. Unless one possesses a full concept image he or she would not be able to sketch that diagram correctly. Relying solely on the analysis of the first task we could have wrongly deduced that a large portion of the preservice and inservice teachers (38.64%/48.81% respectively, as seen from table 1) held a full concept image regarding the concept of parabola. The results described in table 2 point out to the fact that only 1 (3%) preservice and 2 (9.5%) inservice teachers had succeeded in sketching a proper diagram. We can confidently declare that all the participants were acquainted with Venn-Diagrams, since while they were asked to sketch a diagram describing the set of quadrilaterals all the participants could easily do this. Thus it can be concluded that the demonstrated difficulties can be attributed only to their deficiencies regarding the concept image of the parabola. One of the inservice teachers summarized that difficulty as follows: "*it was hard to tell what drags what, since they all designate parabolas, they are all the same. It is just a matter of locating the axis correctly*".

How can the wide gap between the levels of the participants' performance in both tasks be interpreted? As we know, definition no. 1 is compatible for every parabola, while the other three definitions describe only subsets of the set of parabolas. In addition, definition no. 1 forms a geometrical condition while the others form algebraic ones. Naturally, while one deals with a specific concept, he or she may use the possessed concept image and concept definition regarding that concept. While considering the concept of parabola the situation is slightly more complicated. One can consider it either from a geometrical point of view or from the algebraic one without identifying links between them. Thus, it is not always possible to decide whether one possesses a full concept image or not only by examining the interpretation he or she gives to each definition separately. Doing that might produce insufficient evidence. The concept of parabola cannot be fully understood unless the learner identifies links between the two concept images and associate them under the same cognitive structure. Creating the links between the two concept images is an **action** one has to perform in his or her mind. This analysis leads us to complete our discussion using the A.P.O. theory (Dubinsky & Lewin 1986). According to that theory the acquisition of an insight regarding a certain mathematical concept can be characterized by three sequential levels -"action", "process" and "object". Conceiving a concept as an action enables one to refer to it only in associate with its definition. Conceiving a concept as a process makes it possible for one to slightly discharge from that conjunction. At the advanced level one perceives the concept as an object and thus he or she is capable of performing manipulations on it. Producing the mentioned links is such a manipulation. Its performance is possible only as a part of one's ability to perceive the concept of parabola in both contexts (algebraic and geometrical) as an object. Performing the second task successfully demands the possession of a wide concept image, one that integrates both aspects (the geometrical and the algebraic) of the parabola.

Finally, we would like to highlight the strength of using Venn-diagrams as a tool for identifying weather one conceives a certain mathematical concept in a high level or not. While sketching a Venn-Diagram it is necessary to relate to the concept images as objects and to perform manipulations upon them. The extent of which one can successfully build a Venn-Diagram can be used as an indication to the level in which he or she perceives the concept.

CONCLUSIONS

From our study it is clear that inservice and preservice teachers do not possess a full concept image regarding the concept of parabola. Other studies (e.g. Hershkowitz, 1989, Guitiérrez & Jaime, 1999, Stump, 1999) had shown that they also do not possess full concept images regarding other concepts such as the altitude of a triangle or a slope. It seems reasonable to believe that there are plenty of other concepts in which preservice as well as inservice teachers do not possess full concept images. Additionally, we could see from our study that it cannot be assumed that the teachers' experience would influence their ability to develop full concept images by their own.

As a consequence many doubts should be raised regarding the teachers' ability to implement the reform's recommendations.

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METHODOLOGICAL PROBLEMS IN ANALYZING DATA FROM A SMALL SCALE STUDY ON THEORETICAL THINKING IN HIGH ACHIEVING LINEAR ALGEBRA STUDENTS

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Abstract

The paper gives an outline of a research on theoretical thinking in a group of 14 high achieving linear algebra students. The research was instigated by our hypothesis that one of the reasons why, even in the most 'friendly' environment, students' understanding departs in many ways from the theory is that students try to grasp the theory with a practical rather than theoretical mind. We were interested in knowing if the highly successful students in linear algebra, i.e., those most likely to have a 'good understanding' of the basic linear algebra concepts, indeed think in ways that can be characterized as strongly theoretical. The paper proposes a characterization of theoretical thinking and discusses certain quantitative methods of analyzing the students' mathematical behavior in an interview based on mathematical and epistemological questions.

Introduction: the research question

Our purpose in this paper is to briefly outline our recent research focused on theoretical thinking in a group of high achieving linear algebra students and point to some questions related to the methodology of this research.

Our research was instigated by the need we felt to refine our hypothesis that one of the main reasons why certain students' understanding of linear algebra substantially differs from the theory is that these students try to grasp the theory with a practical rather than theoretical mind (Sierpinska 2000). Taking the contrapositive form of our hypothesis, and *assuming that highly successful students in linear algebra are likely to have a 'good understanding' of the basic linear algebra concepts, we asked ourselves if the thinking of these students could be characterized as 'strongly theoretical'.*

The research of this question called for (a) a definition of theoretical as opposed to practical thinking on which our judgment of students' mathematical behavior could be based; (b) a research instrument capable of externalizing students' thinking in some observable form; (c) analytical means for the clarification of what we meant by 'strong' or 'weak' theoretical thinking tendency in a student or a group of students.

Research procedures

We interviewed 14 students who obtained A grades in the first university linear algebra course (vector spaces and linear transformations). Twelve of these students took the second linear algebra course (Jordan canonical forms and inner product spaces), and 6 of those obtaind A grades also in the second course.

Our interview was aimed at revealing what we considered, at that time, the features of theoretical thinking in the students. However, our analysis of the students' responses led us to refine our definition of theoretical thinking, and eventually we found ourselves analyzing the students' behavior against a set of criteria that was slightly different from the one that guided us in the design of the interview.

The interviews were audio-recorded and transcribed. For each student, a 'story' was then written of his or her mathematical behavior, highlighting behaviors that would attest to the theoretical or practical features of the student's thinking. These stories were revised several times, as our definition of theoretical thinking was refined. A third step was the question-by-question description of the students' behavior, first individually and then as a group. The fourth step was to dress a 'profile' of the theoretical thinking in the group of students, from the features that were the most strongly represented in the group to those that were the weakest.

Theoretical framework: a definition of theoretical thinking

Mathematics education has always been concerned with theoretical thinking in its various aspects and forms. However, there has been a renewed interest in this kind of thinking in the past years (e.g., Steinbring 1991; Boero, Pedemonte, Robotti 1997). Boero et al.'s definition of theoretical knowledge was inspired by Vygotski's distinction between scientific and everyday concepts.

So was, at the beginning, also our definition. However, the concept could be derived from such classics as Aristotle, and his epistemological categories of empirical knowledge, art, craft, productive science, and theoretical science. 'Theoretical science' aims at the development of wisdom or 'knowledge of certain principles and causes' (*Metaphysics*, Book I (A), 982a). Aristotle proposed that the motor of theoretical science is the need to know for the sake of knowing rather than knowing for the purpose of carrying out some other action (ibid., 983a). In recent mathematics education research, Steinbring's work (ibid.) refined this feature, postulating the 'self-referential' character of theoretical knowledge.

Inspired by, among others, these ideas, we assumed that theoretical thinking is a voluntary mental activity which is 'self-serving' or thinking for the sake of thinking, goal-directed, and 'self-referential' or seeking meaning and validity within itself. To support its self-referential character, theoretical thinking is *analytical* and *systemic*, i.e., it distances itself from experience by filtering it through languages and conceptual systems, which it invents and uses as tools, as well as studies as objects in their own right. Aware of the existence of these 'filters', theoretical thinking has no aspiration to certainty. It considers its claims as *hypothetical* and tries to make their assumptions as explicit as possible. Rejecting any a priori assumptions about what is plausible or realistic, it aims at identifying and discussing all possible cases and implications within a system. It is concerned with the problems of *validation* of its claims. In particular, it *questions the scientific procedures* that may have become routine in solving certain type of problems and invents alternative methodologies of research and validation. The motor of such investigations is *wonderment, and doubt* about the existing explanations, irrespective of any practical use that the obtained answers may have.

The scope of this paper does not allow us to include a justification of the relevance of the above mentioned features of thinking for the learning of linear algebra, but this justification constituted an important part of our a priori analysis of the project.

For the purposes of our analysis of the students' behavior we codified the assumed features of theoretical thinking (TT), as follows:

1. **TT is reflective,** i.e., TT1.1 self-serving, TT1.2 self referential, TT1.3 voluntary
2. **TT is analytic,** i.e., TT2.1 mediated through language which is both an object of reflection and invention; TT2.2 aware of the conventional/symbolic character of language and mathematical notations and graphical representations in particular; TT2.3 aware of the possibility of inventing/designing an artificial language; TT2.4 sensitive to syntax and mathematical syntax in particular, and especially to the quantification of variables; TT2.5 sensitive to the logical rules of drawing conclusions and negating statements.
3. **TT is systemic**, i.e., TT3.1 relational; TT3.2 has a definitional approach to meanings; TT3.3 has a systemic approach to validation; TT3.4 uses systemic categorization (see Bruner, Goodnow & Austin, 1960, p. 5-6).
4. **TT is hypothetical**, i.e. TT4.1 is aware of the conditional character of mathematical statements; TT4.2 is concerned not only with the plausible or the realistic but also with the hypothetically possible; TT4.3 believes in the relativity of truth.
5. **TT is concerned about validation**, i.e., TT5.1 is fuelled by doubt and uncertainty and hence considers validation as an important problem; TT5.2 considers proofs in mathematics as necessary for the establishment of knowledge
6. **TT has a critical attitude towards standard procedures**, i.e., it problematizes procedures and underlying concepts, does not take them for granted and does not accept them just because they have a stamp of authority.

The research instrument: the interview questions

There were seven questions in the interview. Question 1 asked the students to classify a set of 5 algebraic expressions into at least two groups according to their own criteria. Question 2 asked the students to comment on a flawed definition of linear independence of vectors. Question 3 cited a test question in which typographical mistakes were made: *Let **u**, **v**, and **w** be vectors in a vector space V over **R**. Show that the vectors **u** - **v**, **u** - **w**, and **v** + **w** are linearly dependent.* The students were asked to describe how they would approach the problem and then carry out their plan. They were expected to notice the flawed formulation and propose a correction. In Question 4 the students were shown two graphs in log-log base 2 scales, both looking like straight lines and asked if they think that these graphs represent linear functions. One of the graphs could represent a linear function and the other could not. In Question 5 the students were given 5 statements about a certain class of numbers called 'brillig numbers' (odd prime + 2) and asked to pick one they would consider as best suited for (a) a definition, (b) an explanation. They were also asked to tell if another statement about these numbers was true or false and to comment on whether they found these numbers interesting. In Question 6 a four-element set was given, T = {1, 2, 3, 4} in which an operation called 'vorpal' was defined by a Cayley table (the rows were:[[1,1,4,1], [2,2,2,4], [3,3,1,1], [4,4,4,3]]). The students were asked several questions about this operation; in particular, if the operation has a right (left) hand zero element. They were asked also if it is possible to define an operation in the set T with distinct left and right hand zero elements. Question 7 aimed at identifying the high achieving students' epistemological profile, i.e. their declared attitudes towards the basic epistemological questions related to the nature of scientific truth, the ways of arriving at scientific truth, and the ways of validating scientific statements.

Analysis of the interviews

In this report we shall focus on the third and fourth steps of our analysis and the encountered methodological problems.

We found it necessary to represent the TT features, as formulated in our definition, by features of the students' behavior in responding to the interview questions. For example, the feature TT2.5, i.e. sensitivity to the logical rules of drawing conclusions and negating statement) was assumed to underlie three kinds of behavior, which we called 'theoretical behavior' and labeled 'TB':

TB2.5a : Analytic-logical sensitivity to logical connectives - in the 'Linear dependence typo' and 'Vorpal' questions, the theoretically thinking student would correctly use and negate the connectives 'and', 'or' and 'if ... then'.

TB2.5b : Analytic-logical sensitivity to circularity in reasoning - in deciding whether the graphs in the 'log-log scales' question represent linear functions, the theoretically thinking student would not assume that the functions are given by equations of the form y = mx + b and then try to prove that the graphs represent linear functions.

TB2.5c : Analytic-logical sensitivity to implications - in the 'Brillig numbers' question, the theoretically thinking student would distinguish between a statement with the necessary and sufficient conditions for a brillig number and a statement with only a necessary condition.

This analysis led us to a list of 33 TB features. We represented our analysis of the students' responses in two tables: the Question-by-question table (Q-b-Q) and the Feature-by-Feature table (F-b-F). In Q-b-Q, each question had a rubric, in which we collected information about how the students behaved on the features assumed to be revealed in this question. The behavior of a student with respect to a TB feature was coded as a vector [a, b], where a and b could be either 1 or 0. The vector tt = [1, 0] represented the fact that the student's behavior consistently had the TB feature; pt = [0, 1] - consistently the PB feature or the practical opposite of the TB feature, tt&pt = [1, 1] - a mixture of TB and PB features, and ø = [0, 0] was assigned whenever there was no evidence of either. The behavior of an individual student through the whole question was then represented by a coordinate-wise sum of the vectors obtained for each feature observed in the question: a vector [x, y]. In order to have a measure allowing to compare the individual students' behavior on a particular question, we used the number x/(x+y)*100% which we denoted by itt%. This number was not defined if both x and y were 0, i.e. if there was no evidence of any of the TB features in the behavior of the given student in the question. Had such case occurred, we would have discarded the data about the student, or about the question, from the study. Such case had not occurred, however. In Q-b-Q, for each TB feature in a question we calculated also a group index. The students' names were listed in a column. The individual student's itt% on a question was calculated by adding the vectors in the rows of the table. The group index for each TB feature in the question was computed, similarly as itt%, but applied to the vectors in the columns of the table: if the sum of the vectors in the column was [x, y], then gtt% = x/(x+y)*100%. The gtt% for the whole question was computed by adding the summative vectors for individual students. The indeterminate case of x=y=0 could not occur because we considered only those TB features for which there was some evidence in the group.

One could ask why not describe the group's behavior with respect to a TB feature just by counting the students in the tt, pt, tt&pt and ø categories. For example, for the feature TB2.4c (Sensitivity to the form of definitions), these numbers were, respectively, 9, 4, 1, 0. For the feature TB3.2c (Definitional approach to meanings in a graphical context) they were 7, 0, 7, 0. We could say that there were less students in the tt category for TB3.2c than for TB2.4c, but we would not be able to compare the strengths of the features on just this basis, especially that there were less students in the pt category in the latter feature than in the former. On the other hand, using the gtt% indices for the features did give us some ground for comparison. For both features, gtt% = 67%, so we could say that they were equally strongly represented in the group.

To illustrate the way we computed the itt% and gtt% indiced in Q-b-Q we include the results from the 'Classification' question (Table 1). As only three features were observed in this Question, the table fits within the frame of this short paper.

In our analysis, we formulated interpretations of the outcomes of our computations. Here is a sample of such commentary:

> In Question 1, most students were not thinking of letters in algebraic expressions as names for concrete objects in concrete contexts: the group, as a whole, had a rather good analytic-representational approach to variables in linear algebra (71%). The students in the group were nearly as likely as not to be systemic in tasks of classification (65%). The group's sensitivity to the syntax of mathematical expressions was low.(38%).

In trying to grasp the students' theoretical behavior in the whole interview using Q-b-Q, we decided not to sum up the horizontal scores and then use an index similar to itt%, because such counting could privilege those TB features that were observable in more than one question over those that could be seen in one question only. A better statistic was, we thought, the average of the itt% obtained by a student in the seven questions. The list of average itt% for the students in the same order as in the table was [37, 36, 84, 50, 72, 67, 70, 56, 87, 62, 67, 70, 57, 74].

	Q. 1	'Classification'					Question 1 totals		
	TB2.2a		TB2.4a		TB3.4a				IttQ1%
O1	0	1	0	1	0	1	0	3	0
O2	0	1	1	0	0	1	1	2	33
O3	1	0	1	0	1	0	3	0	100
O4	1	0	0	1	1	0	2	1	67
V1	1	0	0	1	1	1	2	2	50
V2	1	0	1	0	0	1	2	1	67
V3	1	0	0	1	1	0	2	1	67
V4	1	0	1	0	1	0	3	0	100
S1	0	1	0	1	1	1	1	3	25
S2	1	0	0	1	1	0	2	1	67
S3	0	1	1	0	1	0	2	1	67
S4	1	0	0	0	1	0	2	0	100
N1	1	0	0	1	1	1	2	2	50
N2	1	0	0	1	1	0	2	1	67
	10	4	5	8	11	6	26	18	61
	gtt2.2a	71	gtt2.4a	38	gtt3.4a	65	gttQ1%	59	(aver)

Table 1: The Question 1 part of the Q-b-Q table.

The rubrics of F-b-F table organized along the TB features. If a feature was observable in several questions, it was counted only once. This analysis treated the interview as a whole, not as a set of questions. In this case, we thought it reasonable to compute the summative itt% index, adding the vectors horizontally. The list of summative itt% scores for the students was [45, 45, 84, 53, 76, 75, 62, 57, 86, 60, 74, 64, 56, 74]. Knowing the grades in the second course of the 12 students' who took both courses ([70, 50, 90, 63, 70, 85, 80, 60, -, -, 85, 77, 90, 85]), we could compute the correlation between these grades and the two lists of itt% scores, i.e., the average itt% from Q-b-Q and the summative itt% from F-b-F. The correlation was 0.71 in the former case and 0.67 in the latter. While both were quite high, it appeared that the average itt% from Q-b-Q was a better predictor of a student's success in the second linear algebra course. A question for us was why it would be so. We thought that perhaps the reason was that the Q-b-Q results were structurally similar to the results of a test, where the students' performance is also evaluated on a question-by-question basis.

The analysis in F-b-F served different purposes in our study. This table appeared to give a better picture of the students as a group, independently of their individual successes in the courses. It allowed ranking the features from the strongest to the weakest and raising questions concerning the instruction in the courses. If a TT feature was weakly represented in the group it may mean that it is not a necessary condition for the success in the course. The question is, however, if this feature can be considered as negligible from the point of view of educational objectives and values.

Taking the averages of the gtt% indices obtained for the categories of 'reflective', 'analytic', 'systemic', 'hypothetical', 'concern with validation' and 'critical' features of TT, we obtained the following ranking. The strongest feature in the whole group of students was systemic thinking (gtt% = 74%). This feature was even stronger in the group of the 6 students who achieved As in both courses (80%). Next best in the whole group was reflective thinking (72%), but the high achievers in both courses scored only 67% on reflectiveness. This implied, in particular, that these students were approaching problems more as students, trying to be effective in producing a solution that would satisfy the interviewers, than as theoretical thinkers whose aim would be to understand the problem and its implications for the sake of knowing alone. The next ranking feature in the high achievers in both courses was the concern for validation and a spontaneous need for proof (76%, compared to 65% for the whole group). Analytic thinking was also quite strongly represented in this subgroup (73%, compared to 66% for the whole group). Next came the critical approach to procedures (67%, compared to 59% for the whole group), and reflectiveness was the weakest, as mentioned above.

The question of the necessary conditions of high achievement led us to taking into account not only the itt% and gtt% indices but also the numbers of students in the tt, pt and tt&pt categories. If, for a given TB feature, one or more of the high achievers in both courses scored pt = [0,1], would this mean that the feature is unnecessary for high achievement? There were 11 such features. However, in many of these features, there was only one student in the pt category or (boolean 'or') the gtt% index was high. Perhaps only those features should be considered as unnecessary for the high achievement for which the number of students in the pt category was more than 3 *and* the gtt% was low? In this case there would be only two such unnecessary features: (a) approaching a problem not just as a student seeking to produce an acceptable solution, but with the goal of understanding the problem as part of a domain of knowledge; (b) being sensitive to mathematical terminology and trying to be articulate in formulating and communicating ideas. Ironically, these two features are perhaps those about which we care the most as mathematics educators.

Questions for further research

The quantitative approach we used in our research raises many questions. One of them is: would the ranking of the theoretical thinking features in the group of students be different if a different index of the strength of a feature was used? Indeed, if we look at the numbers of students in the tt, pt, tt&pt, and ø categories and apply a method of 'implicative analysis' suggested in the work of Gras (1992) then a slightly different ranking is obtained. Which of these rankings better reflects the theoretical tendencies in the group? Which one should we trust more? But, even more fundamentally, to what extent can we trust the outcome of any one of these methods in view of the fact that the assignment of the tt, pt, tt&pt, and ø vectors to individual students was based on our subjective interpretation of their behavior? These and similar questions led us to a certain skepticism about the relevance of statistical analyses in the domain of advanced mathematical thinking. We recognized, however, that, without the moderate quantitative approach that we used in our research, all we could do would be to tell 14 x 7 fascinating stories about the 14 students' behavior on the 7 questions, flavored perhaps with epistemological and historical reflection. But maybe these stories could have a bigger impact on our teaching of linear algebra than the dry numerical data? The question remains open: What is the relevance of each type of research?

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THE AESTHETIC IS RELEVANT

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Many would agree that we need to make more mathematics relevant and interesting to students, yet most recommendations for increased relevance have ignored the aesthetic dimension of student interest and cognition. In this paper, I argue that the aesthetic dimension plays a central role in determining what mathematics is personally or epistemologically relevant to children. I present an example of a learning environment that attempts to explore this dimension—both mathematically and pedagogically—and then briefly describe a small study that examined the responses of middle school students to this environment.

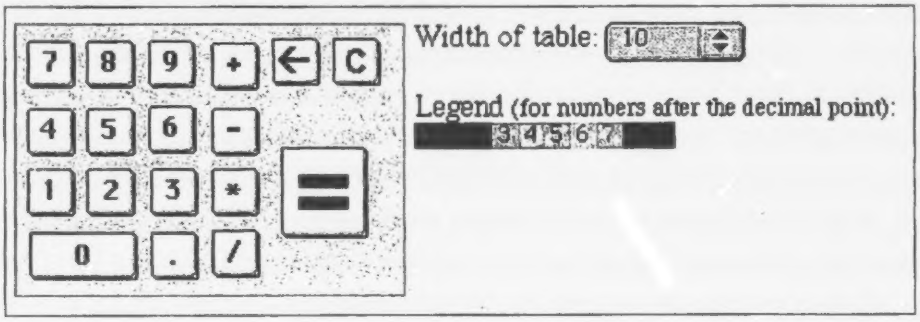
Recently, researchers have argued that all human abstract thinking is metaphorical, based on our sensory-motor experiences (Lakoff and Johnson, 1999) and that humans possess an innate aesthetic sensibility that acts as one of our primary meaning-making capacities (Dissanakye, 1992; Wilson, 1998). This conception of aesthetics is not limited to the formal, detached, and objective judgements of beauty and elegance. Rather, an aesthetic response is a cognisance of fit, of structure or order, perceived in part as being intuitive and recognised at an emotional level as being pleasurable. It is reflective in the sense of resulting from an awareness of the perceiver in relation to the environment. The role of the aesthetic in mathematics has been explored by many mathematicians (e.g. Penrose, 1974; Poincaré, 1956 Tymoczko, 1993). The emerging picture is that aesthetics is involved in: (a) motivating the choice of certain problems to solve; (b) guiding the mathematician to discovery; and (c) helping a mathematician decide on the significance of a certain result (Sinclair, 2000). In a challenge to traditional epistemologies, some researchers have argued that the aesthetic is in fact a mode of cognition used by scientists and mathematicians (Burton, 1999; Papert, 1978; Weschler, 1978). Based on these claims, and more generally on aesthetics' perceived role in learning (Dewey, 1933; Eisner, 1985), a growing number of educators have argued that aesthetic considerations should be of primary importance in children's learning of mathematics (e.g. Brown, 1993; Silver 1994; Whitcombe, 1988). However, adequate understandings of how aesthetic considerations play into mathematics learning have yet to be developed.

I propose that aesthetically rich learning environments enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies, and to experience pleasure and pride. They are characterised by two facets; first, they legitimise students' expressions of innate sensibilities and subjective impressions—they "work with" such perceptions rather than exclude or deny them. Second, they uphold Dewey's (1933) sense of the fourfold interests of children: communicating, finding things out, making things, and expressing themselves artistically. These dual facets—of perception and of action—permit children to become absorbed in and identify themselves with some object or idea, to become interested.

I wanted to explore the possibility of creating an aesthetically rich learning environment that would make accessible to middle school students the pattern possibilities of the real numbers and to explore the potential of students' aesthetic engagement with the much abhorred topic of fractions and decimals. I therefore developed an internet-based calculator[1] designed to facilitate the visualisation and manipulation of real numbers.

Description of the Colour Calculator

The Colour Calculator (CC) is a regular calculator that produces numerical results, but that also outputs its results in a colour-coded table. Conventional operations are provided; the division operation allows rational numbers while the square root operator allows irrational numbers. Each digit of the result corresponds to one of ten distinctly coloured swatches in the table, as shown in Figure 1. The calculator operates at a precision of 100 decimals digits, and thus each result represented by a (long) decimal string and an array or matrix of colour swatches. It is also possible to change the dimension, or the width, of colour table.

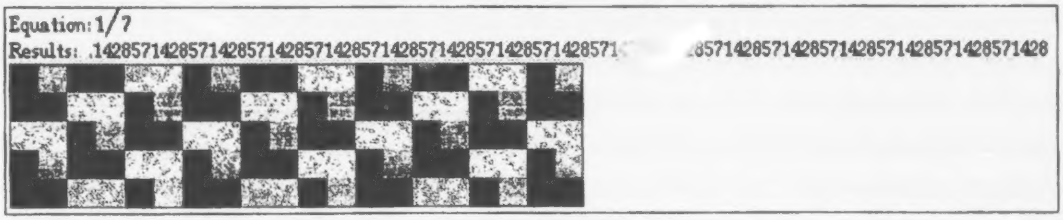


Width of table: 10

Legend (for numbers after the decimal point): 3 4 5 6 7

Figure 1. The Colour Calculator

In this following screenshot, 1/7 has been typed into the calculator with the table width set at 20. The associated table of colour has been generated:



Equation: 1/7

Results: .1428571428571428571428571428571428571428571 8571428571428571428571428571428

Figure 2. The Colour Calculator outputting 1/7

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[1] The calculator is part of a larger project called Alive Maths (on-line at http://math.ai.iit.nrc.ca) which, as an internet -based environment is not only platform independent and freely accessible by students both at school and at home, but allows students to chronicle—in writing and through activity recording—their discoveries and questions on their own personalised web pages. This work was funded by SchoolNet through the National Research Council of Canada.

Using the button that controls the width of the table of colours, the student can create different colour patterns, some which highlight different aspects of the number's period.

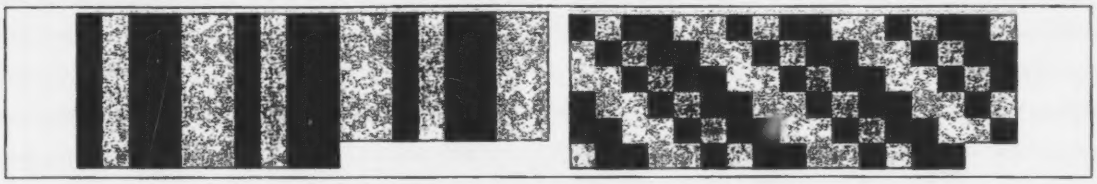

Figure 3. Different tabular representations of 1/7: widths 18 and 17

I designed the environment with three hypotheses in mind:

1. The pattern-rich table of colours patterns resulting from rational number calculations would surprise and engage the students.
2. The students' sensitivity to visual patterns would prompt and facilitate their sense-making of some characteristics and relationships of rational numbers
3. The CC would provide a setting in which students could develop more positive relationships with fractions and decimals.

The structure of the CC mathematical environment emphasises two aspects of learning. The first is to encourage students to make sense of mathematical ideas such as fractions, which they often find almost repelling, using some of their aesthetic sensitivities such as symmetry, repetition, rhythm, and pattern. This type of sense-making is part of the cognitive processes that students use to understand the form and meaning of objects and ideas. The second is to facilitate a process—one of exploration, research, and discovery—that potentially gives rise to a chain of sensory and emotive responses (Dewey, 1934).). This process is initiated by surprise (or novelty) and ambiguity. It culminates in the grasping of new knowledge that has been experientially developed.

Methodology

I conducted structured task-based interviews with 15 middle school students, 8 male and 7 female, of mixed ability (as rated by their regular classroom teacher). The students were all in grade 8, and came from lower to middle class, small town backgrounds. The interviews were task-based in that each student worked through a mathematical task using the CC as I observed and asked questions. They were structured in the sense of my facilitating the problem-posing and problem-solving process for each student. Each interview began with the student reading the instructions for the CC out loud. The student then started on the task while I asked a series of questions designed to elicit some of her thought processes as well as to guide her through the exploration. I occasionally intervened to provide guidance, following a set sequence of prompts that were only given when I judged that the student could no longer progress either in identifying a problem or solving it. The interview continued until the student had concluded at least one exploration; that is, until the student had resolved one problem. Following this, I asked each student to reflect on their experience, first asking them what they thought about what they had just done, then asking them to compare what they had just done with their other mathematics activities, and finally asking them how they felt about the open-ended nature of the activity. Each interview lasted between 20 and 30 minutes.

The interviews were all audio-taped and then transcribed. In addition, while interviewing, I kept notes of their facial and bodily reactions as they interacted with the environment, particularly at the beginning when they tried their first fraction and also when they were approaching the resolution of their problem.

Findings

I will discuss each of the three hypotheses presented above about how students would interact with the CC environment. At an obvious and almost trivial level, every student expressed that they had never seen fractions or decimals like this, together and with colours—many of them realised for the first time that a fractional and its corresponding decimal are the *same*[2], one student noting "you never see them together like this." Every student also expressed how different this type of mathematical activity was from their regular classroom work, one explaining that "you actually have to *do* things" while another observed that "you have to *notice* things."

Of the 15 students I interviewed, thirteen of them showed obvious physical signs of surprise, which they expressed either through one or more of the following actions: widening their eyes, sitting upright or moving forward, making a sound such as "ooh," or saying some form of "wow." One student, whom I will call Nadia, showed no physical surprise at all, and answered "I don't see anything" when I asked her what she saw in the table of colours. Nadia was either completely insensitive to the patterns in the table or, because of her timidity and lack of confidence, she may have been under too great of an affective barrier to even attempt to engage. The other student who showed no physical reaction was Cameron, a very ends-oriented student, who remarked flatly: "there are lots of colours and patterns there."

Of course, initial surprise is only desirable if its effect is to engage the student in sense-making; that is, if it prompts the student to try to understand something about what they are seeing. This was easiest to observe with the more articulate students who provided a running commentary of their thought processes, like Sean:

> Okay. Ah. It looks like an abstract painting. Not exactly like a math problem. I'm trying to figure out how it calculates that. Uh. Well, it says that the results are 0.142857 and it repeats. So this is a repeating pattern. I can see it because the red sticks out and the purple, and ooh the green. They kind of go in a

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[2] It is interesting in fact to recall that a regular calculator replaces its input with its output so that a student calculating 1/7 on the calculator never actually sees both the fraction 1/7 and its decimal expansion simultaneously. Though I am sure that the students think that a fraction and its decimal are equal, they seemed struck by an ontological equivalence.

> diagonal which shows a standard repeating pattern but I'm trying to figure out how things are working. So the number corresponds to the colour…

There were a few other students who provided such spontaneous descriptions of their thought processes, but most of the students had to be prompted to share their thoughts and perceptions. All the students quickly made the connection between the table of colours and the colour legend (Figure 1), and between the decimal number and the table of colours. A few of the students failed to see the connection between the fraction and the table of colours, needing some further experimentation to be able to conceive of them as the same number. However, beyond suggesting these obvious relationships, I wanted to know whether the CC environment would produce a generative engagement: for example, would the students wonder *why* the 1/7 fraction produced the table of colour or *why* the table showed the patterns it did?

I judged a student to be generatively engaged if, after their initiation to the CC, they made observations or took actions that indicated an emerging question or conjecture. For example, Ann's observation that "every seventh box is a purple" indicated a conjecture that the period of 1/7 is 6, and was followed by her experimentation with the width of the table (which, perhaps not surprisingly, she first tried at 7 before realising she really wanted 6). Sean's immediate experimentation with 1/3, then 1/2 indicated an emerging question of how other fractions will contrast with 1/7. Julie took a slightly different approach by experimenting first with the width of the table of colours, describing a width of 7 as "it's like a staircase" and a width of 3 as "it's doubled up," indicating an emerging question about the types of possible patterns. She went on to characterise diagonal patterns as those that were one more or less than the width that makes the colours of the table line up. Four other students each embarked on explorations similar to the three described above.

The other students either paused, waiting for instructions or guidance, or asked me whether I wanted them to make the colours line up (as was suggested in the instructions to the activity). These students, either because of their ends-oriented approach, their lack of confidence, or a lack of interest in the activity, did not quickly become generatively engaged. Four students required some guidance and prompts, as if they needed to know what was interesting or significant enough to pursue. After they had formed a question or conjecture, they were able to experiment and all but one of them added a personal variation to their experimentation. For example, Robert started by following my prompt of figuring out what kinds of numbers are non-terminating, but then decided to investigate what kinds of fractions gave solid tables of colours, discovering that n/9 (for $0 < n < 9$) would always give a solid table in the colour corresponding to n.

I now turn to my second hypothesis about whether the students' sensitivity to visual patterns and engagement would prompt and facilitate their sense-making of some of the characteristics and relationships of rational numbers. There were two types of sense-making exhibited by the students. The first type was around the characteristics and relationships of rational numbers that these students had encountered or "already learned" in their regular mathematics classes. The second type was around the characteristics and relationships of rational numbers that were new to them, and mediated by the CC environment.

Of course, not all the students made the same inquiries and discoveries; In fact, the wide range of inquiries and discoveries made by the revealed much of the students' existing understanding of fractions and decimals. Within the first type of sense-making, the majority of the students realised, some to a greater extent than others, that fractions aren't just the canonical 1/2, 2/3, 3/4, 1/10 numbers they have often encountered during "fraction class," but that they can have a denominator greater than 10, and that they can even be *any* integer over *any* integer, as Steve's question shows: "You mean I can put any number on the bottom?" Several students also expressed surprise at seeing the fraction and the decimal at the same time—as I mentioned above—and seemed to gain a new understanding of their equivalency, as Alice concluded: "they mean the same number." Related to this understanding of equivalence, a few of the students became intrigued with trying several equivalent fractions to see what the table of colours would depict, allaying any small doubts they were having that 1/2, 5/10, 20/40 were really the same number. A few of the students were somewhat fluent at the outset with decimals (i.e., knowing that 1/2 is 0.5 and that 1/3 is 0.33…) but most of the students seemed to have very little sense of which decimal would result from a given fraction, even with fractions whose denominators were multiples of 10. This is perhaps due to the situatedness of their fraction-decimal knowledge in classroom worksheets but it would be interesting to see what impact their brief exposure to fraction-decimal pairs has on their future classroom work with fractions and decimals. These findings highlight some of the basic conceptions with respect to fractions and decimals that students rarely have a chance to develop, yet that are almost assumed to be part of their ability to operate on fractions, convert them, and estimate them.

I now turn to sense-making of the second type. Since many of the students experimented with changing the width of the table, they were able to see what the period of a fraction is, how long the period of 1/7 is, and how any multiple of the period of the fraction makes the colours in the table line up. These are not typically the kind of rational number characteristics and relationships taught in school curricula, but are ones that were both accessible and interesting for this group of students in the CC environment. Other than making these common realisations, the students embarked on quite individual investigations. The different investigations were entirely student generated in that I only proposed questions during the interviews that had already been posed by other students in this study. Here is an incomplete list of the topics explored by the students, to various degrees of generality: What values of the width of the table would create diagonal patterns? What values of the denominator yield non-terminating decimals? What values yield terminating decimals? How is the period of the fraction related to its denominator? When does the decimal only start repeating after a certain point? What kinds of numbers neither terminate nor repeat? How can you get a solid red (or blue or green) table of colour? What is the effect when you square a fraction that has a certain period?

This environment certainly prompted the students to make new understandings of fractions and decimals and in particular, to explore characteristics and relationships they are not usually encouraged to explore. The CC environment highlighted some of the incomplete fraction and decimal understanding that students have and allowed them to gain a new understanding of what a fraction *is*, as opposed to what you can do to fractions—add them, generate equivalent ones, etc. Additionally, the CC appears to be an environment in which students are interested and motivated to discover certain things about numbers, using fractions and decimals, that are different than what is emphasised in current school curricula. The ideas explored by these students are not easier than the ones we typically emphasise, but, in this CC environment, they are perhaps more relevant to students' personal and epistemological interests.

This brings me to my third hypothesis, of whether the CC would provide a setting in which students could develop a more positive relationship with fractions and decimals. The only data I collected that is useful in verifying this hypothesis is the students' reflections at the end of their interviews. In these reflections, I asked them how they compared what they had just done with their usual mathematics activities. I found it difficult to determine the cause of their unanimous beliefs that this environment provided them with a more positive experience. Comments such as it's "fun because you can work with patterns," or "good because it helps you out more," or "creative because you can make patterns" or "fun because you don't just have to look at numbers" suggest that the colourful patterns were enjoyable but do not ascertain whether the students have a different relationship with fractions and decimals now than they did before. Some students may also have had positive experiences just because they like working on the computer or because they like having an adult's attention and help. And still, for others, the fact that they weren't set up for failure at the outset (as often is the case in mathematics class) may have made their experiences more enjoyable. That this third hypothesis remains unclearly substantiated is due both to the paucity of data in this particular study and partly to the methodological challenges of assessing students' emotional responses.

Conclusions

A majority of the 15 students called upon their aesthetic modes of cognition to explore and make sense of the visual patterns depicted by the CC. An even larger majority of the students initially became engaged either through surprise, novelty, or perceptual attraction, prompting them into a varying degree of sophisticated mathematical meaning making about fractions and decimals. These are promising findings given that each student had less than half an hour to interact with this aesthetically rich learning environment.

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CHALLENGING A PURELY MATHEMATICAL PERSPECTIVE ON TEACHERS' COMPETENCE

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This paper has two main parts. First it describes some recent trends in research on mathematics teachers (sections 2 and 3). One trend concerns the development of relatively concrete pieces of methodological advice; another is to re-conceptualise the mathematical knowledge that teachers need. Both these trends are based on analyses of school *mathematics and of classroom interactions. Second, I present a study of three novice teachers (section 4). One conclusion of this study is that the teachers' activity is often directed at broader educational motives than facilitating the students' learning. Therefore the efforts at redefining relevant mathematical qualifications for teachers should not be pursued at the expense of general pedagogical emphases in teacher education. Rather there is a need to ground both general pedagogical perspectives, pedagogical subject matter knowledge and mathematical qualifications in analyses of classroom practice.*

1: AN EXTENDED VIEW OF TEACHING PRACTICE

Current initiatives in mathematics education present the teacher with very different challenges in comparison with 20 years ago. I have previously argued that the role of the teacher in reform classrooms - inspired by fallibilism, constructivism, and socio-cultural theory - may be summarised as one of *forced autonomy* (Skott, 2000): The teacher is required to manoeuvre autonomously and independently in order to support individual students and to orchestrate small-group and whole class discourse so as to facilitate individual and collective conceptual development by balancing communal involvement in processes of for instance experimenting, conjecturing, reasoning, generalising, formalising, and refuting with more traditional teaching-learning processes. In this sense the teacher has been recognised to move to centre stage of curriculum enactment and is expected to become involved in instantaneous decision-making on the basis of his or her reflective activity. For instance (s)he is required to

- develop and flexibly use a wide range of experientially and mathematically rich tasks and contexts, some of which have open beginnings and/or open ends;
- to interpret the students' current understanding and potentials for learning and to decide on the types of support needed by individuals and groups of students;
- consider when and how to introduce small-group interaction, either to provoke individual cognitive disequilibria through social interaction or to pave the way for the creation of small communities of mathematical practice;
- capitalise on students' contributions to the discourse by instantaneously evaluating their pedagogical and mathematical potential and if appropriate to involve the rest of the class to become involved in developing taken-as-shared concepts, procedures, and meta-mathematical understandings.

The majority of the teacher decisions corresponding to these requirements cannot be made de-contextually in more than very general terms. Consequently the domain of on-the-spot teacher decision making has expanded and the notion of teaching practice is extended beyond the teaching methods in the narrow sense of the term, i.e. beyond the set of observable teacher actions.

The recognition that many of the most important educational decisions have to made by the teacher in the classroom seems to be one reason why - until recently - the literature has remained remarkably silent with regard to methodological advice: If the specific interactions between teacher and student(s) are essential to the types social or cognitive of support required, any attempt to provide a general set of suggestions for classroom teaching may be in vain. As a consequence the recommendations for teachers have in practice often degenerated into a caricature of what they are expected *not* to do (use whole class approaches, stand at the board, use routine tasks, etc.). Instead the situation of forced autonomy and the inherent emphasis on teachers' reflective activity has fuelled a large amount of belief research, that has focussed on their meta-mathematical conceptions and views of the teaching-learning process[i]. One rationale behind this latter approach seems to be that the teachers' views of mathematics and of the teaching-learning process plays a fundamental role for the ways in which they cope with the extended conception of teaching practice (e.g. Schoenfeld (1992); Ernest (1989)).

2: THE DEMANDS OF FA: A METHODOLOGICAL PERSPECTIVE

Over the last few years the lack of recommendations on how to cope with classroom interactions has to some extent been remedied, and a number attempts have been made to indicate how teachers may proactively support student learning in ways that comply with the views of mathematics and learning that dominate the reform.

Cobb, Boufi, McClain, and Whitenack (1997) described how the teacher's introduction of symbolic records of the students' suggestions provides him or her with opportunities to provoke a collective meta-cognitive shift taking the students' previous activities as objects of the continued discourse. In doing so the teacher supported individual students' learning, facilitated the development of taken-as-shared concepts and at the same time contributed to the emergence of a meta-mathematical conceptions and norms for action compatible with the reform.

Chazan and Ball (1999) challenged the understanding that teachers' non-interference is the most dominant feature of reformist teaching. They suggested that teacher reflections on the mathematical value of the topic in question in relation to the students' future learning, the direction and momentum of the discourse, and the social and emotional tone of the classroom form the basis for decisions of for instance inserting disagreements in order to create a productive learning environment.

Stephan (2000), using Toulmin's model of argumentation, argued that the teacher should elicit warrants and backings from the students in order that they substantiate their initial claims. Stephan suggests that profound taken-as-shared mathematical understandings may emerge as a consequence, while reform oriented visions of mathematics and of its teaching and learning is maintained.

Skott (2000) suggests to explore the potential of intentional methodological discontinuities (IMDs) as a way of coping with situations in which the intended mathematical focal point of a classroom interaction is discarded, for instance as a funnelling type of interaction emerges. IMDs require the teacher to play an active part in the classroom discourse by breaking with the dominant methodological and organisational framework of the situation in question, and by doing so to ensure a continued emphasis on its potential for student learning.

It is a common feature of these suggestions that they highlight and specify the teacher's role in mathematically qualifying the classroom discourse. In other words, they are all based on the assumption that the teacher's activity in terms of support of the students' individual and collective mathematical learning is essential, and that even though teaching requires competent, on-the-spot decision-making some methodological recommendations are needed that facilitate it. In this sense the suggestions mentioned may be seen as attempts to present relatively concrete pieces of methodological advice that are in line with the meta-mathematical priorities and the approaches to learning that dominate the reform. The suggestions do obviously not relieve the teacher from the obligations of forced autonomy, i.e. from becoming involved in the types reflective activities described above. It cannot be decided de-contextually when and how to introduce a meta-cognitive shift, insert a disagreement, call for a backing to an argument, or propose a methodological discontinuity. The teacher has to make these decisions based on the specific set of interactions in the classroom in question. But the recommendation to consider doing so is in itself a relative concretisation of the reform's demands, as it introduces possible focal points for the teacher's reflective activity.

These suggestions for practice, then, share two characteristics. First, they recommend teacher activities that are methodologically oriented, but based on the set of meta-mathematical understandings and conceptions of learning that frame the reform. Second, they do not do away with the teacher's obligations in a situation of forced autonomy, but they may assist him or her in handling these obligations more competently and with greater ease or deeper comprehension.

3: A MATHEMATICAL PERSPECTIVE ON TEACHERS' COMPETENCE

Recently a few publications have emerged that address the issue of the mathematical qualifications required, if the teacher is to play role of flexibly supporting student learning. These studies, then, view teaching competence from the perspective of the teachers' subject matter knowledge.

Ma's (1999) study, at first sight a comparison between the qualifications of American and Chinese teachers, convincingly argues that teachers at the elementary levels need *profound understanding of fundamental mathematics* (PUFM). PUFM is more than conceptual understanding of the topic in question. The teacher must also be aware and make use of connections between different concepts and procedures in different domains of mathematics both at a given educational level and in a longitudinal sense; (s)he should be able to use multiple perspectives on a mathematical domain and various approaches to the solution of a problem; and (s)he should guide students' mathematical activity by use of basic but powerful mathematical concepts and principles. The essence of the argument is that one of the most significant obstacles to substantial mathematical learning on the part of the students is poor understanding of *school* mathematics on the part of the teacher. The teacher, then, needs profound understanding of fundamental mathematics rather than superficial acquaintance with more advanced mathematical topics, in order to facilitate significant student learning.

In a somewhat similar sense Ball and Bass (2000) focus on the mathematical qualifications of the teachers and challenge the idea that the main teacher related obstacle to reformist classroom practices is pedagogical or meta-mathematical. Using a terminology that acknowledges the challenges of forced autonomy, they claim that:

> "... teachers need mathematical knowledge that equip them to navigate these complex mathematical transactions flexibly and sensitively with diverse students in different lessons" (p. 94)

Their argument is that the divides between teaching methods, mathematical qualifications and teaching practice, institutionalised in most teacher education programmes, need to be bridged, and that this may be achieved by "grounding the problem of teachers' content preparation in problems and sites of practice" (p. 101). When doing so they also build on the research on what Shulman (1986, 1987) has called *pedagogical content knowledge*. Pedagogical content knowledge is "that special amalgam of content and pedagogy that is uniquely the province of teachers" (Shulman (1987), p. 8), "... the ways of representing and formulating the subject that make it comprehensible to others" (Shulman (1986), p. 9). However, Ball and Bass argue that this type of knowledge does not suffice as a basis for the teacher's instantaneous decision making. What the teacher needs, they claim, is

> "a kind of mathematical understanding that is pedagogically useful and ready, not bundled in advance with other considerations of students or learning or pedagogy" (p. 88)

Both Ma and Ball & Bass, then, point to the teacher's mathematical qualifications as the main obstacle to the enactment of reformist intentions in mathematics classrooms, and attempt to redefine the required qualifications. They do so on the basis of investigations of teacher competence related to *school* mathematics and of an analysis of what it may take in terms of teacher qualifications for the conceived potentials for student learning to materialise. In other words, the argument is that significant student learning requires teacher qualifications in mathematics, that must be defined not with reference to mathematics in its own right, but to the contents and envisaged interactions of mathematics classrooms at school level. This may be interpreted as compatible with the methodological recommendations described in the previous paragraph as it specifies the types of mathematical qualifications required for the teacher to successfully employ the types of teaching methods suggested: the meta-cognitive shifts, mathematical disagreements, the warrants and backings and the methodological discontinuities. In short, these studies present a long-needed reversal of a dominant approach to teaching competence. This tradition asks what it takes to transform a set of mathematical ideas (concepts, procedures, conceptions) based in traditional college or university courses to classroom practice. The two studies referred here start with a perspective on *school* mathematics and on classroom practice. They then use this perspective as a basis to redefine the mathematical qualifications for teachers that are deemed necessary for the realisation of the reform, for instance as conceived in the methodological recommendations described in the previous paragraph.

However in doing so the two studies disregard the need to integrate a general pedagogical perspective in the mathematical qualifications of the teachers. In the remaining part of this paper I shall argue that by doing so they in effect endanger the enactment of the very reform initiatives they seem to pursue.

4: INCLUDING A PEDAGOGICAL PERSPECTIVE ON TEACHING COMPETENCE

The study described in this section aimed to understand how novice teachers deal with the situation of forced autonomy. The study followed three teachers for 2 to 3 weeks each. The teachers were selected because they presented strongly reformist priorities of school mathematics both in a questionnaire before and in research interviews after their graduation from college. They all described the students' activity in terms of investigations and experimentation; they conceived mathematics as a way of approaching and posing problems; and they presented their visions of teaching in terms that reflected intentions of being unobtrusively supportive in relation to student learning. In short, the school mathematical priorities of these teachers were strongly inspired by the reform, and they all seemed confident to take on the responsibilities inherent in the extended notion of teaching practice (section 1).

The teachers' classrooms were videotaped and if possible I had an informal discussion with them after each lesson, asking a few questions based on my field notes. Also a final interview was conducted with each teacher, and he was asked to comment on a number of clips from the video recordings. These clips were selected in order to exemplify situations in which the teacher's role appeared to be crucial for the further development of the classroom interactions and for the learning potential.

In the case of all three teachers the classroom interactions often developed in ways that resembled their school mathematical priorities. For instance, they frequently invited the students to explore open problems - often using manipulatives - and to present their own hypotheses for further investigation. In some cases the continued investigations would be the responsibility of a one or a few students, while in others the teacher would attempt to involve the whole class in it. Also the teachers all tried to support student learning by asking them to explain and reword their current understandings and by taking these understandings as the starting point for their own contributions to the interaction. When doing so the teachers generally used everyday language when discussing new concepts and procedures and only later introduced standard mathematical terminology that in turn was used to focus the students' attention on particularly important aspects of the concepts in question.

However, there were also episodes in which the classroom interactions were at odds with the teachers' professed school mathematical priorities. In some of these the teachers' themselves referred to mathematical insecurity on their part as the main reason why the interaction developed the way it did. For example one of the teachers, Larry, - after having been shown an episode from his grade 5 classroom - explained why he in effect avoided pursuing an idea developed by one of his students:

> "For one thing I hadn't considered the question beforehand, and if I am to pursue something like this at the instant, I at least have to have some idea about where I want to go, and if I'm too much in doubt about the direction, if I haven't understood it properly myself, it becomes difficult to convey it to the children." (From the final interview with Larry).

There were, though, also a number of episodes in which the learning opportunities were strongly inhibited by the Larry's contributions to the interactions, but in which mathematical insecurity did not appear to be the main reason. For instance Larry, who teaches at a very conservative private school, often struggled to reconcile his own educational priorities and those of the school. In particular he often referred to school's emphasis on covering the syllabus and preparing for the next test as incompatible with his own intentions. At other times the classroom interactions and Larry's comments on them indicated that his contributions to the discourse were directed primarily towards manifesting his own professional authority rather than to facilitating the students' mathematical learning. In these situations he became very direct in his explanations and at times took over the students' suggestions.

Christopher, a 28-year-old teacher working at a municipal primary/lower secondary school, was also asked to comment on episodes in which his contributions to the interaction seemed particularly significant, especially when they appeared in dissonance with his explicit images of school mathematics. Also in his case, mathematical insecurity appeared as *one* reason why he did not always exploit the mathematical potential of students' questions and comments. However, there seemed to be several other and more frequent such reasons, reasons that - in a sense similar to the situation for Larry - were related to a shift in the motives of his activity: He was sometimes more concerned with building students' self-confidence by ensuring that they provided an acceptable solution to a textbook task than with supporting their mathematical learning, and in consequence he got involved in funnelling types of interaction that in effect depleted the task in question of its mathematical contents. Also his activity was sometimes primarily directed towards managing the rather noisy classroom in which many different (groups of) students simultaneously asked for his assistance. And - like Larry - he sometimes tried to manifest professional and mathematical authority in ways that seemed counterproductive to student learning.

John, the last teacher in the study, works at small village school with only 120-130 students. Like the others he was asked to comment on episodes in which his school mathematical priorities were challenged. In some of these episodes the interactions with the students developed very much along lines compatible with his expressed views of school mathematics, while in others this was clearly not the case. Describing his reaction in some of the latter ones he says:

> "There are some children in here, some of the weak ones, with whom I've had to choose [...] especially with Louise, I've had to say to myself 'If only she acquires a system [of how to

> solve the tasks], then it doesn't matter, if I've provided her with it, because at least she can follow what goes on'. I've chosen that. And she does [follow]. So up to now she is part of the team. She largely makes the same tasks as the rest, although she finds mathematics very difficult." (From the final interview with John).

John was more concerned that Louise remained part of the classroom community than with providing the types of support he found best suited to facilitate her mathematical learning. In other situations he tried to integrate his support to particular students' learning with concerns of for instance avoiding to challenge what he conceived as their weak and vulnerable self-perceptions and of taking their family background into consideration. Asked if he ever experienced a conflict between his intentions teaching mathematics and of taking broader educational considerations into account he said:

> "No I don't think so. Because I can't imagine being a teacher without taking all the other things into account. You know. But that's my attitude to being a teacher. [...] [these other things] are always part of being a teacher, and I don't consider it a conflict. It is just another type of professional challenge." (From the final interview with John).

An important characteristic of the episodes described above is the simultaneous existence of multiple motives of the teacher's activity. In each of them the teacher's intention of facilitating mathematical learning is challenged, as the energising element of his activity is changed to for instance complying with the dominant school culture, supporting students' self-confidence, managing the classroom, or manifesting professional authority[ii]. The teacher, then, is often playing a very different game than one of teaching mathematics, and consequently the most dominant obstacle to the enactment of their reformist school mathematical priorities is not insufficient mathematical qualifications on their part.

5. SUMMARY AND CONCLUSIONS

In this paper I have outlined some of the responses on the part of the research community to the pressures on teachers as a result of forced autonomy. One response is to develop of a set of relatively concrete pieces of advice on how to proactively support student learning. Such advice suggests possible focal points for the teacher's reflective activity in the classroom without relieving him or her from the obligations of forced autonomy. Another response is to investigate teachers' mathematical qualifications, and recently it has been suggested that the most significant teacher related obstacle to the enactment of the reform is the teacher's (insufficient) mathematical competence.

These two sets of responses are similar in that they are both based on analyses of interactions in mathematics classroom and of *school* mathematics rather than on general pedagogical theory or on mathematics per se. They may, then, be seen as attempts to develop conceptions of teachers' pedagogical content knowledge and subject matter knowledge from the practices of mathematics classrooms.

The study described in section 4 also investigated teacher-related potentials for and obstacles to student learning on the basis of classroom observations. The results of that study point not only to the teachers' mathematical and pedagogical content knowledge as relevant factors, but to the ways in which the teachers sometimes shift the objects and motives of their activity to much broader educational issues than those related to mathematics. This change of the teacher's activity turned out to be one of the most important challenges to his intention of unobtrusively supporting the student's learning. It follows that teachers' pedagogical knowledge is not irrelevant for student learning, and it should - just like the pedagogical content knowledge and the subject matter knowledge - be re-defined on the basis of analyses of classroom interactions and integrated with the other two types of knowledge. The second quotation from Ball and Bass, then should be turned upside down (cf. section 3). Maybe what is needed, is exactly 'a kind of mathematical understanding that is [...] bundled in advance with other considerations of students and learning and pedagogy'.

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[i] In Zeitschrift für Didaktik der Mathematik DM 96/4 Törner and Pehkonen list 764 papers in belief research in mathematics education. This indicates an interest in the field that does not seem to have diminished since then.

[ii] I have previously called such episodes *Critical Incidents of Practice* (CIPs). For a discussion of CIPs based on the classroom observations of and interviews with Christopher see (Skott, in press).

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PROMOTING MATHEMATICAL THINKING: A PILOT STUDY FOR INNOVATIVE LEARNING ENVIRONMENTS

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***Abstract**: The aim of this pilot study was to experiment and test some novel innovative teaching methods in practical teaching situations with primary level pupils (10 year-olds). In the experiment on learning environments four different teaching methods was implemented with that aimed primarily at using of collaborative communication, supporting pupils' meta-cognition and improving problem solving processes. The development of teaching methods used in the study were based on the earlier research on word problems, the research on meta-cognition connected with them, and research done on the use of open problems. The results of the pilot study show that with the use of such a many-sided learning environment and the teaching methods used it is possible to support the development of pupils' meta-cognition and problem solving skills.*

Background

The theoretical background for the pilot study was a compound of theories on mathematical thinking and and learning with have been developed in the current constructivist tradition (e.g. Davis & al. 1990). Particularly our approach is based on theories that emphasise the role of communication, interaction, and meaningful tasks in supporting the development of pupils' meta-cognition and problem solving skills. A teacher's role in mathematics lessons is seen primarily as an activator of pupils' thinking skills, to help pupils to understand mathematical structures and ideas as well as to help pupils when developing their own mathematical knowledge (Schoenfeld 1985, Sfard 1998, Lehtinen & al.1999). The central element in developing the teaching methods of the pilot study is the activation of a pupil's own thinking.

Mathematical thinking should be distinguished from mathematical contents and techniques. Burton (1984, 35) states that "mathematical thinking is not thinking on mathematics, but a thinking style which is a function of special operations, processes and dynamics characteristic to mathematics". Of them especially the processes of mathematical thinking are interesting for problem solving. An individual's meta-cognitions are regulating elements of his thinking (cf. Schoenfeld 1987). In mathematics, meta-cognitions are often connected to problem solving skills (e.g. Schoenfeld 1985). An important component of meta-cognitions is self-regulation (cf. De Corte & al. 1998). To practice problem solving is seen to promote the development of pupils' higher order thinking and understanding (e.g. Verschaffel & al. 1999). It is stated in many sources that i.a. verbal elaboration of mathematics to be learned (e.g. Cockcroft 1982, NCTM 1996, 2000) and a pupil's own experiences in problem solving (e.g. Schoenfeld 1985) promote the development of his meta-cognitive skills. For example, self-regulation can be promoted with proper problems and their treatment (De Corte & al. 1998). Just with the aid of many-sided communication in the pilot study, it is searched for different verbal methods with which a teacher could support the development of his/her pupils' meta-cognitions and problem solving skills.

Research on mathematical thinking will often be operationalized through problem solving (e.g. Burton 1984). Since in the middle of the 1980's it was realized that teaching of heuristics was not a successful solution, alternative approaches were searched for (e.g. Schoenfeld 1985). Among other methods, it was developed for teaching problem solving an approach that emphasized creativity as an important part of problem solving (e.g. Mason 1982). During the last ten years, the so-called open problem solving has become popular which has near connections to creative problem solving (Pehkonen 1997a) and also to the Japanese "open approach" (e.g. Nohda 1991) and the use of investigations (e.g. Wiliam 1994).

Creativity and logic are central elements in mathematical problem solving (e.g. Mason 1982, Pehkonen 1997a). Recently one less known component of problem solving – problem posing – has been pushed up (e.g. Silver 1995). It belongs also to the open understanding of problem solving (cf. Pehkonen 1997b). A teacher's role is of paramount importance when selecting proper tasks and problems for mathematics lessons. The selection of problems and their application properly in lessons is a difficult task in mathematics teaching. The role of the teacher is very important when constructing a proper atmosphere in class where pupils can frankly investigate, make mistakes, share their success und failure, and exchange their ideas with each others without fear that they will be critically assessed (NCTM 2000). With the aid of such a learning environment it is possible to educate pupils who trust in their own abilities and who are willing to participate and investigate problems.

Hakkarainen & al. (2000) state that in order to be able to help intensive and pedagogically intentional learning with the aid of information technology, it would be very important to locate computers in classes and integrate working with computers with many different working methods and disciplines during the working phase in classroom. Computers should not be seen as separate tools for learning of an individual pupil, but rather as a method of learning in a co-operational learning environment (e.g. Reusser 1991, Lehtinen & Repo 1996). Computers should not be used as a substituting tool when striving for mathematical understanding and intuition, but their use should promote and strengthen processes of mathematical understanding (NCTM 2000).

The focus of the paper

The pilot study in question experiments and investigates the possibilities and limits of new and innovative teaching methods for promoting mathematical thinking and reasoning skills when used in the teaching practice of 10 year-old pupils. Furthermore, the focus lies in the possibilities of developing pupils' problem solving skills, pupils' communication skills, pupils' added awareness of their meta-cognitive skills and abilities as well as pupils' motivation. Based on the results of the pilot study, the usefulness of different teaching methods in the learning environment used are evaluated for the implementation of the main experiment. Teacher students who implemented a part of teaching form an additional factor in research. A side objective of the pilot study is to clarify the possibilities to integrate such research projects into the teacher education, in order students would get experiences from a new kind of mathematics teaching.

Implementation

The pilot study on promoting mathematical thinking and reasoning skills was implemented in the teacher training school in Turku in the spring 2000. A forth-grade class (10 year-olds, N=17) formed the experiment group; the class has a special emphasis on mathematics and computers. This class has four lessons mathematics pro week during the experiment period of ten weeks there were altogether 40 experimental mathematics lessons. During the intervention special emphases was given on promoting pupils' thinking and reasoning skills, communication, added awareness on their own knowledge and skills (i.e. meta-cognition) and motivation. The teacher of the class, Mr. Jari Sorvari (one of the authors) with five student teachers was responsible for the practical implementation in class. The teaching methods used in the pilot study were planned together by the authors, in the way that a teacher in a usual school could also implement them alone. The researcher wrote beforehand written plans for all experiment lessons. Each student teacher got in advance the written lesson plan for his future lessons with instructions. Before each lesson, there was still a brief discussion on the lesson plan with the student teacher, if needed. During the discussion, possible difficulties or lack of clarity in the lesson plan were dealt with. Each student teacher implemented during his week of practice four experimental mathematics lessons. The personal teaching style of each student teacher was not influenced in any other way.

The pilot study was built of four different teaching methods which were implemented always in the same weekday and which aimed to promote pupils' mathematical thinking. Additionally, computers were used in routine exercises. The method called *Scattered-Minded Moses* was based on the teacher's own loud-thinking, modelling of word problems, and conscious "thinking mistakes" during the solving process before the class. Pupils' task during the lesson was to perceive these mistakes that were then discussed together. With the learning game *Quest of the Golden Chalice* (Vauras, Rauhanummi, Kinnunen, & Lepola, 1999) we tried to develop pupils' solution skills in word problems and meta-cognitive skills as well as to improve their calculation skills and to promote mathematical communication in class. With the aid of the method *Concrete Problem Solving*, we tried to build creative, open and motivational learning situations with hand-on material. Problem solving gave an opportunity to develop pupils' thinking skills and logical reasoning. The method *Concept Lessons* concentrated in mathematical concepts using problem-oriented approach and instructional discussion. Beside these four methods *Computers* were used to intense routine exercising Computers offered for the teacher a possibility to work with a smaller group in class.

Data was gathered in the pilot study by using multiple observation methods Since we wanted with a small number of participants to find out the appropriateness of different teaching methods, observational methods were stressed in data gathering. There were teachers' field notes (both the researcher's and the student teachers') on implemented lessons, the student teachers' own notes and experiences of mathematics lessons, videos on the part of the lessons and pupils' interviews. Observations were unstructured, and thus, the field notes are based on each observer's own personal view on teaching situation.

Results

Based on the analyses of the video records of the lessons in which the method *Scattered-Minded Moses* was used, it seemed that the teacher-pupils interaction functioned well, especially in short word problems. Pupils were able to follow excellently the teacher's modelling and thinking process in these problems. Almost all pupils could perceive the teacher's "mistakes" in loud-thinking processes and amend them during the common discussion. The use of the method seemed really to motivate pupils to follow their teacher's loud-modelling process. Since the teacher did not make mistakes in each problem, pupils were compelled to follow very keen their teacher's thinking process, in order to perceive his mistakes. The motivational level seemed to be really high when solving so-called short word problems with the help of the loud-modelling method. Whereas in the case of long problems, similar effect was no more so clear to observe. Difficulties in long problems seemed on one hand to be partly connected with the complexity of the teacher's own thinking process, and on the other hand with the fact that pupils were simply not able to concentrate and to comprehend the structure of the long problems.

The learning game *The Quest of the Golden Chalice* is planned for small-group teaching or remedial teaching of third-graders (9 year-olds). In this study, the learning game was accommodated with small changes in rules to normal teaching in mathematics. The changes in rules done seemed, according to video analysis, to produce some especially interesting observations. Differently as in normal case, the pupils played the games during the experiment in pairs. This seemed to change the nature of the game. When playing in pairs, the communication between players (pupils) increased very much. The teacher's role as a game leader was also changed. Since the pupils played in pairs, main instructional discussion was done within the pairs, and not between the teacher and the pupils as usual. The teacher's role was mainly to act as a discussion leader. The communication within pupil pairs seemed to increase and develop, when the pupils gathered experiences with the game. Pupil pairs advised each other and could give those pupil pairs in trouble good hints for solutions of problems. Furthermore, pupil pairs' loud-modelling of solutions seemed to get more clear with time. The additional changes in rules done in the middle of the experiment seemed to still add spontaneous use of drawing, in order to clarify one's own solution process. Drawings seemed to clear and strengthen pupils' thinking processes, when they explained and described their solutions to other players.

In the analyses of *Concrete Problem Solving*, observations made were based on the researcher's own perceptions as well as student teachers' observations and field notes. On the ground of these we can state that instructional discussion had a meaningful share in the problem solving lessons with hands-on materials. Concrete materials offered a tool to "externalize" different phases of the problem at hand, and their possible problematic points. This gave an opportunity to consider also difficult topics with the aid of common instructional discussion. The pupils were very motivated to solve problems using concrete hands-on materials. Such material used were match sticks, soma cubes, multilink cubes, fraction cards and normal playing cards.

Working with matches seemed to interest the pupils very much. In addition, the problems used supported well the objectives set for the use of concrete materials. An interesting observation was connected to the behaviour of two pupils, when giving a home problem. The home problem was a continuation to the following problem dealt with in class. A square sequence should be constructed with matches, where the starting point was a square of four matches. The home problem was formulated, as follows: "How many matches are needed to construct a sequence of 500 squares? Try to find a possible formula / solution not using the help of matches." The giving of the problem was just finished, when two boys in class wanted to give the solution: *"It is an easy piece, Mister. Those matches will be needed altogether 1501, since in the first square there are four matches, thus 3*500+1 is 1501."*

During the *Concept Lesson*, the teacher had an opportunity to work with a smaller pupil group. Half of the class worked at the same time with computers on routine tasks connected to the topics to be learned. Based on the observations done we could state that during the Concept Lessons the interaction between the teacher and pupils could be intensified. The time the teacher used for a pupil doubled. Furthermore, it seemed that materials used for concretization motivated pupils. Working with a smaller group made it also possible that the level of pupils' understanding was more effective secured. The big amount of instructional discussion should be also mentioned. A special emphasis seemed to be in discussions on different solution alternatives.

In this study, six *computers* were used in classroom. The programs used were two in Finnish, of which the first concentrated on routine tasks of mathematics to be learned, and the second contained mainly problems of proper level. Since there were only six computers at hand, and half of the group (N=17) at the same time, a part of pupils were compelled to work in pairs on one computer. Observations made during the computer working phase showed that on one hand to transfer such amount of mechanical exercises on computer seemed to be a functioning solution. On the other hand, the programs used did not offer enough purposeful and useful tasks, in order to implement instruction effectively, especially there was a lack of problems for individual differentiation. In addition, the program did not contain characteristics needed for following effectively an individual's learning process.

The communication and interaction within pupils pairs was very active almost all the time of the experiment period on computers. The instruction given and the awareness that the computer will remember mistakes seemed to activate the discussion within a pair. It seemed that the interaction between the pupil pairs could have acted as a feedback system, and the pairs did not need the computer's feedback system when solving the problems. Observations on those pupils working alone were slightly different. They seemed to base their work more on the computer's feedback system than their mates working in pairs.

Discussion

In Finland, a teacher is compelled to work, especially in growing counties, with bigger and bigger classes. This results that time allotted to help pupils individually, e.g. in mathematics lessons, will be decreased all the time. Therefore, one is compelled to search new ways and methods to organize learning environments. There is a need to develop different ways of group working which will support effectively the development of pupils' mathematical thinking. Research results and observations gained during the experiment showed that the methods used in this study are worthwhile developing further. With the aid of the teaching methods developed through research, we should be able to find functioning working methods for the class, and through using them to conduct pupils toward higher order mathematical knowledge and understanding.

The objective of developing such learning environments is to find out pedagogically purposeful teaching practices. Some research questions which can been seen as results of the pilot study are, as follows: Is it possible to reach deeper and better level learning with such a learning environment? Could it be shown that some individual teaching method would support some special area of learning? In the light of the pilot study, none of these questions could be answered exactly, but the study showed these questions to be sensible to ask in the main experiment.

The benefit of the pilot study lies just in its concrete context. The experiment in real teaching situation and the data gathered during it made it possible to implement the learning environment with different teaching methods at one time. On one hand, such an implementation gave much valuable information on the advantages and difficulties of combining individual teaching methods. On the other hand, we could experiment and plan the learning environment as a whole. Thus, we gained information on of what kind of changes in classroom, teaching organization, working order etc. should be taken account of when implementing such a learning experiment.

All the student teachers considered the participation to such an experiment very important. In the following, there is a comment in one student teacher' notes which comment reflects the feeling and thinking of all participative student teachers: *"I am* *glad of the possibility to participate the mathematics intervention. For me mathematics has always been a challenging topic, since my best method of learning has been doing, but mathematics has been traditionally very abstract and theoretical. It was absolutely wonderful to have an opportunity to experiment that mathematics, and especially challenging problem solving, could be taught so concretely. A pupil has a possibility himself to do, to experiment and to have insights."* (student teacher #2)

Such kind of developing projects on learning environments should be connected to teacher education also more generally. Thus we may construct important links, e.g. in mathematics, between research and practice. To see this connection is not necessary always self-evident. This view of the gap between research and practice has been described very hitting in a comment of a student teacher: *"The mathematics project which I was allowed to participate seemed to be very interesting. In the model used, a teacher was a guide for learning instead of a knowledge transmitter. This was at last the concretization, which was long heard in the speech of the theoreticians. The ultimate aim was to get pupils aware of their thinking processes. In the matter of fact, we are in the core of learning (a pupil understands his thinking), if I have condensed rightly. This starting point awakes me a desire to know more."* (student teacher #1) The combination of learning theories, didactics and teaching pedagogy would add student teachers understanding on the teaching / learning process.

What were the pupils experiences in the learning environment? One third of the pupils could see nothing special done during last weeks in class. Another third of the pupils described the classroom working very many-sided. All pupils thought that mathematics has been nice and interesting. On one hand, some did not like mathematics, when there were too difficult problems to solve. On the other hand, many pupils experienced mathematics interesting such when having difficult tasks to solve. This view is described very touching in one pupil's interview: *"difficult tasks are really nice to solve, ... such where is some idea and not only a mere calculation, ... such where you should yourself pick the numbers, ... such straight calculations are dull"* (pupil #38). Finally one student teacher's view on the behaviour of the class during the last weeks of the experiment: *"In this class, it was self-evident that questions like "Why?" and "What did you think?" were presented. I had a good feeling: pupils' pre-knowledge and beliefs were activated, and these were the starting point for further development."* (student teacher #5)

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REPRESENTING CHANGE

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We report on data from a summer institute for high-school students, part of a 13-year longitudinal study of the development of mathematical ideas. The students were invited to discuss the motion of a cat, given 24 time-lapse photographs taken in less than a second. As students explain, justify, *and convince others, a re-examination of previous explorations, including prior reasoning, is often triggered. The student data are analyzed through detailed examination of how students work with a variety of representations in order to build arguments, through extended social interaction.*

The work presented here is part of an extended longitudinal study of the development of mathematical ideas of a focus group of students, beginning in first grade.[2] The members of the focus group are now first-year university students. The objectives of the study, now in its thirteenth year, are: to provide in-depth case studies of the development of proof making in high-school students; and to investigate the relationship of students' earlier ideas and insights to later justifications and proof building; to trace the origin, development, and use of representations of student ideas, explorations, and insights relating to explanation, justification, and proof building. Within the context of the learning community, additional goals are to investigate the nature of teacher intervention in the growth of student mathematical ideas, and to study individual cognition in the context of the movement of ideas through the community of learners.

Significance. It is generally accepted that traditional approaches to "teach" students how to reason mathematically (for example, proof making) have not been successful. Most secondary teachers know this; few elementary teachers have been able even to address the issue, given the curricula with which they currently work. Robert B. Davis (1994) distinguished between what mathematics students should learn—that is, what "content" or "topics"—and what kind of knowledge students need to develop, domain by domain.

> If one takes seriously the various new suggestions about the teaching and learning of mathematics—if, for example, one takes seriously the NCTM Standards (1989)—then one is faced with asking teachers to play a quite new role... It will not be easy for teachers to shift to the new role—working alongside students, trying to be aware of the student's thinking, working to help the student modify that thinking in an appropriate way... (Davis, 1994, p. 17)

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In our view, specific, detailed knowledge about how students build mathematical ideas is central to successful teaching. To build instruction based on knowledge of student thinking requires detailed research into how students reason and communicate under particular conditions.

Theoretical Framework. As students engage in mathematical investigations, they frequently reconsider prior constructs[3] while attempting to make sense of new experience (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997). As they explain, justify, and convince others, a re-examination of previous explorations, including prior reasoning, is often triggered (Maher & Speiser, 1997). In this way, learners come to emphasize certain features, for example, of the problem situations under study. Challenges to students (by each other and perhaps the teacher) to explain their ideas may lead to modification and/or rejection of some prior knowledge, or to generalizations which can be supported by convincing arguments. Further, emerging theories can be subsequently modified, extended or refined, often through long-term critical discussions by the learners.

The theoretical framework that guides this study comes from research on what students do when they do mathematics (Davis, 1984; Davis & Maher, 1990; Davis & Maher, 1997), recent work that traces the intricate and complex pathways for individual students' understanding within a larger community (Maher & Speiser, 1997), Kieren and Pirie's Dynamical Theory for the Growth of Mathematical Understanding (Kieren, Pirie & Reid, 1994; Pirie & Kieren, 1994a, 1994b), and some recent work by Dörfler (2000) on the construction of meaning in the course of social interactions.

To investigate the history, development and use of learners' arguments, we identify *events*, i.e., connected sequences of utterances and actions which invite explanation by us, by the learners, or by both. An event is called *critical* when it demonstrates a significant advance from previous understanding (also recorded as events), in the context of an emergent narrative. We may refer to critical events (moments of insight) as conceptual leaps (see, for example, Maher & Martino, 1996b). They are obvious and striking moments, in which learners demonstrate compelling intellectual power, to each other and to us, by having wonderful (Duckworth, 1996) ideas and putting them across in forceful (Maher & Martino, 1996a) ways. In large measure, we see education as creating situations which elicit critical events and then support reflection on their consequences.

Following the same learners for several years, under conditions where instruction has built carefully on what learners have already built, helps us to understand more clearly what students might do cognitively, when proposing justifications that reflect their growth of understanding. It is clear that we need to learn much about the cognitive processes involved in making proofs, especially as students, over time, advance toward higher mathematics.

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[3] More precisely, learners can retrieve and critically examine previously built notations and constructions as new knowledge enters.

Background. We build on an existing and extensive data base, and on considerable experience in the conduct of studies of this type. Further, our work connects to and draws from other investigations by members of a broader community of researchers with whom we have substantial common ground. Long-term studies of the type that we conduct (and with the depth of data collection we accumulate on the same learners) are almost unprecedented in mathematics education. While there are aspects of this research that are significantly different from other studies, important related research includes work by Susan Pirie, Tom Kieren, Leslie P. Steffe, Andrea diSessa, Alan Schoenfeld and his colleagues, and Jim Hiebert. Further, there are similarities between our presentation of tasks developed by David Page, and methods of Paul Cobb and Erna Yackel. Our interpretation of student performance, in some ways, resembles work by Susan Pirie and Tom Kieren, although, in other respects, our approach is different. Our research supports and then extends investigations of learning which have emphasized the central and essential role of social interaction in the growth of learners' understanding.

In the school year 1998-99, when the research subjects were third-year high-school students, after-school explorations were designed to elicit the construction of mathematical models, the making of connections, and the construction, not only of specific proofs, but of the idea of mathematical proof, at increasing levels of abstraction. In the context of this study, models and proofs were found to emerge from concrete explorations which include demands for careful and convincing explanation. In practice, the drive for explanation requires learners to support their ideas, which often triggers connections across different problem contexts.

In order to help prepare for calculus, the student subjects, during the July 1999 Summer Institute, were offered two extended explorations which drew deeply on precalculus mathematics.[4] We discuss the second exploration here.

Toward Calculus: the Catwalk task. Calculus, historically, arose to study change and motion. In the task we now describe, called Catwalk, one challenge is to build connections between local rates of change and total changes, based on real-world data.[5] The Cat task was designed originally to expose some of the complexity inherent in the use of mathematics to examine motion, and so to open opportunities for students to discuss the issues raised. Work on this task by college calculus students, and by a study group of university faculty, has been reported in three papers by Speiser and Walter (1994a, 1994b, 1996). The questions which arose seem very challenging.

At Kenilworth, the Catwalk was tried with high school students for the first time. In the context of the present study, the task was chosen to elicit rather than simply illustrate the construction of important underlying concepts, such as the distinction, and connections, between average and instantaneous rates of change.

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[4] Only one of the 17 student subjects (Victor) had studied any calculus before the Cat activity.

[5] We emphasize that this example illustrates only one facet of how calculus is useful for such problems, but does so in depth.

> **The photographs** (Muybridge, 1957, Plate 124). These consist of 24 frames of a single cat, entitled Cat in Walk Changing to a Gallop.[6] The photos were made in 1880 by Eadweard Muybridge, using 24 cameras, activated successively at intervals of .031 second. They show the cat against a background grid, composed of lines spaced 5 centimeters apart. Every tenth line is darker. The 24 photographs show the cat over a total time of action of .71 second.
>
> **The task** (Speiser & Walter, 1994a, 1994b, 1996). Project the Muybridge photos from the overhead. Distribute copies of the photos to the students. Discuss briefly how speed is measured. Then pose the following two questions: How fast is the cat moving in Frame 10? How fast is the cat moving in Frame 20?

The teaching team at Kenilworth was led by Walter.

Data. The data we present are from July 15, 1999, from the fifth hour of work on Catwalk. Just before the first segment below, the students had built a linear representation of the cat's position, in the form of a horizontal line of masking tape, 65 meters long, in the hallway where the students had their lockers. Along the line, two students marked locations for the cat's position (scaled by a factor of 50) in each frame. Beating a makeshift drum,[7] several students[8] ran the course, striving to reach each mark just as the drumbeat sounded.

In the first data segment, attention focuses on the whiteboard, where Romina's scatter plot of velocity vs. time has been projected. Earlier, she had plotted this graph in her calculator, using data found with members of her group by measuring the Muybridge photos. Matt has just walked to the whiteboard.

Matt: This is time and this is velocity. This is …[points with a marker to the first few scatter points] Each of these…little dots here, these are all his velocities at a certain time… So … [begins to draw a line connecting all the scatter points] If you see how the line goes, that's …
Victor: This is …
Matt: … his climb in velocity.
Victor: … acceleration graph
Matt: Yeah. This is, like… This is, like, his acceleration.
Victor: Alright. That's, uh, vel… its velocity, uh, over time. So then its, like, uh, you know, dt over t. You know, its, like…
Matt: Acceleration.
Victor: … dt squared… Right?
Matt: It shows acceleration.
CW:[9] So, you said you saw that as an acceleration graph, Matt. Could you … could you tell me a little more about that?

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[6] The photographs are reproduced in (Speiser & Walter 1994a, 1994b).

[7] Roughly 70 beats per minute.

[8] And two of the investigators. At the given tempo, keeping pace was strenuous. Most participating students were athletes. They chose the scale themselves, after rejecting a shorter model.

[9] Charles Walter.

Matt: It just …that … It just basically shows you, like, how far … how fast. Like, you can tell, like, [indicating the segment connecting the first seven points] from here to here he goes, his acceleration goes down, and [indicates the steep segment through the next four points] from here to here, it starts to skyrocket, up like that. Then [indicates the next six points] he evens out for a while, then it goes down a little bit and [refers to the next five points] then it skyrockets up again.

CW: Well, where is the acceleration?

Victor: Right there where he lands, he's slowing down, right?

Matt: These lines. [Draws arrows pointing to the two sharply increasing segments of his graph.] This line here and this line here. Where it starts to slant up.

Magda: … he's going at a constant speed.

Matt: At these … At these ones here [indicates the two relatively flat segments] he's going at a constant speed.

It might be helpful to consider first the surprising range of representations now in play.[10] Reference to videotaped evidence,[11] as well as to observers' field notes[12] leads to the following list of nine representations, just for the segments we discuss. For each representation, we define a code in boldface, to facilitate analysis

1. The Muybridge photographs. **Ph**
2. Graph of position as a function of time. **P-T**
3. Romina's graph of velocity, as a function of position, projected on the overhead. [The horizontal axis, in effect, is a scaled replica of the marked line on the floor.][13] **V-P**
4. Romina's graph of velocity, as a function of time, projected on the overhead. [The scatter points for the first few frames, for example, are now further apart than in the previous graph.] **V-T**
5. References to standard mathematical or scientific notations or formulas. **F**
6. Gestures. [For example, Victor's hand, tracing the graph of speed against time, moving from left to right as seen by the viewer.] **Ge**
7. Onomatopoeia. [Sounds which mimic actions.] **O**
8. Mime, enacting motion. **M**
9. Drawing and writing, at the whiteboard, on projected images. **DI**

In the first data segment, above, we see V-T, F, Ge and DI. Romina's graph has anchored Matt's discussion, while Victor's attempt to introduce a calculus notation has been set aside. Matt focuses on changes in the cat's velocity, by drawing on the whiteboard onto which Romina's graph has been projected. Almost immediately after running the linear model in the hallway, momentary speed changes seem especially evident. In our second segment, hardly four minutes later, several students respond to an observer's question about how many steps the cat has taken in the photographs.

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[10] By representation, we mean *presentation*, but in the very particular sense that the students, by inventing a notation, for example, simultaneously create a structure which the notation represents *for them* by presenting it to them. Our usage is similar to terminology of Dörfler (2000).

[11] The session was filmed by four cameras: three following the students group discussions, and a fourth, hand-held, operated by a documentary film-maker, to record some of the flow of conversation, especially between the student groups.

[12] There were six observers, two per student group.

[13] Square brackets denote researchers' comments.

Brian: No way he can take four steps in a …three fourths of a second. [referring to the total time of action for the cat photos, .0713 sec.]

Romina: Tch, I don't know. Maybe he did only take, like, a step.

Brian: Three fourths of a second is fast. That's [pauses] starting … now. Now! [Timing three fourths of a second.]

Benny: [At the overhead projector, adjusting the display of the cat photos.] There we go. Now, if you look at the, at the first step. [points to the cat in the first frame.] He begins it, he begins his step right here, and if you look, [points successively to frames 2 through 6] he is lifting his leg up gradually, gradually, gradually, gradually and he begins to put his foot down here and then his foot touches base right here [frame 6]. That's one step! Now, [points to frame 7] he begins another stride right here. Gradually lifts his back leg up. You can see. This, this is the foot we're going to judge it by, the back leg. Then [pointing sequentially to frames 8 through 12] he lifts his back leg up again, then he lifts it up higher, higher, higher, then he bends it, bring it down, touches base again [frame 12]. That's his second step! Then as his speed picks up, it starts to take him less time to take a step. So, [points to frame 13] in this, in this third one, his, his other leg, his front leg, picks up, picks up, picks up. We see he touches base here 'cause he's picking up speed, so he touches base earlier, at a quicker time. So, that's his third step! So then he's pickin' up his back leg, back leg, back leg, back leg, back leg. Back leg touches base here and then he begins another stride, right here.

Matt: Basically he takes, like, four steps.

Benny: Four steps.

Victor: All right.

Benny: [Starts back to his seat but stops.] You ask? [retrieves the cat transparency and walks slowly to his seat, his voice trailing off as he moves] Its gone. Its gone, gone, gone, gone! [Laughter.]

Matt: [At the overhead, which projects the scatter plot onto the whiteboard.] Like, this is what, this is his first two steps here, where he's walking at a constant speed. Then all of a sudden, on that third step he gains his speed and this [indicates the flat segment in the middle of the graph] is when he lands. And then that's his speed again.

Benny: True.

Matt: When he pushes off.

Benny: And did you remember what we did with the other graph? Its coming back to me. Yes!

Matt: Oh yeah!

Benny: You remember what we did with the other graph? [Refers to the position-time plot derived earlier from Romina's data.] When we weren't moving, the line was flat. [His hand traces a line in the air, resembling the first segment of the position-time plot.]

Matt: Uh huh.

Benny: So when his foot comes down, he's not movin' at that time. So, that's why the line is flat. That's why you see that gradual line.

Matt: And then when he pushes off again, he accelerates.

Benny: Acceleration then. [Leans to his left. Laughs.]

Matt: Oh, man.

Building from the first segment's discussion, the incremental changes in velocity, as graphed, have been connected to steps taken by the cat, as seen in the photographs. Again, the anchor seems to be V-T, together with Matt's added line (DI), which now extends beyond the given interval of time. Benny works directly from the photos (Ph), summarizing with an onomatopoeic "Gone, gone, gone, gone, gone," as if to sound the rhythm of the cat's steps: first slow, then faster (O). Matt shadows Benny at the scatter plot (V-T) for velocity, checking every step against the graph. Then, suddenly, Benny connects the cat's low velocity before frame 10 (walking) to a prior discussion of position vs. time (P-T). In each of these two data segments, at least four different representations were in play. In the second segment, however, ideas built in the discussion of velocity connect both to the cat photos and the earlier discussion of position. The narrative the students build seems to demand not one but several different visual, numerical and graphic anchors.

Discussion and Conclusion. The next morning, student discussion would begin about frame 10, where the largest change in interval velocity (the spring into the gallop) had already been identified. In this way, grounded in a *global* picture of the motion of the cat, triggered in part by the experience of running, the two questions given in the task were broadened, deepened and then reshaped, to yield a powerful and detailed argument that the velocity at Frame 10 cannot be known, based on what might be gathered from the photographs.[14]

These students build their understanding through collective effort, in which students continually interpret and reframe each others' claims and contributions. This process of interpretation and reframing leads to clearer and more widely shared conclusions. Our data show how students jointly build from prior knowledge, with little teacher intervention, either before or during the discussions we present. Our analysis suggests that mathematical proficiency requires shared experience in social situations that promote both careful reasoning and reflective reconstruction of past knowledge. The students' work presented here is notable for the number and variety of presentations, as well as for its powerful response to the task's questions. We find these students' work astonishing.

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[14] For Frame 20, the situation is different. As several students emphasized, the cat is in the air, the change in interval velocity around Frame 20 is quite slight, and so an average should (unlike Frame 10) suffice.

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CONFUSIONS BETWEEN DECIMALS, FRACTIONS AND NEGATIVE NUMBERS: A CONSEQUENCE OF THE MIRROR AS A CONCEPTUAL METAPHOR IN THREE DIFFERENT WAYS

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Many students confuse decimal numbers, fractions and negative numbers. Data, some of which is new, is provided to support this observation. Interview data also identifies other confusions between number lines and number-line hybrids and between zero and one. These observations are explained by drawing attention to the use of the mirror as a conceptual metaphor in three different ways for understanding the number system. It underpins the usual positive/negative number line, links natural numbers and their reciprocals and operates in a pseudo number line related to place value columns. Students mentally merge the components that are the images under the analogical mapping of the same mirror feature. This extends recent work on metaphors in mathematics itself to their role in understanding mathematics.

INTRODUCTION

Sometime ago I (the first author) was sitting at the back of a class watching an excellent Year 8 lesson on ratio and scale. The students measured small plastic model animals and had to find the size of the real animal, given the scale factor. The girl alongside me correctly calculated that the length of the pig would be 0.9m, but then seemed puzzled. I asked her to show me how big the real pig would be, expecting her to indicate a length along the desk with her two hands. However, after a long pause, she pointed out the window to the left and said that it would be "a long way, out there". Apparently, she was confusing 0.9 and –9. At the time, I thought that the confusion was surprising but that it was simple to explain. Mathematics lessons must have made so little impact on this girl that she had not learned adequately the meanings of the two symbols: the dot for the decimal point and the dash for the negative. Rather like an English speaker beginning to learn French might confuse an acute or grave accent, she confused the meanings of the dot and the dash. I now reject this simple explanation and propose that this confusion is in fact deep, arising from the use of the mirror as the common conceptual metaphor that underpins comprehension of negative numbers, decimals and fractions and place value. The phrase *conceptual metaphor* is used in the sense of Nunez (2000), who describes this as the "cognitive mechanism by which the abstract is comprehended in terms of the concrete" (p I-6).

In the following sections, we first summarise evidence that confusion between decimal numbers, fractions and negative numbers is common and report some new quantitative data. This establishes that there is a phenomenon needing explanation. Second, we report evidence from interviews conducted by the second author which provide clues to the reasons for the decimal/negative number confusion and display a surprising confusion of zero with one and fundamental problems with number lines. Third, we explain these results by demonstrating how the conceptual metaphor of the mirror is involved in three different ways in understanding the number system and propose that students mentally merging these three "mirrors" explains the observations outlined above. These ideas are explained in more depth and further evidence is presented in Stacey and Helme (submitted). Whereas previous work on conceptual metaphors has concentrated on mathematics as a discipline (e.g. Nunez, 2000), this paper examines conceptual metaphors in students' thinking.

Unless specifically qualified, in this paper the terms *decimal numbers* and *fractions* always refer to mathematically positive numbers. The term *negative number* is used in two ways: in the standard mathematical sense and also to indicate a number "less than zero", which some of our interviewees wish to distinguish from standard negative numbers. Space does not permit a full discussion of this.

CONFUSING DECIMALS, FRACTIONS AND NEGATIVE NUMBERS

Confusions between decimals and fractions

Misconceptions about the meaning of decimal numbers have been documented in many parts of the world and widely studied. The task of comparing the size of two or more decimals (e.g. identifying which of 2.4 and 2.375 is the larger) has been found to be very revealing and has been used as the basis for studies of misconceptions (Resnick et al, 1989; Stacey and Steinle, 1999). One innovation introduced by Stacey and Steinle in developing their decimal comparison test was to include decimals of equal length. It had not been expected that many students would make errors when selecting the larger from a pair of decimals such as 0.3 and 0.4 or 2.64 and 2.57, but a significant number of students made errors on all items of this type on the test. Stacey and Steinle were interested to identify patterns of thinking that significant groups of students were using consistently on all items. They realised that students who were consistently incorrect on comparisons where the longer decimal is larger, consistently correct on comparisons where the shorter decimal is larger and consistently incorrect on equal length comparisons (such as 2.64 / 2.57) may have been interpreting decimals as reciprocals of whole numbers or as other fractions. (Note that this is not the "fraction rule" of Resnick et al, 1989.) For example, students may be identifying 0.3 as something like one third and 2.64 as something like two and one sixty fourth or as two sixty fourths or similar. Evidence that some students think in this way comes from production tasks, as used by Irwin (1996) and others. Table 1 shows how the incidence of this type of thinking varies from Grade 5 to Year 10 (ages about 10 to 15). This is previously unpublished data, based on a test of 24 carefully chosen comparisons from the longitudinal study reported by Stacey and Steinle (1999).

Table 1. *Percentage of students consistently interpreting decimals as reciprocals*

Grade Level	5 (N=963)	6 (N=1465)	7 (N=2297)	8 (N=2102)	9 (N=1645)	10 (N=1066)
Percentage	7.2%	4.8%	5.2%	7.1%	4.3%	3.3%

Confusions between decimals and numbers less than zero

The patterns of responses that arise from interpreting decimals as reciprocals can also arise from interpreting them as negative numbers. In Table 1, it is expected that the students in Grades 5, 6 and 7, who have not met negative numbers at school, would predominantly be confusing decimals with reciprocals. However, older students may have either confusion (and note the increase at Grade 8, when students do a lot of work on negative numbers). In order to explore this possibility, a later decimal comparison test included a group of three direct comparisons of (positive) decimals with zero: the comparison of 0.6 with 0, of 0.22 compared with 0 and 0.00 compared with 0.134. This test was given to 553 teacher education students, at various stages of their training, from four universities in Australia and New Zealand. The results are reported by Stacey et al (in press). In summary, 73 students (13%) made at least one error on the three comparisons with zero and 50 (9%) made either two or three errors. This was markedly higher than the percentages of students making at least one error on the other types of comparison items (generally about 7%). The item most likely to be correct was the comparison 0.00 with 0.134. The presence of the additional digits encouraged or permitted use of a digit-by-digit comparison strategy. As one student said "*It's the decimal point* (i.e. in 0.00 but not in 0) *for some reason makes the zero seem much more like a zero…the fact that there is a one in that tenths position indicates that it* (i.e. 0.134) *is larger… the decimal point has obviously made it easier for me to see.*"

Only a handful of students (about 1%) consistently answered all items on the test according to the reciprocal/negative pattern of thinking described above and also made at least two errors on the comparisons with zero. In fact, most of the students making errors on the comparisons with zero tested as expert on the remainder of the test. This behaviour indicated that about 1% of the teacher education students completely identified decimal numbers with negative numbers and about 7% could order non-zero decimals, but thought that some such as 0.6 and 0.22 were less than zero.

The teacher education students were asked to select comparisons which children would find difficult and to explain why. The explanations canvassed two possibilities. The first was that children might think that decimals are negative or less than zero (first two quotes below) and the second explanation (3rd and 4th quotes) is that zero is bigger than a decimal number because it is a whole number.

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> *"0.22 may be mistaken for a negative number below zero."*
>
> *"Some kids have trouble with the notion of zero. 0.22, though less than one, is greater than zero. [Children] might see it as negative or less."*
>
> *"Children are taught that the ones column is larger than the tenths column so assume 0 is bigger than a decimal."*
>
> *"Because '0' is a whole number (to the left of the decimal point) whereas .7 is a decimal number, they may choose 0, as it's not seen as a 'decimal', a smaller number."*

Children's difficulties making comparisons of decimals with zero have also been reported elsewhere. Irwin (1996) reported that some 11 to 13 year-old children placed decimal numbers starting with zero (e.g. 0.5, 0.1) below zero on a number line. Irwin concluded that these attempts at ordering were consistent with a system that pivots around zero as equivalent to the decimal point, rather than around one.

Interview results

In order to find explanations for the data summarised above, individual interviews were held as soon as possible after testing with 7 volunteer students from one of the universities, who had made errors in the zero comparison items. Both types of explanations outlined above were given for the incorrect answers. Sometimes a student's answers contained elements of both. Jocelyn explained her wrong answers this way and drew the number line in Figure 1:

> *" I think I was thinking that zero is equal to one. So I was thinking half of one is less than zero. I was thinking that 0.5, for example, was half of zero, so was thinking it is less than zero. I was visualising a number line with 0.5 on the left hand side of zero".*

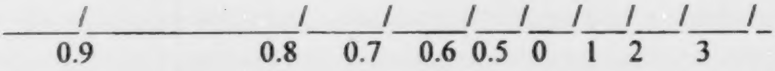

Figure 1: Jocelyn's number line showing that small decimals are less than zero.

She commented in her interview that this line made no logical sense, because she could see that the numbers on the left were approaching one. Jocelyn believed that the decimal point must have acted as a trigger for thinking the number was less than zero:

> *"When I think of a fraction, decimal points, I always think of .5, and instead of thinking .5 as half of something, it's half of—it has to be less than zero because it's, I don't know why I thought that. I suppose maybe because it's got that point, the decimal point ……in some way it's less than zero because it's got a point there."*

Stuart said that he might have been thinking of 0.6 as a negative number. When asked to draw a number line, his first attempt was not a line as such but the numbers 10, 0 and 0.1 set out from left to right. Next he drew a line with zero in the middle but with positive numbers on the left and negative numbers on the right. When he went on to explain his thinking, he also appeared to be confusing zero with one:

> *"I know 0.6 is a portion of one. I may have been thinking along the lines of 0.6 is less than the whole number zero. Is zero a whole number? I don't even know . . . . . I'm looking at whole numbers as being positives and decimals as being negatives . . . . . decimals aren't, they're just fraction amounts."*

Lisa was another student who may have been confusing the number line with the place value columns. When she explained her response to the comparison of 0 with 0.6 she said, *"I was thinking of the number line...I was thinking it was on the right hand side of zero, so going into the negative area for some reason*". She drew a number line left-right inverted, and placed 0.6 between 0 and -1 "*Normally I draw a number line in this direction* (indicates conventional number line) *but just when I was thinking of the decimals then I immediately drew it as a negative in this direction*". Later in the interview Lisa stated that she also thought of zero as a whole number:

> *" I can remember that I was thinking that 'this is a whole number' and that this is a fraction of a whole number. And that a fraction of a whole number must be smaller than zero which is a whole number."*

When the interviewer discussed the idea of zero being a whole number with Anna, she summarised: "*Logically 0.6 has to be part of a whole, part of one, but I guess it's like zero is being turned into one and parts are trying to be made out of nothing really.*" In a revealing instant, as she spoke about zero, she made a circle with her hands, which seemed to indicate simultaneously the shape of the symbol for zero and a unit (perhaps the classic pie of introductory fraction teaching) which could be divided into parts.

THREE APPLICATIONS OF THE MIRROR METAPHOR TO NUMBERS

The data above leads us to seek explanations for the facts that some students think decimals (and fractions) are negative numbers or otherwise less than zero and some students confuse zero and one. In addition, some students confuse decimals and reciprocals, but we feel that this is well enough explained as persistence of an undifferentiated primitive idea that decimals represent the fractional parts of numbers. We propose that the confusion and interference arise from the use of the conceptual metaphor of the mirror in three different ways for understanding numbers. Recent developments in cognitive science represent a move away from the traditional role of reasoning as primarily propositional, abstract and disembodied to viewing it as embodied and imaginative. From this perspective, mathematical reasoning entails reasoning with structures that emerge from our bodily experience as we interact with our environment.

Lakoff and Johnson (see, for example, Johnson 1987) demonstrate how reasoning with metaphors is fundamental to human thinking and communication by pointing out how everyday language uses common ideas as metaphors to convey abstract concepts. Nunez (2000) and Lakoff & Nunez (1997) apply these ideas to mathematics demonstrating reasoning through metaphors, such as "numbers are points on a line"; "variables are boxes with numbers inside" or "an equation is a balance". These instances view a less familiar target situation (numbers, equations, variables) through the lens of a familiar, concrete source situation (lines, balances, boxes).

The key feature of metaphor is that one domain is conceptualised in terms of another. We propose that aspects of numbers relevant to the problems outlined above are conceptualised in terms of a mirror. A mirror as a conceptual metaphor has three basic components: the real objects, their images (reflections) and the mirror position (some sort of line of symmetry/balance point/pivot/axis). In order to make a conceptual metaphor, the relations between these basic components must also translate from source to target. With a mirror, each real object has its own clearly identified image. The images share many of the features of the real objects and have the same spatial relation to each other, although with an inversion so that in the image world, things are "the other way around", but otherwise the same. There is, however, a critical asymmetry between the real object and the image: the image is a reflection of the real object, the real object is not a reflection of the image. The real object is primary and the image exists in relation to this, not in its own right. In our more extensive paper (Stacey and Helme, submitted), we link this asymmetry of opposites to the linguistic phenomena of positive and negative terms and marked and unmarked adjectives (Clark and Clark, 1977).

The conceptual metaphor of the mirror with natural numbers functioning as the real objects is used in three different ways in understanding numbers as displayed in Table 2. First, in the classic number line, the positive and negative numbers are balanced around zero. The images are the negative numbers {-1, -2, -3, -4, ...} (with later extension to other numbers). In formal mathematical terms, these images are the additive inverses of the natural numbers. Second, in an instance less commonly perceived spatially, the positive numbers and their reciprocals are balanced around the natural number 1. The images are the unit fractions {1/2, 1/3, _, ...} (again with later extension). In formal mathematical terms, these images are the multiplicative inverses of the natural numbers. We contend that this is also conceptually a mirror, because of the way in which the (unit) fractions are conceptualised in terms of the whole numbers and their basic relations (such as size) are similar but inverted.

The third spatial arrangement with a mirror as conceptual metaphor relates to the number line drawn by Stuart. Here the real objects are not quite the natural numbers but the values of the place value columns {ones, tens, hundreds, thousands, . . . }. Their images are the "fractional" place value columns {tenths, hundredths, thousandths, . . .}. The evidence from the interviews leads us to believe that the spatial arrangement of the usual place value numeration is seen by many of our students as some sort of "number line" along which numbers are distributed. This model, which like the other two stretches out infinitely in both directions, has whole numbers of increasing value on the left side of the decimal point and "decimal numbers" of decreasing value on the right. The mirror position is unclear to many students, who may see it as the decimal point, rather than the ones column.

Table 2. *Comparison of features of three mirror metaphors.*

Aspects of mirror metaphor	**Positive/negative mirror metaphor**	**Reciprocals mirror metaphor**	**Place value mirror metaphor**
Mirror position	0	1	Ones column (not decimal point)
Real objects	Natural numbers, represented by points or positions on the line.	Natural numbers, represented by points or positions on the line.	Values of places {tens, hundreds, etc} or vaguely numbers without decimal part
Images	Negative numbers, represented by points or positions on the line.	Unit fractions, represented by points or positions on the line.	Values of places {tenths, hundredths, etc} or vaguely decimals with zero integer part
Direction of increasing size	Increasing to right (monotonic)	Increasing to right (monotonic)	Increasing to left (but not really monotonic)
Extent of "number line"	From – **infinity** to +**infinity**	From 0 to + **infinity**	From long decimals to long whole numbers

The two confusions above can now be seen as confusions between the different targets of the metaphorical mappings of the same source features. Students who think decimals (and fractions) are negative numbers are merging the different images of natural numbers. They may have merged the images in the first two columns or the first and third. Confusion of 0 and 1 is merging of the mirror positions, and also relates to the decimal point as the significant "divider" (mirror position) between whole numbers and others. Merging the number line and place value "columns", produces a hybrid where the "whole number" part of the place value system is placed on the "positive" side of the number line and the "decimal" part of the place value system on the "negative" side of the number line. As Lisa commented: *"I get my number lines mixed up"*.

CONCLUSION

The discussion above outlines an explanation of the confusions between decimals, fractions and negative numbers that are certainly common amongst school students and teacher education students. In summary, the basic elements of the explanation are:

(i) that the natural numbers are the primary elements from which concepts of other numbers are constructed,

(ii) that the metaphor of the mirror is involved in the psychological construction of fractions, negative numbers and place value notation for decimal numbers, although in three different ways,

(iii) that the observed confusions result from students' merging (confusing or not distinguishing between) the different targets of the same feature of the mirror metaphor under the different analogical mappings.

Two of the mirrors are recognised within the formal mathematical system (additive and multiplicative inverses), but the place value mirror and its associated "number line" is only a psychological construct. It is hoped that this example will further the exploration of the role of conceptual metaphors in students' mathematical thinking.

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ON THE RELUCTANCE TO VISUALIZE IN MATHEMATICS: IS THE PICTURE CHANGING?

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This study examined the perceived and actual role of visual representation use as a possible heuristic in advanced mathematics problem solving by experts (mathematicians) and novices (undergraduate students). While both groups perceived visual representations as a useful tool and frequently attempt to use them, further analysis showed that students have little training associated with the use of visual representations.

The role of visualization in mathematical problem solving remains an active question in educational research. For centuries, visual tools such as diagrams, graphs, and sketches were considered to be indispensable in the work of mathematicians (Rival, 1987). Reports on the work of expert mathematicians have provided anecdotal evidence of the use and value of diagrams and other visual tools in the research work. Pólya (1945) for example, argued that the use of visual representations is an essential element in problem solving and offer specific advises to novices as to how to use visual representations in their own problem solving. But, despite the obvious importance of visualization in mathematical activity, deceiving visual clues often led mathematicians to false beliefs, resulting to the tendency to consider visual representation use as only an informal part of mathematicians' work. Little empirical work has been done towards the better understanding of the processes related to the use of visual representations by experienced problem solvers. Detailed accounts of individual practices are still relatively rare with the personal accounts of a few mathematicians like Pólya and Hadamard being notable exceptions and bearing the limitations of self-reports. Only recently have mathematics educators begun to study empirically and systematically the nature of mathematicians' practices.

In the realm of education, research studies indicated that advanced students are often reluctant to use visualization to process mathematical information (Eisenberg & Dreyfus, 1986, 1991; Vinner, 1989) and that, whenever possible, students choose a symbolic framework to process information and to approach problems rather than a visual one. Eisenberg and Dreyfus present a review of the literature with several cases in which college calculus students repeatedly resist the use of visual representations in solving problems. Yet, more recent studies in advanced mathematical problems suggest that the picture may have been changing in the past decade; Partly due to the changing curricula and attitudes toward the use of diagrams, mathematics students now appear to be interested in using visual representations (George, 1999; Gibson, 1998; Stylianou & Dubinsky, 1999).

Further research regarding use of visual representations is warranted. This study, whose purpose includes an investigation of the ways that both expert mathematicians and students reasoned with visual representations in solving advanced mathematical problems, aims to provide further insight into this issue. Specifically, this study investigated (i) the perceived role of visual representations as a possible heuristic in advanced mathematical problem solving by experts and novices, and (ii) the frequency of visual representation use during the actual problem solving (iii) the relationship between these two aspects of problem solving behavior. The study participants were 10 mathematics professors (experts) and 10 college mathematics students (novices).

I. Perception of visual representation utility

In the first part of the study, the role of visual representation use as a problem-solving heuristic by experts and novices was ascertained. Experts and novices were asked to categorize a set of 24 mathematics problems according to their similarity in their solution process. The study aimed at finding out whether "draw-a-figure" is viewed as a viable strategy when solving a problem.

Experts produced a total of 13 categories. These included some well-established strategies such as use of induction and contradiction, or use of similar problems; they also produced three categories which were closely related to the use of visual representations: "geometry/analytic geometry", "algebra and analytic geometry" and "draw-a-figure". The first category, "geometry and analytic geometry", was used by 8 experts primarily to group together problems which were geometric in nature (i.e., they used topic as their classification criterion). The second and third categories, "analytic geometry" and "draw-a-figure", were used to describe problems which relied strongly on the use of visual representations (that is, problems whose solution can be facilitated by the use of a visual representation). The 5 experts who produced these categories argued that the problems they classified as "draw-a-figure" are different from problems they classified as "geometry" in that the problems in the draw-a-figure category were not necessarily geometry problems. These are problems which reside in other topics or areas of mathematics (such as algebra and calculus) but visual representations can be helpful in the solution process.

Novices produced a total of 23 categorizations for their groupings and 6 of these are closely related to the use of visual representations: "geometry/analytic geometry", "algebra and analytic geometry", "draw-a-figure", "circles", "area", and "physical constructions". The "geometry and analytic geometry" category was used by 9 novices primarily to group together problems which were geometric in nature. Three novices used the "geometry and analytic geometry" and "draw-a-figure" categories to classify a small number of problems which they perceived as problems which are not geometry problems but which required the use of visual representations. Finally, 3 of the novices produced categories which focused on contextual features of the problems such as "circles", "area" and "constructions".

Data was analyzed using a cluster analysis – a process which allows for the arrangement of objects into clusters. The problems within a cluster are more homogeneous than they would be if they were compared to problems that belong to other clusters. The information provided by the clustering process was transformed into dendograms to facilitate the interpretation of the data. Figures 1 and 2 show the dendograms that were constructed using the experts' and novices' classification data

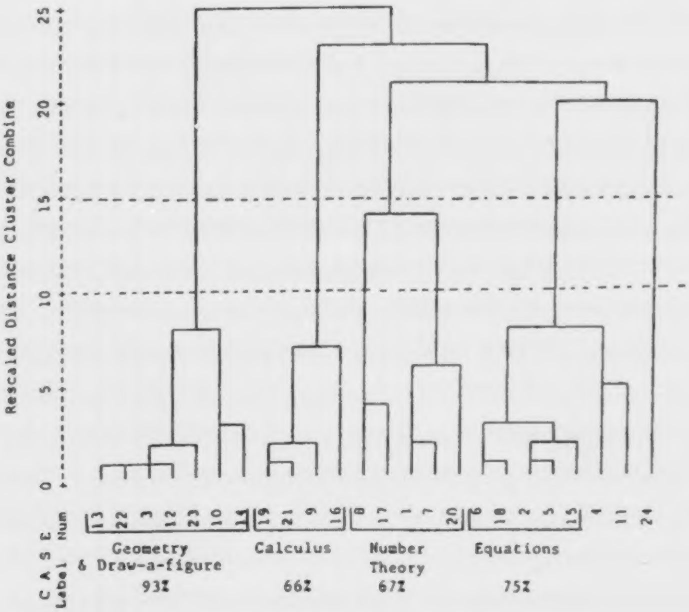

Figure 1: Experts' classification data

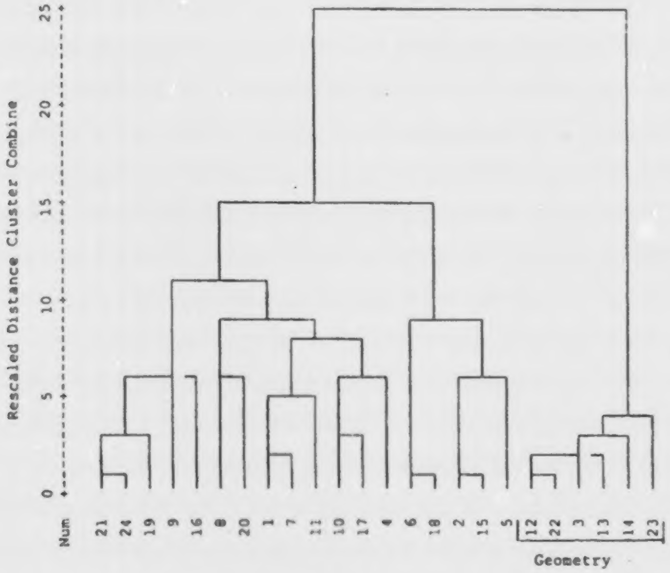

Figure 2: Novices' classification data

respectively. The 24 mathematics problems are represented as equidistant points along the horizontal axis. The measure of similarity, which is plotted as an ordinate, is the distance between problems belonging to different categories. Problems in the dendogram with greater similarity are connected at lower ordinal values.

The expert dendogram in Figure1 contains a large cluster which was labeled as "Geometry and Draw-a-figure." This cluster contains both geometry problem and problems for which the use of visual representations would facilitate their solutions even though they may not be geometry problems. The novice dendogram in Figure 2 also includes a "Geometry and Draw-a-figure" cluster. In fact, this was the only cluster on which novices agreed to a large extent on the labeling of the category.

In a summary, when asked to classify a set of problems using mathematical similarity (that is, similarity in the way problems would be approached, and in the strategies that would be utilized during the solution process), both experts and novices produced categories which related to visual representation use. Therefore, the results showed strong evidence of both experts' and novices' perception of visual representation use as a viable strategy in mathematical problem solving. We may conclude then that visual representation use is perceived by expert and novices to be a viable heuristic in advanced mathematical problem-solving.

II. Frequency of visual representation use

In the second part of the study, the frequency in the use of visual representations was determined. Subjects were asked to solve five of the 24 problems and their written solutions were coded with respect to evidence of diagram use. The coded data were converted into numerical scores for analysis purposes by assigning a 0 for the solution of a problem for which there was no evident use of a visual representation and a 1 for each solution that contained evidence of visual representation use. For each participant a visual representation use score was then computed, by summing scores across the five problems. Thus, visual representation scores ranged from 0 to 5.

The frequency of visual representation use by the two groups was determined by examining experts' and novices' visual representation scores. A summary is shown in Table 1. In general, most experts used visual representations when solving the five

Table 1: Distribution of Visual Representation Use Scores

	Visual Representation Score					
	5	4	3	2	1	0
Number of Experts	6	3	--	1	--	--
Percent	60%	30%		10%		
Number of Novices	--	5	4	1	--	--
Percent		50%	40%	10%		

problems. Ninety percent of the experts used a visual representations in solving all, or all-but-one of the 5 problems, and no expert used a visual representation on fewer than 2 problems. Similarly, novices also used visual representations to a great extent, though somewhat less frequently than experts. Only 50 percent of them used visual representations in solving all-but-one of the 5 problems.

The mean score for the entire sample was 3.90, while experts' and novices' mean scores were 4.4 and 3.4 respectively. A t-test comparison with an alpha level of 0.05 revealed a statistically significant difference in the frequency of visual representation use by the two groups ($t=2.65$, $p<0.05$). Thus, this analysis also suggests that experts were more likely than novices to use visual representations.

These results gave prevalent evidence that both experts and novices frequently attempt to use visual representations in the form of diagrams, figures and graphs when solving advanced mathematics problems. The frequency in which experts utilized visual representations in their written solutions of mathematics problems in this study provides empirical evidence for the anecdotal reports of expert mathematicians claiming that visual representation use is an essential element in their mathematical problem solving. With regard to novices' use of visual representation, the results showed that advanced undergraduates, similar to experts, frequently utilized visual representations within their written solutions to advanced problems.

This result seemingly contradicts earlier findings by educators (Eisenberg & Dreyfus, 1986, 1991; Vinner, 1989) who documented a reluctance on the part of advanced students to use visual representations in mathematics. One way of interpreting the results of this group of studies is, as Vinner commented, to see that they reflect the current situation in mathematics learning (especially at early levels) where success is essentially measured by routine problems which do not require visual ability; students give up meaningful learning and prefer to memorize fomulae and algebraic techniques since experience has shown them that this is an effective prescription for success in standard tests. This interpretation is consistent with Lean and Clements (1981) who admit that "in [their] study the mathematical variables were measured by tests which did not require the solution of difficult, unfamiliar word problems" (p.294). The problems used in this study are different in that respect; they were chosen to not resemble standard textbook tasks, so that both novices and experts would have to make an effort to first understand and then solve the problems.

Finally, the frequency in visual representation use by novices may also be explained, in part, by recognizing the difference in time and curricular trends between the empirical work conducted in the early eighties and this study. A report by the Mathematical Association of America (Tucker & Leitzel, 1995) assessing the reform efforts in American higher education institutions showed that more than half mathematics departments were engaged in some sort of reform efforts and concluded that calculus reform, in particular, is gaining widespread acceptance. This suggested that undergraduate students receive different instruction using different curricula than students did approximately one decade ago. Reform calculus textbooks present a large number of visual representations. Further, they often encourage or expect students to use graphing technology in the form of hand-held calculators, or computers. In short, undergraduate students in the past few years have been exposed to the use of visual representations to a larger extent than their counterparts a decade ago. The novice in this study were undergraduate students who received "reform" calculus instruction. Therefore, the contradiction between the findings of studies conducted a few decades ago and this study may be explained in part by differences in experiences students have been having with visual representations as part of their mathematics curricula.

III. Perception versus use

This study focused on experts' and novices' perception of visual representation viability in problem solving, and the two groups' actual use of visual representations during problem solving. Yet, the "obvious" question of the relationship between the two has not been addressed: How consistent are subjects' perceptions of the usefulness or viability of visual representation use in advanced mathematical problem, with their actual use of visual representations during problem solving?

The results from the first part of this study showed that both experts and novices perceive visual representation use to be a viable strategy in problem solving. Both groups' classifications indicated that they perceive visual representations to be useful not only when solving geometry problems, but also for problems from other areas of mathematics. This result indicated that visual representation use is part of the declarative knowledge of both experts and novices.

The results from the second part of this study suggested that the two groups' perception and actual use of visual representations was consistent; Both groups constructed visual representations relatively frequently. Based on these results we could argue that the "picture" of particularly novices with respect to visual representation use has been changing; Novices are no longer reluctant to visualize. On the contrary, novices are well-aware of the utility of visual representations in mathematical problem solving and are, in fact, eager to use visual representations in their own problem solving. Since the problems that were given to subjects to solve were given previously to them in the sorting task, we can compare the number of experts and novices who initially perceived visual representations as a potentially useful tool for these problems, and the number of experts and novices who actually used visual representations during problem solving.

Figure 3 shows the contrast between perception and actual use of visual representation for each of the two groups. For experts, it is clear that their actual use of the visual representations during problem solving was higher than their perception of the possible utility of visual representation. For novices, on the other hand, the pattern was reversed; more novices thought a visual representation would be useful, than those who actually used visual representations during problem solving.

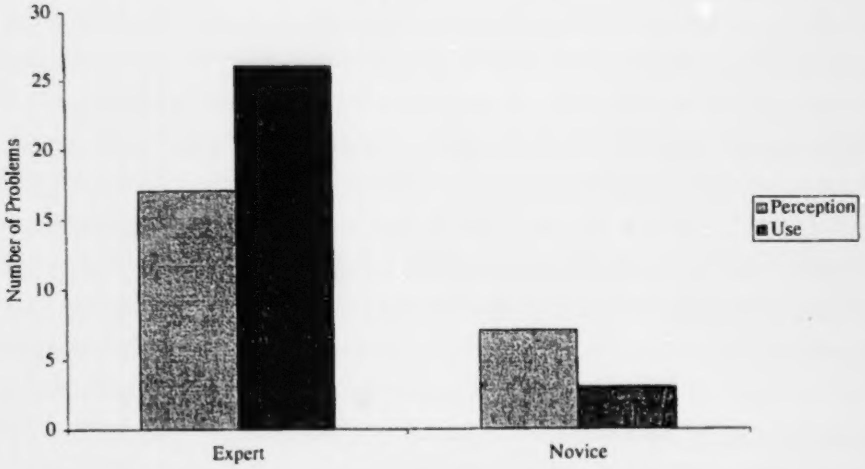

Figure 3: Perception versus use of visual representations

With respect to experts' visual representation use, the above comparison confirms the earlier conclusion that experts perceive visual representation use as a very useful tool in problem solving (declarative knowledge). Additionally, experts have strong procedural knowledge attached to this heuristic; once an expert decides that a visual representation would be a useful tool in solving a problem, even if visual representation use was not the first tool that came to the expert's mind, the expert is very likely to pursue the use of this visual representation, and, as the results for the second research question showed, experts know how to make use of visual representations as problem solving tools (procedural knowledge). Novices, though, appear to be very different; even though they appear to value visual representation use, and to have developed an ability to foresee the potential viability of visual representation use for a certain type of problem (relatively strong declarative knowledge), novices still lack the skill to use visual representations (lack of procedural knowledge).

Closing comments

Pólya (1945) and Schoenfeld (1985) argued that visual representation use is an essential element in problem solving and offered specific advises to students of as to how to use visual representations in their own problem solving. Pólya introduced the use of visual representations as one of the main problem-solving "heuristics", and Schoenfeld's subsequent work supported and extended Pólya's discussion of visual representation use as a problem-solving strategy.

The results of this study gave prevalent evidence that both experts and novices perceive visual representations as a useful tool and frequently attempt to use them when solving problems, suggesting that the "picture" in advanced mathematics instruction may be changing. However, further analysis clearly showed that the changes may only be covering the surface; students may be willing to use visual representations but have little training associated with this skill. Recognition of the willingness and at the same time difficulties identified in this study can lead mathematics educators to make more explicit and informed decisions about visual representation use in curricular materials and instruction, providing opportunities for students to become successful mathematical problem solvers.

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MOVING BEYOND PHYSICAL MODELS IN LEARNING MULTIPLICATIVE REASONING

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A key stage in learning multiplication and division is a capacity to move beyond reliance on physical models of problem situations and to form mental images to seek solutions. Some longitudinal data are presented to suggest that young children (5 to 8 years) progress through identifiable key stages in learning multiplicative concepts. One of these key stages is represented by movement away from a need to refer to physical models, and some children progress to this stage within the first three or four years of schooling. A clear finding is that teacher interventions facilitate this progression.

This paper reports results from the Early Numeracy Research Project[1] (ENRP) that examined the effect on student learning of a whole school approach to improvement of teaching and learning (Hill & Crevola, 1998). As a measure of student learning, the project collected data across nine domains of mathematics, one of which was multiplication and division. The data suggest that a key stage in the learning of multiplicative concepts, termed here *abstracting*, presents a significant barrier to many students, but that this barrier can be overcome with teacher support. This key stage, *abstracting*, is characterised by students moving beyond a need to create physical models as a prerequisite to solving multiplicative problems. It is conjectured that the necessary steps include developing a conceptualision of multiplication and division that allows students to deal with different situational contexts (e.g., partition and quotition) and generalising the concepts in a way that prepares them for future learning (Sullivan & Beesey, 2000).

It seems, for example, that there are many students in the later primary years (ages 9 to 12) who can cope with multiplication and division concepts with natural numbers, but who experience difficulty not only with multiplication and division of decimals but also with the very nature of fractions and decimals (e.g., Baturo, 1997). It is possible that the semantic complexity of the question forms and associated physical models used to assist the learning of multiplicative concepts in the early years themselves contribute to these difficulties (e.g., Mulligan & Mitchelmore, 1997; Verngaud, 1988). This paper suggests that students will develop more robust conceptualisations of multiplication and division if teachers pose problems that gradually but explicitly remove physical prompts or supports, and encourage students to form mental images, in multiplicative situations.

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[1] The project, titled *Early Numeracy Research Project* (ENRP) was established in 1999 by the (then) Victorian Department of Education as a collaborative venture between Australian Catholic University, Monash University, the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria. The ENRP Project is directed by Doug Clarke, and the team includes Barbara Clarke, Jill Cheeseman, Ann Gervasoni, Donna Gronn, Marj Horne, Andrea McDonough, Pam Montgomery, Anne Roche, Glenn Rowley and Peter Sullivan.

Early Learning of Multiplication and Division

Over the past decade, there has been considerable attention to research on multiplication and division concepts in early mathematical learning (e.g., Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan & Mitchelmore, 1997; Steffe, 1994; Wright, Mulligan, & Gould, 2000). Studies focused largely on the analysis of counting, calculation and modelling strategies from children's solutions to problem solving tasks.

There has also been some emphasis on the importance of developing conceptual structures for multiplication and division (Greer, 1992). In longitudinal analyses of young children's intuitive models for multiplication and division problems, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Students acquired increasingly sophisticated strategies based on an equal groups structure, and calculation strategies that reflected this.

The acquisition of an equal-grouping (composite) structure is at the heart of multiplicative reasoning. For example, a composite is a collection or group of individual items that must be viewed as one thing. To understand multiplication and division the child needs eventually to co-ordinate a number of equal sized groups and recognise the overall pattern of composites of composites, such as "three sixes". Steffe (1994) described the demand on students as follows:

> For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit. (p. 19)

The key issue is that co-ordinating these two composite units is complex, and physical models can help initially. Clearly students must move beyond physical models partly because such models do not easily represent all multiplicative situations (e.g., Greer, 1992), and partly because these models become less feasible with large numbers and inappropriate with rational numbers. We suspect that some teachers avoid these difficulties by using limited situational contexts of multiplication and division and continuing to rely on physical models, generally restricted to repeated addition. Since the form of the models is likely to be representative of the problem structure, we argue that it is preferable to encourage students to create models or mental images of a variety of multiplicative situational contexts and to use these models or images in solving the problems. It seems desirable to pose tasks that specifically remove elements of physical models, even within the first three or four years of schooling, and to emphasise movement towards use of mental images.

The Data Collection

In order to explore various aspects of numeracy learning, the ENRP project created a framework of key "growth points" that can be thought of as conceptual signposts on the road to children's development as mathematical thinkers. The focus of interest here is on what the data can tell us about the learning of multiplicative concepts in the early years of schooling.

The source of data was a one-to-one interview over a 30 to 40 minute period with every student in the first three year levels in 35 trial schools at the beginning and end of the school year. Note that the Australian school year is February to December.

Although the full text of the interview involves around 50 tasks (with several sub-tasks in many cases), no student moves through all of these. Given success with a task, the interviewer continues with the next task in the given mathematical domain as far as the student can go with success (see Clarke, Sullivan, Cheeseman, & Clarke, 2000 for a fuller discussion). Many of the interview questions invited the children to solve problems using small plastic teddy bears.

Interviews were conducted by the classroom teacher, who was trained in all aspects of interviewing and recording. As well as moving carefully through the 18-page interview schedule, the teacher completed a four-page Student Record Sheet.

Some Growth Points and Items for Assessing Multiplication and Division

The ENRP project developed sets of growth points in nine domains of mathematics, one of which is multiplication and division. The multiplication and division domain includes seven growth points, only four of which are relevant here.

Growth Point 0 – Not apparent
Not yet able to create and count the total of several small groups.
Growth Point 1 – Counting group items as ones
To find the total in a multiple groups situation, refers to individual items only.
Growth Point 2 – Modelling multiplication and division (all objects perceived)
Can successfully determine totals and shares in multiplicative situations by modelling.
Growth Point 3 – Abstracting multiplication and division
Can solve multiplicative problems, where objects are not all modelled or perceived.

These are presented as a conjectured sequence of development. It is accepted that students can follow different pathways in their learning, but nevertheless the intention is to describe a learning trajectory (Cobb & McClain, 1999) of the majority of the students.

The four questions that addressed these aspects of multiplication and division are shown in Figure 1. It is noted that these questions address only two of the four multiplicative situational contexts proposed by Greer (1992). Multiplicative comparison (I have 3 times as many as you), and Cartesian product (2 cones, 3 flavours, how many possible combinations?) were not included.

23. Teddy cars
Put four matchboxes in a line.

24. Sharing teddies
Show the child the picture of four "teddy

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Here are four teddy cars.
Please put two teddies in each car.
a) How many teddies is that altogether?
b) Tell me how you worked that out.
c) *If the child appear to be counting all, ask*: Could you do that another way, without counting them one by one?

25. Dots task

Here are some dots. *Show card (4 x 5 array of dots) for an instant.* I'm going to hide some. *Cover the bottom 3 x 3 section.*
a) How many dots are there altogether on the whole card?
b) How did you work that out?

mats". Put out 12 teddies.
a) Here are 12 teddies. Share the 12 teddies between the four mats so that there is the same number of teddies on each mat. How many teddies go on each mat?
b) How did you work that out?

26. Teddies at the movies

Here comes another story.
a) 15 teddies are sitting in rows at the movies. The teddies are sitting in three equal rows. How many teddies are in each row?
b) How did you work that out?

Figure 1: Multiplication and division questions.

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Materials were provided for the first three questions. The interviewers asked the first two questions for all students, but only proceeded to the latter two if the responses to both the first two were correct.

The instruction provided to the coders for this domain was to rate a student at the:

- *Counting group items as ones* growth point if they responded to the *Teddy cars* and *Sharing teddies* questions correctly;
- *Modelling* growth point if they used a non-count-all strategy in *Teddy cars* and answered *Sharing teddies* correctly; and
- *Abstracting* growth point if responses to *Dots task* and *Teddies at the movies* questions were correct, although count by ones strategies were not allowed.

Indicators of Student Growth

To examine the way the growth points portray the nature of the increasing sophistication of the students' strategies, the following presents a profile of students' achievement over the grade levels. Table 1 shows the percentage of students at their highest achieved growth point both by grade level and overall in the March 1999 interview. "Prep" refers to students about 5 years old, in their first year of school; Grades 1 and 2 are the next years.

Table 1: Students (%) Coded at the *Multiplication and Division* Growth Points (March '99)

	Prep (*n*=1237)	**Grade 1** (*n*=1233)	**Grade 2** (*n*=1168)	**Total** (*n*=3638)
Not apparent	71	37	12	41
Counting group items as ones	24	26	14	21
Modelling	5	36	66	35
Abstracting	0	1	7	3
Basic strategies +	0	0	1	0

There are significant numbers of students at each of the first three growth points overall and by grade level. It can be inferred that the first four points, at least, are necessary to describe the growth of such students.

To consider whether the growth points represent a sequence, it is appropriate to consider the way the students develop. Given that the questions were asked in such a way that if students made an error in an early item, they were not asked the latter ones, it is not possible to draw inferences on the sequence merely from the percentages of students answering the questions correctly. To allow consideration of the growth, Table 2 presents the ratings of all students in November 1999.

Table 2: Students (%) Coded at the *Multiplication and Division* Growth Points (November '99)

	Prep (*n*=1257)	**Grade 1** (*n*=1225)	**Grade 2** (*n*=1170)	**Total** (*n*=3652)
Not apparent	26	8	2	12
Counting group items as ones	27	12	6	16
Modelling	44	73	61	59
Abstracting	1	6	18	8
Basic strategies +	0	2	14	5

An indicator of the sequential nature of the growth points is the extent to which students progress from one point to subsequent points. Note that a better sense of the growth of the students can be gained by comparing Tables 1 and 2 than merely by comparing across grade levels within either table because the comparisons are between the same groups of students.

At each of the three levels, students progressed through the growth points. Few students in either Prep or Grade 1 progressed to *Abstracting*, and only one third of the Grade 2 students reached that point by the end of the year.

The *Abstracting* Barrier

The project team examined whether the conjectured growth points are sufficient to describe growth within each domain or whether more growth points are needed in between. One possible indicator of the need for an additional growth point could be that students take too long to move from one point to the next. It can be noted that the points as conjectured represent quite major growth stages since it takes the group, on average, just over 12 months to progress one growth point.

Of a total of 3410 students, 841 were rated at *Modelling* in both March and November. This represents 24% of all students, and 70% of the students rated at *Modelling* in March were still rated at that level in November. This implies that these students were able to represent the *Teddy cars* question and skip count or use other multiplicative strategies for calculating the total, and to represent and solve the *Sharing teddies* task, but were not able to answer both the *Dots task* or *Teddies at the movies* in a non count-by-ones manner.

Not only is the next growth point, *Abstracting*, an important goal for most students, it seems also to present a significant barrier. There are a number of components of this barrier. These could include the problem structure, the semantic subtlety of words like each and between, the calculation demand, and the need for the students to form some sort of mental image of the problem statement.

To explore further the nature of the development needed between these two growth points, the following tables present some characteristics of the 841 students who were rated at *Modelling* in both March and November.

One of the possible contributors to the barrier is the counting demand of the tasks. To examine this, Table 3 presents the responses of the students on the *Counting* growth points, from the same interview, for the 841 students rated as *Modelling* on the *Multiplication and Division* domain in both March and November.

Table 3: The *Modelling* students (%) at each *Counting* growth point (n=841)

	March	**November**
Not yet able to count to 20	1	0
Can say number sequence to 20	2	0
Can count a collection of 20 objects	50	13
Counts forwards and backwards by 1s	19	7
Counts from 0 by 2, 5, 10	26	54
Count from x by 2, 5, 10	2	25

These data suggest that over three quarters of these students are able to skip count and one quarter are able to skip count from variable starting points by November. While the *Sharing teddies* task prompts counting by 3, it seems that these *Modelling* students are able to calculate at a level sufficient for either the *Dots task* or *Teddies at the movies*. This suggests that the difficulty with those tasks may be related to the way the students interpreted the questions or their capacity to form the necessary mental images. To explore this further, Table 4 presents the growth points for *Addition and Subtraction* for these *Modelling* students.

Table 4: The *Modelling* students (%) at each *Addition and Subtraction* growth point (n=841)

	March	**November**
Not apparent	1	1
Count all	27	5
Count on	52	39
Count back	18	42
Basic strategies	2	12
Derived strategies +	0	2

To be rated at *Count on*, students find the total of nine teddies that are screened and four that are shown. This appears to require a similar imagining of the elements of the problem as the *Dots task*. Most of these *Modelling* students seemed able to do this.

To be rated at *Count back*, the students answer two questions about subtraction situations posed as stories but not modelled (8 – 3; 12 – 9). This seems to require similar interpreting and imagining of the representation as the *Teddies at the Movies* task. Of the *Modelling* students, over one quarter were able to do this.

It is possible that it is not so much the abstract dimension of the task, or the need to form some mental image, as it is the multiplicative conceptualisation that creates the barrier for these *Modelling* students. In other words, it might not be imagining generally that is required, but imagining of particular multiplicative situations.

A key issue, of course, is the extent to which the particular growth point *Abstracting* is defined and measured appropriately, and whether it has implications for teaching. On one hand, it is possible that there is an interim step between *Modelling* and *Abstracting*. On the other hand, it may be that the step is appropriate but the apparent barrier is as much an artefact of the curriculum and teaching approaches, or even that this is a single step but it takes time.

There are two further investigations that are appropriate to explore these possibilities. The first of these relates to the nature of any interim steps between modelling and abstracting. In examining the questions, it seems that the two division questions represent the growth being posed by the framework. However the growth between the two multiplication questions might have provided some unintended hurdles.

A second possible investigation relates to whether the issue is related to curriculum and teaching approaches. It appears that teachers do make a powerful difference. For each teacher, the number of students who moved from *Modelling* or below to *Abstracting* or above over the course of the year (March to November) was counted. Table 5 presents the number of students per grade, for just the straight Grade 2 classes and the Grade 1 and 2 composites, who progressed beyond this *Modelling* barrier over the year.

Table 5: Number of students per grade moving beyond *Modelling*

Number of students	Grade 1/2 classes	Grade 2 classes
Above 13	0	4
11 or 12	0	0
9 or 10	4	1
7 or 8	5	3
5 or 6	11	3
3 or 4	15	3
1 or 2	14	2
0	4	0

Clearly there is a broad spread. In some Grade 2 classes more than half of the students crossed the barrier, whereas in others it was only a few. It would be interesting to examine the approaches of the more successful of the teachers, in terms of the number of students progressing beyond *Modelling*, and whether this is a result of specific or intended actions on their part. Certainly these data suggest that the barrier is not impenetrable for students at these levels.

Summary and Implications

This paper reported one aspect of a project investigating the learning of mathematics in the early years of schooling. Data from individual interviews with over 3000 students confirmed that the conjectured growth points in multiplication and division represent key stages or goals for students. It seems that the *Abstracting* growth point represents a significant barrier, and that, to achieve this growth point, students need to move towards solving problems without using physical models. Further, students may need experiences both with forming mental images to solve problems, as well as with various multiplicative situational contexts. For students who can solve multiplicative problems by modelling, some specific activities prompting visualisation of multiplicative situations, broadly defined in groups, and arrays, multiplicative comparisons and Cartesian products seem desirable (see Sullivan et al. (2000) for examples of such tasks). Tasks that explicitly remove the materials seem desirable as a first step. It also seems that some teachers are much more successful than others in terms of the number of students who cross the *Abstracting* barrier.

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The Struggle Towards Algebraic Generalization And its Consolidation

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The activity of abstraction is central to mathematization. In the past, it has been discussed but generally not studied experimentally. The following study exemplifies a way for tracing processes of mathematization. We extend the nested model of abstraction elaborated by Hershkowitz, Schwarz, & Dreyfus (in press), to study successive activities of two Grade 7 students collaborating to solve algebra tasks in technological learning environment. The analysis demonstrates the consolidation of abstractions among the activities.

Theoretical framework

Mathematical activity, like any other human activity, is embedded in a socio-cultural environment (e.g., Voigt, 1995). This view is increasingly accounted for by the mathematics education community, which sees learning as a culture of mathematization in practice. Such an enculturation gains from alternating collective with individual activities, analytic with reflective stages, and integrating intra with inter-processes that are at the root of mathematical development (Hershkowitz & Schwarz, 1999a, see related ideas in scientific enculturation in classrooms in Woodruff & Meyer, 1997). A crucial issue concerns the relationship between construction of shared knowledge and the contribution of the individual (Hershkowitz, 1999). This issue is "hot", especially when the analysis does not focus on sole activities but on a series of activities.

Like in several other studies conducted in the CompuMath project (e.g., Hershkowitz & Schwarz, 1999a, 1999b; Schwarz & Hershkowitz, in press), we adopted the *activity theory* perspective (Leonte'ev, 1981), as a framework for studying different forms of practice of the individual and of the group within an between activities. The *activity theory* is a descriptive tool appropriate for socio-cultural analysis. The unit of analysis is not the individual human action but the activity as a whole, that is, "the minimal meaningful context for understanding individual action" (Kuutti, 1996, p. 28). An activity is a chain of actions done (cooperatively or individually) on the same object. Participant's motives determine their actions. Artifacts mediate actions on objects. Activities are under continuous change and development, where parts of previous activities are often embedded in following ones.

Mathematisation is central in mathematical enculturation. Abstraction is at the heart of mathematisation (Freudenthal, 1991; Gravenmeyer, 1995). To study abstraction experimentally, Hershkowitz, Schwarz, and Dreyfus (2001) gave an operational definition of abstraction: an activity of vertically reorganising previously constructed mathematical knowledge into a new structure. The suggested model is based on three observable epistemic actions, which are nested in each other: *Constructing* (*C*) is the central action of abstraction. It consists of assembling knowledge artefacts to produce a new structure with which the participants become acquainted. The action of *Recognising* (*R*) a familiar mathematical structure, occurs when a student realises that the structure is relevant to the problem situation on which participants are engaged. The *Building-With* (*B*) action consists of combining existing artefacts in order to comply with a goal such as exploiting a strategy or justifying a statement. The *RBC* model of abstraction will be used in this article to trace the construction of new mathematical knowledge between different activities.

Collaborative problem solving in an interactive setting takes many forms. Kieran and Dreyfus (1998) recognised different types of peer interaction. The types designate the interaction itself and the same pair may adopt various types of interaction during the same setting (Kieran, 1999).

The study

We focus here on the work of two Grade 7 students who participated in a one-year algebra course. The basis for the selection of these two students was that they were used to talk to each other. Five activities were chosen out of an algebra course. The algebra course, an introductory course in the Compu-Math project (Hershkowitz et al, in press) consists of a sequence of activities organized around problem situations. Students had a spreadsheet program (Excel) at their disposal. The tasks in the algebra course were designed to give opportunities to students' construction of structures of mathematical concepts (algebraic variables and models) and of various mathematical processes (hypothesizing, making generalizations, testing hypotheses, interpreting representational information, solving and justifying). In the present study, we examine one socio-cultural setting in which these constructions took place. That is, we observe (a) the types of interactions while collaborative work is taking place and (b) the construction of a shared knowledge of the pair and, the contribution of each participant, as well as what is left of it in the individual. All these aspects are examined within and between five activities as one continuum along the academic year.

All activities were open - no guidance for solution was provided, neither instruction for making use of the Excel. The tasks in each activity were of increasing difficulty.

The work of the pair in the class was videotaped and written work was collected. The videotapes were transcribed. Following Chi (1997), the protocols were divided into "cognitive segments". The dimension of interaction was considered and analyzed as well (Resnick et al., 1993; Hershkowitz, 1999), so that the chronological flow of the interaction and its logical flow might be seen clearly, including the underlying assumptions and motives of the students.

In this presentation we will analyze selected parts from the last activity (The *Sequences of Dots* activity) in which the creation of new structures of knowledge

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takes place while students collaborate together. Conclusions concerning knowledge constructing and ways of interacting will be drawn.

The *Sequences of Dots'* activity.

This activity took place at the last month of the year while students were quite familiar with the spreadsheet, and were accustomed to work pair with dedicated peers. The activity consists of 6 tasks, two of which we focus on here. In each of the first two tasks students observed a sequence of dots (see Figure **1a** & **1b**), were asked to discover a pattern for the number of dots in the shapes of the sequence, and to express it algebraically. Such a kind of activity was quite new to students.

Figure 1 – a. the first three shapes in Task 1; b. the third and fifth shapes in Task 2.

Possible path for solving the tasks.

Note: Students may obtain a sequence of numbers describe a given pattern on a spreadsheet by using one of the following generalizations methods: (a) Relating recursively to the previous number in the sequence (usually appearing in the previous cell of the same column) (b) The explicit generalization - using the position numbers (usually appearing in the same row in an adjacent column). Whenever possible, most students tend to use the recursive method, in which they consider locally the difference between two consecutive numbers of the sequence. With the spreadsheet tool, the recursive strategy, which is primarily local, may turn to be global when the dragging operation is used, thus leading to a recursive generalization. It is obvious that educators valorize more explicit generalization - the position number method - in an algebra course, because it articulates the algebraic method of modeling which is general, and can be used in learning equations and functions.

Each of the first two tasks in the Dots activity was designed to promote generalizing of the dot pattern into an explicit algebraic expression, using the position number method (b). The design of the task supports the connection between a specific *counting method* of the dots number in the specific elements of a given dotted sequence and the corresponding algebraic expression. For example, in the sequence in Figure 1b one may see a central dot and four "arms" each of which containing n dots, leading then to the expression $1 + 4n$. Alternatively, one may count the horizontal "arm" with $2n + 1$ dots, and the vertical "arm" with $2n + 1$ dots, and subtracting one dot counted twice. This counting method is reflected in the expression $2*(2n + 1) - 1$. In short, various *counting methods* lead to different expressions, which are all equivalent. Linking together the counting method and the algebraic expression may support and help student at an early stage of instruction to generate symbolic generalizations.

In the first sequence (Figure **1a**), the three first shapes are presented, to help students to familiarize with the idea of sequence of shapes. In the second task (Figure **1b**), two non-consecutive shapes are presented. Our experience from experimental classes, showed that presenting a sequence of consecutive elements leads students to express the pattern they could generalize through a recursive method.

Analysis of the collaborative work of a dyad on the first and second task

In this part we will describe shortly the work done by a pair of students Avi&Ben on the task. We will present selected parts from the full transcript of their work in this activity.

Avi (A) & Ben (B) started working on the first task of the activity (Figure **1a**). They made several trials to generalize the given sequence of dots. Some of the trials were wrong. Finally they counted the first sequence of dots systematically in a correct way, generalized it through the use of verbal informal explanations, but failed to generate an algebraic expression. They moved on to the second sequence, and initiated a correct counting method. Yet, as Ben himself mentioned it, they did not generate an algebraic expression. Following the intervention of the teacher (T), they succeeded in finding an expression using the position number method. Then, they voluntary went back to the first task, counted it globally, using a correct method -- in a sense an application of the method they used in the second task, to generate also an explicit algebraic expression, using the position number method.

We go through some episodes to clarify the above general description:

In the first episode Avi is about to find a correct way to count the number of dots in each shape of the first sequence (Figure 1a) using the position number method.

A52 *I think I found something here*
B53 *What?*
A54 *In the first we add 4, in the second we add 6, in the third we add 8, in the fourth we add 10. Its place A…*
A55 *I don't know how to write this expression. I found the pattern: the first add 4, the second, you take the second and add to it 6, the third, you take the third and add 8, the fourth, you take the four that is its place and add 10.*
…
B87 *I know what can we do*
A88 *What?*
B89 *=A1+, NO, it will not do it. We need something that each time what we are adding will grow by 2.*

We see that Avi generates a systematic counting method (A54-A55), and expresses it by using the positing number. But, he does not know how to formulate it as an algebraic expression (A55).

After a while (B87-B89), Ben tries to write Avi's idea on the spreadsheet and fails as well.

The two boys move to the second task (Figure **1b**). So far, Avi was dominant. We will see in the next episode that this dominance will vanish. Both students observe the shapes in Task 2, and try to find how many dots there are in the 20th shape.

A105 *How do you know that there are 81?*
B106 *What? Because always on each line there are three at each side, or one, it depends on n, it depends on the number, it depends on the order.*
A107 *If it is 3 you add 1?*
B108 *No, here, this is the third place, so I have here 3, 3, 3, 3 and here another one.* [Ben waves his hand successively to the left, right, up, down, and finally points to the central dot.]
.....
B114 *Did you find the expression?*
A115 *No.*
T118 *Can you describe how the fourth shape is going to look like?*
B119 *Yes, four, that's like, four from every side, and here, an extra dot.* [pointing to the central point]
T120 *Four on every side and...*
B121 *And one point in the middle.*
T122 *And at the 200th place?*
B123 *Then it will be, how much is 199 divided by four?*
T124 *Wait, how the shape is going to look like at the 200th place, not the one that is made of 200 dots.*
A125 *Yes, a dot and another 200 on each side.* [draws in the air, lines at every side. Ben acquiesces]
T126 *A point in the middle and...*
A127 *200 on the right, 200 on the left, 200 up and 200 down.*
B128 *No*
T129 *Do you agree? Yes or not?*
B130 *Oh, yes, yes.*
T131 *So how would the n shape look like?*
AB132 *Oh, one plus n times 4.*
A133 *No, a dot and n here, n here, n here and n here* [show with his hand]

As we can see here, Ben is aware that he is invited to *construct* an algebraic model of the pattern. He fully understands of the pattern governing the sequence of shapes (B106, B108). He explains it to Avi, and yet, cannot find the algebraic expression (B114). We can see that Ben *Recognizes* the structure of the given shape in the sequence as being symmetric, consisting of four equal sides with one dot at its middle. Ben can *Build-With* it numerical solutions for various elements (B119, B123). The intervention of the teacher (T124) pushed the students to work on the shape at the 200th place. The students cannot use direct counting strategies anymore and are led to imagine the 200th shape, which is a significant step towards generalization. In this intervention, the teacher helps them to consolidate the pattern constructed before. Ben is even able to answer a question requiring "backwards" numerical thinking (B123), but he needs the support of the teacher in order to express the algebraic expression explicitly with the position number method. Both peers show understanding of the expression (AB132). The mediation of the teacher makes it possible for them to make the necessary *Construction* that leads from the verbal model to the algebraic model based on the position number method.

In the next episode, the two boys are going back voluntarily to the first task, in which they failed generalizing the given pattern algebraically:

B155 *So here, that's what I'm saying, each time it's like, here we have to add four, here six and here eight.*
...
T173 *The counting method that was used there didn't give you any idea? What was the idea there?*
B174 *Oh, I get it. I know what. It is like one of the lines, and we can double ,*
A175 *Oh, I found it, here we add one, one here and one here.* $=A1 + 3$
[writes the expression in the computer and drags down], *oh, no!*
B176 *OK, I found it*
A177 *Yes?*
B178 $N + (N + 1) * 2$

We can see that Ben capitalizes on Avi's idea in their last attempt of generalizing the sequence (B155). Following an additional intervention from the part of the teacher (T173), Avi phrases a wrong generalization, combining together generalization by recursion and generalization by position number (A175). Ben implements successfully the *Construction* he made by himself in the second task, and generates a correct explicit algebraic expression, based on the position number. The knowledge, which was constructed in the second task, was *recognized* here, in a similar mathematical task, and was used for *Building-With* the needed generalization.

Conclusion remarks

We described here a pair of students working in collaboration in order to solve a problem. Both of them were determined to solve it, listened to each other, and tried to explain their ideas to each other. Kieran and Dreyfus (1998) designate this interacting style as *inhomogeneous*, each of the student trying to make an effort to understand his partner's thought (Trognon, 1993).

The analysis of the collaborative work of Avi&Ben is an additional example in which the RBC nested model of abstraction (Hershkowitz, Schwarz, & Dreyfus, in press) can be used. Here as well we can see that the first two actions (*Recognizing & Building-with*) are nested at the *Construction* action. Hershkowitz, et al. did not show that newly created knowledge structures are consolidated, as they are used as artifacts in further activities. In the present we succeeded to show how such a transition may happens; where the previous *Constructed* knowledge is becoming *Building-with* in the second cycle of the pair work on Task 1. The importance of this example is beyond the specific knowledge that was constructed and used later on in the activity. It shows that the suggested RBC model can be used as a methodological tool, by which consolidation of abstracted knowledge can be observed and investigated.

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Cognitive Mechanism of Constructive Activity Development

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In this paper, with respect to works of L. Vygotsky, A. Leont'iev, P. Anokhin and others, I will analyze the key concepts of the modern activity theory: structure of activity; zones of cognitive development; and instructional sequence of cognitive development. I will also consider comparative issues on the relationship between activity theory and constructivist approach. Based on this analysis, the conception of constructive activity[*] *will be considered as an integrated theory. The main philosophical idea of this integration is that if constructivism intends to understand the world, and activity theory appeals for changing the natural and social reality, then conception of constructive activity aims on changing the reality through understanding.*

First, what is activity? "Activity is a specific form of the societal existence of humans consisting of purposeful changing of natural and social reality" (V. Davydov, 1999, p. 39). Activity theory rooted in philosophy, psychology, sociology, and anthropology. The major beliefs of activity theory are: human activity changes nature; human beings change their own nature by activity (how people work implies how people think); social, cultural, and historical processes are important for understanding of individual's development.

Activity theory has its own view on the knowledge construction. There is no such a debate "whether knowledge can be transmitted by a teacher or it can be constructed by the learner" in the activity theory. The main principle of activity theory is *knowledge can only be constructed through the activity.* Studying the concept of activity in Soviet psychology J. Wertsch mentioned: "According to them [Leont'iev and his fellow researches], neither the external world nor the human organism are solely responsible for developing knowledge about the world. They argue that the key to the process is the activity in which the human agent engages" (J. Wertsch, 1979, p. 38). Some constructivists already paid attention to this principle. Activity, in whole, and mental actions, in particular, might be a starting point in the integration of constructivism and activity theory. L. Steffe and H. Wiegel (1996, p. 486) write: "Mental operations form strong connecting links between Soviet activity theory and constructivist approaches."

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[*] The term "constructive activity" was first introduced by E. von Glasersfeld (1987).

Comparing the two philosophical ideas, I would underline that if constructivism intends to understand the world, activity theory appeals not only for understanding but changing the natural and social reality. Though the activity theory "was born" as a philosophical and anthropological doctrine, nevertheless, it had and continues to have tremendous influence and importance for educational practice. Below I will consider "three elephants" or three main educational issues of modern activity theory: structure of activity, zones of cognitive development, and instructional sequence of cognitive development.

Structure of Activity

The issue of structure and components of activity is still remaining one of the unsolved problems of activity theory (V. Davydov, 1999). According to the fundamental work of A. Leont'iev on activity theory "Activity. Consciousness. Personality" (1977), the structure of activity includes such components as motives, goals, conditions, actions, and operations. There are three pairs of components in this sequence: activity – motive, action – goal, and operation – conditions. In other words, activity is motivated, actions are goal-directed, and operations are depending on conditions. Leont'iev explains that "the difference between actions and operations emerges … in the case of actions involving tools. For example, one can physically dismember a material object with the help of a variety of tools, each of which defines a way (an operation) for carrying out the given action (the dismemberment)" (Leont'iev, 1977, p. 107). Analyzing Leont'iev's structure of activity V. Davydov mentioned: "If we examine this structure, we notice the absence of the means of solving a problem. It seems clear that this component should be added" (Davydov, 1999, p. 45).

Actions usually come from an emerging problem or task (e.g., real-life situations, inquires, surprise situations, puzzles, paradoxes, sophisms) which reflects a cognitive conflict or intellectual difficulty and encourage student's curiosity. The way for carrying out goal-directed actions in the problem solving process is called an *operation*. The set of goal-directed actions and conditionally determined operations we call *a technique*. The set of techniques to accomplish the activity we call *a method*. Technique corresponds to actions and operations, and method corresponds to activity. Techniques and methods are means or instruments of solving a problem. Activity systems represent different aspects and domains of social and cultural life. Culture as a set of activity systems is relevant to social needs, activity is relevant to motives, techniques and methods – to instruments of solving a problem, operations – to conditions of problem, and actions – to goals of given problem (figure 1).

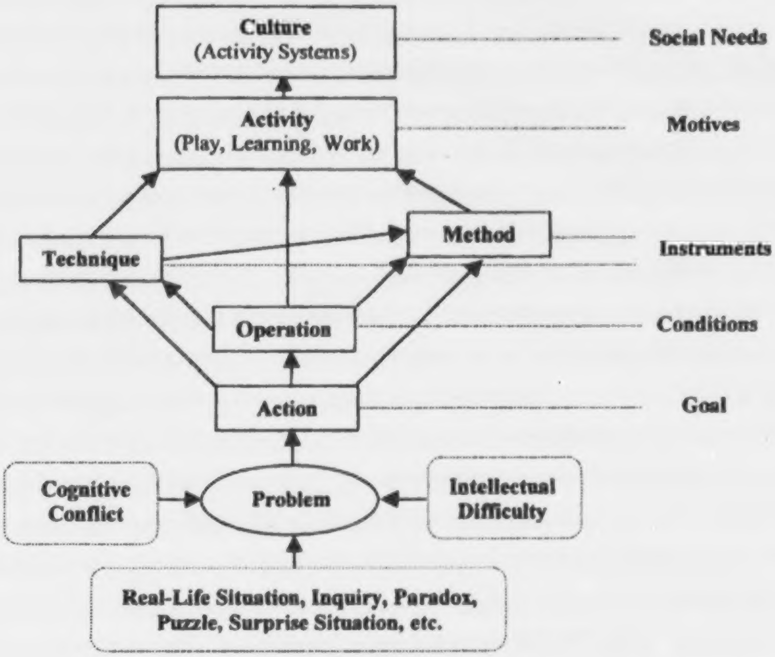
Culture
(Activity Systems)
Social Needs
Activity
(Play, Learning, Work)
Motives
Technique
Method
Instruments
Operation
Conditions
Action
Goal
Cognitive Conflict
Problem
Intellectual Difficulty
Real-Life Situation, Inquiry, Paradox, Puzzle, Surprise Situation, etc.

Fig. 1. Structure of activity

Zones of Cognitive Development

This concept is another "elephant" of activity theory. Vygotsky noted that the possibilities of genuine education depend not so much on the already existing student's knowledge and experience (level of actual development) as on the characteristics that are in the zone of proximal development. He wrote: "Pedagogy should be oriented not toward yesterday, but toward tomorrow in child development. Only then will it be able to create, in the process of education, those processes of development that are at present in the zone of proximal development" (Vygotsky, 1996, p. 251). Zone of proximal development is the distance between what child knows and his potential for knowing with the help of "more knowledgeable other". It is necessary to stress that in Western pedagogy the main attention is paid to the ZPD (zone of proximal/potential/nearest development) though Vygotsky considered ZPD as one of the domains between the lowest and highest levels of cognitive development. "We always should determine lowest threshold at which instruction may begin. But it is not the end of the deal: we should be able to determine the upper threshold of instruction as well. Only between these thresholds instruction might be fruitful" (Vygotsky, 1996, p. 251). The lowest threshold is the level of actual development (LAD) which contains the student's actual knowledge, skills and experience. Then follows the zone of proximal development (ZPD) which aims on cognitive change basically connected with the guided development of student's understanding. "The ZPD is the locus of social negotiations about meanings, and it is, in the context of school, a place where teachers and pupils may appropriate one another's understandings" (D. Newman, P. Griffin, M. Cole, 1989, Foreword by S. White). There is one more zone after ZPD. When Vygotsky wrote about the upper threshold, he didn't mean that it is equal to ZPD. Till now nobody has paid attention to this important fact in Vygotsky's work. It is a new zone - zone that goes beyond the development of understanding. It is a zone of formation of student's creativity. Whereas in ZPD the functions of comparison, reproduction, assimilation, and coping are of primary importance, in a new zone the functions of construction, generation, and creation are most important. This upper threshold of instruction and cognitive development we call a zone of advanced development (ZAD). If ZPD is the interpsychological dimension where social activity and interpersonal dialog is taking place, ZAD is the intrapsychological dimension where advanced individual activity and intrapersonal dialog is going on.

Activity cannot be understood as simple internalization of ready-made standards and rules. S. Rubinshtein (1973) stressed that human activity presupposes not only the process of internalization but also the process of externalization when humans create new standards and rules. So, if the psychological outcome of ZPD is internalization, for ZAD – it is externalization. According to L. Vygotsky the guidance is crucial in helping student move from LAD to ZPD. We cannot say the same about student's transfer from ZPD to ZAD. In other words, if ZPD is a domain of guided cognitive change (understanding), ZAD is a zone of student's individual (independent) activity. Therefore, we consider ZAD as a domain of higher cognitive achievement and creativity which student may reach in the process of intense individual studies (figure 2).

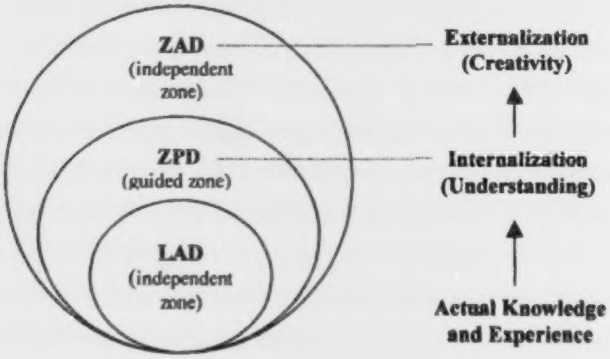

Fig. 2. Zones of Cognitive Development

Sequence of Cognitive Development

Human activity is conscious and it exists in both collective and individual forms. L. Vygotsky argues that determination of individual consciousness and cognition might be presented by the following sequence: *collective (social) activity – culture signs/symbols – individual activity – individual consciousness*. That is why, for Vygotsky and his fellows, it is very important to examine the transformation of all aspects of this schema in the study of the sequence of cognitive development. Based on this schema, P. Gal'perin and N. Talyzina (1968) worked out the conception of orientation and stepwise formation of intellectual actions that includes the sequence of following stages: orientation, "hands-on" actions, thinking aloud, inner speech, and mental actions. *The conception of orientation* is a central idea in Gal'perin and Talyzina's (G&T) theory. There are actions with complete and incomplete orientation bases. For operating in the zone of proximal development complete and detailed orientation is efficient. At the same time, theoretical and experimental studies (O. Tikhomirov, 1999, p. 349) show, that in accomplishing creative activity and operating in the zone of advanced development incomplete but systems orientation is required. "Hands-on" actions is the "doing" stage in G&T model which include manipulations, modeling, physical actions, and experiments with external representations. During this stage "public" (peer and small groups learning, whole group discussion, etc.) discourse is crucial. This corresponds with "collective (social) activity" in Vygotsky's schema. Then "private" discourse (e.g., thinking aloud, inner speech) accompanies the development of student's mental activity. Needless to say, that two of these activity schemata complement each other by pointing out different aspects. In our research, we integrated Vygotskian and G&T models in order to design *a cognitive structure of constructive activity development.* This structure includes the following stages:

1. ***Orientation stage:*** cognitive entrance to the problem/activity/thematic unit guided by instructor (who defines the level of completeness of the orientation base), creating an intellectual difficulty (cognitive discomfort) by using challenging problems, real-life projects, paradoxes, misconceptions, surprise situations, fallacies, etc. From psychological point of view this stage reflects such elements of afferent synthesis as actualization and motivation.
2. ***Hands-on stage:*** approaching the problem using standard external representations given by instructor or chosen by students (manipulatives, physical objects, visual tools, etc.) as well as expression of initial students' representations through a "public" discourse (e.g., exchange of students' ideas, peer learning, small groups, whole class discussion).
3. ***Minds-on stage:*** a "private" discourse (e.g., students' thinking aloud, inner speech) and development of students' conceptual understanding. In this stage

the primary attention should be on making inter- and intra-subject connections and development of students' internal representations (e.g., mental images, schemata, metaphors).

4. ***Generalization stage:*** reflection, extension, and construction. Focus should be on development of students' creativity using general methods for advanced problem solving and reasoning. It is also a stage of efferent synthesis and evaluation. If orientation stage reflects the level of actual development (LAD), hands-on and minds-on stages reflect the ZPD, then generalization stage is the zone of students advanced development (ZAD).

The advantage of this model is that it contains an attempt to create a sequence of instructional stages for cognitive development. One of the main disadvantages of constructivism is an absence of such kind of instruments. Unfortunately, the point of confusion for teachers who are trying to implement a constructivist approach is that there is no clear procedure how to do it. Hopefully, an integrated conception of constructive activity "takes care" of this confusion.

Another concern is a difference in views on the role of representations in cognitive development. Constructivists P. Cobb, T. Wood, and E. Yackel (1992) reject the "representational view of mind". They ignore basic principles of theories of L. Vygotsky and J. Piaget according to which representations (culture signs and symbols) are necessary tools in the process of transition from collective (social) accomplishment of an activity to individual accomplishment. Sign systems "are the real bearer of human culture, the means by which individual activity and individual consciousness are socially determined. The incorporation of signs into the structure of a mental function (mediation through signs) links that function to culture. On the one hand, a sign is always supra-individual and objective since it belongs to the cultural world, but on the other, it is individual since it belongs to the mind of particular person" (V. Davydov, V. Zinchenko, 1993, p. 102). Therefore, an involvement of multiple representations into the process of constructive activity plays a significant role in mediation between an external reality and student's internal cognitive growth.

We also consider the cognitive structure of constructive activity development as *a functional system*. The main feature of functional system is its invariance. P. Anokhin and A. Luria stressed an importance of functional systems in brain research: it could sustain functioning regardless of partial damage of the brain. Approaching a situation of cognitive change, D. Newman, P. Griffin, and M. Cole (1989) "view the activity in the ZPD as constituting a functional system". At the same time they erroneously consider that "the zone of proximal development is a functional system for cognitive development" (ibid.). They wouldn't have come to this conclusion if they had taken into account an existence of ZAD along with ZPD and the structure of functional system which includes afferent synthesis, decision making, anticipation, goal-oriented action, efferent synthesis, and evaluation (Anokhin, 1978). Because then it would become evident that functional system is a cognitive mechanism for the ZPD and ZAD construction, opposite to what D. Newman, P. Griffin, and M. Cole thought it to be.

Outcomes of the experiment on implementation of the conception of constructive activity in teaching and learning of secondary mathematics show important cognitive changes on students' motivation, attitude, and confidence. The experiment took place from 1990 till 1996 in number of Russian high schools (Moscow, Kazan) with total enrollment of 650 students. After the experiment students' self-evaluation showed that 88% of them considered their progress on high level, 8% - on average level, and only 4% of students responded that there was no progress in their learning of mathematics. 85% of students reduced their anxiety and obtained confidence in mathematical problem solving and reasoning. 76% of students changed their attitude from negative to positive toward mathematics. Before the experiment only 16% of students were interested in mathematics, but after it 84% of students were ready to continue study mathematics on the advanced level (M. Tchoshanov, 1996, p. 148).

Conclusion

Current stage of development of psychology of mathematics education is characterized by growing new learning theories and pedagogical approaches (e.g., constructivism, situated learning, etc.). There is an emerging necessity of integration of "new" theories with more traditional ones. It will create a polyphonic pedagogical environment and help to avoid artificial contradictions between relatively close pedagogical theories. The key advantage of polyphonic pedagogical systems before monophonic ones is that they can accumulate new integrative quality, which will allow improving education in a global community. Based on this idea, we considered an integration of activity theory and constructivist approach with respect to development of conception of constructive activity. The key pedagogical idea of new conception is that *knowledge can only be constructed through the oriented activity*, which follows the sequence of instructional stages: orientation, hands-on and minds-on actions, and generalization. The basis of orientation (complete or incomplete) determines the level of cognitive development: beginning with understanding in the zone of proximal development up to creativity in the zone of advanced development. The pedagogical strength of this conception is that some details of learning process may change but the pedagogical functional system, as a cognitive mechanism of constructive activity development, remains intact.

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Representations as Conceptual Tools: Process and Structural Perspectives

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Much has been written about the apparent duality of many mathematical concepts as processes and objects. Further, the importance of the representational forms in which mathematical concepts are presented has been analysed. In one of these representations, the symbolic form, the idea of a procept linking process and object forms, has proven a valuable concept. However, this analysis has been lacking from other representations. In this paper we look at the characteristics of the process and object view of some other representations with a view to beginning a classification of interaction with them and give examples to indicate the validity of the classes.

Process and Object

Many mathematical concepts take the form of objects, and these may arise in a number of different ways. Tall, Thomas, Davis, Gray & Simpson (1999, p. 239) have distinguished three types of object construction: perceived objects; procepts; and axiomatic objects. The second of these has been given considerable attention over a number of recent years, with researchers describing in detail both the distinction between the dynamic process and static object view of mathematical concepts, which Sfard (1991), calls an operational and structural duality, as well as the manner in which the former is transformed into the latter in the mind of the learner. Sfard (*ibid*) proposes that processes are *interiorised* and then *reified* into objects, while Dubinsky and his colleagues (Dubinsky & Lewin, 1986; Dubinsky, 1991) talk about processes being *encapsulated* as objects and have imbedded this in an Action-Process-Object-Schema or APOS theory, for the construction of conceptual mathematical schemas. The term *procept,* as used above, arose in the work of Gray & Tall (1994) to describe the use of mathematical symbols to represent a process (which the symbols may invoke) or a concept (which they may represent), or, depending on the context, the viewpoint, and the cognitive aim of the individual.

These theoretical ideas have proven useful, with widespread applications in algebra, calculus, and advanced mathematical thinking, as described by Tall (2000), and others (e.g., Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Clark, Cordero, Cottrill, Czarnocha, Devries, St. John, Tolias, & Vidakovic, 1997).

Representations of Mathematical Objects

We note that in the discussion on the nature of mathematical concepts and their description in terms of process and object, and especially in the case of procept, there has been a firm emphasis on the symbolic representation. This is understandable of course, because of the tremendous power of symbolism, and algebraic symbolism in particular. However, a key component of schemas is the representation of conceptual processes and objects in a number of associated but different ways (Kaput, 1987, 1998). How then do representations other than the algebraic symbolic relate to the process-object conceptualisation of much of mathematics? If this is a valid view of the underlying mathematical concepts then there should be corresponding perspectives for these. In particular, the graphical and tabular representations which are so common in secondary mathematics should be amenable to such an analysis. In the symbolic representation it has been proposed that the object view enables one to take the object and perform an action on it, using it for example in a further process. An example of this is the construction of groups of functions. But this is due to the power of the symbolic notation, mentioned above. What is the corresponding position with tables and graphs, and other representations, and how would we recognise this shift in perspective for them?

We argue here that a representation can be seen as a multi-faceted construction which assumes different roles depending on the way that students interact with it. When an image is on a computer screen, for example, Mason (1992) has suggested that students can be *looking at* the images or *looking through* them depending on the focus of their attention. In this sense we say that students can interact with a representation in at least two different ways, by observing it or acting on it. The observation can be at a surface level, looking at, or at a deeper level, looking through. For example, looking at a representation a student may comment on a property of the representation itself but by *looking through* it students may use it to assist them to notice properties of the conceptual processes or object(s) represented. This is in line with the *property noticing* of Pirie & Kieren (1989), who talk about how images can be examined for specific or relevant properties. An example occurs in the paper by Ainley, Barton, Jones, Pfannkuch, and Thomas (2001), where, looking at a graphical representation on a spreadsheet of five data points, two of the students comment on surface features of the representation, saying, for example, that "It's a hill" and "It's like a mountain there" (p. 3). In contrast, if they had been looking through the graphical representation they may have commented on properties of the function, such as its maximum or minimum values.

While this idea of observation of representations is important, in order to build rich cognitive structures more is needed. When he goes beyond such acts of observation of a representation and performs an action on it, *doing* in the sense of Mason (1992), in order to obtain further information or understanding from it, then we maintain that the representation becomes a *conceptual tool* for that student. The metaphor of a tool is appropriate here since it is not the shape of a spade, or its properties that make it a tool, but the actions of a person using it. Such *learning from* activity has been described as *construing* by Mason (1992) and it is this type of activity which gives rise to a *conceptual representation tool*. However, thinking back to the process/object views of concepts, the ways in which a student observes or acts on a particular representation will depend on whether they have a process or an object view (or both) of the concept(s) it represents. For example a student may use a table or a graph and perform linear interpolation on values obtained in order to approximate an intermediate value of the function. Whether the student sees the function as the sum of the discrete results of an input-output process or as a function object may not be clear. In contrast, to be given the graph (or table of values) of a function *f(x)* and being asked to draw the graph of, say, the function *f(x+1)*, when there is no specific function given, may require a structural or object view of the function.

Table 1 *Possible Modes of Interaction Between Student and Representation*

Interaction	Concept View	
	Process	Structural/Object
Surface Observation	Process Surface Observation (PSO)	Structural Surface Observation (SSO)
Observation of Conceptual Properties	Process Property Observation (PPO)	Structural Property Observation (SPO)
Action on the Representation	Conceptual Process Representation Tool (CPRT)	Conceptual Object Representation Tool (CORT)

This gives a matrix of six different possible modes of interaction with a conceptual representation, as delineated in Table 1. We have tried to exemplify each of these modes below, choosing to talk about a structural rather than an object view of concepts in this context, since it seems to convey the idea better. One should not get the impression from the discussion so far that because it is the primary source of examples that these concepts are only applicable to the learning of function. An example from group theory may help support this contention. Figure 1 contains a representation of the klein four-group, namely the multiplication table of the set {1, 3, 5, 7} under the operation of multiplication modulo 8.

× mod 8	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Figure 1. A multiplication table representation of the klein four-group.

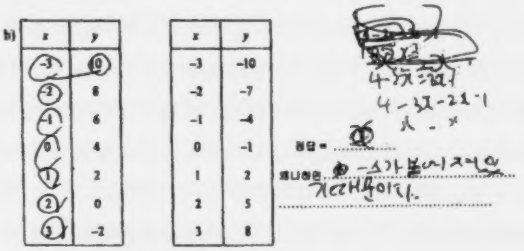
A student with a process view of a group as the sum of the combinations of elements under the law of composition may interact with this representation by (for example): spotting that combining elements in the set gives rise to only elements of the set, since these appear in the body of the table (with no reference to closure) (PSO); noticing that every element is self-inverse (PPO); using the table to verify that $ab=ba$ for all $a, b \in \{1, 3, 5, 7\}$ (CPRT). Alternatively the student who has a structural view of a group may interact in quite different ways by (for example): noticing that there is a leading diagonal of ones in the table (SSO); observing that there is a subgroup {1, 3} or that the symmetry of the table means that the group is Abelian (SPO); demonstrating by rearranging the columns of a table (we assume here the necessity of this) that the group is isomorphic to the group represented by a multiplication table of the matrices:

$\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \quad \pm\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ (CORT, action to compare two object structures).

It needs to be said at this point that it is often difficult, even with student interviews, to be sure whether a particular student has a process or object view of a given concept. However, we can still say whether he/she is acting on the representation and using it as a conceptual tool rather than simply observing it or even for abstracting properties. In cases like this we may call such usages *surface observation (SO), property observation (PO)* and *conceptual representation tool (CRT),* leaving out the process or object distinction.

Experimental Exemplars

Space does not allow us to exemplify from our research each one of the six categories here, but we will try to present sufficient examples to allow the reader to see how they arise. What we have found in our recent research is that students often interact with function graphs and tables of values in a *process-oriented* manner (Thomas, 1994), as series of discrete pairs of values or points. In Figure 2 we see an example of a student who, in trying to solve the equation $4–2x = 3x–1$ has a process perspective of the two given tables, stating (translated from Korean) "because the values have (–) on the table". This is a surface property of the representation and so the interaction is a PSO.



b)

x	y
–3	10
–2	8
–1	6
0	4
1	2
2	0
3	–2

x	y
–3	–10
–2	–7
–1	–4
0	–1
1	2
2	5
3	8

4-2x=3x-1
4-3x-2x-1
정답 =
까닭은

Figure 2. Student A's process surface observation (PSO) use of tables.

This student has been asked to use the table as a tool to solve a linear equation, but because of his process-oriented perspective he was unable to relate the input-output values to the solution of an equation. In contrast, some students, after they had experienced the different related representations using calculators were able to solve these equations. Student B for example, who although unable to solve the equation correctly in the algebraic representation, making several errors, is able to use the tabular representation as a conceptual tool (CRT) to solve the equation (Figure 3), stating that the solution is $x=1$, $y=2$ because "when $x=1$ they are consistent with each other." We cannot be sure whether he has a process or a structural view of the tabular representation.

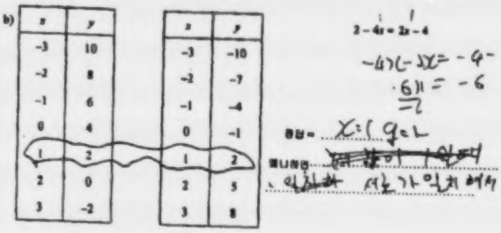


Figure 3. Student B's use of tables as a conceptual representation tool (CRT).

Other students were also using the tables as a CRT were able to give similar reasons for the same type of solution. We have also come across evidence that some students appear to see graphs in process terms, as a sequence of discrete points which happen to be joined with a curve. In Figure 4 we see an example of such thinking.

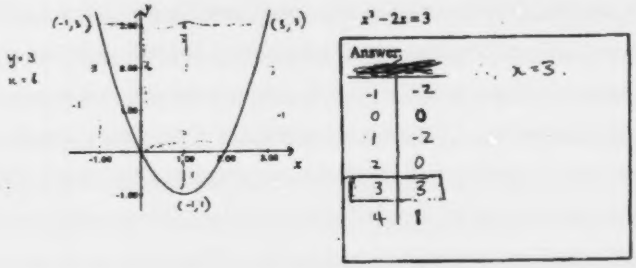
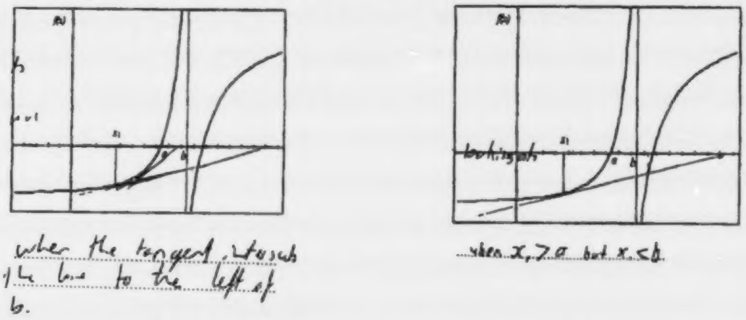


Figure 4. Student C's graph used as a conceptual process representation tool (CPRT)

Here student C, asked to solve $x^2 - 2x = 3$ using the graph, has acted on the graph in a process manner (CPRT), executing an inter-representational shift to produce a table of a discrete set of 6 integer-valued points. The answer happens to be incorrect in this case simply because of an error finding $f(-1)$.



when the tangent touch the line to the left of b.

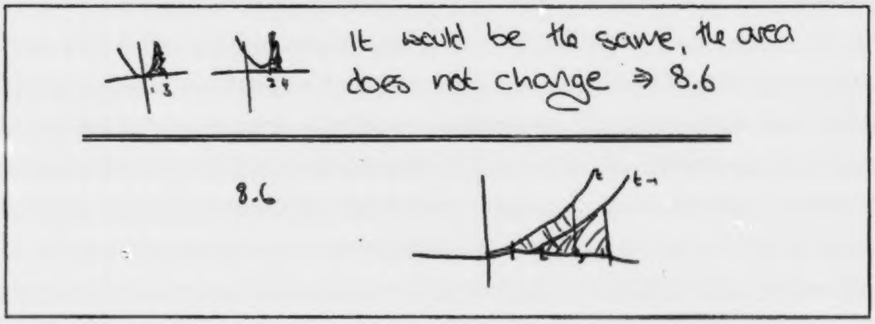
when x,>a but x<b

Figure 5. Two examples of using a given graphical representation as a CRT.

In Figure 5 students D and E employ the graphical representation of a function, whose symbolic formula is not given, as a CORT in order to explain conceptual ideas, namely, how the successive approximations in the Newton-Raphson method approach a root a, and when x_1 is a suitable first approximation for the root a of $f(x)=0$. These students are thinking conceptually using the graph and have no need to carry out a process. Students have also been able to operate conceptually on a graphical representation involving the relationship between the area under the graph of a function and its symbolic representation when the function undergoes a transformation which can be described as parallel to one of the axes, i.e., $x \rightarrow x \pm k$ or $f(x) \rightarrow f(x) \pm k$. Figure 6 shows two examples of students' work where they have interacted with a graphical representation they have constructed to answer a question with a symbolic presentation, namely:

If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$.

The first student has drawn separate graphical representations of the unknown function $f(t)$ (possibly using $y=t^2$), and the second has put both graphs on the same axes, but they have both acted on these by clearly marking the area represented by the symbolic definite integrals and operating on this area as a structural object, to answer successfully the conceptual question about the transformation. Once again These students are using the graph conceptually with no need to carry out a process. These are therefore examples of CORT interactions with the graphical representations.



It would be the same, the area does not change ⇒ 8.6

Figure 6. CORT use of graphs for area conservation under a transformation $t \rightarrow t-1$.

Discussion

We believe that the start we have made here on a classification of interaction with various mathematical representations of concepts has potential benefits for the teaching of mathematics. It is the teacher that is the key to benefits emanating from any theoretical position, and the types of interaction proposed above suggest possible ways in which teachers could address student learning. One approach they could be encouraged to consider is to construct lessons which build meaningful uses of different representations of concepts into modelling activities based on real world problems. Lesh (2000) has suggested that helping students to be able to construct conceptual tools that are models of complex systems in such a way that they can mathematise, interpret and analyse using these tools is a key goal of mathematics teaching. We strongly agree with his further statement that "… representational fluency is at the heart of what it means to "understand" many of the more important underlying mathematical constructs" (p. 74). Such 'fluency' includes the ability to interact with these representations, using them as conceptual tools, but doing so, as Kaput (1998, p. 273) suggests being aware of the potential "inadequacy of linked representations and the strong need to provide experiential anchors for function representations."

A second consideration for teachers of ways to approach the building of representational fluency is to consider use of the graphic and super-calculators, since the primary representations needed in schools arise naturally, in a dynamically related way, in the context of these machines (Kaput, 1992). For example, in many classrooms students may initially learn about functions only through the symbolic representation, becoming immersed in algebraic manipulations and equation solving. Only some time later will they approach the graphical solution of equations. By then it may be much harder for many students to build inter-representational links, if they are constrained to a process view of function. They may employ a surface, procedural method of solving equations graphically by drawing the graph of each function and reading off the x–value(s) of the point of intersection without engaging with deeper relationships of the four different representations: algebraic, tabular, ordered pairs and graphical. To build rich relational schemas based on these external representations, it seems a good idea that, where possible, students should interact with the sub–concepts of one–to–one, independent and dependent variable, etc. in each representation in close proximity, exploring the links between them.

The third implication for teachers is one of assessing diagnostically the type of thinking with which their students are approaching representational use. The term *versatile thinking* was used by Tall and Thomas (1991) to refer to the complementary combination of the sequential/verbal-symbolic mode of thinking and the more primitive holistic visuo-spatial mode, in which the individual is able to move freely and easily between them, as and when the mathematical situation renders it appropriate. However, with the theoretical stance we have presented here we can now enlarge this concept of *versatile thinking* and say that this would include the ability to move between the PSO, PPO, and CPRT modes of interaction with any representation and the SSO, SPO and CORT modes of interaction as and when each is considered applicable. Thus a useful goal for teachers would be to try assess the extent of their students' *versatile thinking* and aim to assist them to build it further so that they are not limited to a purely process approach to mathematics, important though that is.

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THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF THE QUOTIENT: A TEACHING EXPERIMENT

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Abstract: The conceptual development of the Quotient in four children in the US was studied through a series of parallel individual teaching experiments. At the beginning of the study, children viewed whole number division and fair sharing as two different domains: Division and fractions. By requiring children to confront fractional and division situations involving similar contexts, quantities and symbolization, children moved from a Whole Number Quotient scheme to a Fraction-as-Division scheme. At the end of the study, children were able to think about fractions in terms of division and were able to solve problems involving division of fractions.

Introduction

Research on understanding children's rational number concepts has been of keen interest to mathematics educators in the last two decades. Children's solutions to problems involving fractions and ratios have shown that children use additive schemes to solve such problems (Lamon, 1993; Resnick and Singer, 1993), and that children have difficulty in conceptualizing the rational number as *a* number (Mack, 1990, 1995; Behr et al., 1985). Children's difficulties with rational numbers can be attributed to the fact that rational numbers take different meanings across different contexts (Kieren, 1976; Behr, et al. 1983). Rational numbers are used to represent and control part-whole relationships; they are fundamental to measuring continuous quantities; in particular, they are involved where quantities are divided; and they are used in multiplicative comparisons of two quantities. Therefore, when we consider the variety of interpretations of rational numbers, it is not surprising that children have difficulties with rational number concepts.

One of these personalities that rational numbers takes on is the quotient interpretation (e.g., Kieran, 1993). Building phenomenologically from the contexts of fair sharing, under the quotient interpretation, rational numbers become quotients of whole numbers. More specifically, a rational number can be defined as the yield of a division situation. Considering this meaning, the relationship between division as an operation and fractions as quotients becomes evident. It is reasonable to assume, therefore, that interpreting rational numbers as quotients of division situations requires one to have a sound understanding of the division concept. Moreover understanding of the division concept beyond whole number partitioning should address the relationship between quotients as numbers and the operation of division as embodying a multiplicative relationship.

Although the quotient is considered the phenomenological source of rational numbers (Freudenthal, 1983; Streefland, 1991; Kieren, 1993), the question of how children conceptualize the quotient subconstruct still remains unanswered. Within the quotient subconstruct, any fraction can be seen as the result of a division situation that requires the distribution of the quantity represented by the numerator equally into the number of groups represented by the denominator. For example, 6/4 can be thought as the quotient of 6 things distributed into 4 equal groups.

Children first learn the concept of division partitioning whole numbers—e.g., fair sharing—and they express quotients as whole number partitions. Division situations with remainders are found to be difficult because instruction fails to engage students in the interpretation of resulting fractional quotients appropriately within the given problem context, instead putting off fractional quotients for later study. However, in research on division, there is a lack of study addressing the relationship between rational numbers and division of whole numbers and whether this traditional separation of whole number quotients-with-remainder and fractional quotients is pedagogically sound practice. Moreover, developmentally, there is little research that documents plausible trajectories by which children come to see the two as isomorphic conceptually (see Behr, Harel, Post, & Lesh, 1992, p. 308 which illustrates the limits of our current understanding in rational number research).

At any rate, current curricular offerings in the United States present whole number division and fractions as separate and distinct topics. Division is treated as an operation one performs on Whole numbers, and fractions are taught almost exclusively as Part-Whole concepts. Only in the late middle grades, when an understanding of quotient becomes necessary for dealing with algebraic entities such as rational expressions, are children expected to connect these two heretofore distinct topics. At no point in the preceding 7 or 8 years is explicit attention focused on teaching fractions as division, nor on teaching division as expressing a rational number: A quotient.

The purpose of this research was to study four fifth graders' conceptualizations of the quotient subconstruct under instructional conditions that expressly required them to confront and connect the isomorphisms inherent in thinking about fractions and in thinking about division. Specifically, it was intended to understand the transitional understandings children constructed as they moved from division of whole numbers to the description of a fraction as the quotient in division situations. Therefore, a special focus was put on how children conceptualized the remainder as the fractional part of the quotient in whole number division and how they moved to the conceptualization of a fraction as a quotient itself. In short, the research was designed to understand how children come to understand fractions as division, and concomitantly division as a number.

Method

A set of four parallel individual teaching experiments were conducted in studying the development of four fifth graders in an inner city school in the southwest United States. The primary goal of this teaching experiment was not aimed at uncovering how children thought about a given set of tasks, but at determining how children went about making sense of, and structuring their experience towards learning novel content, and in particular, the sense that they made in confronting the subdivision of the remainder of whole number division. The teaching experiment lasted approximately 8 weeks.

One of the students was Anglo, the rest were Hispanic and bilingual. Subjects consisted of two boys and two girls. A clinical interview at the beginning was conducted to determine baseline data and another clinical interview at the end was conducted to determine changes in children's schemes over the entire teaching experiment. Between the clinical interviews, children participated in teaching episodes. The researcher constantly modified the tasks to draw finer-grained details of schemes children were constructing at a particular moment of time (e.g., Steffe & Thompson, 2000).

Each teaching episode lasted approximately 60 minutes. Tasks were presented either in the form of a word problem or a division number sentence. The general strategy we employed was to present word problems that involved fair sharing contexts, which we felt would have the highest probability of eliciting quotient thinking in children, wording the problems to guide the students to both fractional and whole number division solutions. By pairing mathematically isomorphic problems that afforded these different interpretations, we forced students to confront their common qualities, and to draw connections between symbols in their respective notational forms. With symbolic problems, we paired problems that involved the same quotient relationship, but that utilized different notation (e.g., 13/4 vs. $13 \div 4$). In particular, we presented problems that required students to deal with the partitioning of the remainder in whole number division, or reconceptualizing the unit in the case of improper fractions.

Each child was interviewed individually. Children were asked to explain their thinking and show their work as they solved the problems. Interviews were audio-recorded. The researcher also took notes during the interviews. Therefore, there were three main sources of the data: Audio-recording of verbal interaction, student's written work and field notes of the researcher. All recordings were transcribed fully and analyzed qualitatively. Each child formed an individual case. After multiple reading of the data, children's schemes of quotient subconstruct were constructed and generalities across cases were established.

Results

The results of this study indicated that children constructed four schemes that embodied their progress in conceptualizing the quotient. These schemes appeared to follow a developmental order of 1) Whole Number Quotient, 2) Fractional Quotient, 3) Division-as-Number, and 4) Fraction-as-Division when children progressed toward an understanding of fractions as quotients. Based on the regularities in conceptual development evidenced by the children, and supported through the sequence of teaching episodes, a schematic illustrating how children coordinated these schemes to come to a more full understanding of the quotient subconstruct was generated (see Figure 1). More specifically, it describes children's progress from a disconnected understanding of fractions as representing part-whole comparisons and division representing a whole number operation, toward a more connected understanding of fractions as quotients of division situations.

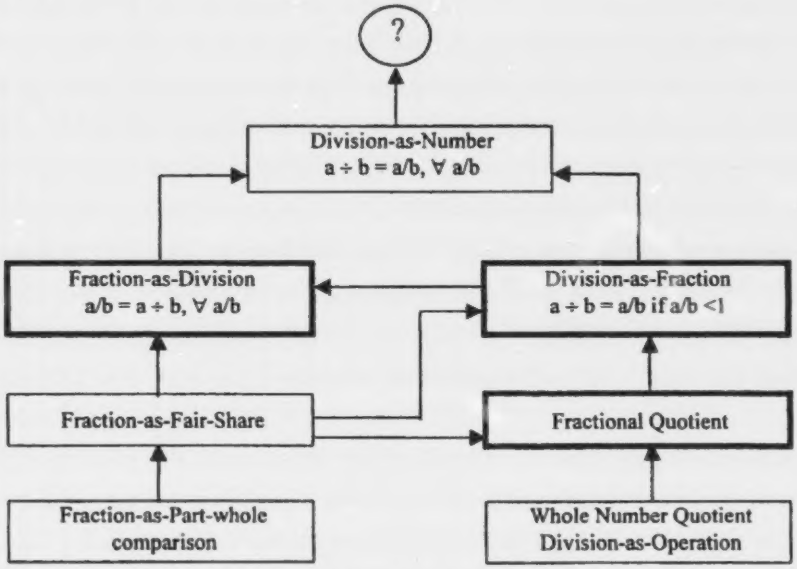
?
Division-as-Number
a ÷ b = a/b, ∀ a/b
Fraction-as-Division
a/b = a ÷ b, ∀ a/b
Division-as-Fraction
a ÷ b = a/b if a/b <1
Fraction-as-Fair-Share
Fractional Quotient
Fraction-as-Part-whole comparison
Whole Number Quotient
Division-as-Operation

Figure 1. Schematic Illustrating Children's Developing Connections Among Fractions and Division Schemes

In the rest of this section, each scheme will be described in detail.

Fractions-as-Part-Whole Comparisons. At the beginning of the teaching experiment, all four children conceived of fractions as representing part-whole relationships exclusively. When asked to describe the meaning of a number presented in fractional form (e.g., ½), they attempted to describe the given fraction in terms of part-whole relationships by partitioning a whole into the denominator and coloring the number of parts specified by the numerator. Beside part-whole comparisons, they were unable to provide a different explanation for the given fraction. Because in part-whole comparisons, the unit whole is limited to one, three of the children had difficulty in identifying the whole to explain a given improper fraction. One of these students, Juan, referred to these fractions as "bad fractions."

Whole Number Quotient Scheme. The Whole Number Quotient scheme appeared when children tended to provide whole number quotients for all division situations. It was also evidenced by their inclination to write quotients with remainders even when they converted the remainder to a fraction. In cases when the problem situation allowed the equal distribution of the remainder among the groups, they avoided writing fractional quotients as answers to division number sentences. It appeared that when they conceived of a situation as division, they provided the quotient with remainder even when they realized the possibility that the remainder could be further subdivided.

Fraction as a Fair Share. The Fraction-as-Fair-share scheme was illustrated by children's ability to solve given fair sharing problems in which the answer was less than one. In such situations, children were able to find the answer by partitioning the given quantity. However, when they were asked to write a number sentence representing their solution, they were unable to do so. Transcripts indicate that children did not initially conceive of fair sharing situations as cases of division even though they easily found the fractional quotients through partitioning. It was also evident that they held a conception of division in which the dividend is always bigger than the divisor. In such conception, the quotient of a division situation is never less than one. This was also supported by children's tendency to reverse the dividend/divisor order in solving symbolically represented problems in which the dividend was less than the divisor.

Fractional Quotient Scheme. The Fractional Quotient scheme (which appears to be a key transitional phase) was characterized when children started to symbolize fractional quotient situations as answers to "division problems." Children were comfortable with fractional answers when solving partitive word problems. However, for symbolically represented problems, they provided fractional quotients only after the problem was put in a partitive context. They started to conceptualize fair sharing situations as division because they recognized that fair sharing situations involved the creation of equal "groups" (i.e., partitions) similar to whole number division.

Division-as-Fraction Scheme. The Division-as-Fraction scheme was illustrated in children's ability to anticipate the quotient of a given division situation without applying any analytic way of solving it. Children came to this understanding after they started to symbolize fair-sharing situations with an answer less than one. This notion can be summarized as: If any a quantity was divided into b equal groups then the resulting quotient is always a/b. Hence, when a fair sharing problem was given, children first anticipated the answer using this rule. Once the answer was anticipated, they then were able to solve the problem conceptually. When children didn't get the same answer as the one they had anticipated, they had to verify whether their answer and the anticipated result were equivalent or not.

Fraction-as-Division Scheme. The Fraction-as-Division scheme was revealed when children reasoned about fractions in terms of division situations. For example, this scheme entails that ¾ can be considered as 3 things divided into 4 equal groups, equivalently 6 things divided into 8 equal groups and so on. Children began to use this reasoning as a parallel explanation to Part Whole to explain what a given fraction meant. Under this scheme, children were able to reverse the Fraction-as-Division scheme. In other words, without probing, they were able to realize that if any division situation could be represented as a fraction, the reverse was also true.

Division-as-Number Scheme. This scheme was illustrated by the children's ability to generalize the two way relationship between a division situation and the fraction representing its quotient to all division cases. In other words, $a \div b$ was equal to a/b for every $a>b$, where $b \neq 0$. However, only Kate constructed this scheme. The rest of the children did not generalize this relationship to all division cases. These children tended to write given any improper fraction as a mixed number so this hindered their ability to recognize this relationship.

Discussion

Previous research has shown that young children deal with quotient situations- fair sharing- easily (Lamon, 1996; Kieren, 1988). In a fair sharing situation, a typical behavior is to partition quantities and write the resulting fractions as the quotient of a given situation. From their partitioning behavior, it is concluded that children conceptualize those fractions as quotients. The results of the present teaching experiment, however, suggest that ability to partition quantities into its parts didn't necessarily indicate that children actually conceptualized the resulting fractions as quotients. This was evidenced by the general reluctance in symbolizing quotients less than one as division number sentences.

Children were reluctant to symbolize the situations in this manner for a variety of reasons. First, for them, "division" always yielded a whole number quotient with or without a remainder. It was not that they didn't recognize the partitionability of the remainder, it was the fraction, the result of the partitioning that they didn't conceive of as a part of the quotient. Typically, in their minds, "fractions" resulted from fair sharing situations which involved cutting or splitting of whole quantities into parts less than one, whereas "division" resulted from partitioning quantities into groups with number greater than one. Second, students' resistance to consider a fraction as a quotient was due to the the fact that children had such a strong understanding of the Part-Whole subconstruct which, from prior experience, was the only way they were used to thinking about a fraction.

Moreover the existence of an understanding of fractions representing fair shares and part-whole relationships and division as a whole number operation as separate entities does not necessarily imply that children will connect these understandings into a coherent quotient scheme without intervention. Rather, the results of the teaching experiment showed that some explicit connections have to be made between these concepts. By making the commonalities in context and notation between whole number division and fraction situations problematic , children need to be encouraged to reflect on the equivalency of fair sharing situations and whole number division. Using a common symbolization; children's conceptions of division can be challenged. In this teaching experiment, division number sentences were used to show that the result of a division operation is a fraction represented by the common divisor/dividend notation).

The findings of this study suggest that a fragmentary approach to teaching quotients, which arranges instruction into two distinct and separate conceptual dimensions, i.e., division as an operation and fractions (as primarily part-whole quantities, but also applying to simple fair sharing), may lead children to develop a fragmentary understanding of the quotient. Instead, the present cases suggest that providing common problems in both fractional and division forms, and confronting children with the basic equivalencies of these forms may be fruitful in developing a more powerful understanding of the quotient subconstruct. In addition, the commonalities among cases in this study suggest that instruction on quotient should be cognizant of how the linkages between division as an operation and fractions as numbers are made. In particular, the order children followed as they moved towards the quotient subconstruct suggests that both whole number division and fraction instruction can proceed fruitfully in concert, intertwined but not parallel.

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INVESTIGATING YOUNG CHILDREN'S STRATEGY USE AND TASK PERFORMANCE IN THE DOMAIN OF SIMPLE ADDITION, USING THE "CHOICE/NO-CHOICE" METHOD

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Abstract. *In this study we investigated young children's strategy use and task performance in the domain of simple addition, using the "choice/no-choice" method. Second-graders, divided in 3 groups according to general mathematical ability, solved 25 problems in 3 different conditions. The results showed use of multiple strategies, adaptive strategy choices, and group differences in strategy choice and strategy execution in parallel with differences in task performance. Furthermore, the use of the "choice/no-choice" method revealed that freedom of choice enhances task performance, and that "retrieval"--although mastered only marginally by children of this age--is a highly efficient strategy to solve simple addition problems up to 20.*

1. Theoretical and empirical background

During the last decades, researchers have intensively studied the strategies people use to solve cognitive tasks, the way people choose between these strategies, and the changes that take place in these processes during lifetime. A major finding of all these studies is that young children, as well as adolescents and adults, make use of, and choose adaptively between, multiple strategies to solve cognitive tasks in diverse domains, including mathematics. The theoretical and methodological ideas of Siegler (1996; see also Lemaire & Siegler, 1995; Siegler & Lemaire, 1997) deepened our understanding about this topic. In his "model of strategic change", Siegler distinguishes among four dimensions of strategic competence and strategic change, namely: (1) strategy repertoire, i.e. the repertoire of strategies a person uses to solve a task, (2) strategy distribution, i.e. the relative frequency with which each strategy is used to solve a task, (3) strategy effectiveness, i.e. the speed and the accuracy with which each strategy is executed, and (4) strategy selection, i.e. the adaptiveness with which each strategy is chosen. According to this model, at the beginning of the learning process, the learner frequently chooses rather primitive "back-up" strategies (like, for instance, counting), which he or she executes rather ineffectively (i.e. slowly and inaccurately). With experience, the learner uses more efficient "back-up" and "retrieval" strategies, which he or she executes ever faster and more accurately, and also more adaptively.

Furthermore, Siegler proposes the use of the "choice/no-choice" method to obtain unbiased information concerning both the efficiency of the strategies an individual uses and the adaptiveness of the strategy choices he or she makes. This method requires testing each subject under two types of conditions. In the "choice" condition, subjects can freely choose which strategy they use to solve a series of problems from a given task domain. In the "no-choice" condition, the experimenter forces them (experimentally) to solve all problems by means of one particular strategy. The number of "no-choice" conditions can vary according to the number of strategies available to the subject, research interests, technical possibilities, etc.

Taking into account Siegler's theoretical and methodological ideas, we aimed at investigating which strategies 6-7-year-old children with strong, medium, and weak mathematical abilities use to solve simple addition problems up to 20, and how adaptively they choose between these strategies, in relation to task performance. In order to get an accurate picture of both the efficiency of the strategies used and the adaptiveness of the strategy choices, we used the "choice/no-choice" method.

2. Method

Subjects were 77 second-graders from two Flemish schools in the beginning of the school year. Based on their overall scores for mathematics in the first grade and on the second-grade teacher's judgement, subjects were divided in three groups according to mathematical ability (strong, medium, and weak, further referred to as, respectively, the S-, M-, and W-group).

All children were asked to solve a series of 25 simple addition problems up to 20 in three different conditions. These 25 problems were constructed from the 49 possible pair wise combinations of the integers 3 to 9. The problems belonged to five different problem types (with five problems in each type): one type of easy additions up to 10 (T1; e.g. 3 + 4 = .), and four types of additions up to 20: (1) problems with a large first addend and a small second addend (T2; e.g. 9 + 3 = .), (2) problems with a small first addend and a large second addend (T3; e.g. 3 + 9 = .), (3) "tie sums" (T4; e.g. 7 + 7 = .), and (4) "almost tie sums" (T5; e.g. 7 + 6 = .). Problems were presented on a computer screen and the computer registered the reaction time (RT, with an accuracy of 0.01 sec) as well as the answer.

All children solved the problems in three different conditions. In the first condition, the "choice" condition (= condition CHO), children solved each problem by means of the strategy they preferred. Meanwhile their problem solving behaviour was observed by the experimenter. Immediately after solving each problem, children were asked to report verbally which strategy they had used. In the second condition, children were explicitly instructed to solve all problems with one particular strategy, namely "adding up to 10" (= condition ADD). Note that, as a consequence of the obligatory use of this strategy, the five easy additions up to 10 (T1) were not administered in this condition. To further enhance children to use the strategy "adding up to 10", the computer presented the problems in the following format: X + Y = X + (. + .) = . In the third condition, the maximum solution time was limited to 2 seconds, to force children as much as possible to solve all problems by "retrieval" (= condition RET). All children first solved the problems in condition CHO (day 1). Half of the children solved the problems in condition ADD on the second day, and ended with solving the problems in condition RET on day 3. For the other half the order of the two "no-choice" conditions was reversed.

Generally spoken, we expected a variation in strategy use in condition CHO, in the sense that all children would use different strategies to solve the problems in this condition (see Siegler, 1996). Next, we expected group differences in strategy choice and in task performance (i.e. accuracy and speed of problem solving), in favour of the S-group (see Geary, 1990). Finally, we expected differences in strategy use and in task performance between the different problem types and between the three conditions (see Siegler & Lemaire, 1997).

3. Results

Strategy use

As expected, all children used multiple strategies to solve the problems in condition CHO. The number of strategies used varied from 2 to 8 different strategies, ranging from counting strategies like "counting all starting from 1" or "counting on starting from smaller/larger" to "adding up to 10" and "retrieval". As shown in the table below, "retrieval" and "adding up to 10" were the two most common strategies (respectively, 42.29% and 37.51% of all strategies used in this condition), followed by the use of a counting strategy (14.60%). As expected, we observed clear group differences in strategy choice ($\chi^2(6) = 268.51$, $p = .001$), in the sense that S-children used the "retrieval" strategy more frequently than M- and W-children, whereas W-children solved more problems by means of counting than S- and M-children.

		"Retrieval"	Adding up to 10	Counting	Other	Total
S-group	**N**	274	210	11	30	525
	%	52.19	40.00	2.10	5.71	100.00
M-group	**N**	360	383	88	44	875
	%	41.14	43.77	10.06	5.03	100.00
W-group	**N**	180	129	182	34	525
	%	34.29	24.57	34.67	6.48	100.00
Total	**N**	814	722	281	108	1925
	%	42.29	37.51	14.60	5.61	100.00

Accuracy

Scores were analysed using a 3x3x4[1] ANOVA (group x condition x problem type). The scores (maximum score = 5) of the three groups of children on the four different problem types in the three conditions are given in the table below. All differences reported are significant at the 1% level.

As expected, an effect of *group* was found: S-children scored higher than M-children, who scored higher than W-children. We also found an effect of *condition*: As expected, scores in condition CHO and in condition ADD were higher than scores in condition RET, but there was no difference in score between condition CHO and condition ADD. The effect of *problem type* was also significant: Subjects scored highest on T4 problems; the scores on T2 problems were higher than those on T3 and T5 problems, which did not differ mutually. Finally, an interaction between *condition* and *problem type* was found: There was no difference in score on the four problem types in condition CHO and in condition ADD, whereas T4 problems were answered more accurately than T2, T3 and T5 problems in condition RET.

	S-group			**M-group**			**W-group**		
	CHO	**ADD**	**RET**	**CHO**	**ADD**	**RET**	**CHO**	**ADD**	**RET**
T2 **(e.g. 9 + 3)**	4.81	4.95	1.86	4.80	4.83	1.29	4.52	4.29	0.33
T3 **(e.g. 3 + 9)**	4.76	4.62	1.38	4.74	4.34	1.23	4.14	3.81	0.05
T4 **(e.g. 7 + 7)**	4.90	4.95	3.29	4.54	4.91	2.80	3.57	3.90	1.52
T5 **(e.g. 7 + 6)**	4.86	4.86	1.05	4.63	4.60	1.03	3.76	4.00	0.38

Speed

Reaction times were analysed using a 3x2[2]x4 ANOVA (group x condition x problem type). The reaction times of the three groups of children on the four different problem types in condition CHO and in condition ADD are given in the table below. All differences reported are again significant at the 1% level.

First, an effect of *group* was found: As expected, the S-group answered the problems fastest, whereas the W-group answered the problems slowest. Second, we found an effect of *condition*: As anticipated, lower RTs were observed in condition CHO than in condition ADD. Next, we found an effect of *problem type*: T2 problems were answered as fast as T4 problems; T2 and T4 problems

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were answered faster than T3 and T5 problems, which did not differ in RT. Furthermore, there was an interaction between *group* and *condition*: The difference in RT between condition CHO and condition ADD was biggest for the M-group, and smallest for the S-group. Finally, we observed an interaction between *condition* and *problem type*: There was no difference in RT between T2, T3 and T4 problems in condition CHO, whereas T5 problems were answered much slower. This pattern was not observed in the ADD condition: In this condition, RTs on T2 and T4 problems, T3 and T5 problems, and T4 and T5 problems did not differ, whereas we did find a difference in RT on T2 and T3 problems, T2 and T5 problems, and T3 and T4 problems.

	S-group		**M-group**		**W-group**	
	CHO	**ADD**	**CHO**	**ADD**	**CHO**	**ADD**
T2 **(e.g. 9 + 3)**	4.13	5.70	4.27	7.44	6.76	9.86
T3 **(e.g. 3 + 9)**	4.38	8.02	4.61	11.68	7.99	13.08
T4 **(e.g. 7 + 7)**	3.90	6.23	4.05	9.41	7.40	11.40
T5 **(e.g. 7 + 6)**	6.04	7.18	6.62	9.58	10.16	12.56

4. Conclusions

In line with earlier studies concerning young children's strategy use in the domain of simple addition (e.g. Geary, 1990; Siegler, 1996), we observed a rich variation in strategy use (in the "choice" condition) as well as considerable group differences in strategy choice and strategy execution, resulting in group differences in task performance. Furthermore, a first global analysis of the data in the three different conditions revealed, first, that children in general made adaptive strategy choices in the CHO condition: They obviously chose those strategies that allowed them to answer the problems in a relatively fast and accurate way. Second, the overall difference in RT between condition CHO and condition ADD demonstrated that freedom of choice enhances (at least partly) task performance: Forcing children to solve all problems in a standardised and stereotyped way (by means of the "adding up to 10"-strategy) influenced their response time negatively (although it did not influence their accuracy). Finally, the data obtained in condition RET indicate that children of this age are typically not (yet) able to solve simple addition problems up to 20 by means of "retrieval" (except for the "tie-sums"). Nevertheless, when a child succeeded in responding by "retrieval" in the RET condition, the answer was mostly correct. Ongoing more fine-grained and individualised comparative analyses of the nature and the efficiency of the strategies used in the different conditions will shed further light on the adaptiveness of children's strategy choices.

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[1] Taking the central topic of the study into account, namely second-graders' strategy use and task performance on simple addition problems up to 20, data concerning *problem type* 1 were not included in the analysis.

[2] Because of the strict time limit in *condition* RET, data concerning condition RET were not included in the analysis.

On "How to Make Our Ideas Clear": A Pragmaticist Critique of Explication in the Mathematics Classroom

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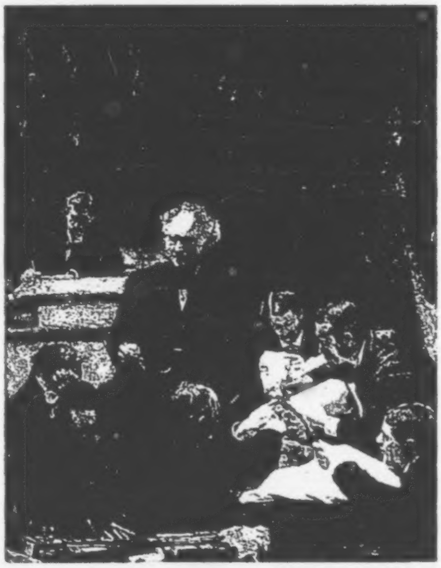
Instruction in the mathematics classroom often involves a paradigmatic explication of material. Children are asked to follow along the reasoning of an "experienced" adult who actually acts on the part of the children. Knowledge and understanding of the material by the children are measured by the ability of the children to catch up with the reasoning of the teacher. In this paper I intend to argue that paradigmatic instruction is based in false interpretation of a pragmatic logic as the translation of ideas into action. I will support my argument with discursive analysis of a short interactional sequence between teacher and children in a sixth-grade class in a primary school in Greece.

At Easter last year I was busy organizing a conference for primary school teachers. A local school—the only one with an amphitheatre large enough to accommodate all the teachers—offered to host the event. While the principal was giving me a tour, he happened to describe another in-service session on mathematics education that had taken place recently in that same amphitheatre. Participating teachers, he told me, observed a tutor teaching a class, which, for the purposes of the session, had moved to the amphitheatre.[1]

Later on, I stood alone in the amphitheatre, trying to imagine the scene: the tutor teaching the class, while the teachers looked on. My musings brought to mind two of Thomas Eakins' most famous paintings, *The Gross Clinic* and *The Agnew Clinic*. In the painting reproduced here the master surgeon and his assistants operate on a patient. While several students observe the operation, the patient's mother sits in the corner, writhing in agony. The master surgeon is depicted in a state of apparent withdrawal from the act of operating, still holding his surgical blade, his facial expression revealing intense thinking.



Thomas Eakins, *The Gross Clinic*, 1875-76

Ostensibly, paradigmatic instruction follows a pragmatic logic since the master-teacher must translate ideas into action. Even if the "experts" might be said, in this way, to bridge the gap between theory and practice, the fact remains that "novices" are taught to observe the center of activity from a distance. In the case of the in-service session, paradigmatic teaching contributed not only to the formation of methods for teaching the content of mathematics, but also, even if it was not explicitly stated, to the formation of the essence of belief concerning how the learning and teaching of mathematics should be approached in the classroom.

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[1] Readers should not perceive this description as prototypical of current in-service training programs in Greece, even though it was chacteristic in the past. Nevertheless, the mentality it reflects still prevails in beliefs on learning and teaching, albeit in more subtle ways.

In this paper I will analyze an episode from a mathematics lesson in a sixth-grade class in Volos, Greece. The purpose of my analysis is to bring attention to certain linguistic phenomena that characterize social encounters and, therefore, are partly responsible for framing the interactional space of the encounter. Elaborating on Peirce's *pragmatic maxim*, which locates the utmost clearness of ideas in "all the conceivable practical bearings we conceive the object of our conception to have" (Peirce, 1991: 169), I differentiate a 'pragmaticist' framework for mathematics education from an instrumental one, which presents itself as 'hands-on'.

From paradigms to the production of a "Practionary"

> "Well, that's what we do first. We open the little trunk where we keep the materials for the circle, we empty it, and what do we have? We have a center, a diameter, a curve ... Everything I might need I take out. Same thing as if we wanted to install a faucet. I go and take the toolbox..."
>
> [6th grade teacher addressing his class] (Volos, Nov 2000)

The phrasing of Peirce's *pragmatic maxim* can easily mislead. In his famous essay on "How to Make Our Ideas Clear" (1878), Peirce identifies the first stage in the clarity of ideas as familiarity with the common uses or nature of an idea. Standard dictionaries provide us with linguistic definitions, which for Peirce comprise the second degree of clarity. It is tempting to consider, then, the possibility of composing a kind of super-dictionary of praxis—a *practionary*—that would furnish the reader with descriptions of all the conceivable practical effects a thing could have in experience, which is Peirce's third stage in the clarity of ideas. The possessor of such a "practionary" could acquire a clearer understanding of an idea simply by emptying its "little trunk"—to use the metaphor of the teacher, whose pedagogical style I will describe below. For Peirce, though, knowledge of the practical effects of an idea can only be achieved through personal hands-on experience with an idea or a thing (Parker, 1998; Peirce, 1991).

Peirce's discussion of the three grades of clearness of ideas and the notion of *practionary* are all too relevant for mathematics education. A number of researchers have stressed, for instance, the fallibility of the assumption that concepts are acquired mainly through their definitions (Sfard, 2000; Vinner, 1991). A definition is a configuration of signs that assists in establishing the meanings of other signs. The definition of an idea is an exercise of power over the idea: its utterance conveys an appropriation of complete knowledge of the idea. Even within the deductive theory of mathematics education, signs are depleted of meaning, if considered without reference to their anthropological milieu (Cobb, 1990). Characteristic of the irreducibity of mathematical concepts to strings of words are the notions of "concept image" (Tall and Vinner, 1981) and "figural concept" (Fischbein, 1993).

The processes of symbol manipulation present in traditional school mathematics is often linked with a mechanistic instruction of "basic skills" and ideas (Romberg and Kaput, 1999). Even though research suggests otherwise (Hewitt, 1997; Voigt, 1995), teaching is still considered, or at least approached, as an act of explication. In his critique of the "explicative order," Jacques Rancière describes explication as a series of reasonings used in order to explain a series of reasonings that already exist within the material being taught. If a student cannot understand the first series of reasonings, why should we assume that he or she will understand the teacher's reasonings. And, if the teacher's reasonings themselves need to be explained, we can see how, in Rancière's words, "the logic of explication calls for the principle of a regression ad infinitum" (1991: 4).

Over-reliance on the act of explication inevitably places emphasis on what Jackobson defined as the poetic function of language, a focus "on the message for its own sake" (1960: 356). Indeed, what speakers say is always evaluated according to aesthetic rules, in other words, for its efficacy, in "moving" an audience. The master-teacher of the in-service session described above had to employ a series of reasonings in order to explicate to the students the material that he himself had chosen. The outcome of the series of series of reasonings, including those of the attending teachers, judged by the standards of the explicator, created the illusion of an instance of successful mathematics teaching. The ability of the master-teacher to "move" the audience of teachers made this session a successful instance of paradigmatic teaching—judging from the story that was told to me by the school's principal.

I suggest that teaching, conceived as an act of explication, represents an attempt to produce a *practionary*, a false substitute for Peirce's third step in the clarity of ideas. Reaching or, to be more precise, approaching this third stage is, thus, trivialized and reduced to cramming practices or a series of reasoning, stored in a "little trunk." The appropriation of knowledge by the students is measured by their ability to follow and/or reproduce these practices, even if this requires, in many cases, an act of collusion by all the engaged parties (McDermott and Tylbor, 1995).

Explication as action? Analyzing instances of explication

I now turn to the description and analysis of an episode from a mathematics session from a sixth-grade class in Volos, Greece. The educational system in Greece is nationally standardized. Mathematics syllabi are prescriptive in the sense that they allocate a specific number of hours to the teaching of each area. Instruction in the classroom is restricted to the textbook provided by the state, with teachers usually teaching "from the front."

With its 140,000 inhabitants, Volos is the fifth largest city in Greece and the third largest port. The primary school that I visited is in a lower middle-class area far from the center of the city. My almost daily visits to the school began in October, 2000, though I was familiar with the school from a project I had carried out there two years before. It was then that I first met with the children of this year's sixth-grade class. Their teacher had changed, though. I was not surprised to find a male teacher; reflecting gender hierarchies and stereotypes, the few men among primary school teachers tend to be overwhelmingly represented in the higher grades of primary school. My presence in the classroom was limited to observing and audio-taping the mathematics sessions. I was conscious of the fact, though, that even as a bystander I was co-constructing the course of events (Goffman, 1981).

Linguistic anthropologists view language as a set of practices "which play an essential role in mediating the ideational and material aspects of human existence, and, hence, in bringing about particular ways of being-in-the-world" (Duranti, 1997: 4-5). Delineating types of organization that characterize the interactional space of a mathematics classroom will highlight how particular ways of "being-in-the-world" are formed, sustained and altered.

The extract that follows comes from a session on geometry. The class is discussing the concept of perimeter, having already described the characteristic features of a square. Perimeter is a concept formally introduced in the fourth-grade, so the current lesson served as an opportunity for the teacher to freshen up children's memory of the concept. In the transcript that follows, I interpolate brief explanatory comments and theoretical elaborations. (A description of the transcription symbols are explained below.)

(The class has already discussed equal angles and equal, parallel, and perpendicular sides in a square. The teacher had drawn a square EZHK on the board.)
1 Teacher: **NOW**. (3.0) We are looking for (2.0) the perimeter of the square EZHK . WHAT INFORMATION do we have for this unknown quantity you see here?
2 Child 1: Sir!
3 Teacher: Yes Elena.
4 Child 1: We know the length of all sides.
5 Teacher: Well **LENGTH** (0.5) TOTAL LENGTH (0.5) of all sides. Splendid!
[
6 Child 2: Sir it also has equal sides.

The teacher takes the turn to speak as soon as he receives the first response (line 4) from the class. He restates what Child 1 has just said, addi[illegible] phrase "total length," an expression that frames the concept of perimeter (line 5). At the same time, he speaks louder in an assertive tone, underlining the significance of what he had just mentioned for the wanted answer. As soon as Child 2 chips in another piece of information (line 6), the teacher interrupts the process once again to restate the task and suggest how to proceed (line 7). Child 2's comment concerning the equality of sides is ignored.

7 Teacher: = What we have to do is take this information, what we know (0.5) and apply it here in this certain (0.5) square.
[ ]
8 Child 3: Sir! Sir!

9 Teacher: = TOTAL length (0.5) of **ALL** (0.5) sides. What element do you think I get from this information that I can match up here?

The teacher refuses to give to Child 3 permission to speak (line 8). He repeats and emphasizes the words "total" and "all" using them as cues that would evoke further information about the concept of perimeter. Child 3's hesitant and faintly-uttered responses in line 10 and 12 below indicate the bafflement of the class as to what the aim of the process is.

10 Child 3: °°Perimeter? Perimeter? ° That the::
11 Teacher: It says (0.5) **length** (0.5) **of** (0.5) **all** (0.5) **sides**.
12 Child 3: We know how many: °are all the sides:: *((almost like asking))*
13 Teacher: BRAVO! How many are all the sides. = Where are they? Here. (1.0) We write them down again (12.0) *((teacher writes on the board))*
I made the BEST OUT (0.5) of the element that **says** (0.5) SIDES. Did~I~know? = I~did~know. = I~took~them~and (1.0) I~placed~them~there. = **WHAT~ELSE~DOES~THIS~INFORMATION~TELL~US?** = It~says~sides~only? = It~says~perimeter (1.0) **TOTAL** (1.0) **length**. = **A::::** that's right! Addition.
[ ] [ ]
14 Child 4: Total (1.0). Total (0.5) with addition.

Sensing the hesitance of the class, the teacher interjects another explication in line 11. This time he pauses briefly between words, implying that "this is it!" Child 3 strives to follow the teacher's line of argument. Even though her response is expressed in almost a questioning mode (line 12), the teacher congratulates her and proceeds with yet another explication of the argument so far (line 13). Next he asks the children to think of other characteristics of the square that would be relevant in finding its perimeter. Child 4 (line 14) interprets correctly the cues given by the teacher and takes the process a step further. Notice that in line 13 there are no "transition-relevant points," i.e. moments when a change of speaker may take place (Sacks, Schegloff, and Jefferson, 1974). Besides speaking too fast, the teacher leaves no interval between the end of a prior unit of speech and the next piece of talk.

15 Child 5: Sir we have done these before.
16 Teacher: Everything's fine?
17 Child: Yes.
16 Teacher: **What else** did we discover today or better did we mention again? = WE KNEW IT BUT WE SEE IT AGAIN (0.5) FOR THE SIDES:
17 Child 6: That these *((pointing towards the board))* are:: opposite to one another.
18 Child 4: That these:: are:: parallel and do not meet.
[
19 Teacher: They are parallel. Yes. **This element** (3.0) can it be used? (6.0) At~the~point~where~we~are~now? = In order to see let's say =
20 Child 4: The parallels?
21 Teacher: **YES**. That they are parallel (2.0) does it interest us in finding the total length?
22 Child 6: Eh sir::
23 Child 4: Ho-? How much are all the parallels? *((she wants to say how long))*
24 Child 7: **SIR the parallels are equal!** *((the child does not express himself adequately))*
25 Teacher: **A::::!** *We have another element.* = Look here. = That sides:=

26 Child 6: Are equal.

In line 16, the teacher seeks reassurance that the class comprehends. More information is needed in order to calculate the perimeter, though. Children 4 and 6 suggest that the missing link is the fact that the sides of a square are parallel to one another (lines 17 and 18). The teacher interrupts (line 19) making an implicit statement that the concept of "parallels" is not the information missing. Child 4's question in line 20, though, indicates that the class is not following the process consciously. Children appear to be choosing information at random from the "little trunk." The teacher attempts again to lead the argument away from the information "parallel sides" (line 21), and Child 6 with the help of Child 7 seem to be making the required connection (line 23 and 24).

27 Teacher: **They are** (2.0) equal. Therefore
[
28 Child 4: We can multiply.
29 Teacher: **BRAVO!** We move to the property that says that multiplication~is~a~**SHORT:?** (1.5)
30 Class: °Addition.
[
31 Teacher: Addition. When~the~terms~**ARE:?** (3.5)
32 Class: ° Same.
33 Teacher: **THE SAME.** That's righ-. = Instead of saying 4~and~4~and~4~and~4~I~say 4 times
4. = Or 3~and~3~and~3~and~3 (0.5) I~say 4 times (1.0) 3. Is that so? If here then
(1.0) instead of saying side EZHK I put a general name (2.0) *α ((he writes a on all sides of the square))* for the sides (0.5) of the square α ZH α and KH α (0.5) and this
(1.0) and the other (1.0) = *((pointing to the figure on the board))*
34 Child 4: **4~TIMES α** (4.0)
35 Teacher: With a 4 times α then I define:
36 Child 4: The::: the:: perimeter!
37 Teacher: **BRAVO!** I define the magnitude that is called perimeter (1.5) of a square.

Concluding remarks

Participation in any social scene requires a minimum consensus on what is getting done (Gumperz, 1992; McDermott and Tylbor, 1995). On a micro-perspective level, teacher and children appear to be discussing already familiar concepts in an attempt to deepen or widen their understanding of the concepts (see lines 15 and 16). Despite this observation though, the fact that children's responses, like those in lines 14 and 18, are overlooked, indicates that the teacher ranks the correct use of information and the pious following of a process higher than understanding the concepts. On a macro-perspective level, children are disciplined into following the series of reasonings used by the teacher in explicating the process of finding the perimeter of a square with side α.

As McDermott and Tylbor suggest, "it is possible to live lies without having to tell them" (1995: 281). Judging by Child 5's response in line 36, we remain puzzled as to whether the teacher actually moved through this series of reasoning alone or not. A fair amount of collusion was involved in this process of explication, both by the teacher and the children, in the sense that a particular state of affairs was sought that could be accepted as an indication of understanding and, therefore, of closure and successful or satisfactory teaching (see lines 5, 12, 13, 16-17, 29-30, 31-32, 37).

The confusion of the class concerning the task at hand, along with their hesitant decision to follow the teacher's explication of the task, may suggest a form of resistance to the way of "being-in-the-mathematics-classroom" proposed by the teacher. One cannot force reasoning to follow a specific order of steps. What the teacher does is lay out his own interpretation of what "being-in-the-mathematics-classroom" might mean. Knowledge and understanding are then measured by the degree to which the students collude in the teacher's mode of explication. Analyzing modes of explication can show us the subtle ways that states of "being-in-the-mathematics-classroom" are being sustained and reproduced. A discursive analysis of conversational exchanges in the classroom may be one tool for understanding this process. Further research would entail connecting these to a broader fabric of social structures and relationships.

Transcription features

Boldface indicates emphasis signaled by changes in pitch and/or amplitude.
A *left bracket*, connecting talk on separate lines, marks points at which one speaker's talk overlap the talk of another.
A *right bracket* marks the place where the overlap ends.
Colons indicate that the sound just before the colon has been noticeably lengthened.
A *dash* marks a sudden breaking-off of a particular sound.
A *degree sign* (°) indicates that the talk following it is spoken with noticeably lower volume.
An *equals sign* is used to indicate that there is no interval between the end of a prior unit of speech and the next piece of talk.
Capitals indicate increased volume.
Numbers in parentheses mark silence in seconds.
Text in *italics* between *double parentheses* mark the author's comments.

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A FRAMEWORK FOR ASSESSING QUESTIONS INVESTIGATING THE UNDERSTANDING OF PROBABILISTIC CONCEPTS

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Research has suggested that two well-regarded tests of probabilistic understanding measure different constructs. Here a framework is proposed for classifying questions on probabilistic understanding and used to analyse the structure of the tests. It provides a way of predicting some differences found by post factum *statistical analysis. This suggests that it may have more general validity for evaluating instruments assessing probabilistic understanding.*

Probability has come into many school curricula at much the same time as schools have been expected to raise the quality of their assessment procedures. Here a framework is proposed for classifying questions on probabilistic understanding, and is used to help explain some research findings about two well-regarded tests.

Some mathematics educators argue, with some validity, that written assessment instruments are inappropriate for assessing complex concepts like probability. Alarcon (1982), for example, has shown that, when probabilistic concepts are examined in discussions and then in a written questionnaire, many students change their responses. These arguments are not addressed here because the external demands on schools for increased assessment mean that we need to be able assess the quality of those instruments which are likely to be used, and the principles outlined are equally applicable to more interactive modes of assessment.

Two Instruments Assessing Probabilistic Understanding

Green (1982a, 1982b) reported a study of 3000 children from Years 7 to 11 in the East Midlands of England. He hoped that his carefully developed assessment instrument could be related to Piagetian levels as had been done for other school topics by the Concepts in Secondary Mathematics and Science Project. He obtained informative responses to a wide variety of questions, but was unable to link all of them to a Piagetian stage-model. Only eighteen of the original, quite diverse, fifty items satisfied the requirements for validity and reliability, so these constituted the sub-test he used to establish his Piagetian stage-model.

Soon after, Fischbein & Gazit (1984) developed two instruments to assess the effect of a programme of instruction in probability with about 300 Israeli pupils from Years 5 to 7 in the experimental and control groups. The first (A) contained procedural questions directly related to the instruction, the second (B) comprised eight questions specifically designed for the experiment to reveal several well-known probabilistic misconceptions which were potentially present. The authors found that the questions were able to detect changes in understanding, although sometimes the effect of instruction seems to have led to decreased understanding.

Subsequent Use of These Instruments

Both tests have been used by subsequent researchers (e.g., Watson, Collis & Moritz, 1994; Glencross & Laridon, 1994) as source of questions, sometimes with minor modifications. Green (1986) used his experience to develop a new test to assess understanding of randomness, but has not re-used his original test. However, Izard (1992) administered Green's eighteen items to about 1100 students in Hungary, Brazil, and francophone Canada and found a general confirmation of Green's work but with some variations. He considered that the test had acceptable test reliability and item fit, but that the decision rules for grading the open-ended questions needed further examination. Surprisingly, he did not make comparisons between the different countries.

All these researchers seem to have believed that their questions were valid tests of the concepts being investigated, although Green was concerned that his statistical analysis had removed some of what he saw as his more interesting questions. Later Godino, Batanero & Cañizares (1994) presented both these tests of primary probabilistic reasoning to the same group of 251 Spanish children in Years 6 to 8, so the opportunity arose to examine just what the questions were assessing.

Godino et al. constructed a detailed analysis of the skills and understandings being tested in both tests (using all fifty of Green's items). They argued that if Green's test were a test of probabilistic reasoning then the whole test, and perhaps its reduced form would have high predictive value for results from some other such test. They found significant correlation between some of the Israeli questions and Green's test, but not for all comparisons, and concluded, *inter alia,* that the Israeli test contained components of probabilistic reasoning not included by Green. A factor analysis of the Israeli test found two factors (one based mainly on qq. 6, 7 & 8, and the other on qq. 2, 3, 4 & 5), the second of which did not correlate well with Green's results. A factor analysis of Green's test produced fifteen factors, thus confirming its diversity of coverage. They concluded:

> The multitude of factors included in Green's test and the low predictive value of the "probabilistic level" and of the other scores in the said test, with respect to the success in Fischbein & Gazit's test, suggest that a critical review is needed to consider the probabilistic knowledge of the subjects as a linear structure.
>
> More research is needed to explore in-depth the nature of probabilistic reasoning and its structure. At the same time it shall be necessary to compile and analyse the banks of items that make up a representative sample of [Primary Probabilistic Reasoning] and of the universe of appropriate contextual variables … .

A Framework for Classifying Questions on Probabilistic Understanding

At the same time as Godino et al. were doing their work, J. Truran (1994) was preparing a framework, of which a modified version is presented in Table 1, for class-

Table 1
A Framework for Classifying Random Probability Functions and Ways of Encountering Them

		1	2	3	4
A	Type of RG	Disc	Coin	Die	Urn
		5	6	7	8
		Cards	Contiguous Spinner (†)	Non-contiguous Spinner	Electronic
		9	10	11	12
		Human	Asymmetric Solids	Many interacting forces	Miscellaneous

		1	2	3	4
B	Place of RG in Culture	Own Culture	Unusual	Different	
C	Previous Practical Experience with RG	> 7 Days	2–7 Days	< 1 Day	
D	Previous Theoretical Experience with RG	> 7 Days	2–7 Days	< 1 Day	None
E	Operator of RG	Not Mentioned	Self	Other—Present	Other—Absent
F	Style of Response	Oral	Written	Multiple Choice	Non-Linguistic
G	Number of Elementary Events	2	Small	Large	Very Large
H	Number of Events	2	3–5	6	> 6
I	Structure of RG	Symmetric	Slightly Asymmetric	Very Asymmetric	Deceptive
J	Knowledge of Structure of RG	Known	Unknown		
K	Reward	None	Hypothetical	Actual	

	Type of Question		**Type of Probabilistic Situation**
α	Prediction of Outcome	I	Single Trial
β	Prediction of Set of Outcomes	II	More than One Trial
γ	Selection of Outcome	III	Previous Results
δ	Statement of "Likely" Outcome	IV	Previous Predictions of Results
ε	Comparison of RGs	V	Concurrent Operation of Another RG
ζ	Fair Allocation of Payout for Bets	VI	Previous Experience with Similar RGs
η	Examination of Sequences of Outcomes	VII	Changes in RG from Trial to Trial
θ	Linguistic Questions of Technical Knowledge	VIII	Long Term Reward Maximisation
ι	Listing of Outcomes		

(†) Spinners may be further divided in three categories:
(a) those where only the pointer is free to move;
(b) those where only the sectors are free to move;
(c) those where both pointer and sectors are free to move (as in a roulette wheel).

ifying random generators (RGs) and questions about them. He saw this as useful for assessing the comprehensiveness of a test or unit of work, and it now also seems to be able to fulfil some of the gaps identified by Godino et al. It has three parts—the first describes the nature of the RG and its relationship to the student, the second the type of question being asked, and the third the type of probabilistic situation in which the question is used. For simplicity the classification deals only with outcomes from a single RG, not with compound RGs. It has been based on many research findings about the types of probabilistic situations which seem to influence students' responses. There is not space to list all these findings here, but, for example, Category E, which considers who actually operates the RG, follows in part from Zaleska's (1974) finding that responses may differ according to whether the subjects or the experimenters actually draw the balls from an urn. It has been tested for its ability to classify more than fifty questions developed by a wide range of researchers, including Green, but not Fischbein & Gazit, and the modification presented here summarises the experiences gained in this preliminary testing.

To help to clarify the complex detail of Table 1, the classification of one of Green's questions is presented in Table 2. The question is:

> When an ordinary 6 sided dice is thrown which number or numbers is it hardest to throw, or are they all the same?
>
> Answer..

The right hand column summarises of the meaning of the alpha-numeric terms to clarify the links with Table 1. The only potential ambiguity here is the classification "δ"—statement of "likely" outcome. Here the student is being asked for an "unlikely" outcome, and this meaning is implied by the use of quotation marks.

Table 2
Analysis of Green's Question 4

A	3	Die
B	1	Own Culture
C		Unknown
D		Unknown
E	1	Operator: not mentioned
F	2	Written
G	2	Small Number of Elementary Events
H	3	6 Events
I	1	Symmetric
J	1	Known
K	1	No Reward
	δ	"Likely" Outcome
	I	1 Trial

Analysis of the Two Tests

All the items of Green (G) and Fischbein & Gazit (FGB) concerned with a single RG have been classified using this framework, and the results are summarised in Table 3 with Green's items first. His test seems to have 53 items, not 50 as mentioned above, and it has not been possible to explain all the reasons for the discrepancy. The term "u" refers to "unknown" and "na" to "not applicable"—cases where the questions are not dealing with operations of a single random generator. Those questions which comprised Green's final statistically reliable form are in bold, apart from G10 and G26 (a) which do not fit in the framework. This makes nineteen items, not eighteen: both parts of G3 were probably taken as one item.

Table 3
Classification of Questions of Green and Fischbein & Gazit

Question	RG Type	RG Nature											Question	Situation
		A	B	C	D	E	F	G	H	I	J	K		
G1	disc	1	1	u	u	1	3	1	1	1	1	1	δ	I
G2	**urn**	4	1	u	u	4	3	3	1	1	1	1	δ	I
G3 answer	**spinner**	6	1	u	u	1	3	2	2	1	1	1	ε	I
G3 reason	**spinner**	6	1	u	u	1	2	2	2	1	1	1	ε	I
G4	**die**	3	1	u	u	1	2	2	3	1	1	1	δ	I
G5	**coin**	2	1	u	u	1	3	1	1	1	1	1	δ	III
G6 (a) answer	urn	4	1	u	u	2	3	2	1	1	1	1	ε	I
G6 (a) reason	urn	4	1	u	u	2	2	2	1	1	1	1	ε	I
G6 b answer	**urn**	4	1	u	u	2	3	2	1	1	1	1	ε	I
G6 b reason	**urn**	4	1	u	u	2	2	2	1	1	1	1	ε	I
G6 c answer	**urn**	4	1	u	u	2	3	2	1	1	1	1	ε	I
G6 c reason	**urn**	4	1	u	u	2	2	2	1	1	1	1	ε	I
G6 d answer	**urn**	4	1	u	u	2	3	3	1	1	1	1	ε	I
G6 d reason	**urn**	4	1	u	u	2	2	3	1	1	1	1	ε	I
G6 e answer	**urn**	4	1	u	u	2	3	2	1	1	1	1	ε	I
G6 e reason	**urn**	4	1	u	u	2	2	2	1	1	1	1	ε	I
G7 (a) (i)	language	-	1	u	u	-	3	-	-	-	-	-	θ	-
G7 (a) (ii)	language	-	1	u	u	-	3	-	-	-	-	-	θ	-
G7 (a) (iii)	language	-	1	u	u	-	3	-	-	-	-	-	θ	-
G7 (a) (iv)	language	-	1	u	u	-	3	-	-	-	-	-	θ	-
G7 (b) (i)	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G7 (b) (ii)	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G7 (b) (iii)	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G7 (b) (iv)	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G7 (b) (v)	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G8	coin	2	1	u	u	1	3	1	1	1	1	1	β	I
G9	**die**	3	1	u	u	4	2	2	3	1	1	1	ζ	II
G10 – G12	na													
G13	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G14	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G15	language	-	1	u	u	-	2	-	-	-	-	-	θ	-
G16	language	-	1	u	u	-	3	-	-	-	-	-	θ	-
G17 answer	spinner	7	1	u	u	1	3	2	1	1	1	1	ε	I
G17 reason	spinner	7	1	u	u	1	3	2	1	1	1	1	ε	I
G18	**urn**	4	1	u	u	1	3	2	2	1	1	1	δ	VII
G19 (a)	**spinner**	7	1	u	u	1	3	2	1	3	1	1	δ	I
G19 (b)	**spinner**	7	1	u	u	1	2	2	1	3	1	1	δ	I

		A	B	C	D	E	F	G	H	I	J	K		
G20 (a)	coin	2	1	u	u	4	2	1	1	1	1	1	η	III
G20 (b)	coin	2	1	u	u	4	2	1	1	1	1	1	η	III
G21 (a)	channels	12	2	u	u	1	3	2	4	3	1	1	δ	II
G21 (b)	channels	12	2	u	u	1	3	2	2	3	1	1	δ	II
G21 (c)	channels	12	2	u	u	1	3	2	2	3	1	1	δ	II
G21 (d)	channels	12	2	u	u	1	3	2	2	3	1	1	δ	II
G22	channels	12	2	u	u	1	2	2	4	2	1	1	δ	I
G23	drawing pin	10	1	u	u	4	3	1	1	3	1	1	β	III
G24	births	12	2	u	u	4	3	1	1	?	?	1	δ	II
G25	urn	4	1	u	u	4	3	u	1	u	2	1	δ	III
G26 (a)–(d)	na													
FGB1	urn	4	1	u	u	1	2	2	2	1	1	1	β	II
FGB2	many	11	1	1	u	4	2	4	1	4	1	2	δ	VIII
FGB3	urn	4	1	u	u	4	2	4	4	1	1	2	δ	III
FGB4	urn	4	1	u	u	4	2	4	4	1	1	2	δ	II
FGB5	urn	4	1	u	u	4	2	4	4	1	1	2	δ	III
FGB6	urn	4	1	u	u	4	3	3	1	1	1	1	ε	I
FGB7	urn	4	1	u	u	4	2	3	1	1	1	1	ε	I
FGB8	urn	4	1	u	u	4	2	3	1	1	1	1	ε	II

Probably not all workers would agree with all these classifications. For example, the channels in G21 & G22 have been treated as single RGs, and for FGB2, which deals with a child's view that entering a classroom right foot first will increase his chance of gaining good marks, it is certainly arguable whether good marks may be seen as an outcome of a random generator. However, the discussion below does not require agreement for each and every classification, so any debate about a small number of difficult classifications should not affect the general conclusions.

Discussion

Table 3 shows that Green's final set of questions are predominantly about comparison of urns, only three items deal with situations more complex than that of considering just one trial, and all but one deal with "likely" outcomes or comparison of RGs. Furthermore, using the classification as a guide to comprehensiveness, shows that although the test covers four different RGs, it does so within restricted contexts and does not address many ideas necessary for a good understanding of probability. So it does not seem appropriate for Izard's international survey unless his concerns were more to do with statistics than assessing probabilistic understanding.

The framework shows clearly just why Green felt that his final test omitted many interesting questions. For example, most of the questions involving asymmetric RGs have been omitted, as well as all the questions directly addressing language and almost all of the questions incorporating outcomes of previous trials

When we consider the FGB questions, we see that the first factor identified by Godino et al. involves all the questions concerned with comparison of random generators (ε), and the second involves questions about "likely" outcomes (δ). Godino et al. see the second factor is concerned with the students' "biases and deep-rooted beliefs"—ideas more likely to be elucidated by questions of this type.

Godino et al. state that Green's test does not contain questions like these second factor questions. But the framework is able to show that G21 does have some similarities, but also differs in not involving hypothetical rewards, large numbers of elementary events, or considering more than one trial. Furthermore, G21 deals with an RG—channels—which may well have been less familiar to students, and is not very clearly explained in the test. So the framework taken with the statistical analysis can assist in clarifying the nature of a statistical claim.

The first six factors identified in Green's test each contain items from different parts of particular questions (in Factor 3, the item "5dr" must be a misprint for "6dr"), sometimes with other items as well. Factor 3 contains G18, which is clearly of a different structure from the other three items, and Factor 4 contains two items of a quite different type. On the other hand, the markedly similar items arising from the comparison of urns in G6 have not all grouped themselves into one factor.

So while factor analysis has identified some factors in both tests which framework analysis might have predicted, it has also highlighted some questions which are surprising linked or not linked, and the framework provides a starting point for examining why these discrepancies might have occurred. For example, although the comparison of urns questions have a standard format they form a single factor in the Israeli test, but separate into different factors on Green's, as well as indicating different Piagetian levels. Green (1982b, p. 338) suspected that children did not consider the problems as being essentially similar. J. & K. Truran (1999) summarised research into such questions and showed that children's responses to slightly different sets of numbers are often idiosyncratic, subconscious, and unpredictable. We do not yet understand the reasons, but it would clearly be unwise, for example, to deduce much from an assessment instrument containing just one "comparison of urns" question, because the development of understanding seems to be non-linear.

The framework approach rests predominantly on an analysis of question format; the statistical approach more on understanding of meaning. Neither is an all-purpose tool, although, as we have seen here, their strengths do overlap to some extent.

Conclusion

Researchers often report new tests for assessing probabilistic understanding, e.g., Batanero, Serrano & Garfield (1996), Reading & Shaughnessy (2000). The framework and statistical analysis can be complementary partners for analysis and building up the item-bank mentioned above. The framework is simple, and applicable *before* administering a test as well as after obtaining statistical results. It can provide a guide to the comprehensiveness of a test or unit of work, which may be of value for analysing the complexity of probabilistic knowledge and deciding whether its acquisition is linear or holistic. While it needs a little modification, further testing, and much extension, this paper has shown how useful a research tool it can be.

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Cultural Activities as Learning Arenas for Children to Negotiate and Make Sense Mathematical Meanings

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Abstract: *The main purpose of this study was to develop a model called the " Cultural Conceptual Learning Teaching Model" (CCLT) that addresses the ways in which children's real experiences and cultural practices can be connected to mathematical classroom lessons and to improve children's understanding of mathematics. Eight second-grade classes participated in this study in Hsin-Chu, Taiwan. The results showed that school mathematics for those who are involved in CCLT model based on cultural activities improved more than for those who were involved in the control group. Children learned more in the transfer of learning in the CCLT group than Children in the control group.*

Introduction

In recent years, many studies have focused on mathematical cognition related to individual competence in daily life context (Bishop & Abreu, 1991; Carraher, 1988; Lave, 1988; Saxe, 1991; Tsai & Post, 1999; Tsai, 2000). A review of children's out-of-school mathematics raises critical questions about how children come to understand mathematics and how they connect informal knowledge out of school with formal knowledge in school. (Hibert & Carpenter, 1992; Millory, 1994; Resnick, 1987). According to Resnick (1987), teachers should concerned about the role of cultural aspects, constructing meaningful ways for students to make sense of the abstract symbols of school mathematics. She emphasized that culture contributes to better understanding in students' learning and it therefore needs to be integrated into mathematics teaching. Central to this study is the view that an understanding of mathematical meaning is the ability to connect different learning environments or situations (Greeno, 1991). Brown, Collins, & Duguid (1987) also emphasized the importance of the relationship among activity, concept and culture and claim that learning must involve all of them. Hiebert & Carpenter (1992) further proposed that children's informal knowledge could serve as a basis for the development of understanding of mathematical symbols and procedures in school setting, regardless of the content domain.

Based on these points of view, this study develops a learning-teaching model called the Cultural Conceptual Learning-Teaching Model (CCLT) (Tsai, 1996) that attempts to combine individuals, activities, concepts, and culture together. The hypothesis of the study stated that establishing a link between children's cultural activities and school mathematics will improve children's learning of mathematics in school and their ability to solve daily mathematics problems out of school. There were three questions raised in this study: (1) Did children achieve differently in school mathematics when they participated in different programs? (2) How did children involve in the CCLT group perform in solving addition problems as compared to control group? (3) How different were children's strategies of solving the word problems between CCLT group and control group?

The Cultural Conceptual Learning Teaching Model (CCLT)

The CCLT (Figure 1) contains three learning environments: construction environment; connection environment; and practice environment; and six learning stages: Play Stage; Construction Stage; Connection Stage; Reapplication Stage; Practice Stage; and Reflection Stage.

Play Stage: Play Stage provides children with an activity of playing monopoly. In this stage, children share, negotiate, and construct their immediate experiences to achieve the emergent goals of arithmetic problems with peers and more advanced children (the expert children).

Construction Stage: In the Construction Stage, the teacher designs a worksheet that has structural objectives that need to be accomplished by students. For example, children need to count the total money they have at the end of game.

Connection Stage: In the Connection Stage, based on children's experiences or strategies, the teacher tries to help children construct a connection between their experiences and concrete materials like ten-based blocks or mathematical symbols and procedures.

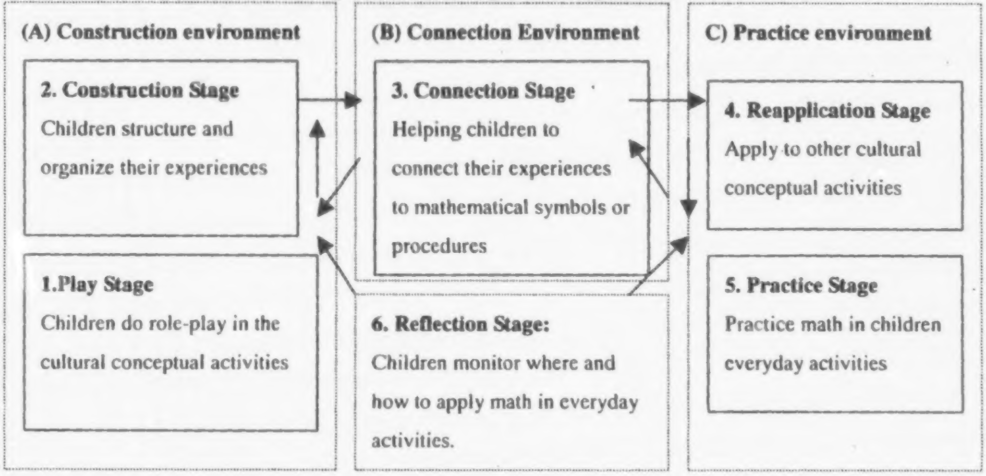
Reapplication Stage: In the Reapplication Stage, the teacher provides another similar or same cultural-conceptual activity for children to reapply to the learned mathematical concept.

Practice Stage: In the Practice Stage, children try to practice school mathematics in everyday situations by using opportunities provided for them.

Reflection Stage: In the Reflection Stage, children are trained to monitor their thinking and to be aware of where and how they can apply school mathematics in everyday activities.

In the CCLT, four kinds of cultural activities are integrated into classroom teaching, these activities include Pick-Ten-Point Game, Counting Lucky Money Activity, Shopping and Selling Toys Activity, and Monopoly Activity. This paper discusses only some findings from the Monopoly Game.

Figure 1: The Cultural Conceptual Learning Teaching Model


(A) Construction environment
(B) Connection Environment
C) Practice environment
2. Construction Stage
Children structure and organize their experiences
3. Connection Stage
Helping children to connect their experiences to mathematical symbols or procedures
4. Reapplication Stage
Apply to other cultural conceptual activities
1.Play Stage
Children do role-play in the cultural conceptual activities
6. Reflection Stage:
Children monitor where and how to apply math in everyday activities.
5. Practice Stage
Practice math in children everyday activities

Methodology

Eight second-grade classes in a school located in the city of Hsin-Chu participated in this study. Four classes were randomly assigned to the CCLT group and the rest were assigned to the control group. Teachers from the control group classes met in a half-day workshop to discuss the arithmetic content of the textbook. On the other hand, teachers from the CCLT group classes met every Friday to design the cultural activities and sharing their experience of teaching.

Three tests, The Situation Test, The Standardized Test, and The Achievement Test, were conducted to examine the effects of teaching approaches and the children's ability to solve everyday task problems. The classroom instruction of each teacher was observed and videotaped. Students' worksheets and journals were also collected and analyzed.

Space allows us to present and discuss only the some findings of the Situation Test problems (some results will be presented in the meeting of PME25 conference).

The Situation Test problems contain two problems. The first problem was presented in a story form, such as "John plays the monopoly game with his friends. When the game is over, John sorts the bills he won and recorded all the bills. In all, there are fifteen ten-dollar bills; six fifty-dollar bills and three one-hundred-dollar bills totally. Could you help John count how much money he won?" The students were asked to count how much money John had and write down the process in which they calculate the amount. The second problem was also presented in the same form but asked them to calculate more and different values of bills as follows: seven ten-dollar bills; two fifty-dollar bills; eight one-hundred-dollar bills; four two-hundred-dollar bills; two five–hundred-dollar bills; three one-thousand-dollar bills; three two-thousand-dollar bills; two five-thousand-dollar bills. Children's strategies and processes of calculation were analyzed and recorded.

Results

The teaching effects. Each part of the results was described in accordance with each research questions. The first part documented the effects of the CCLT teaching model on children's ability to solve the two problems described previously when compared to a regular teaching program.

Table1: Summarization of frequency, or percentage, and test results between the CCLT group the control group of the correct answers to the situational problems

	Treatment		Test	P
	CCLT	Control		
1. Totaling NT$ 750	60.3% (76)	23.1% (30)	χ^2 (1, N = 256) = 36.57	P*** < .001
2. Totaling NT$ 21770	16.7% (21)	2.3% (3)	χ^2 (1, N = 256) = 15.53	P*** < .001

Though by the end of the year second graders are expected to learn to add 2-digit numbers with a sum of no more than 100 and to learn numbers up to 1,000 by counting, the result of table 1 show that children who learn addition through the CCLT model gained more transfer learning than the national standard goal for the elementary school curriculum when compared with the control group. For the first problem, 60.3% of CCLT group children were able to solve the addition problems involving 3-digit numbers. On the contrary, only 23.1% of children in the control group were able to solve them. The difference between these two groups was tested significantly.

For the second problem, 16.7% of CCLT group children were able to solve the addition problems involving 4-digit and 5-digit numbers, while only 2.3% of control group children were able to add the same 4-digit and 5-digit numbers.

This difference was also tested significantly.

Achievements in different levels of addition. Some children couldn't express the processes of the problems s correctly, but they do some parts of the solutions. This part describes the different levels of addition between the children involved in the CCLT group and those involved in the control group when they calculate two problems.

Table2: Summarization of frequency, percentage, and test results between the CCLT group and the control group of the achievement level for addition in the situational problems

	Treatment		Test	P
	CCLT	Control	χ^2(1, N = 256)	
Totaling NT$ 750				
O	7.1% (9)	25.4% (33)	15.525	P*** <.001
2D	92.1% (117)	74.6% (97)	13.938	P*** <.001
3D	86.5% (109)	61.5% (80)	20.646	P*** <.001
Totaling NT$ 21770				
O	10.3% (13)	30.0% (39)	15.314	P*** <.001
2D	88.9% (113)	69.8% (91)	14.155	P*** <.001
3D	86.5% (110)	63.8% (83)	17.525	P*** <.001
3DTH	77.0% (98)	43.8% (57)	29.135	P*** <.001
4D	73.8% (93)	33.8% (44)	41.078	P*** <.001
4DTH	46.8% (59)	13.8% (18)	33.090	P*** <.001
5D	35.7% (45)	10.0% (13)	24.144	P*** <.001

When we analyzed students' calculating procedures, seven levels of addition were identified. Level 0 coded as the symbol (O) is characterized as students who were incapable of solving the given problem. Level 1 is characterized as students who were able to add the 2-digit numbers (2D). Level 2 is characterized as students who were able to add the 3-digit number, but the sum is less than 100 (3D). Level 3 is characterized as students who were able to add the 3-digit numbers with the sum up to several thousand (3DTH). Level 4 is characterized as students who were able to add the 4-digit numbers with the sum of less ten thousand (4D). Level 5 is characterized as students who were able to add the 4-digit numbers, with the sum of more than ten thousands (4DTH). Level 6 represents that students who were able to add the 5-digit numbers, but with a sum of less than several ten thousands (5D).

The data sketched in table 2 show that CCLT group had higher percentages at each level to the control group. In the first problem, the CCLT group standing at the highest level (3D) had significantly higher percentage (86.5%) than the control group (61.5%). In the second problem, the CCLT group at each level possessed a higher percentage than the control group. This difference was tested significantly. Therefore, children learning of addition using the CCLT model based on cultural activities achieved a higher level than the control group based on the textbook.

Strategies used. This part described the comparison of strategies used to solve the situational problems between CCLT group and control group. According to children's solutions, six children's strategies for solving the two given problems were identified.

Table3: Summarization of frequency, or percentage, and test results for group differences of strategies used in solving two problems

	Treatment		Test	P
	CCLT	Control	χ^2 (1, N = 256)	
Totaling NT$ 750:				
O	7.1% (9)	17.7% (23)	6.511	P*<.05
AI	27% (34)	40.8% (53)	5.420	P*<.05
GAI	15.9% (20)	4.6% (6)	8.887	P**<.01
GVAI	20.6% (26)	13.1% (17)	2.651	P> .05
MIGAI	11.1% (14)	9.2% (12)	.248	P> .05
MVGAI	18.3% (23)	14.6% (19)	.618	P> .05
Totaling NT$ 21770:				
O	7.9% (10)	15.4% (20)	3.431	P> .05
AI	16.7% (21)	33.8% (44)	9.968	P**<.01
GAI	16.7% (22)	3.1% (4)	13.410	P***<.001
GVAI	25.4% (32)	22.5% (29)	.336	P> .05
MIAI	9.5% (12)	3.8% (5)	3.327	P> .05
MVAI	23.0% (29)	22.3% (29)	.018	P> .05

O indicates that children were unable to solve the given problem or the strategies can't be identified. AI indicates that children calculated the addition problems with iterating method (one by one). For example, children solved the first problem as 10+10=20, 20+10=30, 30+10=40...140+10=150, 150+50=200...650+100=750. GAI indicates that children calculated the subtotals within each group using iterating method then calculate the subtotals to get the total. For example, 10+10=20...140+10=150; 50+50=100...250+50=300; 100+100=200, 200+100=300; 150+300=450, 450+300=750. GVA indicates that children calculated the subtotals with visualization and then calculated the subtotal. For example, children wrote 150+300=450, 450+300=750. Children understand that fifteen ten-dollar bills equivalent to $ 150, six fifty-dollar are $ 300, and then add up the total. During the Monopoly game, most children made use of the visualization method to count the total automatically. There was no time for children to calculate the answer slowly. MIGAI indicates that children used the multiplication method to calculate the subtotals one by one within each group, and then add up the subtotal. For example, children wrote 10×1=10, 10×2=20...10×15=150; 50×1=50...50×6=300; 100×1=100...100×3=300; 150+300=450, 450+300=750. MVGA indicates that children used the "within each group with visualization method" and then added up the subtotals. For example, children wrote 10×15=150, 50×6=300, 100×3=300, then 150+300=450, 450+300=750.

From table 3, the data show that the CCLT group more frequently utilized the GAI strategy than the control group, while control group used the AI strategy significantly more often than CCLT group. The CCLT group using other strategies had higher percentage than the control group, but there were no significant differences. However, since the Monopoly game was arranged in the last month of the school year; it needed to taker more time for children to complete the Connection Stage and the Practice Stage.

Conclusions

A previous study (Lin & Tsai, 1999) found that children had a rich store of cultural experiences in daily life that can be applied in the classroom. From the results of this study, learning arithmetic through children's cultural activities not only affects children learning of school mathematics but also improve their ability to solve task problems. This evidence is consistent with the effect of the CCLT teaching model in pervious studies (Tsai & Post, 1999; Tsai, 2000). One of possible reason is that this study chose popular cultural activities for classroom teaching; therefore, children brought rich experiences to take and share. Another possibility is that the CCLT model provides a learning environment for children to connect their everyday experiences to school mathematics and to practice the learned mathematics in everyday activities again.

As the Taiwanese national curriculum standards described, second graders are merely expected to be able to solve two-digit additive problems and the place value of the number less than 1,000 at the end of the school year. According to the findings of this study, children learned more in the transfer of learning in the CCLT group than children in the control group. In the CCLT, children not only need to know the mathematics concepts but also need to know how to apply them in the cultural activities. Children make sense the mathematical meanings gradually when they connect their everyday experiences with school mathematics then reapply them in the cultural activities again. However, we need more time to validate this model.

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CAN X=3 BE THE SOLUTION OF AN INEQUALITY? A STUDY OF ITALIAN AND ISRAELI STUDENTS

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This paper describes some findings of a study regarding Israeli and Italian students' solutions to standard (commonly taught) and non-standard (not commonly taught) inequalities. The findings presented here show similarities in students' difficulties, in both countries, regarding $x = 3$ as the solution of an inequality even in cases where they correctly identified $x = 0$ as the solution of the task $5x^4 \leq 0$. We also found that a substantial number of the participants encountered difficulties in solving the inequality $5x^4 \leq 0$, claiming either that the set of solutions was empty or that the set of solutions was $x \leq 0$.

Inequalities play an important role in mathematics. They are part of various mathematical topics including algebra, trigonometry, linear planning and the investigation of functions (e.g, Chakrabarti & Hamsapriye, 1997; Mahmood & Edwards, 1999). They also provide a complementary perspective to equations. Accordingly, the American Standards documents specify that all students in Grades 9-12 should learn to represent situations that involve equations, inequalities and matrices (NCTM, 1989). They further recommend that students would "understand the meaning of equivalent forms of expressions, equations, inequalities and systems of equations and solve them with fluency" (NCTM, 2000, p. 296). To implement these NCTM recommendations it is crucial to consider students' ways of thinking about inequalities.

However, so far, research in mathematics education has paid only little attention to students' conceptions of inequalities (e.g., Dreyfus & Eisenberg, 1985; Linchevski & Sfard, 1991; Tsamir & Almog, 1999; Tsamir, Tirosh, & Almog, 1998). Most of the related articles dealt with teachers' and researchers' suggestions for instructional approaches, usually with no research support. They recommended, for instance, the sign-chart method (e.g., Dobbs & Peterson, 1991), the number-line method (e.g., McLaurin, 1985; Parish, 1992), and various versions of the graphic method (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992; Vandyk, 1990).

Those few studies, which have been published, tended to describe students' reactions to a few inequalities of the types commonly presented in class, and usually reported only one or two difficulties. For instance, studies pointed to students' tendency to make invalid connections between the solution of a quadratic equation and its related inequality (e.g., Linchevski & Sfard, 1991; Tsamir, Tirosh, & Almog, 1998). Other studies related to students' tendency to regard transformable inequalities as being equivalent. They further identified the need to use logical connectives (Parish, 1992), and found the solutions of inequalities with "R" or "ϕ" results extremely difficult (Tsamir & Almog, 1999).

The present study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. During discussions of Working Group 2 at PME22 (1998), and of Project Group 1 at PME23 (1999), it was found that in both Italy and Israel, algebraic inequalities receive relatively little attention and are usually discussed only with mathematics majors in the upper grades of secondary school. Discussions are usually limited, emphasising the "practical" algorithmic perspective of algebraic manipulations. Attention is paid mainly to "How to solve?" instead of "Why solve it this way?" or "How can I be sure that the solution I have reached is the correct solution?" Moreover, in both countries, the two researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities. Consequently, an Italian and an Israeli researcher who attended these conferences decided to collaborate their research in this area.

A collaborative study was designed to investigate students' ways of solving standard and non-standard tasks with similar, underlying mathematical ideas. The students were given six tasks, presented in the manner to which they were accustomed in their classes, i.e., "solve" tasks, designated as "standard tasks". They were also given nine tasks, related to the same mathematical issues, which were presented in a non-customary manner, and designated as "non-standard tasks".

In this paper we focus on 2 of the 15 tasks that were give to the students. Both tasks dealt with the same underlying mathematical situation, i.e., single-value solutions to inequality tasks. The main related research question was: Do Israeli and Italian secondary school students accept the expression $x = a$ as the solution of an inequality – Once, presented in a standard multiple-choice "solve" task, and once as a "reversed order" task, asking whether a given set can be the truth sets (the solution) of any equation or of any inequality, and are the students' reactions to the two tasks consistent?

Methodology

Participants

One-hundred-and-seventy Italian high school students and 148 Israeli high school students participated in this study. Both the Italian and the Israeli participants were 16-17 year old mathematics majors. That is, in both countries we examined students who were aiming to take final mathematics examinations in high school. Success in these examinations is a condition for acceptance to academic institutions, such as universities.

In their previous algebra studies, the participating students had studied the topic of algebraic inequalities, including linear, quadratic, rational and absolute value inequalities. In both countries, the participating students were taught this topic in a traditional way, being presented with different methods for solving the different types of inequalities. For example, parabolas or the number line to solve quadratic inequalities, and "multiplying by the square of the denominator" for the solutions of rational inequalities.

Tools

A 15-task questionnaire was administered in both countries. Italian and Hebrew versions were given to the Italian and Israeli students respectively. The two tasks analyzed here are Task 1 (a non-standard task) and Task 9 (a standard task).

Task 1

Consider the set $S=\{x\in R: x=3\}$ and check the following statement:

S can be the solution of both an equation and an inequality. Explain your answer.

Task 9

Indicate which of the following is the truth set of $5x^4 \leq 0$

$A=\{x: x>0\}$ $B=R$ $C=\{x: x<-5\}$ $D=\{x: 0<x<1/5\}$

$E=\phi$ $F\ x=0$ $G=\{x: x\leq 0\}$

Task 1 demanded proving the existence of a case where x=3 is the solution of an equation, and also proving the existence of a case where x=3 is the solution of an inequality. The easiest way to go about this was by providing suitable examples. This kind of assignment, asking the students to examine the existence of a case where x=3 is the solution of either an equation or an inequality; then, if possible, to provide tasks to match a given solution, was not dealt with in either the Israeli or the Italian classes we investigated.

We expected the first part that related to the existence of a suitable equation to be easy, and the second part, where the students had to examine the existence of a case where x=3 is the solution of an inequality, to be problematic.

Task 9 was a standard task, similar to other tasks presented in Israeli and Italian classes. We assumed that most students would solve it correctly.

Procedure

In both countries, the mathematics teachers of the classes distributed the questionnaires, during mathematics lessons. The students in each of the countries were given approximately one hour to complete their solutions, which usually was enough time. The researchers analysed, categorised and summarised the different solutions. In two additional meetings the researchers decided on possible ways to present the data.

Results

The results will be presented in the following order. First, an analysis of Israeli and Italian students' responses to Task 1, then their responses to Task 9, to conclude with an analysis of the consistency in students' reactions to the two tasks.

Students' Reactions to Task 1

In both countries, none of the students had any problems in correctly responding that $x = 3$ can be the solution of an equation. Most of them accompanied their responses by an example, usually of a first-degree equation, such as $2x-6=0$. This, however, was not the case with the participants' responses to the question whether $x = 3$ can be the solution of an inequality, in both Israel and Italy.

Table 1: Frequencies of students' solutions and justifications to Task 1 (in %)

	ISRAEL N=147	ITALY N=150
TRUE*	**51.4**	**48.3**
Valid explanation	5.4	2.0
A system of inequalities	15.5	0.7
X=3 belongs to the solution	3.3	3.0
Other**	27.2	42.5
FALSE	**48.6**	**51.7**
A solution of inequality is an inequality	19.5	22.0
Other**	29.1	29.7

* Correct response

** Irrelevant or missing justifications

Table 1 shows that in both countries, only about 50% of the students who responded to this task, correctly claimed that $x = 3$ can be the solution of an inequality. Still, most of them did not accompany their claims by any justification and only a few students, Israeli or Italian, gave valid explanations. These latter explanations were usually the presentation of the following example of the quadratic inequality $(x-3)^2 \leq 0$. More prevalently in Israel, but also in a few Italian cases, explained that the claim "$x = 3$ can be the solution of an inequality" is true, because $x = 3$ can be the solution of *a system* of inequalities. Such justifications were often accompanied by an uncomplicated, linear example, such as

$$\begin{cases} 2x-6 \leq 0 \\ x-3 \geq 0 \end{cases}$$

Another type of interesting justification, given by a small number of Israeli and a small number of Italian participants was that "the claim is true, because $x = 3$ can belong to the set of solutions of an inequality." This justification was accompanied by illustrations, such as, $5x - 10 > 0$, further explaining that "the truth set (solution) of this inequality is $\{x: x>2\}$, and 3 is one of the values that satisfies this condition, and therefore $x=3$ belongs to the truth set of $5x - 10 > 0$."

Students' Reactions to Task 9

Only about 50% of both the Israeli and the Italian participants who responded to this task, correctly identified $x = 0$ as the solution of the inequality (see Table 2).

Table 2: Frequencies of students' solutions to Task 9 (in %)

	ISRAEL N=128	ITALY N=168
$X = 0$*	53.3	51.2
$X \leq 0$	23.8	16.2
Phi	17.1	26.7
Other	5.8	0.9

* Correct solution

A substantial number of the participants claimed that the set of solutions was empty (Phi, or 'there is no solution to the given inequality'). Some of them volunteered the explanation that x^4 has an even power and thus it can never be negative, showing that they ignored the "zero-option". Another interesting phenomenon was the Israeli and Italian students' tendency to answer that the set of solutions of $5x^4 \leq 0$ was $x \leq 0$, which was further explained by a number of them, claiming, for instance, "I simply computed the fourth root of both sides of the inequality."

Examining the consistency in students' reactions to Tasks 1 and 9

As can be seen from Tables 1 and 2, and as mentioned before, about half of the participants from each of the two countries claimed that "$x = 3$ can be the solution of an inequality", and about half of the participants identified $x = 0$ as the solution of $5x^4 \leq 0$. That is, about half of the participating students pointed to the possibility of having $x = a$ as the solution of an inequality, either in Task 1 or in Task 9. A question that naturally arose was, were these the same students? That is to say, did the students consistently express their understanding that $x= a$ could be the solution of an inequality in their reactions to both tasks, by responding "true" to Task 1 and "$x = 0$" to Task 9? Table 3 shows that the answer to this question is no.

Only about 29% of the Israeli participants and about 24% of the Italian participants exhibited a general view that $x = a$ can be the solution of an inequality and also correctly reached this type of a solution in reaction to the "solve" drill in Task 9. It is also notable that a similar percentage of each group rejected the option of x = a being the solution of an inequality, and did not reach the correct x = 0 solution in Task 9 as well.

Table 3: Frequencies of consistent and inconsistent reactions to Tasks 1&9 (%)

		ISRAEL N=148	ITALY N=170
CONSISTENT		**57.8**	**48.3**
Task 1	***Task 9***		
True	Correct	29.36	23.9
False	Incorrect	28.44	24.4
INCONSISTENT		**35.76**	**38.2**
Task1	***Task 9***		
False	Correct	22	21
True	Incorrect	13.76	17.2
OTHER*		**6.44**	**13.4**

* Providing no response to at least one of the two tasks.

More than 35% of the participants in each country were inconsistent in their reactions to the two tasks. Part of them correctly claimed that x = 3 could be the solution of an inequality, but did not identify x = 0 as the solution of the inequality in Task 9. More interesting were the inconsistent reactions of about 20% of both the Israeli and the Italian participants. On the one hand, they claimed that x = 3 can not be the solution of an inequality, usually explaining that "an inequality can only be the solution of an inequality". On the other hand, within the same questionnaire they reached an x = 0 solution to the inequality presented in Task 9.

Discussion

Our findings indicate that, as expected, all students in both countries were aware that x = 3 can be the solution of an equation, and that many of them encountered difficulties in identifying the possibility of x = 3 being the solution of an inequality. These findings can be examined by means of the Intuitive Rules Theory, formulated by Stavy and Tirosh (2000). Students expressed the views that "an equation-result can only be the solution of an equation task" or that "an inequality task must have an inequality-solution." These claims are in line with the intuitive rule Same A (equation / inequality relationship in the solution) – same B (equation / inequality relationship in the task).

Quite surprising were the findings showing students' difficulties in responding to the standard "solve" task. In both countries only about half of the participating students identified x = 0 as the solution of the inequality $5x^4 \leq 0$. It seems that, similar to previous studies, reporting "strange" solutions like Phi and R as problematic for students (Tsamir & Almog, 1999), this study identified that the x = a type of solution is also problematic in cases of inequality-tasks, and should further be investigated. A wider analysis of students' reactions to this task, embedded in different theoretical frameworks (e.g., Fischbein, 1987; Arzrello, Bazzini, & Chiappini, 1993; Bazzini, 2000; Maurel & Sackur, 1998) will be provided in the oral presentation.

Most interesting were the findings related to the consistency of students' reactions to the two tasks. We should remember that both tasks were included in the same questionnaire and students were free to move back and forth among the different tasks. In this manner, students' correct solutions to Task 9 could have served as an example for correctly solving Task 1. Still, no student explicitly mentioned Task 9 when correctly responding to Task 1. Furthermore, a non-negligible number of students (Italian and Israeli) responded to tasks 1 and 9 in a contradictory manner. They wrote, "an inequality can only be the solution of an inequality" (Task 1) and then, that $x = 0$ was the solution of the inequality $5x^4 \leq 0$ (Task 9). A possible explanation for this phenomenon could be that sometimes zero is regarded as a special number. Thus, students could accept $x = 0$, but reject $x = a$, when 'a' is other than zero, from being a solution of an inequality. Naturally, further research is needed to investigate such assumptions.

Moreover, our findings call for interventions that deal with the specific issue of algebraic inequalities and with the general issue of consistency in mathematical reasoning. Questions that arise are, for instance, how to introduce inequalities? How to cope with inconsistencies in students' reactions to inequalities? and how to validate the correctness of specific solutions to inequalities? Suggestions for research based instruction will be presented and discussed in the oral presentation. Clearly, the impact of such interventions should be further investigated.

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INVESTIGATING THE INFLUENCE OF THE INTUITIVE RULES IN ISRAEL AND IN TAIWAN

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Mathematics education researchers are always on the lookout for theories with explanatory and predictive powers – i.e., theories that enable the analysis of students' reactions and the prediction of their likely responses to given tasks. One such theory is the intuitive rules theory, which has been described in a series of articles. In this paper we describe a study which was carried out in Israel and in Taiwan to examine the influence of the intuitive rules More A – more B and Same A – same B on Israeli and Taiwanese students' responses to comparison tasks. Our findings confirm that in both countries the reactions to such tasks of students in various grades were indeed influenced by these intuitive rules.

In their work in science and mathematics education, Stavy and Tirosh have observed that students react in similar ways to a wide variety of conceptually non-related tasks (e.g., Stavy & Tirosh, 2000). Although these tasks differ with regard to their content area and/or to the type of reasoning required, they share some common, external features. So far four types of responses were identified, two of which, i.e., *More A - more B* and *Same A-same B*, are related to comparison tasks. For example, when students are told that Tom saves 15% of his salary, and Marry saves 20% of her salary, they tend to claim that Marry saves more money than Tom, because 20 is larger than 15, in line with the intuitive rule more A (percentage)-more B (money). Similarly, when students are told that John saves 10% of his salary, and Daffy also saves 10% of her salary, children tend to claim that John and Daffy save the same amount of money, in line with the intuitive rule same A (percentage)-same B (money).

Based on these observations, the intuitive rules theory has been proposed to explain and predict students' responses to mathematics and science tasks. Many responses that the literature describes as alternative conceptions could be interpreted as evolving from the intuitive rules *More A - more B, Same A-same B*, which are activated by specific external task features (for detailed descriptions see, for instance, Tsamir, Tirosh &. Stavy, 1997; Tsamir, Tirosh & Stavy, 1998; Tsamir & Mandel, 2000).

The intuitive rules theory is based on data collected in the western world. It is, however, very interesting and important from both the theoretical as well as practical points of view to test the universality of this theory. For this purpose a study, parallel to the one conducted in Israel, was carried out in Taiwan, relating to a wide number of comparison tasks. Here we focus on Israeli and Taiwanese students' reactions to tasks related to three topics: vertical angles, temperatures, and the volume of cylinders, with regard to the intuitive rules *More A - more B* and *Same A - same B*.

Methodology

Background

In Israel, the topic of angles is introduced in Grade 4 and the proof that vertical angles are equal is dealt with in Grades 8-9. The topic of volume is introduced in grade 3-6 and formally discussed in Grades 10 to 12, when dealing with three dimensional geometry, trigonometry and analysis (e.g., extreme problems). The concept of temperature is presented in Grades 4 -5 and children are introduced to the terminator as a measuring tool. Further issues related to heat and temperature are and discussed in Grades 7-9.

In Taiwan, the topic of angles is introduced in grade 3-4 and the proof that vertical angles are equal is presented in Grade 9. The topic of volume is introduced in grade 3-6 and formally discussed in Grades 9 and up, when dealing with three dimensional geometry, trigonometry and analysis (e.g., extreme problems). The concept of temperature in presented in Grades 3-6 and children are introduced to the terminator as a measuring tool. Further issues related to heat and temperature are and discussed in Grades 8-9.

A question that naturally arises is – Are Taiwanese children similarly affected by intuitive rules as are Israeli children? More specifically, will Taiwanese students tend to claim, in line with the intuitive rule more A more B that "The larger the arms the larger the angle"? "The more water- the higher the temperature"?; and in line with the intuitive rule *Same A - same B*, that "Same area-same volume?"

Participants and Procedure

In Israel: Two hundred-and-forty-three students from Grades 2, 4, 6, and 9 – sixty nine, 65, 70 and 60 students, respectively, answered a written questionnaire that included a *comparison-of-vertical-angles* task.

One-hundred-and-twenty students, from Grades 2, 3, 5, 6, 7, 8, twenty from each grade, were individually interviewed regarding the *comparison of temperatures* task.

Three-hundred-and-seventy-five students from grades 1-6, and 10, 12, forty students each from grade levels 1-6, 110 10th graders and 29 12th graders. In Grades 1-6 each participant was individually interviewed, and the 10th and 12th graders were given a written questionnaire regarding the *comparison of volume of cylinders* task.

In Taiwan: Nine-hundred-and-sixty-six students from Grades 3, 4, 5, 6, 10 -- 206, 339, 345, 67, respectively, answered a written questionnaire that included a *comparison-of-vertical-angles* task.

Nine-hundred-and-twenty-one students from Grades 2, 3, 4, 5, 6, 10, -- 33, 197, 210, 205, 209, 67, respectively, answered a written questionnaire that included a *comparison of temperatures* task.

On-thousand-two-hundred-and-sixty-nine students from Grades 1-6, 10, 11, -- 28, 33, 34, 117, 443, 428, 65, 121, respectively. All participants answered a written questionnaire that included a *comparison of cylinder volumes* task.

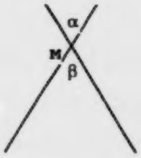
Materials

Comparison of Vertical Angles Task

Consider the following drawing:

Is angle β smaller than / equal to / larger than / angle α?

Explain your answer.

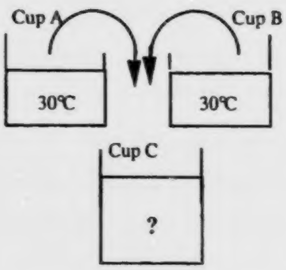


Comparison of Temperatures Task

Consider the following drawing:

The water from Cup A and the water from Cup B were poured into Cup C. What is the temperature of the water in Cup C?

Explain your answer.

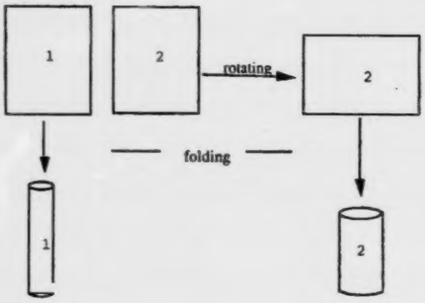


Comparison of Cylinder Volumes Task

Take two identical rectangular (non-square) sheets of papers (Sheet 1 and Sheet 2):

Rotate one sheet (sheet 2) by 90°
Is the area of sheet 1 smaller than / equal to / larger than / the area of Sheet 2?
Explain your answer.

Fold each sheet (as shown in the drawing).
You get two cylinders: Cylinder 1 and Cylinder 2.
Is the volume of Cylinder 1 smaller than / equal to / larger than / the volume of Cylinder 2?
Explain your answer.



The tasks were originally formulated in Hebrew, the language in which they were administered in Israel (see for instance, Stavy & Berkovitz, 1980; Stavy, Tirosh & Tsamir, 1997; Tirosh, Stavy & Ronen, 1996). The mathematics and science educators who conducted these researches in Israel were familiar with the intuitive rules theory. They were responsible for writing, editing, administering the tasks as well as for the analysis of the data.

In order to use these tasks in Taiwan a workshop about the intuitive rule theory was carried out by one of the authors. Forty-five Taiwanese mathematics and science educators participated in ten four-hour weekly sessions, in which they studied the theory and had to carry out a relevant mini study in school classes. The tasks were translated into English, and then one of the workshop participants re-translated it into Chinese. In order to validate the translation, the other participants commented on the translated version. The final agreed upon version was again translated into English and the meanings were verified. Then, the tasks were administered by the workshop participants, in the form of a written questionnaire, in several cities in Taiwan. Students responded in Chinese and the results were analyzed by each of the researchers.

Results

First we present the results related to the intuitive rule *More A - more B*, and then the results related to the intuitive rule *Same A - same B.*

Results related to the intuitive rule more A - more B

Vertical angles

Figures 1 and 2 show that in both Israel and Taiwan the tendency to correctly solve the comparison of vertical angles task increased with age. While among young children only about 30% of the 3rd graders in Taiwan, and about 10% of the 2nd graders in Israel provided correct responses, most Israeli 9th graders (over 80%) and all Taiwanese 10th graders correctly responded that the angles were equal.

Of the participants who did not respond correctly, almost all answered in line with the intuitive rule more A (longer arms, larger enclosed area) -- more B (larger angle).

Comparison of Temperatures Task

Figures 3 and 4 show that both Israel and Taiwan the tendency to correctly solve the comparison of temperatures task increased with age. While among 2nd graders of both countries there were no correct responses at all, most Israeli 8th graders (over 80%) and all Taiwanese 10th graders correctly responded that the temperature remained thirty degrees.

Of the participants who did not respond correctly, almost all answered in line with the intuitive rule more A (amount of water) -- more B (higher temperature).

Results related to the intuitive rule *Same A - same B*

Comparison of Cylinder Volumes Task

Figures 5 and 6 show that in both Israel and Taiwan the tendency to correctly solve the comparison of surface area of cylinders task increased with age, accompanied by an incorrect response to the comparison of volume task. In all cases students claimed in line with the intuitive rule *Same A - same B* that the volumes of both cylinders were equal as they were made from identically sized sheets of paper.

Final Comments

The findings of this study clearly indicate that Taiwanese students, much like the Israeli peers are strongly affected by the intuitive rules *More A - more B* and *Same A - same B*, when presented with relevant comparison tasks. When relating to tasks whose correct answer was not in line with the intuitive rules, two main findings should be highlighted. First, to all three tasks two major types of responses were evident – the correct, and an incorrect response in line with one of the two intuitive rules. Also, the developmental pattern with age was rather similar.

Consequently, we suggest that the intuitive rules affect students' responses in both countries regardless of culture. As mentioned before, additional data were collected with regard to other relevant tasks, and similar findings pointing to the influence of the intuitive rules *More A – more B* and *Same A – same B*, were obtained.

These conclusions should be taken with some caution, due to practical constrains: the methodologies applied were not completely identical. Still we find the picture presented by these preliminary results a very relevant one. Clearly, further research is needed to bolster our knowledge about the universality of the role of the intuitive rules in students' mathematical and scientific thinking

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Fig. 1: Taiwanese students' responses to the vertical angles task

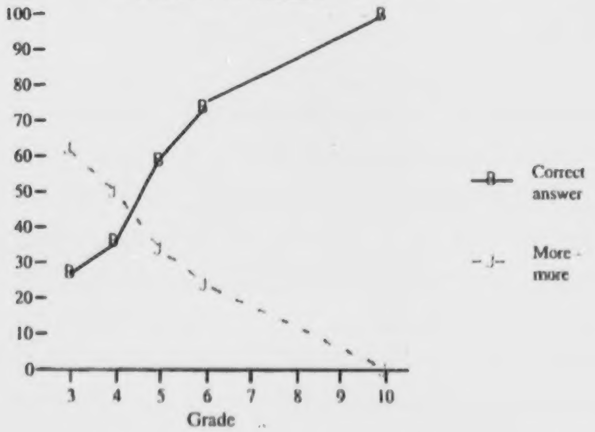


Fig. 2. Israeli students' responses to the vertical angles task

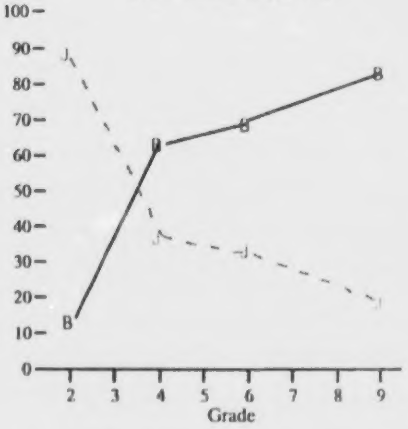


Fig. 3. Taiwanese students' responses to the temperature task

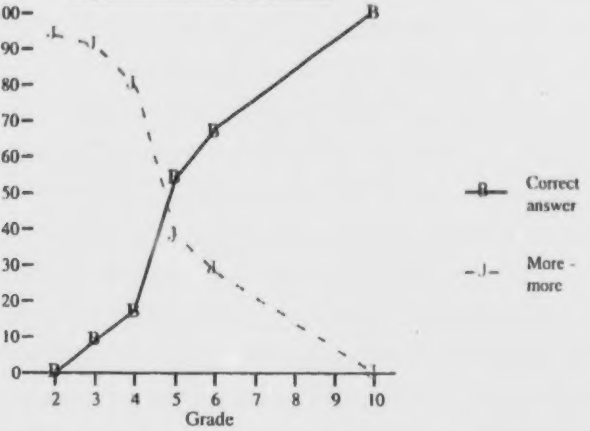


Fig. 4: Israeli students' responses to the temperature task

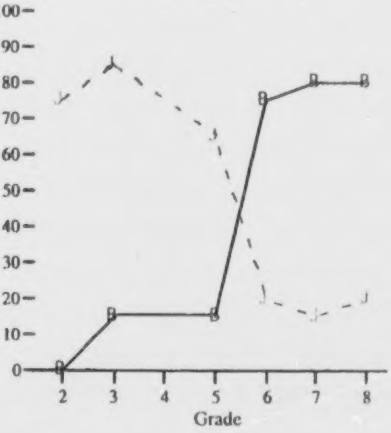


Fig. 5: Taiwanese students' responses to the volume of cylinder task

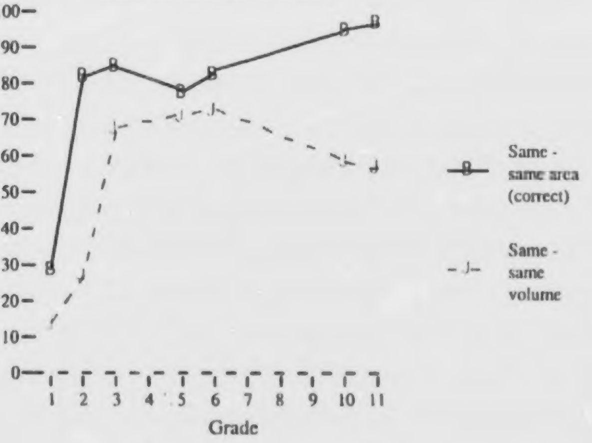
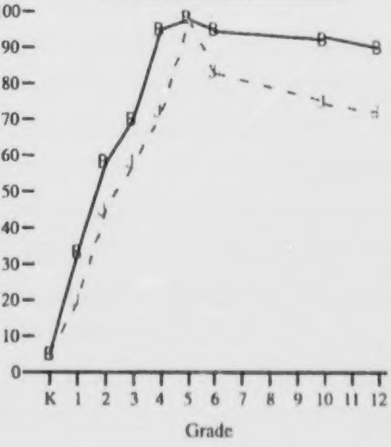


Fig. 6: Israeli students' responses to the volume of cylinder task



RE-EVALUATING ASSESSMENT IN LIGHT OF AN INTEGRATED MODEL OF MATHEMATICS TEACHING AND LEARNING

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Abstract: *This is a preliminary report on a study that examined, in practice, an integrated model of mathematics teaching and learning. The paper addresses a combination of two problems—how credible is a key theoretical distinction about learning a new conception and how does that distinction inform teacher's assessment of students' thinking. . I conducted the study as a whole-class teaching experiment in a 3rd grade classroom over a 4-month period. The analysis indicates that the distinction between a participatory and an anticipatory stage is theoretically sound and practically useful in setting the teacher's goals for and activities of assessing students' thinking. The study highlights how the model provides a new way of thinking about the role for and organization of assessment.*

In this paper I present part of a comprehensive study that examined an integrated model of mathematics learning and teaching in a real classroom setting. Examining the model was a multi-faceted task and reporting on it goes beyond the scope of a single article. Therefore, in this paper I address a combination of two problems—how credible is a key theoretical distinction about learning a new conception and how does that distinction inform teacher's assessment of students' conceptual understanding. This focus highlights the interplay between teacher goals of precisely assessing students' thinking and advancing their understanding. I begin with a brief description of the model and what its examination in practice might consist of, then I present the methods used for the study and an analysis of data regarding the combined focus, and finally I discuss the significance of the study.

Conceptual Framework

The integrated model of conceptual teaching and learning of mathematics (cf., Simon et al., 2000; Tzur & Simon, 1999) is an elaboration of the psychological aspect of the emergent perspective (Cobb and Yackel, 1996). The model evolved as a response to a theoretical problem known as the "learning paradox" (Pascaul-Leone, 1976). This paradox is implied by Piaget's fundamental notion of assimilation. If one can only recognize and respond to aspects of reality by assimilating them into existing conceptions, how is one to ever construct new conceptions? The model untangles the paradox by addressing three questions: what is a conception, how is a new conception formed (mechanism and stages), and how can teaching promote formation of intended conceptions. I briefly describe each of the three below.

The primitive unit of knowing in the model is a dynamic compound, a relationship. The relationship is between an activity and its effect(s) (abbreviated as A-E relationship). The unit is not mainly the activity nor mainly the effect(s) but the dynamic compound consisting of both. Terms such as scheme, concept, mental object, cognitive process, and the like are various notions used by different scholars to refer to A-E relationships. In this paper I will use the term conception and A-E relationship interchangeably.

The primitive unit of learning in the model is the mental mechanism of reflection on A-E relationship (abbreviated as Ref*A-E relationship). The term reflection refers to the ceaseless mental comparison between one's goals for and effects of her activities (von Glasersfeld, 1995). Note that from the learner's point of view, Ref*A-E relationship does *not* need to be directed toward making specific conceptual advances (i.e., it does not imply awareness).

Through Ref*A-E relationship a learner might make a distinction between desired effects she anticipates of an activity and the effects she notices during or after carrying out the activity. This type of distinction, termed *the initial phase*, is thought of as a critical precursor for forming a new conception. However, the initial phase is not considered a stage because no new A-E relationship has been abstracted, yet. If goal and effects do differ, the cognitive system searches for some new recurrence, that is, a new regularity in A-E relationship other than the regularity—existing conception—that triggered the activity.

Forming new, regular A-E relationships is postulated to occur in three stages. First is the *participatory* stage. Knowing at this stage is marked by the learner's ability to relate effects and activity *only* when one is somehow oriented to focus on the activity. Thus, at this stage it is assumed that the learner does *not* know, spontaneously, to call up the activity for the particular goal. The second stage is the *anticipatory* stage. In contrast to the participatory stage, the learner, upon setting her goal in a situation, can spontaneously select the activity she newly related to that goal(s) and figure out its effects. Though the effects cannot yet be known immediately, the learner can spontaneously initiate the activity from within the A-E relationship, generating and reflecting upon its effects. The third stage is the *reified* stage. Unlike in the anticipatory stage, the learner can immediately identify the anticipated effects whereas the activity that generated the A-E relationship fades to the background. To an observer it appears as if the learner uses a type of "idle" knowing, knowing without activity. This misleading appearance is probably one reason why "knowing" is often mistaken with quick recognition of facts. However, an activity of a reified A-E relationship is always implicit and it reappears when, for example, the learner is asked to justify her solution. I use the term *reified* because the third stage seems consistent with Sfard's (1991) notion of reification. The formation of a reified conception completes a learning cycle such that it can then afford a nev distinction among effects, and so on.

Building on the learning process as postulated above and on Simon's (1995) and Tzur's (1999) works, the component of teaching in the integrated model is cyclic in nature and consists of four principal phases:

1. assessing (inferring) learners' current conceptions based on their actions and languag
2. hypothesizing (trying to predict) a learning trajectory for the learners on the basis of what they know, that is, identifying a higher stage and a potential process of change toward that stage;
3. selecting and engaging learners in tasks (problem situations) they are likely to understand and use in service of the intended advance, that is, to assimilate into available conceptions, to set appropriate goal(s), and to initiate appropriate activities;
4. using probing questions and/or comments both to foster reflection on patterns of A- relationships and to re-assess learners' current conceptions, and so on.

Regarding phase 3 in the cycle, four types of tasks that correspond to the four postulated transformations from an available conception into a new, intended conception are proposed. (Note: The name refers to the intended phase or stage.) The teacher uses initial tasks to foster the use of activities available in current conceptions to make new distinctions among effects of that activity. For example, consider children who already constructed whole numbers at least as an anticipatory relationship between the activity of iterating the unit of one and the effect of having established a composite unit of a certain amount (e.g., 3=1+1+1). Then, using paper strips, the teacher can engage learners in using the repeat strategy (see next) to share a paper strip among a given number of people. The repeat strategy consists of four activities that learners can already initiate and use in sequence: estimating the size of one piece, iterating that piece the desired number of times, comparing the whole produced in iteration to the given one, adjusting (re-estimating) the size of the piece, etc. The repeat strategy is likely to foster learners' initial distinction between a piece that is too short, too long, or exactly the size needed. The teacher uses participatory tasks to orient learners' to reflect on and identify a new relationship between the activity and the newly distinguished effects. In the repeat strategy example, a participatory task can be the question, "Why did you make that piece shorter than the previous one?" Such a question might orient learners' reflection on the magnitude relationship between the activity of adjusting the size of the estimated piece and the uniqueness of that size relative to the whole. Moreover, the learners may notice a new (inverse) regularity between the number of times the piece is iterated and its size (e.g., 1/6 is larger than 1/7 because each of the 6 pieces has to occupy more space). The teacher uses anticipatory tasks to foster learners' abstraction of the participatory A-E relationship into an anticipatory A-E relationship. If learners are to advance from the participatory stage, it is critical that anticipatory tasks do not indicate what activity the learners should use. In the repeat strategy example, an anticipatory task might be, "You received 1/7 of a pizza; can you figure out a way to cut another piece of the same-size pizza so that your friend gets a smaller piece than yours? What fraction of the pizza is your friend's piece?" It is the learner who translates the question into another one, "How could I share a pizza among 7 people?" which triggers the activity (repeat strategy) she was already using prior to the initial phase, which fosters mental reprocessing of the activity and its effects. The learner realizes that she has to iterate the piece more than 7 times and regenerates the magnitude relationships in her thinking. The teacher uses reified tasks to foster learners' abstraction of anticipatory A-E relationship into a new, reified A-E relationship. Unlike at the participatory stage, at the anticipatory stage the learner can assimilate abstract symbols and mentally reprocess the anticipatory A-E relationship. Through further reflection, the anticipatory A-E relationship becomes the signified "object" encapsulated within the symbolic, signifying entity and the construction of the new conception is established at the reified stage. In the repeat strategy example, a reified task might be, "What is bigger, 1/4 or 1/7?" because it calls upon the anticipatory inverse relationship between size and number of pieces and fosters further reflection on and reification of that relationship.

Methodology

I conducted the comprehensive study as a classroom teaching experiment (Cobb, 2000) in the context of fractions. I chose fractions because research (cf. Tzur, 1999) identified a developmental sequence of fraction conceptions that could guide my teaching. In a public elementary school in Israel, I selected a 3rd grade classroom that had not yet received any instruction on fractions. Thus, for all 28 students the following two-part description applied. First, they could reason about whole numbers. Some were also able to engage in part-whole numerical reasoning. For example, they could think of 12 as composed of 5 "ones" and 7 "ones," and, when told that a dozen-egg carton already contains 7 eggs, could figure out how many more eggs would fill the carton. Second, as students' responses to a pre-progran problem-solving questionnaire indicated, none used any fraction conception beyond, maybe, primitive, generic notion of half. Thus, I could begin fostering in these students a sense of magnitude of, and operation on, unit fractions (1/3, 1/4, etc.).

As a researcher-teacher I collected data by conducting two videotaped lessons a week, for a total of 26 lessons. I taught the lessons consecutively, on Friday and Sunday, because I wanted to have a sense of continuation while allotting the time needed between lessons for analysis and planning. The ongoing analysis after each lesson focused on students' conceptions and whether or not a conceptual change took place. As part of that analysis I conducted audio-recorded conversations with the classroom teacher and 1-2 other teachers who observed every Sunday lesson. On the basis of my reflections on each lesson I created and/or adjusted tasks for the next lesson. Besides the reflective sessions with the teachers, I systematically documented (audio recorded and/or wrote) my reflections about students' evolving understandings and my past and future teaching.

Upon completion of data collection, I will conduct a retrospective analysis of the teaching-learning process. However, teaching is not over, yet. Thus, the analysis in the next section represents a work in progress—my first screening of data from three lessons that are relevan to the combined foci of this paper. This screening included careful reflection on lesson segments (videotapes) of the three lessons and on chunks of transcripts of my notes (written or recorded) regarding my teaching activities and decisions.

Analysis

I did not set the goal to specifically examine the aspect of assessment from the outset. Rather assessing students' conceptions became a perturbation for me in the practice of teaching. In this section I present how I resolved the perturbation. In the first take at the perturbation my goal of promoting students' learning overruled the goal of making subtle distinctions among students' ways of thinking. In the second take the order was reversed. Through paying attention to considerations of both ways of organizing assessment, a pattern emerges that highlights both the credibility and power (in terms of assessment) of the theoretical distinctio between the participatory and anticipatory stages.

First Take at the Perturbation: Promoting Learning Overrules Subtle Assessment. After a fe lessons in which the students were engaged in using the repeat strategy, I had a rough sense that some already formed a participatory sense of the inverse relationship between size and number of pieces while others did not. However, I made several notes that indicate that I became frustrated because of my initial inability to get to know how each student was thinking. For example, I wrote in reflection on the lesson of November 11, 2000:

> I thought, intensively, about the problem: How to overcome the enormous difficulty I have to get to each and every student and observe what they do, to "interview" them on the spot ... so I can understand how they think. I realized that I need to give them some task that requires written responses that will give me a more detailed information than I currently am able to gather, information that is structured by the model I use.

Consequently, my planning for the next lesson turned to creating a task that would delineate students' thinking. After several hours of thinking and jotting down ideas for different tasks, I designed a set of 7 questions that I hoped would provide insight into students' current thinking about the conceptual aspect at issue—direction of adjustment in size of the estimated piece (make it shorter or longer). I considered this aspect as the conceptual root of the uniqueness of the size of a unit fraction and of the inverse relationship between size and number of unit fractions in a given whole (cf. Tzur, 1999). For example, questions #1 & #2, and #5 & #6, read as follows (figures are not presented):

> Question #1: In Figure 1 you see a paper strip that Pat tried to share among 3 people and underneath it is the piece Pat used to mark the strip [Note: the piece was a bit short]. Please draw, under Pat's piece, another piece that you think will fit better for sharing the strip among 3 people.
> Question #2: In the second attempt to share the strip among 3 people, Pat used a piece that was longer than the piece in Figure 1. Do you think it was smart to do so? ______ Why? ________________
> Question #5: In Figure 4 you see a paper strip that Danielle shared equally among 3 people and underneath it is the piece she used to mark the strip. Please draw, under Danielle's piece, a piece that you think will be appropriate to share the strip among 4 people.
> Question #6: In the previous question, did you draw a piece that is shorter or longer than Danielle's piece? ________ Why? ____

On the basis of my rough assessment that many students did not yet form the intended relationship, I planned a two-part lesson: administering the questionnaire and engaging the students in using the repeat strategy some more (share paper strips among 7 and 11 people). Once the questionnaire was ready, I turned to think about ordering the two parts. My notes indicate how the model informed my decision to do the activity before the questionnaire:

> I was aware that this order will not allow me to evaluate anticipatory conceptions, but decided to do it because I thought that otherwise many students would not be able to participate at all.

The note above is important because questions #1 and #5 above, if asked prior to being engaged in the activity, could be used to make a subtler distinction between students who used only a participatory conception of the inverse relationship and those who already used it in anticipation. Although I was not aware of this at the time, the short note indicates that promoting all students' learning of the conception at least at the participatory stage overruled the desire to make a subtler distinction among the more knowledgeable students. In this sense, the model informed my teaching, and the first take at the perturbation, in that it highlighted a local trade-off between two phases of the teaching cycle, assessing students' thinking and engaging students in tasks that promote their learning. In the case where my rough assessment of the class suggested the lack of even a participatory way of thinking on the part of many students, promoting learning overruled subtler assessment.

Second Take at the Perturbation: Subtle Assessment Overrules Promoting Learning. After the two-part lesson, I read the students' responses to the questionnaire and found that 14 of them knew the direction of adjustment. In my post-lesson conversation with the teachers, I emphasized that due to the order chosen the most I could claim about these students was the formation of a participatory conception. I was aware of and said that there was no way to distinguish if they had formed a higher stage. To promote all students' understanding of the direction of adjustment, we planned to engage pairs of students in using the repeat strategy some more. We paired each of the 14 students whose responses to the questionnaire indicat a participatory conception with another student who seemed not to understand the directio of adjustment. The idea was that both students in each pair could benefit from reflecting on direction of adjustment offered by the more knowledgeable one.

After the next lesson (Friday, 11-17-2000), a short conversation between the teacher and m indicated that both of us assessed, roughly, that all but maybe 2-3 students knew the directi of adjustment at least in a participatory sense. But I was still perturbed:

> The lesson was difficult in terms of my ability to follow and analyze what each child does and thinks While I was trying to help a pair I had to focus mainly on their execution of the repeat strategy and a the time other children would come and pull my sleeves to get my attention. I trusted that the videota will give me the data needed for retrospective analysis, but in terms of monitoring and documenting while interacting with the students I was still unable to follow what's going on.

This time, the ongoing perturbation regarding the lack of precision in assessing students' thinking, along with my indefinite sense that most students advanced at least to the participatory stage, turned into a new, model-rooted perturbation. I wanted to devise a tool and organize the lesson so that I could distinguish students who could use the inverse relationship in anticipation from those that could not. Interestingly, this goal was accomplish semi-accidentally, while I was relating my research proposal of conducting occasional interviews with the idea to let the class do silent reading while the teacher and I work with pairs of students. Below are my notes from Saturday, 11-18-2000.

> Last night I went to a concert. During the concert I was intensively thinking about the continual difficulty I have to assess students' thinking during the lesson and about how to use the organization of the lesson for that purpose. [Initially] I was thinking about my research proposal to conduct occasional interviews as part of the data collection methods. Then, something "clicked"—I realized that the idea to work with one pair at a time actually created the possibility to conduct mini-interviews with students … Thus, I immediately began to think what questions will I ask as well as who among the students I would like to interview. I started with a question that seemed promising in terms of making a distinction between participatory and anticipatory thinking regarding the direction of adjustment: "You have a piece that fits, exactly, for sharing a whole among 6 people and now you ha to share the whole among 7 people. Show me what will you do."

It took only a short, focused reflection to create a set of 3 questions that would enable me t make the intended distinction and to design a form that would enable me to document how each student responded to the question. Moreover, I immediately chose a different order tha in the previous assessment event. Because students would be engaged in reading and not in the repeat strategy, I would first ask the "hard" question where students are not prompted what activity to use and thus I would be able to distinguish those who used the conception i anticipation from those who did not. Then, I would present the questions that give some orientation. The three questions were:

> Question #1: (The teacher presents a drawing of a paper strip marked into 6 equal parts and the piece "he used" to create the sharing underneath it. He gives the students another piece of paper, and says Can you mark (or cut) a piece that could serve in equally sharing the paper strip among 7 people? (Pending students' drawing he asks): Any reason why you made it shorter/longer than the "sixth"?
> Question #2: (The teacher puts, underneath the "sixth," two pieces that he prepared ahead of time, o shorter and one longer than the "sixth" and says): Which of the two pieces would fit better to share the strip among 7 people? Why?

Question #3: (The teacher asks the students to observe how he accurately iterates each of those two pieces and asks): Which piece helped more in sharing the strip into 7 equal pieces? Why?

According to the model, if question #1 is asked after some time during which students did not use the activity that fostered the participatory stage of their thinking (here, the repeat strategy), one can answer it only if one already formed the inverse relationship at least at the anticipatory stage. The reason is that Question #1 does not call upon the size aspect of the relationship; it is the student who must attribute it to the situation. In contrast, Question #2 orients the student to the activity of comparing between sizes of pieces that are shorter or longer than a given piece and hence brings forth the next activities in the repeat strategy sequence, adjustment and iteration. Thus, students who already formed the participatory stage of the inverse relationship could solve Question #2. Question #3 allows an even finer distinction within the participatory stage, by bringing forth the entire activity sequence.

Due to scheduling and time limitations, I conducted the mini-interviews with students over 5 lessons. During those lessons, the class was engaged in non-related tasks while the classroom teacher worked with pairs I already interviewed on a specific way of executing the repeat strategy. Once they used it systematically and equally shared the paper strip among 3 people, she asked each pair: "Next, we have to share the strip among 4 people; should we make the piece shorter or longer than the share of one-out-of 3 people?" In response to her question, all students but one appropriately suggested to make it smaller.

Bearing in mind that only 12 students answered Question #1 in a way that indicated clear understanding of the reason to make the "seventh" shorter than the "sixth," a significant, twofold conclusion emerges. First, organizing the three questions that way resolved my perturbation about assessment. At last I was able to make fine distinctions in students' thinking that corresponded to the first two stages of the model. Second, the very distinction between the two stages was supported by the results of the assessment tool and process that I have designed. In part, this twofold conclusion was made possible because of my model-based choice to give the goal of precisely assessing students' thinking a priority over the goal of promoting students' learning. In turn, this precision allowed for better planning of the next lessons to fit the two sub-groups within the class.

Discussion

In this paper I examined two interrelated aspects of an integrated model of mathematics teaching and learning: the key theoretical distinction between participatory and anticipatory stages and teacher assessment of students' evolving conceptions. The analysis of the researcher-teacher activities to resolve a continual, model-based perturbation and of students' responses to several assessment tools that were devised accordingly supported the theoretical distinction. That is, if students appear to clearly understand a specific conception while using certain activities to solve mathematical problems, but on the next day they revert to lack of understanding, the reason may well be conceptual and not bad teaching or lack of effort on the students' part. For example, about a half of the class reverted to making the "seventh" larger than the given "sixth" (Q. #1, Second Take) even though they clearly knew, while engaged in the activity in the previous lessons, that it must have been smaller. The reason for that change in students' understanding can be explained in terms of competition between two, unevenly formed conceptions. The students have used at least an anticipatory and probably a reified conception of magnitude relationship among whole numbers, but only a participatory conception of inverse relationship between size and number of pieces in sharing situations. Thus, when not prompted for the activity to use, the problem situation was assimilated into the stronger conception of whole numbers and they made the "seventh" bigger than the "sixth" precisely because seven is bigger than six.

The study provided supporting evidence not only to the key theoretical distinction but also to the claim that teaching or assessing anticipatory conceptions is sensitive to orienting prompts. It demonstrated that order of teaching activities matters. Moreover, it provided a way of thinking about the relationship between what the teacher intends the assessment to accomplish and how she should organize it in practice. In particular, the study contributed to better understanding the conceptual goal underneath the potential organization of assessment tools from "hard" to "easy" questions and demonstrated specific ways in which the model can inform teaching (e.g., individual vs. pair work; mini-interviews vs. observation of the entire class, ordering questions). In this sense, the model both provides an articulated "map" of the conceptual terrain to be assessed and guides the teacher's organization and design of assessment tools and activities that fit to where, conceptually, students are.

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A MODEL FOR THE USES OF VARIABLE IN ELEMENTARY ALGEBRA

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This paper presents a theoretical framework we will call the Three Uses of Variable model (3UV model) that can be used as a guideline in the design of tools intended to make diagnostic analysis, to design teaching activities and to analyse textbooks and other teaching materials related to the concept of variable in elementary algebra. Essentially the model consists of a detailed description of the different aspects that underlie a basic understanding of the three main uses of variable in elementary algebra: specific unknown, general number and related variables. We present results that show the possibilities of applying this model in diagnosis, in the analysis of textbooks and in the design of teaching materials.

Antecedents

The development of algebraic knowledge implies the development of the concept of variable. This concept is central as well to the understanding of more advanced topics in mathematics. Its versatility makes it an important object of study. Given the multifaceted character of variable, no matter which starting point is taken, its different uses are very often concurrent within the same problem or situation and this has to be faced during the teaching process. From our point of view, disregarding any of its fundamental aspects in a problem situation may be the cause of many of the difficulties faced by students when working with algebra. In contrast, paying attention to the flexibility of the use of the concept of variable may be the source of a richer understanding.

The Mathematics Education literature, ever since the beginning of the XX century, has highlighted the difficulties students have with the different uses of variable (Thorndike et al.; 1923, Van Engen, 1953; Menger, 1956; Kuchemann, 1980; Wagner, 1983; Matz, 1982; Philipp, 1992; Warren, 1999). Several research studies point out that students encounter tremendous difficulties in grasping the essentials of the notion of variable and in being able to apply flexibly these uses at different levels of abstraction (Matz, 1982; Usiskin, 1988; Trigueros and Ursini, 1999). All these studies have stressed the importance of being able to move flexibly from one use of variable to any of the others, in order to be able to master this concept and work proficiently with algebra.

The understanding of each one of the uses of variable entails the mastering of many different aspects underlying each use at a different levels of abstraction. From our perspective and based on several years of research on the teaching and learning of the concept of variable, an understanding of variables at an elementary level could be described in terms of the following basic capabilities:

- to perform simple calculations and operations with literal symbols;

- to develop a comprehension of why these operations work;
- to foresee the consequences of using variables;
- to distinguish between the different uses of variable;
- to shift between the different uses of variable in a flexible way;
- to integrate the different uses of variable as facets of the same mathematical object.

The 3UV model

Further analysis of students strategies when working with problems related to the different uses of variable and of the mathematical requirements to master this concept led us to the design of a theoretical framework that enable us to analyse students' productions, to design instruments for diagnosis, for research, and to develop didactic approaches. We call this framework the 3UV (three uses of variable) model. It involves a decomposition of variable in its three main uses in elementary algebra: specific unknown, general number and relationship between variables. For each one of these uses, we have stressed different aspects corresponding to different levels of abstraction at which they can be handled. These requirements can be presented in a schematic way as follows:

We consider that the understanding of **variable as unknown** requires to:

U1 - recognise and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem;

U2 - interpret the symbols that appear in equation, as representing specific values;

U3 - substitute to the variable the value or values that make the equation a true statement;

U4 - determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations;

U5 – symbolise the unknown quantities identified in a specific situation and use them to pose equations.

We consider that the understanding of **variable as a general number** implies to be able to:

G1 - recognise patterns, perceive rules and methods in sequences and in families of problems;

G2 - interpret a symbol as representing a general, indeterminate entity that can assume any value;

G3 - deduce general rules and general methods in sequences and families of problems;

G4 - manipulate (simplify, develop) the symbolic variable;

G5 - symbolise general statements, rules or methods;

We consider that the understanding of **variables in functional relationships** (related variables) implies to be able to:

F1 - recognise the correspondence between related variables independently of the representation used (tables, graphs, verbal problems, analytic expressions);

F2 - determine the values of the dependent variable given the value of the independent one;

F3 - determine the values of the independent variable given the value of the dependent one;

F4 - recognise the joint variation of the variables involved in a relation independently of the representation used (tables, graphs, analytic expressions);

F5 - determine the interval of variation of one variable given the interval of variation of the other one;

F6 – symbolise a functional relationship based on the analysis of the data of a problem.

The 3UV model was applied to the analysis of secondary school mathematics textbooks, to the diagnosis of secondary school mathematics teachers conceptions of variable and to the design of school activities involving the different uses of variable at different abstraction levels. In each of these cases an appropriate instrument was designed to serve the specific purpose it was intended for. The instrument was applied and the results were analysed in terms of the 3UV model.

Analysis of textbooks

Method

A grid that takes into account all the aspects involved for the different uses of variable was designed. The grid consists of a table with a column for the number of the lesson, a second column for the theme of the lesson (arithmetic, algebra, geometry, basic statistics, and probability), then each of the uses of variable has its own column subdivided into new columns where the different aspects of the use are taken into account. In Mexico there is an official program for secondary school mathematics education. The books used at this level have to be authorised after a process of evaluation. Three books officially authorised by the ministry of education were analysed. These books correspond to 1st 2nd and 3rd year of secondary education in Mexico. These books consist of 44, 45 and 41 lessons respectively. Each lesson was analysed to see which aspects of the different uses of variable, if any was included. (Benitez, 2001)

Results

The purpose of this analysis was to study the way the concept of variable is included in secondary textbooks, to see which of the uses of variable was stressed and to analyse what kind of relationships were established between the different uses of variable.

The percentages of lessons in which each aspect of the three uses of variable is used in each of the three secondary school textbooks are shown in the following table.

Use	Specific Unknown					General Number					Functional Relationship					
Year	U1	U2	U3	U4	U5	G1	G2	G3	G4	G5	F1	F2	F3	F4	F5	F6
I	77	2	2	77	0	41	27	7	7	4.5	11	7	4.5	7	0	4.5
II	67	15	13	67	11	18	38	4.4	4	13	9	6.7	0	6.7	4.4	6.7
III	73	34	5	73	32	5	56	17	18	19.5	19.5	17.5	10	22	10	19.5

As can be seen in this table, the way in wich variable is used at secondary school is centred mainly in the use of variable as unknown, but its different aspects are not taken into account in a balanced way. The main emphasis is always directed towards U1 and U4. Even if the variable as a general number is considered in some lessons, there is nothing explicit to make sure that the student will develop the ability to differentiate it from the specific unknown, to develop general methods and to work fluently with it. Variable in a functional relationship appears only in few occasions. Even if in several lessons two or three aspects of a specific use of variable or different uses are presented in conjunction, there is no indication helping students to distinguish between them or to shift from one to the other. The 3UV model permits us to see that these textbooks don't provide enough opportunities for the students to integrate the different facets of variable and this lack of flexibility is reflected in the results obtained as they progress through their schooling (Trigueros and Ursini, 1999).

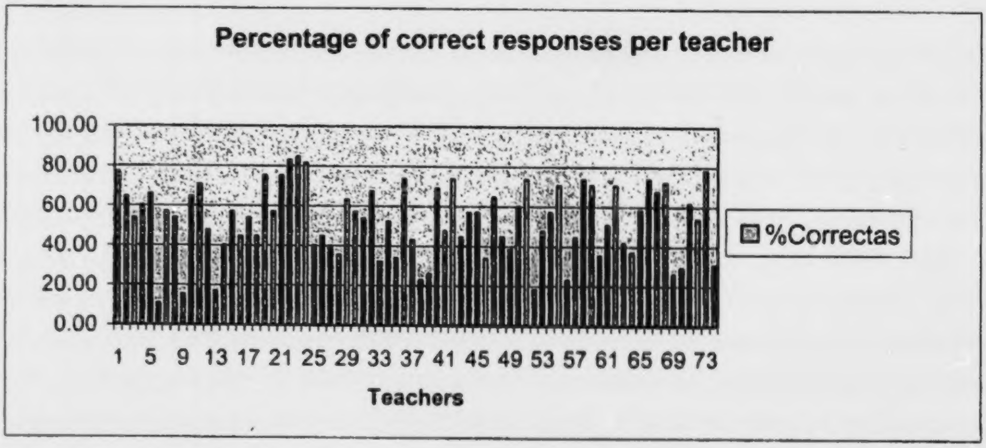
Diagnosis of teachers' conceptions

Method

A questionnaire was used in this case. This questionnaire had been used with students before and the results obtained have been discussed in previous studies (Ursini and Trigueros, 1997). This time the questionnaire was applied to 74 secondary school teachers and their responses were analysed and classified. After reviewing the responses, six teachers were selected to be interviewed. The interviews were also analysed in terms of the model.

Results

The percentage of correct responses per teacher is shown in the following graph



It is striking to see that secondary school teachers results are very similar to those obtained by starting college students (Ursini and Trigueros, 1997) and that their incorrect responses are very similar. Again, none of the questions was answered correctly by all the teachers and no teacher answered correctly all the questions.

High school teachers have problems to differentiate between the uses of variable as unknown and as general number (U2,G2), but in the case of equations they are able to correctly recognise the unknown and find it. Even though their ability to manipulate equations and algebraic expressions (U4,G4) is good, they struggle with the recognition of the unknown and its symbolisation in problem situations (U1, U5). A tendency to avoid the use of algebraic methods and a strong preference for arithmetic approaches was found. Teachers have difficulties to accept open expressions as valid (G2), this is demonstrated by their tendency to complete them by adding an equal sign and thus changing their meaning to that of equations (G2 changed into U2). When an equal sign is present they feel compelled to find a specific result. Their idea of general number can be described more in terms of "any specific number" than in terms of a way to represent a generalisation (G2). They are able to manipulate the general number in different situations (G4) but they always think about it as something that can be determined sooner or later.

The use of variable that teachers have more problems with is variable in a functional relationship. This was also found in the study with college students. Teachers recognise and work with correspondences (F1) and can determine the value of one of the variables when they know the value of the other (F2, F3), however they have trouble with joint variation questions (F4) and their symbolisation (F6). Their description of intervals was always in terms of discrete numbers, that is, they have difficulties with the idea of the continuum in the number line (F5). For them only even and rational numbers are possible as response of several of the problems discussed (Juárez, 2001).

Design of activities

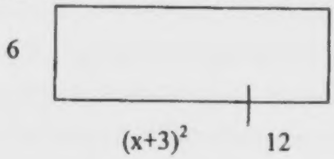
Method

Several activities of different levels of abstraction were developed in terms of the 3UV model. They were designed to take into account the possibility to differentiate and integrate the different uses of variable and to encourage the flexibility in their use. A whole precalculus course for college was designed in terms of the 3UV model.

Results

In order to illustrate the use of the model in the design of activities we show here one of such activities:

For which values of X the area of the following rectangle varies between 168 and 288? If the value of X increases or decreases what happens with the area?



This is not an easy problem. It was proposed for pre-calculus students at college level. The answer to this problem requires the identification of two non contiguous intervals and it implies working with several of the aspects of the three different uses of variable. The variable X has to be recognised as a general number, it has to be manipulated and used in order to obtain new expressions. It has to be recognised that there are unknown quantities involved in the problem that can be determined. Correspondence, variation and determination of intervals are as well involved. Different strategies can be used to solve this problem and students can explore the use of one or another depending of their previous knowledge and mathematical experience.

Some guiding questions that can be used with students in order to help them solve this activity are:

What does X represent in this problem? How many values can it take? The idea of this question is to help the student see the variable X as a general number (G2).

How do you express the area of this rectangle? Students have the tendency to use the memorised formulae for areas without taking into account the specific data given in the problem. This question intends to focus students' attention on the givens and on symbolisation (G5) using the general number and its manipulation (G4) to obtain the expression $6((X+3)^2 + 12)$.

For what values of X the area is equal to 168 and equal to 288? This question can be formulated to help students who use as a starting point for solving an inequality the common strategy that involves solving the equations $6((X+3)^2 + 12) = 168$ and $6((X+3)^2 + 12) = 288$. When students use this strategy they often assume that all the values between the solutions obtained make the inequality true. But this is not always true. With appropriate guide the use of this strategy can help them question its validity. In terms of the uses of variable answering this questions involves posing the quadratic equation (U5), develop the general expressions (G4) and solve both equations (U4). As each equation has two solutions students have to consider that both of them are possible and meaningful in terms of the given problem (U3).

For what values of X the area is bigger than 168 and smaller than 288? For students who used the above mentioned strategy, to answer this question implies the use of the values they already found in order to establish the appropriate intervals ($-9<X<-7$ and $1<X<3$). In other cases, students must form the inequalities (I5) ($168 \leq 6((X+3)^2 + 12) \leq 288$), develop the general expression involved in it (G4) ($168 \leq 6(X^2 + 6X + 9 + 12) \leq 288$) and solve them (I4).

Another possible strategy to solve this question is to relate the expression for the area of the rectangle with a functional relationship between X and area (F1). In this case, answering this question implies defining variation intervals for the dependent variable (F5) and determining, from the graph of the function, the corresponding intervals for the independent variable.

What happens to the area of the rectangle when X varies? The answer of this question involves the recognition of joint variation (F4) of the two variables, determine the values of one of the variables when the values of the other are known (F2). A graphical representation of the relationship can also be used in the solution (F1-F5).

How does the area vary when the value of X increases/decreases? This involves the recognition of the joint variation (F4) and the realisation that while varying X some of the values of the area are a solution to the problem and some of them are not and finding the required intervals for X.

The same problem can be posed using different expressions instead of $(X+3)^2$, that involve different difficulties for the students, for example, for beginning algebra students X or X+2 can be used, and for more advanced students one can introduce more complex expressions such as 1/X.

The detailed description included in the 3UV model is very useful in the design of problems and didactical sequences for algebra courses at different school levels. It can guide the design of didactical sequences to introduce beginning students to the concept of variable so that they can differentiate between its different uses. It can be used to design sequences aiming to foster students' comprehension of variable by introducing relationships between the different aspects of each use and between the different uses, helping them in this way to integrate the different uses of variable in a single concept.

Conclusions

The concept of variable appears in any branch of mathematics. It is widely used also in application of mathematics, but this versatility makes this concept a very difficult one to be mastered by students. The different uses of the concept are on the basis of the difficulties students face when trying to learn algebra.

Research developed during the last thirty years has shown that each use of variable is linked with specific epistemological and didactical obstacles. When algebra is taught taking only one of these aspects as the central focus, the possibility of flexibility and the richness of the relationships between the different uses is lost or obscured and students' understanding of algebra stays limited.

The 3UV model presents a very detailed analysis of each of the main uses of variable in elementary algebra: as a specific unknown, as a general number and in a functional relationship. A lot of research about students' understanding of the concept of variable has been incorporated in the model and it is at the basis of the wide range of possible applications.

The possibility to integrate a collection of related ideas in a single concept, with a specific name, can help students focus their attention on it, manipulate it and use it with ease in applications. Such concepts have rich potential precisely because they carry with them powerful links that enable the users to invoke them to solve problems. If the diverse elements that integrate the concept are not connected sufficiently fluently, it may be very difficult for students to consider such concept as a unit and it may be impossible for them to make links with it. The student can access only some elements of a loosely connected structure, instead of a rich conceptual entity and the possibilities of its invocation are reduced considerably (Skemp, 1970). But identifying the components that have to be linked in that unit can be a very difficult task and cannot be left as something to be done by each teacher. Research is necessary to make this task possible.

The 3UV model is precisely an attempt to fill this necessity in the case of the concept of variable. We have shown here examples of the application of the model to different educational activities: diagnosis, design and analysis of teaching material. These examples show that the analysis of the concept incorporated in the model is detailed enough as to make it possible for researchers and designers to use it in a wide variety of contexts. The possibility to design teaching activities that can help students make the necessary links between the aspects that constitute each one of the uses of variable, and the possibility of identifying them with a specific name may help beginning students to develop a stronger concept of each of the uses of variable. The posterior emphasis in relating the different uses of variable into a single conceptual unity can foster the development of a multifaceted concept of variable.

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ASSESSMENT OF MATHEMATICS ACHIEVEMENTS: NOT ONLY THE ANSWERS COUNT

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Abstract

This paper addresses a study on assessing student learning in mathematics education. The study is connected to a teacher enhancement project at primary schools in New York City. The eventual goal of the project was to improve the students' mathematical achievements. Among other things, this goal includes the use of clever ways of applying mathematics for solving problems. Because standardized tests are not an adequate tool for revealing this kind of student learning a test was developed by means of which the process of mathematization was supposed to become visual. Results gained from this test in 17 classrooms with students ranging from grades 3 through 5 show that strategy-focused assessment is a valuable extension of the regular answer-focused assessment.

Introduction

The dichotomies product-process and answer-strategy play an important role in the assessment of mathematics achievements. If not earlier, this became apparent in the recent worldwide reform of mathematics education. One of the characteristics of the new approach to mathematics education is that it gives more room to processes and strategies. Nevertheless there is yet a large distance between both ends of the respective dichotomies. The answers belong to the product side and the strategies to the process side, and answers and strategies each have their own purposes. The gap is reflected above all in the difference that still exists between standardized tests and tools for classroom assessment.

Although the ways in which students solve mathematical problems are regarded as important information for educational decisions, solution strategies are scarcely covered by standardized tests. More often than not achievement scores are purely related to the number of correct answers. In tests, the answer is usually considered as the ultimate indication of the achievement level. The strategies that the students applied to find this answer are generally beyond the scope of standardized testing and belong more to the field of interviews and observations.

For a long time, only within the context of diagnostic testing, attention was paid to strategies. Data about the process of solving problems, and especially data about *errors*, was used to identify students' misunderstanding and find indications for remedial teaching. Errors in strategies were regarded as "windows on children's internal thought processes" (Baroody, 1987).

As said earlier, the reform in mathematics education has opened the door now for strategies as a more general focal point of assessment. They are no longer restricted to a remedial setting. As expressed in standards and curriculum documents of many countries, the new ideas about mathematics education emphasize that in addition to product information, process information is needed to get insight in the students' thinking. This particularly implies that special attention is paid to the *various ways* by which the students solve mathematical problems. A wide range of alternative assessment tasks and formats has been developed to reveal strategies. The main purpose of this alternative assessment is to inform the teachers about the students' way of thinking in order to provide clues for further teaching.

In other words, there is, nonetheless, a distinction between process and product. Strategies are still considered as a process variable and the answers are still the pivot of all. They are considered as the real output variable that counts for determining the achievement level. The question is, however, how tenable this is. Is this strict distinction not an unnecessary curtailment of the assessment of mathematics achievements?

The present paper cannot answer this question extensively. It only reports about a study in which was tried to use the students' strategies as an output variable, or – adapting Baroody's words – as "windows on achievements level".

The MiTC project

The study was part of the "Mathematics in the City" (MiTC) project. The MiTC Project is a large-scale project on teacher enhancement project funded by the National Science Foundation and the Exxon Educational Foundation (see Fosnot and Dolk, 2001). The City University of New York and the Freudenthal Institute of Utrecht University carried out the project jointly. Over a period of five years, the project worked on systemic reform in mathematics education from pre-kindergarten through grade 5 in five school districts in Manhattan. The project started with ten schools and in the last stage of the project about forty schools were involved.

The two key points that characterize the MiTC project are its model of improving teachers' classroom practice and its idea about mathematics education.

Approach to teacher enhancement

The heart of the project consisted of a very intensive collaboration between staff members of the project and the participating teachers. The mathematics classrooms in the project were co-taught by the project staff and the school teachers. Through joint observations and discussions between teachers and staff members the team worked continually on the development of better ways of teaching. By means of a two-week summer institute in the beginning of the school year, followed by weekly-organized institutes during the year, the teachers were offered opportunities for further deepening their professional knowledge and abilities. The structure of the project was that after a year of participation some of the teachers became staff members, which implied that they coached new teachers in the project. This mushrooming pattern meant that the initial number of forty involved teachers rose up to four hundred.

Approach to mathematics education

The project was strongly related to both the constructivist and the "realistic" view on mathematics education. Moreover, a strong basis was found in the Piagetian and Vygotskian theories on children's cognitive development. These two theories actually constitute the socio-constructivist to mathematics education (see Cobb, 1996).

The socio-constructivist view in the MiTC project is recognizable in the fact that the students are seen as active learners within the social community of the classroom. The mathematics "congresses", in which the students, while seated on a rug, share their thinking and build up new understanding, reflect this approach to mathematics education exceedingly. In addition to this, within MiTC classrooms the Piagetian concepts of assimilation and accommodation play an important role. Teachers have to recognize the present cognitive schemes and structures of the students and must learn how to design problems and situations that can evoke new or re-ordered schemes and structures.

The key beliefs about what should be taught, and how it should be taught, are heavily inspired by Realistic Mathematics Education (RME), which is based on Freudenthal's view that mathematics must be connected to reality, stay close to children and be relevant to society, in order to be of human value (Freudenthal, 1977). Instead of seeing mathematics as subject matter that has to be transmitted, Freudenthal stressed the idea of mathematics as a human activity. Education should give students the "guided" opportunity to "re-invent" mathematics by doing it. This means that in mathematics education, the focal point should not be on mathematics as a closed system but on the activity, on the process of mathematization (Freudenthal, 1968).

Actually this mathematizing ability, including also attitude to mathematize, is the overall core goal of mathematics education. In short, this ability involves that students can use and develop mathematical tools, including models and strategies, which can help to organize and solve real life problems and pure mathematical problems.

Evaluation of student learning

In year five of the project an evaluation was planned. Because of the large-scale character of the project with its focus on change at different levels, i.e. students, teachers, and whole schools, a uni-dimensional evaluation was felt to be insufficient. Instead a multi-focus assessment was designed including macroscopic and microscopic lenses (Van den Heuvel-Panhuizen, Dolk, Fosnot, and Glick, 2000). This approach to assessment was based on the belief that the different perspectives would provide a fuller understanding of the effectiveness of the in-service intervention. The assessment contained the following three foci: student learning, teacher change, and school change. Each of them has various aspects to be examined. The present paper will only deal with the assessment of student learning and will concentrate on student competence deduced from the strategies they applied to solve the problems.

The main question to be answered about the student learning was what the project students gained from the project. To answer this question the achievements of the students from grades 3, 4 and 5 were compared to the achievements of non-project students.

Comparison by means of the standardized tests

The first analysis was done based on the results from the Corrected Terra Nova Math Test (for third-grade and fifth-grade students) and the Corrected New York State Math Test (for fourth-grade students). Although these standardized tests are not entirely in line with the project goals of mathematics education, it was decided to have the test scores from these tests as one criterion for evaluation. If students in reform project do not score well on mandated tests, the reform will fail simply because of political forces at play regarding testing. Moreover, results on sub test items can be helpful in understanding how well the reform project appears to be aligned with standardized achievement outcomes. The comparison of the tests scores revealed that in all the three grades the project students (experimental group) scored significantly better than a group of non-project students (control group). A covariance analysis controlling for entering score at the beginning of the year (which only was possible for grades 4 and 5) showed also significantly better results for the experimental group (Fosnot, Dolk, van den Heuvel-Panhuizen, Hilton, Wolf, and Bailey, under review).

Comparison by means of the MiTC test

From the very beginning of the project it was planned to have an additional evaluation of student learning more aligned with the goals of the project than the regular standardized tests mostly comprised of closed and answer-based items.

For this purpose an alternative test on number sense was developed for grades 3 through 5 using a RME assessment paradigm (van den Heuvel-Panhuizen, 1996).

Assessing mathematizing

Assessment grounded in the RME theory of mathematics education requires that the assessment tool must provide information about the process of mathematization, since that is considered as the overall core goal of mathematics education. The ability of mathematizing involves that students can solve problems by means of mathematical knowledge and concepts that can lead to models and strategies that are adequate for the problem to be solved. This goal is different from the goal of most traditional assessments, where strategies are assessed but only as process, or as a bi-product, or as an additional proof that the student had a clear understanding of what he or she was doing. In contrast, when mathematizing becomes the goal of instruction, the strategies are seen as an outcome, as the goal itself.

Within the process of mathematization several levels can be distinguished, each reflecting a certain level of the students' understanding. Regarding the content domain of number sense related to operations with numbers up to one thousand the mathematization includes concrete or mental activities ranging from carrying out a standard algorithm to making use of number relations and properties of operations in order to find shortcuts and clever and elegant procedures by means of which the answers can come across. Crucial for evaluating the mathematizing activity is to what degree the strategies show a level of maturity that includes both flexibility and effectiveness. Simply carrying out a standard procedure without taking into account the numbers involved is not judged as a high level of mathematizing.

For paper-and-pencil assessment to capture genuine mathematizing it is necessary that the students' own mathematical activity becomes visual on paper. Therefore all problems were put in a work area that the students could use as scratch paper. Another requirement for assessing mathematization is that the tasks allow several levels of mathematizing. The most important requirement, however, is that there *is* something to mathematize.

The following two problems may illustrate the kind of problems that were used in this test. The long addition problem that was in the test for grade 3 (see Figure 1) looks like an ordinary bare number problem and can trigger an algorithmic procedure. A student with number sense, however, will recognize how nicely the numbers fit together, and will adapt her or his strategy to this knowledge.

$$38 + 39 + 40 + 41 + 42 =$$

Figure 1 Problem included in the MiTC test for grade 3

In the problem on the chain of beads (see Figure 2) the students have to figure out the color of the 1000th bead. In addition to this the students have to explain why they are sure about their answer. In this problem, that is from the test for grade 5, the students can apply their knowledge of multiples of three. But other, less advanced ways of working are also possible. Again, this inherent multi-level quality of the problem makes it very suitable for providing information about the students' achievements.

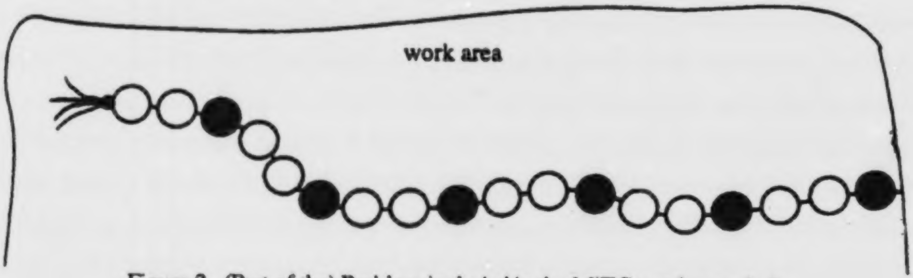

Figure 2 (Part of the) Problem included in the MiTC test for grade 5

Data collection

The MiTC test was administered in 17 classrooms in spring. The classrooms ranged from grade 3 through grade 5. Eight of the classrooms had a teacher from the control group and nine had a teacher from the experimental group. In total four schools were involved, situated in three school districts. The control classrooms were selected to match the experimental ones. In order to avoid large differences between the background of the students and the general school environment, the control teachers and the experimental teachers, in some cases, came from the same school or from the same school district. In both groups two teachers had a mixed-grade classroom. This means that in total 21 grade groups with a total of 17 teachers were involved in the data collection (see Table 1).

Table 1 Number of teachers, students, and grade groups involved in the data collection

	Control group		Experimental group		Total	
	Teachers	Students	Teachers	Students	Teachers	Students
Grade 3	3	61	6	75	9	136
Grade 4	4	72	3	43	7	115
Grade 5	3	55	2	36	5	91
	10 (8)	188	11 (9)	154	21 (17)	342

Together with the development of the test a double coding system was developed: one for the answers and one for the strategies. For the answers, the literal answer given by the student could be coded. The strategy coding was done by means of a two-digit code. The first digit referred to a general description of the strategy and the second digit specified a particular way of working within this category. It is important to explain that the categories and in particular the subcategories are not holistic but task-specific rubrics.

The coding of the students response was done blindly. The coder (1) did not know whether the students belonged to the control group or to the experimental group.

Results

A t-test analysis showed that the average percentage of correct answers in the addition problem (grade 3) did not differ significantly between the two groups (see Table 2). For the beads problem (grade 5) this is slightly different (see Table 3), but on the whole the control students and the experimental students did not diverge with respect to the total number of correct answers (see Table 4). The latter means that if this test would only have focused on the answers, the two groups might have been considered as equal in achievement.

Table 2 Answer and strategy results from the grade 3 problem (see Figure 1)

Grade 3 problem	Control group (n = 61)	Experimental group (n = 75)	
answer	48 % correct	60 % correct	n.s. (t-test)
strategy	5 % tinkering 15 % decomposing 62 % ciphering 18 % other	17 % tinkering 45 % decomposing 20 % ciphering 17 % other	p < .001 (Chi-square)

Table 3 Answer and strategy results from the grade 5 problem (see Figure 2)

Grade 5 problem	Control group (n = 55)	Experimental group (n = 36)	
answer	73 % correct	92 % correct	p < .05 (t-test)
strategy	42 % reasoning 9 % counting on/ multiplying on 4 % long division 7% wrong operation 2 % guessing 2 % other 16 % unclear strategy 18 % no strategy	75 % reasoning 14 % counting on/ multiplying on 0 % long division 3 % wrong operation 0 % guessing 0 % other 8 % unclear strategy 0 % no strategy	p < .05 (Chi-square)

However, the situation with the applied strategies is quite different. Here, a Chi-square analysis showed highly significant differences between the two groups (see also table 4).

In the addition problem (grade 3) the majority of the control group students used column arithmetic. An example of such a ciphering strategy is shown in Figure 3.

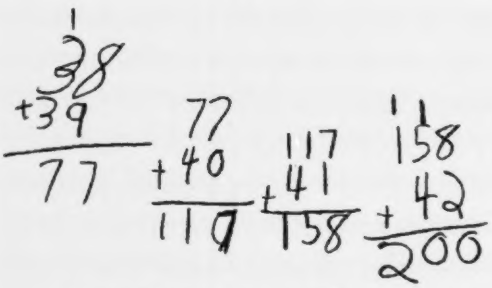
1
38
+39
77
77
+40
117
117
+41
158
1 1
158
+42
200

Figure 3 Example of ciphering strategy

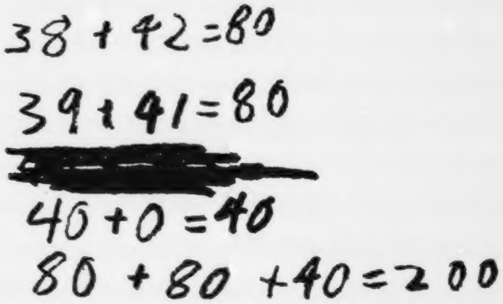
38 + 42 = 80
39 + 41 = 80
40 + 0 = 40
80 + 80 + 40 = 200

Figure 4 Example of tinkering strategy

The students from the experimental group, in contrast, brought their number and operation knowledge into play and applied a smart calculation strategy that is called "tinkering" (see Figure 4) or they used either a stringing strategy (e.g. 38 + 30; + 9; + 40; + 40; + 1; + 40; + 2) or a splitting strategy (e.g. 30 + 30 + 40 + 40 + 40 and 8 + 9 + 1 + 2). The stringing and splitting strategies are summarized by the term "decomposing".

For the beads problem (grade 5) three quarter of the experimental students used a reasoning strategy based on knowledge of number relations. For instance, one student said:

> "I counted 10 and I know 10 go into 1000, so the last color for 10 which is white is the same color for 1000".

Compared to the experimental group, the control group contained more students who, for instance, could not explain their strategy or students who did a long division (see Figure 5) together with the following explanation:

> "There are 2 white beads between each black bead that's why I did 2 ÷ 1000. Then I did 1 ÷ 1000 which gave me my answer".

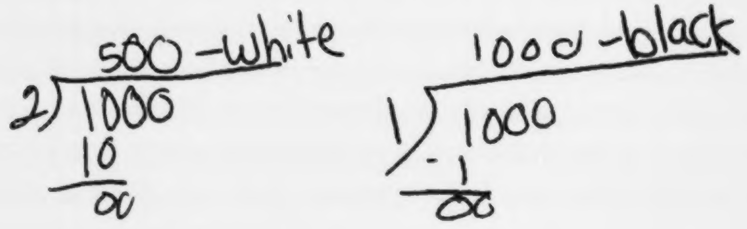
500 -white
2) 1000
10
00
1000 -black
1) 1000
- 1
00

Figure 5 Example of student work that showed a long division

As is shown in Table 4 the difference in strategies was consistent in the study. Sensible tinkering strategies were applied more often by the experimental students (E+) and the ciphering strategies – in problems in which these strategies are not the most adequate solution strategies – were more often found in the work of control students (C+).

Table 4 Answer and strategy results in total for each grade

	Grade 3 11 problems	Grade 4 8 problems	Grade 5 11 problems
Difference in total number CORRECT ANSWERS (t-test)	E + ($p < .10$)	E = C	E = C
Difference in STRATEGIES (Chi-square)			
• tinkering	E + ($p < .001$)	E + ($p < .001$)	E + ($p < .001$)
• decomposing	E + ($p < .001$)	E + ($p < .10$)	E + ($p < .001$)
• ciphering	C + ($p < .001$)	C + ($p < .001$)	C + ($p < .001$)
• other	E = C	E = C	E = C

Concluding remarks

The data need further analysis (e.g. correlation between correct answer and strategy; influence of particular teachers) to conclude what the students gained from the MiTC project. The results so far give strong support to the idea that the mathematics achievements of the students in the experimental group are higher than those of the non-project students. Regarding assessment this study made clear that the measurement of mathematics achievements cannot be restricted to the answers only. This thinking fits with other developments in this area, like the work done by Suzuki (2000) who is developing a new assessment methodology in which the process of thinking in problem solving can be scored. Suzuki used the QUASAR general scoring rubric as a bedrock to develop a scale that can identify characteristics of achievement levels. In contrast with Suzuki's scale, in the MiTC study the categories were more mathematical and task-specific. The future will show what approach will be most helpful for understanding mathematics achievement.

Note

1. The coding was done by Chantal van Rooijen.

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A CASE STUDY OF FOUR GRADE 7 GEOMETRY CLASSES

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Traditionally, geometry at school starts on a formal level, largely ignoring prerequisite skills needed for formal spatial reasoning. This tradition may lead to ineffective teaching and learning. The Van Hiele theory postulates learner progression through levels of geometry thinking, from a Gestalt-like visual level through increasing sophisticated levels of description, analysis, abstraction, and proof. Progression does not depend on biological maturation or development only, but also on appropriate teaching/learning experiences. A Van Hiele-based geometry learning and teaching program was designed and implemented in four Grade 7 classes (133 learners) at two South African schools. The study investigated some factors and conditions influencing the effective learning and teaching of spatial concepts, processes and skills in different contexts. Results suggest that the implementation of a Van Hiele-based learning and teaching program had a positive effect on the learners' levels of geometric thought. Learners who completed the program reasoned on a higher level, gave more complete answers, demonstrated less confusion, and generally exhibited higher order thinking skills than their counterparts who did not take part in the program. For efficacy of this program, a prerequisite is that the teacher should consistently teach from a learner-centered approach.

INTRODUCTION

South Africa has a history of inadequate geometry instruction in primary schools (Van Niekerk, 1997:270). Taylor and Vinjevold (1999:143) state that lessons are generally characterized by a lack of structure, and the absence of activities which promote higher order skills such as investigation and understanding relationships. At present in most schools geometry teaching and learning in the school curriculum starts on a formal level, ignoring prerequisite skills needed for formal reasoning. The effect of ignoring the sequential and hierarchical nature of the learning of geometry causes ineffective teaching and learning because learners are expected to perform without the necessary prior knowledge and/or prerequisite skills (Clements & Battista, 1992:421). It is necessary that the foundation of formal geometry learning (that starts in grade 7) be laid according to a widely respected and acceptable theory, for example the Van Hiele theory.

THE VAN HIELE THEORY

The Van Hiele theory postulates that learners progress through progressive levels of geometrical thought from a Gestalt-like visual level through increasing sophisticated levels of description, analysis, abstraction, and proof (Van Hiele, 1986:39). At the *first level* (Recognition) learners identify and operate on shapes and other geometric configurations according to their appearance alone (Mason, 1997:39). On the *second level* (Descriptive / Analytic) learners are able to recognize and explicitly characterize shapes by their properties (Van Hiele, 1986:40), but can not recognize relationships between classes of figures (Battista, 1994:89) or even redundancies (repetitions) (Spear, 1993:393). Learners at *level three* (Abstract / Relational) can form abstract meaningful definitions (Mason, 1997:39), distinguish between necessary and sufficient sets of conditions for a concept, classify figures hierarchically (by ordering their properties), give informal arguments to justify their classification (Battista, 1994:89), and understand and sometimes even provide logical arguments in the geometric domain (Clements & Battista, 1992:427). At *level four* (Formal Deduction) learners are able to establish theorems within an axiomatic system. They recognize the differences among undefined terms, definitions, axioms, and theorems and are capable of constructing original proofs (Clements & Battista, 1992:428). At the *fifth level* (Rigor / Metamathematical) learners reason formally about mathematical systems, understand the formal aspects of deduction (Presmeg, 1991:9), establish and compare mathematical systems (Mason, 1997:40), and reason by formally manipulating geometric statements such as axioms, definitions, and theorems.

Gutiérrez, Jaime and Fortuny (1991:237-238) theorized that the Van Hiele levels are not discrete and presented an alternative method to evaluate and identify those learners who are in transition between levels. Gutiérrez *et al.* (1991:238-241) quantify the acquisition of a level by representing it with a segment from 0 to 100 thus creating a scale of degrees of acquisition. A division is also made to divide this continuous process into five stages (no-, low, intermediate, high and complete) acquisition characterized by the qualitatively different ways in which the learners reason.

Progression from one level to the next does not depend only on biological maturation or development (Piaget,1970), but also on appropriate teaching/learning experiences (Koehler & Grouws, 1992:123). The effect of appropriate teaching/learning experiences (in accordance with the Van Hiele theory) on South African grade 7 learners in varied context will be investigated. Attention will be given to the (possible) change in geometric thought levels as well as specific details emanating from the implementation of the program. The impact of second language instruction and it's inevitable influence was noted but will not be discussed in this paper but in a later paper.

METHOD

SUBJECTS

The study population consisted of Grade 7 learners (n=221) from a large town in the North West Province of South Africa. A non-randomized experimental-control group design was used with 133 learners in the experimental group and 88 learners in the control group. The experimental group was further divided into two naturally occurring classes, C1 and C3, and similarly the control group consisted of C2 and C4 (naturally occurring classes). Class 1 (C1) and Class 2 (C2) were both instructed in their second language, while class 3 (C3) and class 4 (C4) received instruction in their first language.

PROCEDURE

Before the beginning of the program the two teachers in classes C1 and C3 were trained for one month in the Van Hiele theory as well as the activities developed by the researchers. The activities were implemented after completion of the training and the progression through these activities was continuously videotaped. Despite the initial training and re-training C1's teacher continued with a teacher-centered approach, as she believed that a program such as this would take up too much time, which could result in her not completing the syllabus. The learners in C1 and C3 were randomly arranged into sub-groups. A sub-group was randomly selected to be the group on which the videotaping would focus. The other groups in each class were still included in the videotaping for comparison when analyzing the progress. The learners remained in the same groups for the duration of the study. The mathematics teacher taught the classes while they were being taped for analyses and transcription afterwards. The same sub-groups (school C1: 12; school C3: 13) that were focussed on in the video taping were also used to complete the Van Hiele post-test.

At the same time that the C1 and C3 groups were progressing through the designed program (as developed by the researcher), the C2 and C4 groups continued with their spiral syllabus system where the teacher and the textbook formed the main sources of information with little or no learner involvement in the classroom activities. From the C2 and C4 group, a randomized sample was taken to complete the Van Hiele post-test (from C2: 11 and C4: 13 learners were selected).

INSTRUMENTATION

Post-testing took place after conclusion of (both) programs. The post-test was compiled using selected items of The Mayberry Test (Lewin and Pegg Version) and items from a test developed by the unit for Research in Mathematics Education of the University of Stellenbosch (RUMEUS -1984). The final product (post-test) included 21 items (with some sub-items), on the concept of parallel lines and shapes such as the square, right angle and isosceles triangle. The answers to the items were quantified according to the acquisition scales of Gutiérrez *et al.* (1991:237-241). No pre-testing could be done as the C2 and C4 groups started Geometry teaching already at the beginning of the year (January) while the (experiment and the) C1 and C3 groups started only in May. No interviews were conducted with the learners thus the analysis can not attain the fine grain of qualitative depth that interviews would have provided.

RESULTS AND DISCUSSION

Degrees of acquisition

Investigation through the post-test into the differences of degrees of acquisition in geometric thought provided encouraging results (see figure 1).

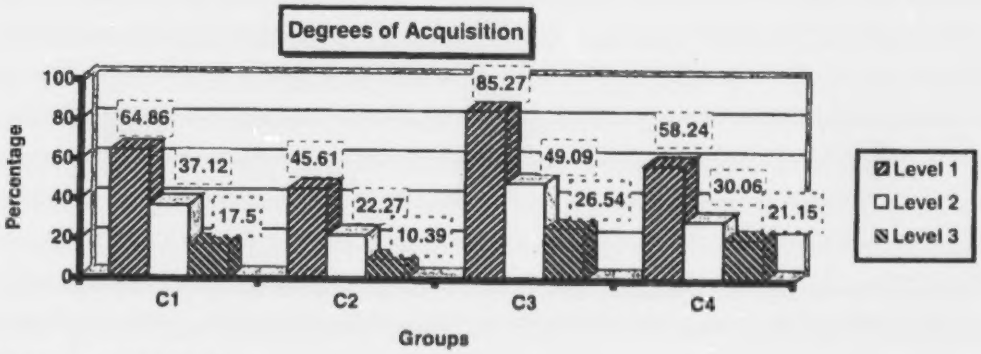


FIGURE 1: DEGREES OF ACQUISITION

The experimental groups consistently achieved higher degrees of acquisition than the control groups, leading to the conclusion that the program did have a positive effect on the acquisition of high(er) degrees of geometric thought.

Class 3 achieved the highest degrees of acquisition in all 3 levels considered. Between C1 and C3 clear differences exist. Possible reasons can include the complex influence of the difference of mother tongue and medium of instruction. A second and maybe more important reason for the difference is the teaching approach followed, with C1's teacher following a content-based teacher-centered approach and C3's teacher following a problem orientated learner-centered teaching approach. C1's teacher persisted in the teacher-centered approach in spite of the training (and re-training) by for example denying learners the opportunity to sort (for example triangles) for themselves and so discover properties. This teacher sorted the triangles herself on the board (with little to no learner participation) and later even provided the properties in the way she wanted them "back" in the test/exam.

- ***Right angles and triangles***

In testing the general acquisition of concepts of triangles, learners were asked to decide if certain figures were triangles and to explain their answers (see table 1). While doing analysis of relevant data concerning triangles, confusion was noticed between right angles and a right-angled triangle in some groups (see table 1). It is not clear whether the incorrect answers concerning a right angle and/or a right-angled triangle were given because of confusion between the concepts (due to second language instruction) or just because of carelessness on the part of the learners. To shed light on this uncertainty more in depth investigation would be necessary.

TABLE 1: IDENTIFICATION AND DRAWINGS OF TRIANGLES AND RIGHT ANGLES

		Groups (with number of learners in each group)			
		C 1 (12)	C 2 (11)	C 3 (11)	C 4 (13)
General identification of triangles					
Answers	Correct	6	3 1[a]	10	8 4[a]
	Incorrect	6	7	1	1
(Reasons:)	3 angles	4	1	7	7
	3 sides	3	1	9	2
	More specific answer	3	1	3	
Identification of right angle and drawings of a right-angled triangle					
Correct identification of right ∠ and correct drawing of right-angled ∇		3	3	9	8
Correct identification of right ∠ but incorrect drawing of right-angled ∇		4	4	1	3
Incorrect identification of a right ∠ and correct drawing of right-angled ∇		1	1	1	1
Incorrect identification of a right ∠ and incorrect drawing of right-angled ∇		4	3	0	1

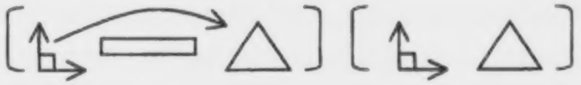
[a] These answers were classified as being correct, but unconvincing or incorrect motivation was given.

Class 1

Learners in C1 who decided that the shapes were not triangles rejected them because the sides were not equal (as for an equilateral triangle). Learners pertinently stated "no, because triangle all side are equal" and "no, triangles it has two equal size at the sides and the bottom one is shorter than two equal size in the sides[2]". Most of the learners who decided that the figures were triangles gave more than one answer that usually consisted of a reference to the angles and sides. In total four learners correctly drew a right-angled triangle. The eight learners who drew the right angled-triangle incorrectly all drew it as a right angle without a third side (∟). Seven learners in total could correctly identify the right-angle but only three could also correctly draw a right-angled triangle.

Class 2

Learners who decided that the figures were not triangles all motivated their answers by referring to the shapes' visual gestalt as the following answers[3] demonstrate: "no it is not triangls because is not the same" and " no Because they don't look like the triangles". Responses such as these demonstrate classical level 1 responses. Only one learner tried to give a more specific answer, but even this learner's thoughts seem scattered: "Yes Because they are all the same but the shape is very different but is all the same way but the shape are not the same[3]". Of the four learners who incorrectly drew a right angled-triangle, one learner tried to draw the triangle but only managed to draw an acute-angled triangle (△). Two learners' confusion surrounding these concepts was clear when looking at their drawings:



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[2] These answers are reported in verbatim from the (written) Van Hiele post-test.

It is noticeable that the learners who incorrectly identified right angles even drew more "strange" figures (▭) that resembled a rectangle or a trapezium, instead of a right-angled triangle. It is disappointing to notice that C1 faired very poorly in correctly drawing a right-angled triangle. Their statistics are as low as C2's and even worse in correctly identifying a right angle and drawing a right-angled triangle. A possible reason for C2's and C1's poor performance could be the influence of second language instruction – an issue for further investigation.

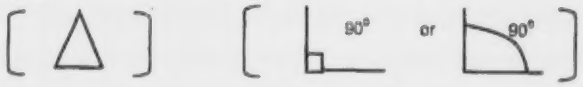
Class 3

Most learners gave a combination of answers[3] that included references to the number of sides and angles needed to be classified as a triangle, for example: "Yes, it has three sides, three angles" and "Yes, Because they all have three sides and they are have three angles they just have different names and sizes." Three learners gave some additional information even when not required to for example: "Yes, There are 3 corners, They don't have to be equal as long Their are 3 sides and 2 acute angles"[3]. An overwhelming number (82%) of learners correctly identified a right angle and correctly drew a right-angled triangle. Most of these learners' drawings also appeared to be "classical text-book" right-angled triangles (◺) while one learner drew the triangle in an "unusual" position (◸). Some of the learners also exhibited more detailed drawings by placing a right angle sign in the relevant angle (◺).

Class 4

Learners judged shapes by appearance as this learner's answer[3] demonstrates: " ... all 4 looks like triangles..."- typical level 1 reasoning as defined by various authors (Mason, 1997:39; Flores, 1993:152; Spear, 1993:393; Presmeg, 1991:9). Motivation for defining a triangle was varied, ranging from answers[3] referring to angle shapes ("it has 3 sharp points") to answers referring to sides ("their sides al have angles....they are sharp"). A learner that indicated that the shapes are not triangles motivated his answer by writing: "the sides are not the same length". It seems to be a common phenomenon that learners perceive a triangle to have equal sides, as some learners in all the groups rejected isosceles and scalene triangles on the grounds that their sides were not all equal.

All four learners who drew the right-angled triangle incorrectly, drew it as an acute angled triangle (△). It stands to reason that these learners may have focussed on the "triangle" part of the question without giving attention to the specification (right-angled). Confusion surrounding right angles in this group seems to be deeper than first discovered as the following drawing shows when learners were asked to draw a rectangle:



In summary it can be concluded that C3's answers proved to be more correct and complete with a total of 19 reasons (from 11 learners) explaining why the figures were triangles (see table 1), compared with C4's 9 reasons (from 13 learners), C1's 10 answers (from 12 learners) and C2's 3 answers (from 11 learners). A synopsis of the data reveals that C3 emerged as to be the top achiever once again with 81,8% of the learners being able to correctly identify a right angle and drawing a right-angled triangle (see table 1).

- ***Spatial Orientation***

Some questions in the Van Hiele post-test require learners to identify specific concepts in a variety of orientations. These concepts (such as a right-angled triangle and isosceles triangle) appear with other concepts in the "classical or conventional textbook form" (◺) as well as in an "unconventional or turned form" (▽). When analyzing the answers to these two questions (see table 2) it was believed

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[3] These answers are reported in verbatim as transcribed from the videos recorded during the class activities.

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and confirmed that recognizing one kind of triangle (right-angled triangle) in a "rotated form" does not guarantee recognition of a different kind of triangle (isosceles triangle) in a "rotated form" (as illustrated in the data shown in table 2).

TABLE 2 IDENTIFICATION OF A VARIETY OF SHAPES IN DIFFERENT SPATIAL ORIENTATIONS

<table>
<tr><th></th><th colspan="2">C1
(12)</th><th colspan="3">C2
(11)</th><th colspan="3">C3
(11)</th><th colspan="3">C4
(13)</th></tr>
<tr><td>Identify right-angled ∇ only in "traditional form"</td><td>7</td><td>1^{a}</td><td>1</td><td colspan="2">5^{a}</td><td colspan="3">0</td><td colspan="3">3</td></tr>
<tr><td>Do not identify right-angled ∇ in "traditional form"</td><td colspan="2">1</td><td colspan="3">3</td><td colspan="3">1</td><td colspan="3">3</td></tr>
<tr><td>Identify right-angled ∇ only in "rotated form"</td><td colspan="2">0</td><td>0</td><td colspan="2">2^{a}</td><td colspan="3">1</td><td>1</td><td colspan="2">1^{a}</td></tr>
<tr><td>Identify right-angled ∇ in both forms</td><td colspan="2">3</td><td colspan="3">0</td><td colspan="3">9</td><td>1</td><td colspan="2">3^{a}</td></tr>
<tr><td>Identify isosceles ∇ only in "traditional form"</td><td>1</td><td>2^{b}</td><td>0</td><td colspan="2">4^{a}</td><td>0</td><td colspan="2">1^{a}</td><td>0</td><td colspan="2">1^{b}</td></tr>
<tr><td>Do not identify isosceles ∇ in "traditional form"</td><td colspan="2">2</td><td colspan="3">1</td><td colspan="3">2</td><td colspan="3">5</td></tr>
<tr><td>Identify isosceles ∇ only in "rotated form"</td><td colspan="2">0</td><td colspan="3">0</td><td colspan="3">0</td><td colspan="3">0</td></tr>
<tr><td>Identify isosceles ∇ in both forms</td><td colspan="2">7</td><td>3</td><td>2^{a}</td><td>1^{b}</td><td>6</td><td>1^{a}</td><td>1^{b}</td><td>3</td><td>3^{a}</td><td>1^{b}</td></tr>
</table>

[a] Learners identified figure in specified manner but also identified incorrect shapes with the correct answer.
[b] Learners correctly identified figure in specified manner, but the answer is incomplete.

Of the four categories for each figure, the last category that entailed identification of figures in both the "traditional" and "rotated" form, is considered the highest category of identification. C3 consistently achieved the highest number of correct answers (when comparing the number of learners in this category with the number of learners in the group) in both of these categories. In the light of the results of C3's learners' recognition of the two kind of triangles (81,8% and 72,7% respectively) a conjecture was formed that if a learner could recognize one kind of triangle (right-angled) in both forms, he/she could most probably recognize any other kind of triangle (e.g. isosceles in both forms. This conjecture was not supported in the data from C4. Only 30,8% of the learners could recognize a right-angled triangle in the "traditional" and "turned" form, but this percentage reached 53,8% in recognizing an isosceles triangle in both forms. It is noticeable that C1, C2 and C4 found recognizing the right-angled triangle problematic, as it is here that they scored the lowest marks of all the figures. It becomes evident that C1 fared as well or as poorly as the classes who did not partake in the study in many aspects, leading to the conclusion that the program is not efficient in raising understanding if it is not implemented using a learner-centered teaching approach.

IMPLICATIONS AND CONCLUSIONS

It can be theorized that reaching higher order thinking in geometry relies not merely on a suitable choice of activities, but also on the active participation of both teacher and learners. The program in a teacher-centered environment (as in class 1) can produce results that are higher than the comparable class, but it is probable that the results could have been more dramatic if the program had been taught in a learner-centered problem solving environment (as in class 3). The fact that class 1 received instruction in their second language could also have had a great impact on their results. The results that were achieved in class 1 (second language education) are comparable with class 4 (mother tongue education) which may indicate that class1 has shown some progress, and to such an extent that their results are equal or higher than the learners who received first language instruction such as class 4 (and whose school is considered to be an advantaged school).

Results furthermore suggest that the implementation of a Van Hiele based learning and teaching program had a positive effect on the degree of geometric thought. Learners who completed the program reasoned on a higher level, gave more complete answers (see table 1), demonstrated less confusion (see table 2), and generally exhibited higher order thinking skills than their counterparts who did not take part in the program. The possible influence of receiving instruction in a second language (class 1) on learning was not investigated, but could have had extensive influence on these learners performances. A prerequisite for achieving success is that the teacher should consistently teach from a learner-centered approach as the program will deliver little or no advantages if the program is presented in a teacher-centered content-based context.

Teachers should take note of the profound part they play in the learning that takes place in their classrooms. The learning profit of their students could (partly) be determined by the teacher's interaction with the learning material and it is therefore necessary for teachers to make sure that their teaching approach and style complement rather than hinder the function of learning material.

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SCIENCE TEACHERS' LEARNING ABOUT RATIO TABLES

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Abstract

In Dutch lower secondary mathematics curricula, the ratio table is used as a tool for developing ratio and proportion calculation skills. Some recent physics and chemistry textbooks use ratio tables when dealing with proportional quantities. This makes the ratio table a suitable issue for co-operation between mathematics and science departments in schools. This article reports a developmental research study to starting science teachers attending the ratio table part of the IVLOS teacher education course. Factors that promote or impede teachers using ratio tables in their classrooms were identified. Main promoting factors appeared to be teaching lower secondary science and consulting mathematics teachers and textbooks. A main impeding factor was senior chemistry textbooks linking the ratio table to cross multiplication, affirming teachers' idea of the ratio table as a trick instead of a tool for promoting understanding.

Context of the study

Ratio and proportion is a core topic in primary as well as in secondary mathematics education. To many pupils it causes various conceptual problems (Streefland 1984, 1985; Tourniaire and Pulos 1985). In Dutch primary and secondary mathematics ratio tables are used as tools for structuring and doing calculations.

In science, many quantities have a proportional character, e.g. speed, density, concentration, concepts that usually are explained taking the mathematics of ratio and proportion for granted. In fact, students face many problems when using proportional reasoning in the sciences, even in the upper secondary (Akatugba and Wallace 1999). Many science teachers complain that students "cannot calculate any more". These problems can only be solved when science teachers and textbooks account for the ratio and proportion tools pupils have learned in mathematics. Recently, some physics and chemistry textbooks are published that do use ratio tables.

Nowadays, a curriculum reform being introduced in Dutch secondary education urges science and mathematics department to co-operate. The reform aims at presenting students a more coherent curriculum and opportunities for autonomous learning. In some schools, participating in the BPS-project of Utrecht University, departments are supported in finding ways to co-operate (Van der Valk *et al.* 1998). As it was noted in the BPS-school that some science teachers in the schools were apt to bring ratio tables in their classes but others objected, it was suggested to focus on ratio tables as a topic for co-operation. That resulted in an inventory of knowledge about the ratio table teacher have to master (Broekman *et al.* 2000) and identification of issues related to applying it into the sciences (Van der Valk *et al.* 2000). These results were used to introduce the ratio table in initial science teacher education.

The ratio table in mathematics education

A ratio table is a tool for structuring and calculating ratio and proportion problems. It can be described by the following characteristics:

1. the table consists of two rows and a variable number of columns
2. the rows have a label, indicating the meaning of the numbers and specifying, if needed, the units used
3. the upper or lower row can be changed: there is no preference
4. the ratio between the numbers in the columns is the same for all columns; this can be used to calculate an empty place in a column
5. so to get the numbers of a column, the numbers of another column can be multiplied or divided by a certain number; proportionately adding or subtracting are possible as well.

					x7	: 2
Spring roll	1	2	3	4	28	14
Guilders	1,25	2,50	3,75	5,00	35,00	17,50
					x 7	: 2

Fig. 1: Example of a ratio table from a Dutch 7th grade textbook.

These characteristics indicate that the ratio table attaches to intuitions young children already have (Lo and Watanaba 1997, Singer *et al.* 1997).

Ratio tables provide students with a structure they can use to find an answer to ratio and proportion problems in several ways. They:

- are allowed freedom which one of the steps that are admitted to choose for filling in numbers into a next column;
- are allowed to use as many steps as (s)he needs or finds useful
- however, have to be careful that the ratio between the numbers in the upper and lower row stays the same.

This is an example of what Dolk (1997) called 'construction space'. It can be seen as a sixth characteristic of the ratio table in mathematics education. Growing insight results in increasing efficiency. An efficient strategy is normalising to one. Thus, every proportion problem can be solved in two steps. For the actual calculating, the pocket calculator can be used. This way of working provides an alternative for 'mechanistic' algorithms like cross multiplication. Middleton and Van den Heuvel-Panhuizen (1995) described how the ratio table can be used in US middle school (grade 5-8).

The ratio table in Dutch science textbooks

In recently published secondary science textbooks (junior and senior level), several versions of ratio tables can be found. They are used to introduce physics and chemistry quantities having a proportional characteristic.

***First way of working*: [use of formula, not transcribed here]**

***Second way of working:* use the ratio table.**

1,000 mol $FeCl_3$	**162,2 g**
***n* ($FeCl_3$)**	**$16{,}4.\ 10^3$ g**

***n* is the number of mol $FeCl_3$**

$n.162{,}2\text{ g} = 16{,}4\ .\ 10^3\text{ g}\ .\ 1{,}000\text{ mol}$

$$n = \frac{16{,}4\ .\ 10^3\text{ g}\ .\ 1{,}000\text{ mol}}{162{,}2\text{ g}} = 101\text{ mol}$$

Figure 2. Text taken from: Antwerpen *et al* (2000) *Curie 1*, p. 129

To show that some ratio tables in science textbook are quite different from the one used in mathematics, one textbook, *Curie*, chemistry for grade 10, is analysed. In *Curie*:

- rows and columns are changed, compared to a mathematics-like table
- there are no more than 4 cells
- labels are missing; the cells contain information, like units and '$FeCl_3$' that should be included in the labels
- no choice of strategy within the ratio table is allowed as only cross multiplication is suggested (however, some choice of algorithm: formula or ratio table).

The ratio table as a topic in IVLOS teacher education

IVLOS teacher education is post-graduate and takes a full year (Wubbels 1992). The participants are pre-service as well as in-service teachers. They do their teaching practice in secondary schools and come to the university for the institute-based part of their teacher education. Main characteristics are its focus on reflection (Korthagen

1993), the integration of theory and practice and its attention to teacher learning (Korthagen and Kessels 1999). A course 'didactics of science and mathematics' is incorporated, focusing on areas of common interest to science and mathematics teachers, the ratio table being one of them.

In the 1999 'didactics of mathematics and science' course, the ratio table class was taught for the first time. It was noted that part of the started science teachers embraced the ratio table as an important solution for the problems they met in their classes, but others refused it. A phenomenon that was also observed in BPS-project schools. Therefore, it was decided to investigate during the 2000 course *what factors impede or promote starting science teachers' willingness to use ratio tables in their teaching*.

Research design and data collection

The research design was based on 'developmental research method' (Lijnse 1995; Gravemeijer 1994). This method is useful to get insight into problems of teaching and learning and to develop ideas how to solve them. Emphasis is mainly on the use of interpretative qualitative methods, which includes classroom observation, content analysis, interviews and analysis of student notes.

We regarded the following as candidate factors:

- teachers' own secondary school experiences with ratio and proportion
- experiencing the 'construction space' possibilities of the ratio table
- discussions with mathematics and science colleagues in the school
- (not) successful experiences with ratio tables or other tools in the classroom.

With these factors in mind, the ratio table part of the 'didactics' course was structured consisting of a 2½-hour class, a pre-class and a post-class assignment that aimed at promoting reflection and integration of theory and practice. The pre-class assignment had three tasks:

- study the way ratio and proportion are dealt with in your student textbooks
- look at the ways your students deal with ratio and proportion
- have a discussion with school colleagues about the ratio table.

At the start of the class, the participants reported about the assignment. They discussed the question 'what is ratio and proportion?' Then a short introduction was given into the way the ratio table is introduced in grade 7 mathematics, along with activities and discussions aiming at teachers experiencing 'construction space' in ratio tables in mathematics as well as in the sciences.

The post-class assignment asked the teachers to read a paper on the ratio table and to write a report about one of three tasks:

- collect students' notes showing ratio tables and give your comments
- teach a lesson on ratio tables in your subject and write down reflections
- prepare a discussion between the mathematics and the sciences departments about fine-tuning on the topic.

The study focused on nine science teachers' (2 female, 7 male) that handed in their pre- as well as post-class reports, 3 physicists and 6 chemists. All of the physicists and 3 chemists taught combined science (physics/chemistry) to junior students. Their reports were analysed by two researchers in discussion with each other. Comparing pre- and post-class arguments resulted in identifying factors that promoted or impeded the use of ratio tables in the classroom.

The implicit premise of the post-class assignment was that ratio tables can help students. One teacher did not agree: he refused to use the ratio table in the classroom because[1] *it will not be a success as I don't believe in it*, and excused himself to the teacher educator, writing: *it seems too much a trick that the students have to follow (I hope I don't offend you, it is just my opinion).* No indications were found that this premise was a problem to others doing the assignments.

Results

Only one teacher referred to her experiences as a student.

In their pre-class reports, four teachers already were positive about the ratio table. Three of them had some experience with using ratio tables in junior secondary science, the fourth had studied ratio tables in a mathematics textbook. Their arguments referred to:

- developmental psychology[2] (3): *[in the ratio table] the relation of the three quantities (m, V and density) is illustrated; this could make somebody with few feeling for formulae, with few sense of abstraction, more sensible to it*
- structuring (2): *To me, it is important that proportional calculations are taught in a structured way*
- students like it (2).

In their post-class reports, all four gave new arguments in favour of using the ratio table in science teaching.

The other 5 teachers, all chemists, related the ratio table with the cross product algorithm: *it is a kind of crosswise multiplication, but because there is no × sign it suddenly is proportional calculating*, three of them referring to senior level chemistry textbooks. In their pre-class reports they argued that students can use the ratio table/ cross product as a calculation trick, e.g.:

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[1] Statements from teachers' reports translated from Dutch, are reproduced in *italics*.
[2] The number indicates to how many teachers it applies.

- *some students just try the trick and hope the answer will be 'a bit normal'*
- *students will fill in figures on the wrong place in the ratio.*

However, four of them knew that it can be a support to students as well:

- *I have heard from a colleague that the number of mistakes made using ratio tables is much less than before. Calculating with formulae apparently is more difficult than filling in a ratio table*
- *Students remarked that using the cross product felt like a trick. They did not understand better how the calculations worked, but made less mistakes.*

Two of these five teachers left the 'trick' argument in their post-class report. The other three, however, left the help-argument. They did not use the ratio table in their teaching, but invited studied senior chemistry students to use it. Studying the results, they were reinforced in their 'trick' argument as all students applied cross multiplication. In their analysis they only used some of the features of ratio tables that were taught in the 'didactics' class. They:

- did not refer to tables having more than two columns (3)
- showed no (2) or few (1) knowledge of strategies alternative to cross multiplication (e.g. of 'reduction-to-one', multiplication factors)
- did not consult mathematics colleagues and books (3)
- reported about disagreement on the use of ratio tables in the science departments (2). One of them *met a difficulty: my mentor teacher did not like the students to use ratio tables.*

The six teachers that got more positive about the use of ratio tables in science:

- consulted maths and science colleagues (5) science textbooks (5) and studied a mathematics textbook (2)
- applied it in the classroom teaching in junior level combined science (4)
- using labels (3): "*now I pay attention to writing the quantities and units in it*"
- using multiplication factors (3)
- applying the reduction-to-one method and so tables with more than two columns (3)
- analysed their students' ratio tables (6)
- experienced success (3): *the students now looked less glassy, asked less questions and showed understanding.*

Conclusions

All starting science teachers initially had the feeling that the ratio table can help students in calculating proportional science problems. However, some teachers feared that it would only support students in doing calculations and would not help them in understanding the proportional character of the science concepts. With three teachers, this fear was not taken away by the ratio table class nor by the assignments because their idea of the ratio table being a 'calculation trick' because of its link to cross multiplication was affirmed:

- by studying upper secondary chemistry textbooks
- by analysing upper secondary students' use of the ratio table.

Indications were found that they were affirmed by colleague teachers in the schools as well, as they met departments that did not agree on using the ratio table in teaching science.

However, with six teachers the idea that the ratio table can support the understanding of proportional science concepts was affirmed by:

- consulting mathematics colleagues and textbooks
- having the opportunity to use ratio tables in lower secondary science classes.

An important factor appeared to be to have some experiences with ratio tables in lower secondary classes before teacher education on ratio tables. Teachers who had, were able to improve their lessons using 'construction space' alternatives to cross multiplication strategy of the ratio table, like indicating multiplication factors with the table, normalising to one. The ones that did not teach the ratio table in the classroom, did hardly apply the construction space aspects of the ratio table doing the post-class assignment, however involved they had been in applying ratio tables during the ratio table class.

To teacher education it is recommended to stimulate science teachers to discuss ratio tables with their mathematics colleagues and to let them reconstruct ratio tables from science textbooks that link to cross multiplication, applying the 'construction space' aspects of the ratio table.

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ARITHMETIC OR ALGEBRA?
PRE-SERVICE TEACHERS' PREFERENTIAL STRATEGIES FOR SOLVING ARITHMETIC AND ALGEBRA WORD PROBLEMS

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Abstract

This study investigated the arithmetic and algebra word-problem-solving skills and strategies of pre-service primary and secondary school teachers both at the beginning and at the end of their teacher training, and the way in which these groups of pre-service teachers evaluated different kinds of algebraic and arithmetical solutions of pupils. The results showed that future secondary school mathematics teachers clearly preferred algebra, even for solving very easy problems for which arithmetic is more appropriate. About half of the future primary school teachers adaptively switched between arithmetic and algebra, while the other half experienced serious difficulties with algebra. Finally, it was found that the problem-solving behavior of the future teachers is strongly related to their evaluations of pupils' solutions.

1. Theoretical and empirical background

Acquiring an algebraic way of reasoning and problem solving is one of the major learning tasks for pupils in the transitional stage from primary to secondary school. However, a vast amount of research has shown that learning algebra creates serious difficulties for a lot of pupils (Kieran, 1992; Filloy & Sutherland, 1996). The focus of the current study is somewhat different; its attention goes to the *mathematics teacher* who has to stimulate and support the transition from arithmetic to algebra.

The starting point of our study was the work of Schmidt (1994; Schmidt & Bednarz, 1997). She states that the complexity of the algebra learning process makes appeal to both primary and secondary school teachers. Primary school teachers should develop in pupils a rich base of mathematical concepts and skills which are the psychological foundations and precursors for algebraic thinking. Secondary school teachers should have a very good understanding of the 'arithmetical histories' of pupils entering secondary education, and be able to show pupils the validity and necessity of the new algebraic way of thinking. At the same time, they should develop in the pupils a disposition to apply arithmetical and algebraic strategies flexibly, taking into account the characteristics of the problem to be solved. In sum, Schmidt claims that primary and secondary school teachers must understand, master and appreciate both arithmetical and algebraic problem-solving strategies themselves, and be able to use them whenever necessary.

These considerations led Schmidt to conduct a study with three different groups of Canadian students at the start of their teacher training. By means of a paper-and-pencil test with typical arithmetic and algebra word problems (see the Method section), and by means of semi-structured interviews, she studied their spontaneous problem-solving behavior, their arithmetical and algebraic skills as well as their related domain-specific beliefs. The *first group* were students who just subscribed to a training to become a remedial teacher (in primary or secondary school). She found that the majority of them had to rely exclusively on arithmetical strategies because they had no good understanding and/or mastery of the algebraic approach. This need to rely always on arithmetic had a negative influence on their performance on the (complex) algebra problems. The *second group* were students wanting to become primary school teachers. Schmidt found that about half of these students were adaptive problem solvers, using arithmetic strategies for easy arithmetic problems, with good success, and algebra for more complex algebra problems, with moderate success. The other half of this second group had a similar profile as future remedial teachers. The *third group* consisted of beginning pre-service secondary school teachers. Here, the vast majority of participants exclusively used algebraic strategies (with good success), even for those problems that could easily be solved arithmetically. This preference was accompanied by a perception of arithmetic as being a primitive, mathematically worthless approach.

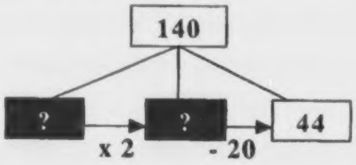
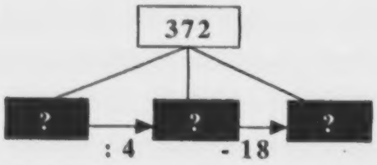
The study presented in this paper replicates Schmidt's study in the Flemish teacher training context, but at the same time elaborates it in two important aspects First we expanded the research group with students arrived *at the end* of their teacher training. Second, we wanted to shed light on the relationship between the student-teachers' pedagogical content knowledge and beliefs, on the one hand, and the quality of their future teaching, on the other hand. Therefore, we also collected data about the way in which the pre-service teachers evaluated pupils' arithmetical and algebraic solution strategies.

2. Method

Participants were 97 pre-service teachers of one typical training institute in Flanders. These future teachers belonged to four different groups according to the specific training they were subscribed to (primary versus secondary school) and the moment in their training (the beginning of their first year versus the end of their third and last year of teacher training). In the group of future primary school teachers, there were 26 participants in the first and 36 in the third year. In the group of future secondary school teachers, these groups contained 19 and 16 participants, respectively.

Two research instruments were administered: a paper-and-pencil test containing 12 word problems, and a questionnaire wherein the student-teachers had to score pupils' solution strategies for six problems from the test.

The paper-and-pencil test contained – in randomized order – six word problems that could easily be solved with a few arithmetic calculations and six more difficult word problems for which an algebraic strategy was more efficient. The problems were generated by means of an analysis method designed by Bednarz and Janvier (1993, in Schmidt & Bednarz, 1997). This method schematizes data in such a way that problems can be characterized as "connected" (two known values can easily be used to calculate a third, and so forth, so that arithmetic solutions are easy), or "disconnected" (a calculation with two known values to generate a third cannot be made, so that an algebraic strategy, which represents all known and unknown data in one equation, becomes more appropriate). In the table below we give an example of an arithmetic and a (semantically equivalent but structurally different) algebra problem.

Arithmetic problem	**Algebra problem**
In our farm we have 140 animals: cows, pigs and horses. The number of cows is the double of the number of pigs, and there are 20 horses less than cows. We have 44 horses in the farm. How many pigs and cows do we have?	In a large company, there work 372 people. There are 4 times as many workmen as clerks, and 18 clerks more than managers. How many people of each group are there then in the company?
	

Student-teachers' solutions were analyzed according to the correctness of the answer, and according to the solution strategy used. For this last scoring we applied a classification schema that was influenced by the findings of several researchers (e.g. Filloy & Sutherland, 1996; Hall, Kibler, Wenger & Truxaw, 1989). The table below presents a global characterization of the strategies for solving algebra and arithmetic word problems distinguished in this classification. For more details and examples we refer to Van Dooren, Verschaffel and Onghena (in press).

Strategies for algebraic word problems		
Algebra: An equation is written and transformed to calculate the unknown.	**Manipulating the structure**: The problem is restructured in a clever way so that it becomes solvable arithmetically.	**Guess-and-check**: The value of one unknown is "guessed", the correctness of the guess is checked. Repeats this until the correct value is found.
Strategies for arithmetical word problems		
Algebra: Same as for algebra problems.	**Manipulating the structure**: Same as for algebra problems.	**Generating numbers**: The missing values are directly calculated by performing the correct arithmetic operations on the known values.

Immediately after finishing the paper-and-pencil test, the student-teachers received a questionnaire with three arithmetic word problems and three algebra problems from this test. Each problem was accompanied by three handwritten correct solutions (one of each category from the table mentioned above). Student-teachers had to give a score on 10 points to express their appreciation of the quality of each of these three strategies. By presenting only strategies that led to *correct* answers, we could interpret differences in scores as evidence for differences in appreciation of the underlying solution strategy. We also asked the student-teachers to motivate their scores in a short written comment. As in the paper-and-pencil test, the order of the word problems (as well as the answers accompanying them) was randomized.

3. Research questions and hypotheses

Question 1: How (well) do pre-service teachers solve arithmetic and algebra word problems?

Our main hypothesis here was that the Flemish student-teachers would have a similar pattern of solution strategies as in Schmidt's (1994) study. More particularly, we predicted that future secondary school teachers would use mainly algebra to solve the word problems (including the arithmetic problems), and that they would do it successfully (**Hypothesis 1a**). Furthermore, we expected to find two groups of future primary school teachers: an "adaptive" group using alternately arithmetic and algebra with moderate success, and another group with a strong preference for arithmetic strategies (and, therefore, a weak performance on the algebra problems) (**Hypothesis 1b**). With respect to the factor "years of teacher training", the following two effects were expected. First, we predicted that the initial differences in strategy use between future primary and secondary-school teachers would increase during teacher training. In the training of secondary school teachers, pivotal attention is given to algebra; therefore, we expected that they will use more algebra at the end of their teacher training. In contrast, in the primary school teachers' training program, large attention is paid to the arithmetical method "manipulating the structure"; therefore, the use of this strategy among pre-service primary school teachers should increase with years of training (**Hypothesis 2a**) Second, we predicted that the test performance of the student-teachers would improve, since the Flemish teacher training does not only aim at the development of the students' *pedagogical* content knowledge, but also of their mathematical knowledge and skills as such (Van de Plas, 1995) (**Hypothesis 2b**).

Question 2: How do pre-service teachers evaluate pupils' solution strategies for arithmetic and algebra word problems?

Our main hypothesis with respect to this second research question was that the way in which the student-teachers solved the word problems themselves would be reflected in their evaluations of pupils' solutions (**Hypothesis 3**). This hypothesis was based on the general claim – which is supported by a lot of research (e.g. Fennema & Loef, 1992; Verschaffel, De Corte & Borghart, 1997) – that teachers content-specific knowledge and skills shape to a large extent their teaching behavior. Therefore, we predicted strong and positive correlations between the number of times a student-teacher used a certain strategy and the average score he gave to a solution that works with that strategy. Furthermore, we predicted that all (solution-strategy related) differences we hypothesized with respect to the first research question would also be present in the student-teachers' evaluations (i.e. differences between primary and secondary school teachers, and between first and third year student-teachers).

4. Results for research question 1

To statistically test the hypothesis concerning the *solution strategies* used by the student-teachers, we performed a 2×2×2 ANOVA on the number of algebraic strategies: type of word problem × group (primary versus secondary school teacher) × study year (1^{st} versus 3^{rd} year). This ANOVA first of all showed that the algebra problems elicited more algebraic solutions (on average 3.65 for 6 problems) than the arithmetic problems (1.75 on average), $Wilks'\ \lambda(1, 93) = 0.47$, $p < .00015$. Moreover, there was a significant difference in the use of algebra between the future primary and secondary school teachers, $F(1, 93) = 11.33$, $p = .0011$. The table below gives an overview of their solution strategies for the arithmetic and algebra word problems, together with the percentage of unanswered problems.

		Type of solution strategy			
Group	**Type of problem**	**Algebra**	**Manipulating the structure**	**Generating numbers / Guess and check**	**No answer**
Primary school	Arithmetic	11.0%	4.7%	78.8%	5.4%
	Algebra	42.5%	20.2%	19.9%	17.5%
Secondary school	Arithmetic	61.4%	1.9%	34.8%	1.9%
	Algebra	93.3%	2.9%	1.4%	2.4%

As expected, the solution patterns of the future primary and secondary school teachers were very different. Among future secondary school teachers, using algebra was the most common method for solving both types of word problems (which is in accordance with **Hypothesis 1a**). The large majority of the future primary school teachers solved the arithmetic problems arithmetically, and, more particularly, by the most straightforward strategy called generating numbers. For the algebra problems, many of them switched to algebra, while several others tried to solve these problems using more cumbersome methods like manipulating the structure or guess-and-check, and left problems unanswered. Again, this is in line with our expectancies (**Hypothesis 1b**). In additional analyses, it was found that there were two subgroups among the future primary school teachers: about half of them exclusively used algebra, the others almost exclusively applied arithmetic. These additional analyses also revealed that this last group was responsible for the unanswered problems.

The ANOVA showed no significant differences between the solution strategies of first and third year student-teachers, $F(1,93) = 1.84, p = .2564$, which means that **Hypothesis 2a** was rejected: contrary to our expectations, future primary school teachers did *not* use more arithmetic at the end of their training, and future secondary school teachers did *not* use more algebra.

To test the hypotheses about the student-teachers' performances on the word problems, another 2×2×2 ANOVA was performed with 'correctness of answer' as the dependent variable. This ANOVA firstly revealed a main effect of the type of problem, $Wilks'\ \lambda(1, 93) = 0.58, p < .00015$, indicating that the average score on the arithmetic problems (5.19 on a total of 6) was much higher than on the algebra problems (3.69). Second, the future secondary school teachers (with an average score of 9.89 on a maximum of 12) scored significantly higher on the word problems test than the future primary school teachers (8.31), but this effect was, as expected, only caused by a different performance for the algebra problems (with respective average scores of 3.19 and 4.57), $F(1, 93) = 15.97, p < .00015$.

The ANOVA revealed no significant main effect of grade, $F(1, 93) = 3.62, p = .0601$, but – again – there were differences on the algebra problems (a significant type of problem × study year effect was found, $Wilks'\ \lambda(1, 93) = 0.91, p = .00038$): third year students (with a mean score of 4.04) performed considerably better than first year students (3.29), and this difference was observed for future primary as well as secondary school teachers. This finding confirms **Hypothesis 2b**. A further analysis of our data showed that the student-teachers became particularly more skillful in the strategy that is envisaged in the educational level of their future pupils: future secondary school teachers became considerably more skilled in using algebra from the first to the third year, while the future primary school teachers showed an increased mastery of manipulating the structure from first to third year.

5. Results for research question 2

According to **Hypothesis 3**, student-teachers' evaluations of pupils' solutions would reflect their own problem solving pattern. The correlations between the future teachers' use of a certain strategy and their appreciation of the strategy, already confirm this hypothesis. All correlation coefficients are positive and significant at the .05-level, varying from 0.30 to 0.46 Further evidence for this hypothesis was provided by a 2×2×2×3 ANOVA on the scores given by the student-teachers, with the same independent variables as for the previous ANOVA's, but with one extra variable: the type of solution (consisting of 3 categories). The table below gives the average scores of the two different groups of student-teachers for the three distinct kinds of solution strategies for the arithmetic and the algebra problems. Since there were no differences between the strategies of first and third year student-teachers on the word problems test, there are no reasons to expect differences here either. Therefore, we will not differentiate in the evaluations of the first and third year student-teachers here.

Group	Type of problem	Type of solution strategy		
		Algebra	**Manipulating the structure**	**Generating numbers / Guess and check**
Primary school	Arithmetic	7.84	6.52	9.28
	Algebra	8.42	7.67	5.91
Secondary school	Arithmetic	9.20	6.60	8.16
	Algebra	9.33	7.18	5.50

The ANOVA showed that – as for the effect on the solution strategies on the word problems test – the overall score of the future secondary school teachers for the algebraic strategy was higher than the primary school teachers' score, *Wilks'* $\lambda(2, 93) = 125.16$, $p < .00015$. Here too, future secondary school teachers had a strong and overall preference for algebra, independent from the problem to be solved. Although the future primary school teachers adapted their appreciations more to the nature of the problem, their score for algebra remained lower than the future secondary school teachers' score, $F(1, 93) = 13.51$, $p = .0004$. In sum, as expected in **Hypothesis 3**, the differences we found in the solution pattern of future primary and secondary school teachers were also present in their evaluations.

6. Discussion

The findings of the present study confirm Schmidt's (1994) concerns about the ability and the inclination of (future) teachers to support their pupils in the difficult transition from arithmetic to algebra. Our findings force us to be even more seriously concerned, because they show that at the end of their teacher training future teachers demonstrated still problem-solving behavior with the same problematic characteristics as the student-teachers who were just starting their teacher training. Moreover, we documented that these problematic characteristics of future teachers' problem-solving behavior have an impact on at least one crucial aspect of their teaching behavior, namely the way they appreciate and score pupils' solution strategies. We doubt whether the subgroup of future primary school teachers experiencing great problems with algebra will have the proper disposition to prepare their pupils for the transition to algebra, but also whether the future secondary school teachers will be empathic towards pupils coming straight from primary school and bringing with them a strong arithmetic background.

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MECHANICAL LINKAGES AND THE NEED FOR PROOF IN SECONDARY SCHOOL GEOMETRY

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Mechanical linkages, which abound in everyday objects as well as in historic "mathematical machines", provide a rich source of geometry appropriate for secondary mathematics. Explaining why these linkages work the way they do offers a rationale for mathematical proof. The results of a small pilot study suggest that, with the emphasis on the underlying geometry, linkages provide a visually-rich, motivating environment in which to encourage students to explore, conjecture and construct geometric proofs.

Recent decades have seen a general decline in Euclidean geometry in Australian school mathematics curricula, with the geometry that remains becoming largely empirical and students having little idea of the significance of proof. One outcome of this has been renewed debate amongst mathematics educators concerning the inclusion of geometric proof in school mathematics, a debate partly driven by the development and introduction into schools of dynamic geometry software, such as Cabri Geometry II™ and The Geometer's Sketchpad®. While concern has been expressed that dynamic geometry software is contributing to the empirical approach to school geometry (for example, Healy and Hoyles, 1999), there is also the strong feeling that the dynamic imagery associated with use of the software has the potential to play a significant part in geometric reasoning.

Although mathematical proof has several functions, one of its roles is verification, that is, conviction or justification of the correctness of mathematical statements. However, while some students may have a cognitive need for proof as conviction, many see little point in proving something which they already 'know' to be true. De Villiers (1998) has criticised the emphasis on the verification aspect of proof in school mathematics, arguing that a more meaningful activity for secondary school students is to focus on proof as explanation. Bell (1976, p. 24) claims that the emphasis on proof as verification/conviction/justification often results in a neglect of other important aspects of proof, noting that "a good proof is expected to convey an insight into *why* the proposition is true" and that "conviction is normally reached by quite other means than that of following a logical proof".

Healy and Hoyles (1999, p. 1), reporting on the project, *Justifying and Proving in School Mathematics*, note that the National Curriculum in the UK

> prescribes an approach to proving … in which the introduction of formal proofs is reserved for 'exceptional performance', and thus delayed until after students have progressed through early stages of reasoning empirically and

> explaining their conjectures. Most of the requirements to explain and justify take place within investigations driven by numerical data.

Garuti, Boero and Lemut (1998) note that students who are actively involved in the conjecturing process in geometry problems are more successful in constructing proofs than students who have been asked to construct a proof without prior involvement in producing a conjecture. They use the term "cognitive unity" to signify the continuity which they assert must exist between the production of a conjecture and the possible construction of its proof. Hoyles (1998, p. 171), cautions against the assumption that the computer and dynamic geometry software will automatically "help to build bridges between the empirical and the deductive", noting that "there remains the question of how to develop a 'need for proof'". She suggests that

> we need to design new learning contexts which require the use of clearly formulated statements and definitions and agreed procedures of deduction but which also allow opportunities for their connection with empirical justification and the conviction this engenders. (p. 169)

The research focus

The proposed research to be conducted in 2001 aims to explore the potential of mechanical linkages, or systems of hinged rods, to engage students in conjecturing and deductive reasoning and to foster a greater understanding of the need for geometric proof. Mechanical linkages occur in many everyday items, for example, umbrellas, folding tables and car jacks, as well as historical "mathematical machines", such as Pascal's angle trisector. Many linkages are based on simple geometry, such as isosceles triangles, similar triangles, parallelograms and kites (see Vincent and McCrae, 2000). The linkages used in the research have been carefully selected because their particular geometry is within the understanding of above-average students in Year 8 (approximately 13 years of age). Dynamic geometry software models of the linkages, which permit accurate measurements and tracing of loci, form a bridge between the concrete and the theoretical, assisting students to visualise the linkage as a geometric figure.

Using both physical models and pre-prepared dynamic geometry (Cabri Geometry II) models, students will be encouraged to conjecture about the behaviour of each linkage, propose a conjecture which seems fundamental to the operation of the linkage, and so prove *why* the linkage operates in the observed way. Such proofs embody the multiple roles of *verification*, for example, that apparent invariants are in fact invariant, *understanding* of geometric relationships, and *explanation*, that is, giving an insight into *why* the linkage works the way it does.

The main research questions are:

1. Does the modelling of mechanical linkages provide a motivating environment in which to introduce secondary school students to conjecturing and proof?
2. Does the static and dynamic imagery provided by dynamic geometry software models of mechanical linkages assist students to produce relevant conjectures?

3. Does active involvement in the conjecturing process facilitate secondary school students' transition to construction of proofs?

Methodology

The participants in the multi-case study research to be conducted in 2001 will be approximately 20 students from an extension Year 8 Mathematics class at a private girls' school in Melbourne. The students are selected for this extension class on the basis of their non-verbal reasoning (Ravens' Progressive Matrices) scores, their performance on mathematics tests throughout Year 7, and teacher recommendations. They have no previous formal exposure to deductive reasoning and could be expected to benefit in some way from the research tasks. Prior to commencement of the research the following geometry will be taught/revised: properties of angles in parallel lines, triangles and quadrilaterals; Pythagoras' theorem; similar and congruent triangles. Pre-testing will include a Proof Questionnaire (Healy and Hoyles, 1999) and a van Hiele level test (Levels 1-4) (see Vincent, 1998).

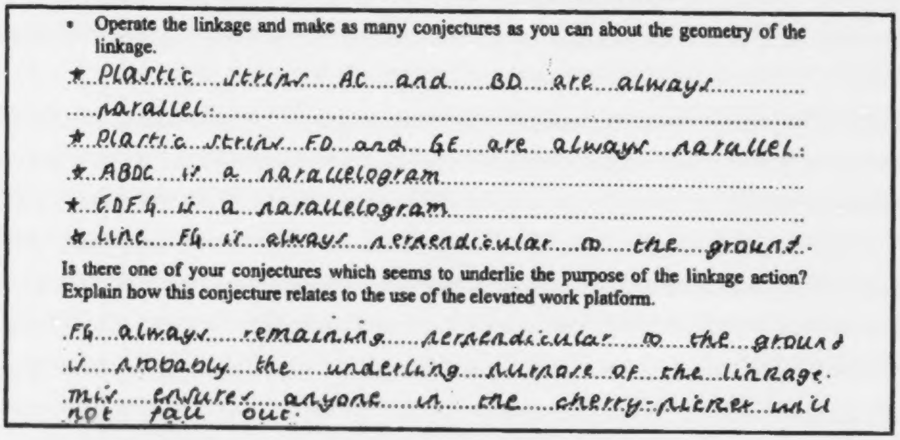
The students will work in matched ability pairs according to high/low Ravens' Progressive Matrices scores and van Hiele levels, and will be videotaped and interviewed during their work on some of the tasks. For each linkage, worksheets will direct the students to operate the linkage, record their conjectures, select the conjecture which seems to underlie the design purpose of the linkage and then construct a proof of that conjecture. Students will also complete a questionnaire for each linkage, requiring them to explain how they obtained the data on which they based their conjectures (for example, physical/Cabri model, measurement, loci of points) and which aspects of the models assisted them in constructing their proofs. Unless directed otherwise, students will be free to work with their constructed physical geostrip models and the Cabri models provided, perhaps, for example, checking the action of the physical linkage again after exploring the computer model. Post-testing will include conjecturing and proof tasks as well as re-testing with the van Hiele test and the Proof Questionnaire (Healy and Hoyles, 1999).

Pilot Study

A small pilot study was conducted with 29 students from the corresponding extension Year 8 class in 2000 with the aim of testing the feasibility of using mechanical linkages to stimulate conjecturing and deductive reasoning. Before undertaking any of the proof tasks the students were asked to give a brief written explanation of what "proof" in mathematics meant to them. While many students were unable to demonstrate any understanding of mathematical proof, 16 students were able to articulate at least one aspect of proof. The collective responses of these students illustrate the many facets of proof, for example, *verification* (10 responses)– "If you have proven something you have given evidence that undeniably shows that something is the truth. Proof cannot be denied"; *acceptance* (2 responses) – "Proof is required for people to believe and use a new mathematical theory"; *explanation* (4 responses) – "Proof is essential in order to explain a solution or idea. It should enable a viewer to comprehend the meaning". These responses may be compared with those of the high-achieving Year 10 students studied by Healy and Hoyles (1999), where approximately half of the sample of 2459 students recognised the role of proof in establishing the truth of a statement, just over one third believed that proof should be explanatory and over one quarter of the sample had little or no sense of proof.

While most of the 16 appropriate responses of the pilot study students focused on the role of proof, several referred to how proof is achieved. However, with the exception of one student who wrote: "Proof in mathematics is like proof in anything: using facts to support an argument. Any answer must be backed up using existing common knowledge and theorems", the responses referred to empirical, rather than deductive, methods. Typical responses included: "Proof in mathematics is when someone may have a theory and they have shown that it actually works by trying out every possibility (though that is very unlikely). They have tried or used something that shows that it works in every situation", "Proof of a theory needs to be tested over and over again to make sure it's true so you don't make a theory that doesn't work". Some responses indicated uncertainty about the generality of a proof, for example, "A proof has to prove that it works most of, if not all of, the time, not just in a handful of cases".

The familiar linkage of the elevated work platform, commonly known as a "cherry-picker", was used to introduce the class to conjecturing in a geometric context. Using plastic geostrips, card templates and paper fasteners, the students worked in pairs to construct the linkage and explore its behaviour. They enjoyed operating the linkage and their observations were accurate, as reflected in their individually-written conjectures (for example, see Figure 1). All students conjectured that there were two parallelograms in the linkage (see Figure 2) and that the safety cage remained vertical (or its base remained parallel to the ground). Most were able to represent the linkage as a geometric diagram, although there was variation in the degree of accuracy of portraying the lengths of equal links and in the positioning of pivot points (for example, see Figure 2), despite the parallelogram conjectures.



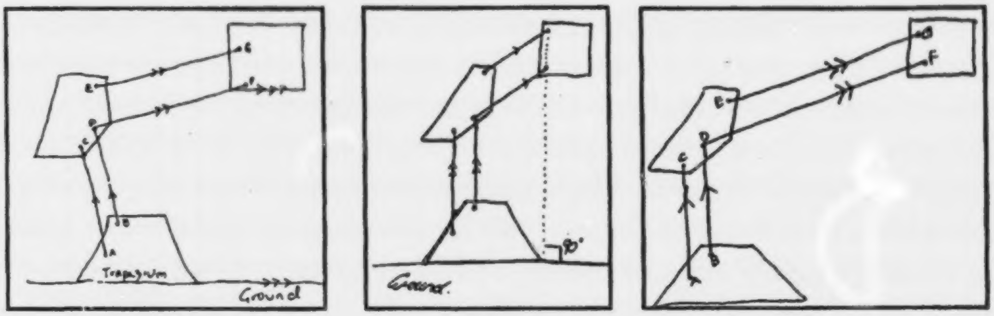
- **Operate the linkage and make as many conjectures as you can about the geometry of the linkage.**

★ Plastic strips AC and BD are always parallel.
★ Plastic strips FD and GE are always parallel.
★ ABDC is a parallelogram
★ EDFG is a parallelogram
★ Line FG is always perpendicular to the ground.

Is there one of your conjectures which seems to underlie the purpose of the linkage action? Explain how this conjecture relates to the use of the elevated work platform.

FG always remaining perpendicular to the ground is probably the underling purpose of the linkage. This ensures anyone in the cherry-picker will not fall out.

Figure 1. Student A's conjectures for the "cherry-picker" linkage.


Trapezium
Ground
Ground
90°
F

Student A Student B Student C

Figure 2. Students A, B and C: drawings of the "cherry-picker" linkage.

Tchebycheff's linkage (see Figure 3) was then introduced to the students as a means of demonstrating that visual evidence could sometimes be misleading. The linkage consists of a crossed quadrilateral where A and B are fixed and AB = 4 units, CD = 2 units and AC=BD = 5 units. The students constructed the linkage from geostrips and were directed to trace the loci of various points on the strips. After conjecturing that P, the midpoint of CD, moved in a straight line parallel to AB, they were provided with a Cabri construction of the linkage. By tracing the locus of P in the Cabri model and accurately measuring the distance PX (Figure 3) the students realised that the motion of P was only approximately linear. In this case empirical evidence was accepted, as the mathematics involved in determining the path of P was beyond these Year 8 students.

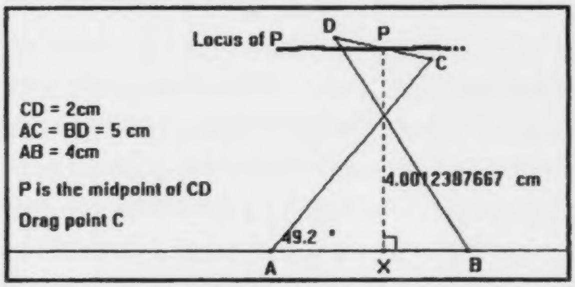

Locus of P
D
P
C
CD = 2cm
AC = BD = 5 cm
AB = 4cm
P is the midpoint of CD
Drag point C
4.0012387667 cm
49.2 °
A
X
B

Figure 3. Cabri model of Tchebycheff's linkage.

The students were then introduced to the car jack linkage, shown diagrammatically in Figure 4, where AB=PB=BC, A is fixed and P moves horizontally.

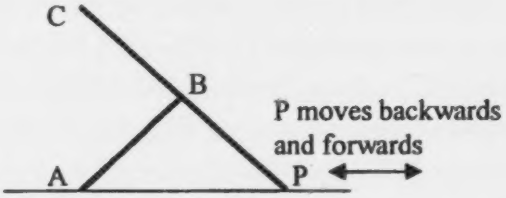

C
B
P moves backwards and forwards
A
P

Figure 4. Diagram of the triangular car jack linkage.

Students D and E, who worked with the actual jack (see Figure 5) while the other students constructed geostrip models, explained their observations of the jack:

> We found that when you draw a line from this point [attachment point for car] to this point [point directly below attachment point] this line is always perpendicular to the ground. So even when it's moving up and down, the .. er … sort of line down here is always perpendicular even though it's attached and this here [base of jack] holds it in place and so the part where the screw is is always the same. This part [pivot at centre of long arm of jack] rotates.

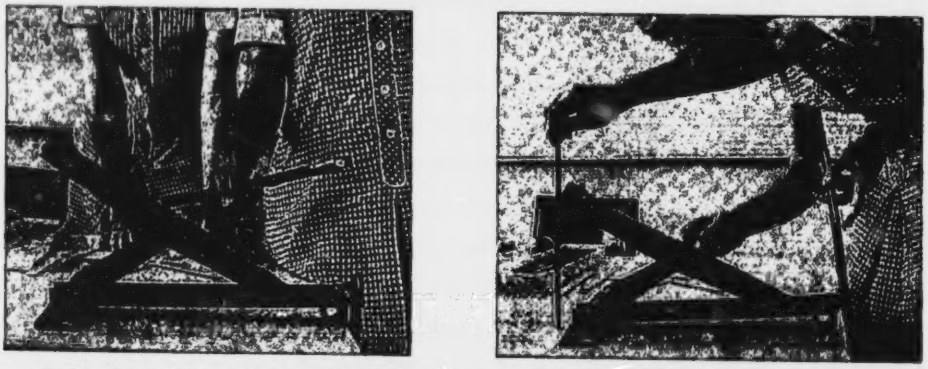
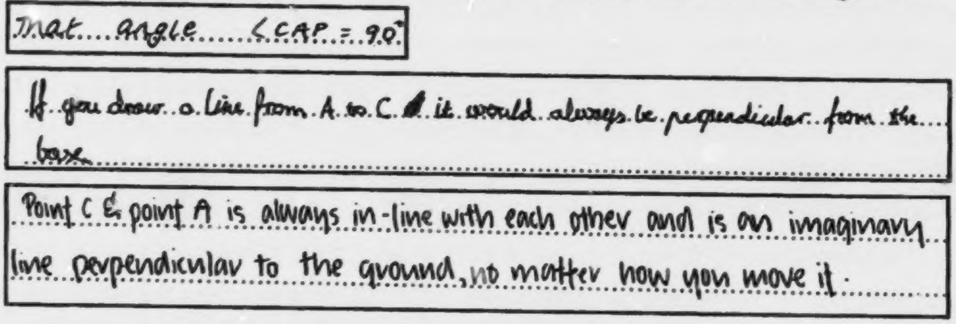


Figure 5. Students D and E: exploring the operation of the triangular linkage car jack.

Figure 6 shows the written conjectures of three students, A, F and G, each implying the same property of the linkage but expressed in different ways. Student A has focused on angle CAP (see Figure 4) without reference to a line AC, whereas students F and G observe that a line from C to A, drawn in the case of student F and imagined by student G, would be perpendicular to the ground.



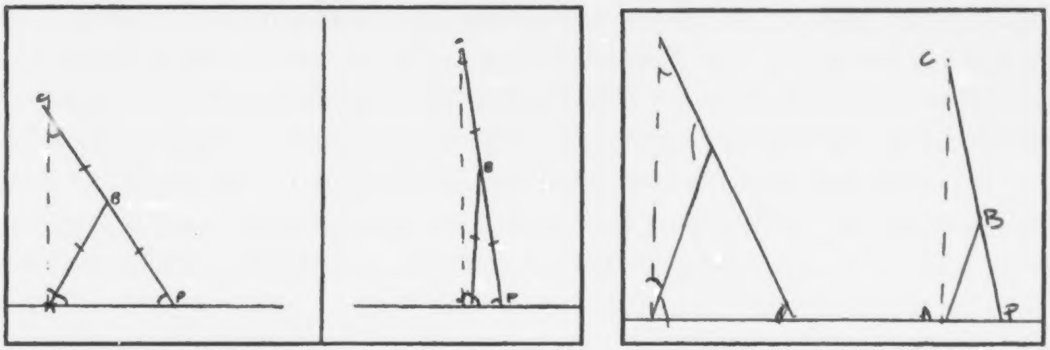
That angle ∠CAP = 90°

If you draw a line from A to C it would always be perpendicular from the base

Point C & point A is always in-line with each other and is an imaginary line perpendicular to the ground, no matter how you move it.

Figure 6. Students A, F and G: conjectures about the operation of the jack.

When asked to draw two different positions of the linkage, some students, for example, student H (Figure 7), accurately demonstrated that point C moved vertically upwards as P moved closer to A and that the lengths PB, BC and AB were preserved. In contrast, student I shows neither the increase in height of C or preservation of lengths.



Student H Student I

Figure 7. Different positions of the jack linkage as drawn by two students.

Investigation of Tchebycheff's linkage had demonstrated that visual evidence, particularly if based on crude measurements, could not be trusted so the students were aware of the need for proof. Some students recognised that the linkage contained two isosceles triangles, but they were uncertain how to proceed from there, as illustrated by the recorded explanation by students B and J:

> The angle BAP [see Figure 4] equals the angle APB because that's an isosceles triangle. And the same thing happens with the angle BCA and BAC because that's an isosceles triangle. Therefore angle CAB and BAP should both [together] equal 90 degrees. But we need to work out exactly why and that's what we're still trying to figure out.

The failure of these students to construct a proof may be due merely to lack of familiarity with proofs, but may also be due to failure to recognise ΔACP. Figure 8 shows the proof constructed jointly by students A and H (written by student A). Having commenced their proof with angles ABC and ABP, they realised that they could instead use the angles of ΔACP. Their equation: $2x+2y=180$, thus led them to the proof that ∠CAP is a right angle. Student A (and H) had conjectured that ∠CAP was a right angle (see Figure 6) and they had now proved their conjecture. It may be significant that students F and G, whose conjectures referred simply to CA being perpendicular to the ground, were unable to construct proofs.

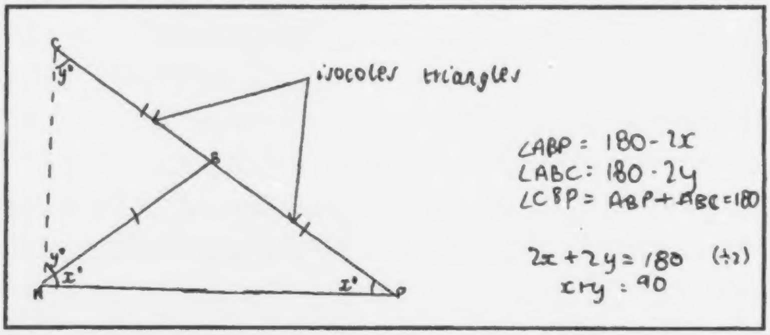


Figure 8. Students A and H: proof that ∠CAP is a right angle.

Conclusion

The results of this very brief pilot study are encouraging and suggest that Year 8 is an appropriate level for introducing able students to geometric proof. The students enjoyed working with the mechanical linkages, were able to produce conjectures relevant to the operation of the linkages and were motivated to try to find a geometric explanation for their observations. Not all students, however, were able to progress from conjecture to construction of proof when working with the physical (geostrip) models of the car jack linkage. This may be due to lack of familiarity with the concept of geometric proof, but may be related also to how a geometric figure is perceived. Further research may demonstrate whether the feedback from dynamic geometry software models can facilitate visualisation and analysis of geometric figures and assist the progression from conjecture to proof, particularly with more complex linkages. It is also important to investigate whether any positive effects of working with the linkages and with dynamic geometry software transfer to students' deductive reasoning and proof construction in pencil-and-paper proof tasks.

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SOLVING EQUATIONS WITH NEGATIVES OR CROSSING THE FORMALIZING GAP

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Abstract

We designed an experiment in teaching the solving equations to 8th grades students. This experiment shows that students encounter many difficulties not really when passing through equations with the unknown in both members but rather when passing through equations with negatives. A clinical study was carried out with 13-14 year old pupils to investigate which obstacles students must overcome to solve equations with negatives. This article presents the principal results from experiment and from interview. The results show that the presence of negatives in the equations represents a cut with arithmetical knowledge and requires from students a formal reasoning in both semantic and formal aspects. The different levels of conceptualization of negative numbers proposed by Gallardo (1994) must be reached by the students in order they can give sense to the equations (with negatives) themselves and to the solving methods.

INTRODUCTION

In the research literature, we found authors such as Kieran (1981), Filloy and Rojano (1984), Sfard (1991), Colomb (1995), Linchevski and Herscovics (1996) who stressed cognitive obstacles met by students in their learning of first degree equations with one unknown : the algebraic sense of the equal sign, the presence of the unknown in both members (didactical cut), the complementarity of the procedural/structural conceptions of the expressions, the operations performed on the unknown, ... In order to help the students to overcome these difficulties, we conceived a set of situations aimed to learn how to solve first degree equations with one unknown. These activities have been tested in two 8th-grade classes (aged 13-14). They are built on students' arithmetical knowledge and lead them to evolve towards algebraic methods, using among others the balance model. A particular attention was paid to the 'didactical cut' (Filloy and Rojano, 1984) – which occurs when the unknown appears on both sides of the equations – from a semantic as well as from a formal point of view. Our observations show that this demarcation point was passed through successfully for the resolution of equations resulting from the balance model, with such a structure : $ax + b = cx + d$. However, the presence of negative literal or numerical terms in the algebraic equations raised a lot of problems. In order to analyze more deeply the students reasoning in that context, we carried out several interviews with some pupils of the 8th and 9th grades.

This article reports on the results of the experiment as well as on interviews. It also presents reflections about the student cognitive processes when they are required to solve equations with negatives.

LEARNING SITUATIONS

Presentation of the situations

The whole of the activities proposed to the students of two 8th grade classes has been organized in two phases:

1. The activities about equations with the unknown in one member (arithmetical equations). The main objectives of these activities are to lead the students to use their arithmetical knowledge in solving equations (substitution, inversion of operations and cover-up) and to initiate the first notions related to the concepts of equation, solution and unknown. In this article, we will not discuss about the results of the first phase.
2. The activities about equations with the unknown in both members (algebraic equations). They are aimed to learn the formal method based on the equality properties (to perform the same operation in both members). Three situations have been designed :

   Situation 1 : Solving a problem : This activity consists in solving a problem of which modeling leads to an algebraic equation. At that stage, no new solving method is introduced. In the state of the students' knowledge, the solution can only be found by trials/errors. This situation is aimed to make students aware of arithmetic methods limits.

   Situation 2 : Balances : That situation introduces the formal method based on the equality properties with balance-based activities. Students are required to find the unknown value of a weight present on both pans.

   Situation 3 : Formalization : That step is aimed to systemize the formal process without the balance support. A set of 4 algebraic equations is presented without any context.

Results and comments

Data have been collected through observations in the classes and through students' productions analysis. They reveal two phenomenon's we would like to stress on :

1. The interest of the balance model

   Activities have been performed one after the other without raising any particular conceptual problems by the children until the formalization situation. When the balance situation has been got over, all the children were able to solve algebraic equations with additions, by performing the same operation in both members. Students productions analysis helped us to observe that the balance model offered a good mental picture of the required operations and the related concepts (sense and properties of equality). The recent interviews we propose hereafter confirm those results : after seven months of learning, students easily reactivate these techniques without needing any recall. A particular interest of using concrete models, like the balance model, is that students are able to reactivate at any moment that self-evident picture.
2. The algebraic equations with negatives : beyond the balances

   On the other hand, the next step of formalization (situation 3) was very difficult for a lot of students. As long as both members were composed of an addition and that the unknown value was a positive whole number, students solved it quite easily. They mentally used the balance model. But when the equation was composed of subtractions, a lot of difficulties appeared. We identified two different error origins:

a) *'The detachment from minus sign preceding a numeral or literal term'*. That type of difficulty - identified by Herscovics and Linchevski (1991) - produced the following errors :

Some students simplified like this the given equation members:

$$\begin{array}{ll} -3x+6=2x+16 & 2-3x+6=2x+18 \\ -2x\downarrow \qquad \downarrow -2x & -2x\downarrow \qquad \downarrow -2x \\ 1x+6=16 & 2-1x+6=18 \end{array}$$

We can see that apparently, the students do not take into account the minus sign before $3x$, no matter it is presented as a number attribute (first example) or as an operation sign (second example).

b) *Subtracting in order to neutralize a negative expression*

In order to cancel a negative numerical (or literal) term, some students use subtraction. An equation such as $3x - 4 = x + 9$ was solved like that :

$$\begin{array}{l} 3x-4=x+9 \\ -4\downarrow \qquad \downarrow -4 \\ 3x=x-5 \end{array}$$

We can imagine two hypothesis to explain that error. The first one may come from an abusive generalization of the balance model. Stimulated by their previous success with the balance process, the children use the subtraction in order to cancel an expression, just like they did in order to withdraw some weight from the pans. The second hypothesis refers to the first type of error mentioned here above. It could be related to the inability of some students to consider the sign before the expression. For them, it is not –4 that is needed to be cancelled, but 4, the sign '–' before 4 is not taken into consideration.

Authors such as Filloy and Rojano (1984) and Linchevski and Herscovics (1996), who experimented learning situations about the same theme, also say that similar processes appear with their subjects when these are facing equations with negatives. Filloy and Rojano (1984) think that difficulty is related to the fact the students do not succeed in generalizing their knowing stemmed from their experiences with the model. These authors bring into question the use of those concrete models since they do not improve significantly the students competencies. On the basis of our analysis, we assert, on the contrary, that the balance model is a useful instrument to help students to understand notions such as the properties of equality and the related techniques.

The introduction of negatives puts the equation solving at an abstract and formal level. In that case, we can no longer consider the equation terms just as weights needing to be withdrawn from the pans (what would it mean indeed to withdraw some weights $-3x$ or -4?). New obstacles have indeed to be overcome. Balances are not designed to overcome that type of obstacle. We have to consider transition activities in order to help students to leave the model while keeping the principles it introduces.

CLINICAL INTERVIEW

Presentation of the interview

We interviewed 7 students: five amongst them were in the beginning of the 9th grade and had learnt during the previous school year how to solve equations in the frame of our experiments. Two other students were at the beginning of the 8th grade and had not yet learnt equation solving. They were all of average ability.

The topic of the interview was to go deeper in the analysis of the data we collected during the previous experiments. The questions concerned mainly:

- Equations with the unknown in one member : i) $a - x = b$ (with $x < a$); ii) $a - x = b$ (with $x > a$), iii) $-a - x = c$ and iv) $-x = a$. Numbers were whole numbers.
- Equations with the unknown in both members.
- Numerical operations, with for instance, ' $237 + 89 - 89 + 67 - 92 + 92 = ?$ ' (Herscovics & Linchevski, 1991).
- Expressions reductions, with for instance, ' $19n + 67 - 11n - 48 = ?$' (Linchevski and Herscovics,1996).

Results

The most significant results obtained with this interview are the following :

in arithmetic equations with negatives :

Here is an example of each type of equation presented in the interview :

i) $12 - x = 7$; ii) $4 - x = 5$; iii) $-4 - x = 10$; iv) $-x = 7$.

1) Giving some sense to equations such ii), iii) et iv)

In order to solve those equations, all the students tried to give them some sense. No student of the 9th grade did solve spontaneously those equations through a formal method. We can reasonably assume that the numbers simplicity as well as the resolution of the first equation (i) $12 - x = 7$, incited the students to use arithmetic methods based on a concrete meaning.

'Inhibitory mechanisms' (Gallardo & Rojano, 1994) then appeared. On the one hand, most students were amazed before an equation such as (ii) : 'It is impossible to subtract a number to 4 to obtain a bigger result''. On the other hand, that difficulty to give some sense to those equations led to the inability to generalize the inversion of operations, even if it was widely used for (i). Either children did not think to use that method for (ii) et (iii), or they thought they were using it when adding 4 and 5 in $4 - x = 5$ (ii) and adding -4 and 10 in $-4 - x = 10$ (iii).

After having hesitating before 'the subtraction which makes bigger', the students who explicitly asked the question like that : 'By which number have I to substitute x in order to obtain a bigger result?' succeeded in finding the solution, remembering the rule 'minus by minus gives plus'.

Only one student solved $12 - x = 7$ with the substitution method ('which is the number which, when subtracted to 12, gives 7?'). The same student solved spontaneously and easily items (ii) and (iii) with the same method.

This observation supports our hypothesis (Fagnant & Vlassis, in press), according which the substitution process involves thinking structures that help students to conceptualize more easily the notions involved in the equations. Kieran (1988 and 1990) also supports that position.

2) Solving '$-x = 7$'

That equation was the most difficult. No student could solve it spontaneously. Five of them began by giving 7 as a solution. Some of them understood it was not possible since they had then $-7 = 7$, but they could not find the solution.

This difficulty has of course its origin in the absence of sense given by the students to an equation like $-x = a$. But why is that particular equation more difficult to solve for the students than equations such as (ii) and (iii)? We think the main reason is that students consider $-x$ statically, just as -4, for instance. That representation leads to confusion in the minus signs. For them, the '–' sign before the x is the same that the '–' sign in -4. In numeric cases, the numbers written behind '–' are 'naturals', so the students consider that x can only be a natural.

Some students who proposed the correct solution $x = -7$ explained that the solution could not be -7 because if it was, it should be written $--7$. When we proposed to other students the solution -7, they answered that the '–' was already written, they refused our argument and suggested $x = 7$ again. Cortès (1993) also indicates that in that type of equation, the unknown considered by the student is not x but $-x$.

In order to give some sense to an equation such as $-x = 7$, it is required the students can make the distinction between the minus sign and x. Two different thinking procedures can be considered:

- Considering the expression $-x = 7$, in a procedural manner, like $-1 \, . \, x = 7$. This raises the following question 'Which is the number that, when multiplied by -1, gives 7'. During the interview, we tried to involve the students in that way of thinking. But none of them seemed to understand where we wanted them to go.
- The equation $-x = 7$ can be considered with the idea of opposite numbers. That perspective leads to express the equation like this : '$-x$ means it is needed to take the opposite of the number x ; thus, which is the number of which the opposite is 7'. That perspective makes possible the idea that x could be a negative. Since it is required to take the opposite of a number, that number can be a negative. That second way of reasoning seems to be easier for the students. It helps them to keep their static conception of the expression while leading them to the correct solution. That method was used to find the solution by the two students who answered correctly.

in algebraic equations (for the 5 students of the 9th grade)

We were surprised to observe that algebraic equations with negatives raised fewer difficulties than the arithmetical ones. The students indeed did not try to give some sense to the algebraic equations to solve them. The presence of the unknown in both members seemed to act as a starter of the solving method consisting in performing the same operation in both members. The students did not meet any particular difficulty in the application of that method. But the presence of negatives nevertheless produced the following errors:

4) *'The detachment from minus sign'*

One student made that error. But no student did subtract a negative term to cancel it. That observation is surprising when we consider the results of our previous experiments. We'll need to interview more pupils to analyze more deeply the problem.

5) *Going from $-ax=b$ to $x=-b/a$ (like in $-5x = 10$)*

It is the most important difficulty we observed in the algebraic equations. We noticed two main obstacles for the students who apply completely the method consisting in performing the same operation in both members:

- Finding the right operation which links -5 with x. Whereas going from $6x = 7$ to $x = 7/6$ seems easy (the students explicitly explain they divide by 6 in each member), it is surprising to observe they do not know which operation they have to perform in order to transform $-5x$ into x. Some of them think they have to make $+5$ in each side. Others do not know at all what to do. It seems that the multiplication sign between -5 and x is not evident when the coefficient of the unknown is a negative number. The students are not able to decode easily the expression $-5x$ in terms of $-5 . x$.
- Dividing by a negative : Once the students succeeded in decoding correctly the expression, another obstacle appeared : they had some difficulty to accept the idea of dividing by -5.

in operations with numerical or literal terms

6) *'Detaching from the minus sign'*

- For two students, the expression $6 + n - 2 + 5$ is simplified in $-1 + n$. They justify it by explaining that if $2 + 5 = 7$ then $6 - 7 = -1$.
- A student makes $237 + 89 - 89 + 67 - 92 + 92$ like that : he crosses out +89 et –89 and makes : $237 + 67 - (92 + 92) = 304 - 184 = 120$

7) *Considering the sign which follows the numerical or literal terms*

- For two students, the reduced expression of $6 + n - 2 + 5$ is '$13 + n$' because $2 + 5 = 7$ so $6 + 7 = 13$. They consider the sign '+' which follows 6.
- In $19n + 67 - 11n - 48 = ?$, we find the following errors, the same ones already identified by Linchevski et Herscovics (1996) (3 students) :

  '*Jumping off with the posterior operation*': in order to reduce the expression $19n + 67 - 11n - 48 = ?$, some students grouped $19n$ and $11n$ by making an addition rather than a subtraction. The considered sign is the '+' sign which follows 19n.

  '*Inability to select the appropriate operation for the partial sum*':

  For some students, $19n + 67 - 11n - 48 = 8n - 19$, because 67 is followed by the '–' sign.

  We attribute these three different types of errors to the same trend as the detachment from the minus and which consists, over all in that particular case, in paying more attention to the sign which follows the operation rather than to the one which precedes it.

According to Linchevski and Herscovics (1991), the difficulties related to the priority rules of the operations. Our interviews do not confirm that hypothesis. The students did not propose any explanation of that kind. We tend to explain those difficulties by the arithmetical practices of the students. According to Vergnaud (1989), it seems that there is confusion between the numbers without signs (measures of size or quantities) and the numbers with signs (quantification of transformations and relations). In the primary school, over all the components of an operation, that is the numbers without signs (we make the operations with meters, francs, etc.) are stressed on, but not so often are the relations intervening in the operations. In the same way, we can also mention the ambiguity of the minus, which can be considered in a procedural way, as a sign of the operation to be performed, or in a static way, as the attribute of a number. It seems that, in this particular case, the students do not consider the second possibility.

8) Confusing the algebraic rules

A student wrote $19n + 67 - 11n - 48 = 8n - 19$ with a quite distinct justification from the other students'one. To our question 'Why -19?', he answered '$+67 - 48 = -19$ because plus by minus gives minus '.

DISCUSSION

The operations with the negative whole numbers present a lot of difficulties for the students. With the introduction of negatives, a formal reasoning, which goes most often into contradiction with the arithmetical knowledge, becomes necessary. In case of solving equations, that difficulty turns out to be still more important because that context makes it necessary for the students to have perfectly integrated the distinct levels of conceptualization of the negatives stressed by Gallardo (1994) : subtraction, signed number (plus or minus sign is associated with the number), relative (or directed) number (idea of opposite and symmetry) and isolated number (result of an operation or solution of an equation). The consideration of the various negative dimensions is needed to give some sense to the equations with negatives themselves, as well as to the formal solving procedures. For instance, the conception of 'relative number' enables to give sense to an equation such as $-x = a$, or to the procedure of neutralization of a term (canceling a numerical or a literal term); the idea of 'signed number' is essential to avoid errors of 'detachment of the minus sign', ...

Moreover, with the presence of negatives, algebraic equations resolution can no longer be considered concretely : for example, it becomes impossible to maintain the 'subtraction' idea (withdrawing some weights from the balance pans) to neutralize a term. The letter has to be given a mathematical interpretation and no longer an intuitive one, as an object (a weight) in the meaning of Kuchemann (1981). Students have to be taught to abstract, from the concrete manipulations, the general mathematical method needed to solve all types of equations with one unknown.

Solving equations with negatives, means thus crossing the formalizing gap. This transition cannot be left in the students' hands. It needs a teaching performing explicitly the transition towards the abstract concepts involved by the formal solving methods. The data collected through the studies presented here above will help us to modify our learning sequence. Further experiments will be carried out.

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TRENDS (1995 – 1999) IN PERFORMANCES OF DUTCH GRADE 8 STUDENTS IN TIMSS AGAINST THE BACKGROUND OF THE REALISTIC MATHEMATICS CURRICULUM

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In 1993, a new curriculum was established in junior secondary schools in the Netherlands, based on the principles of Realistic Mathematics Education (RME). Yet, teachers and students needed time to adjust to the new curriculum. In this article, data from the TIMSS test are studied, by distinguishing between test items that match and those that do not match the RME curriculum. Trend data (1995–1999) of Dutch students' achievement on these two distinct sets of items and data of teachers' approval of the test items suggest that the attained curriculum is approaching the intended curriculum at a very slow pace.

The RME core-curriculum in the Netherlands for junior secondary schools

Three decades ago Hans Freudenthal and his colleagues started to transform the mathematics curriculum with a treatise, which is generally known as *Realistic Mathematics Education* (RME). It is characterised by the understanding that mathematics is an integral part of real-life. Another component is the importance of enabling students to make mental images (Freudenthal, 1973, de Lange, 1987, van den Heuvel-Panhuizen, 1996).

In 1993 the "W12-16-project" established a core-curriculum based on RME for all students at Dutch junior secondary schools. The new curriculum emphasised data modelling and interpreting (through tables, graphs and word-formula), visual 3-d geometry, approximation and rules of thumb, the use of calculators and computers and other topics considered relevant to daily life of the new generation of the 21st century (Kok, Meeder, Wijers & van Dormolen, 1992). National assessment was adjusted to the new content approach. Generally, test items in the RME core curriculum describe an appealing daily life situation (often with authentic photographs to enliven imagination) followed by questions that integrate different mathematical content areas. The test items contain *horizontal mathematization* (Treffers, 1987) whereby realistic situations are modelled and reversibly the model is interpreted in its context. Several integrated mathematics topics can be combined and any test item is expected to keep students' concentration alive for approximately 15 minutes (Dekker, 1993). As for the format of questioning, multiple choice items do not match RME, because the world of real-life hardly ever offers four ready-made alternatives from which to choose.

With TIMSS data, implementation aspects of this new curriculum are studied below.

The TIMSS items matching a heterogeneous set of curricula

TIMSS (Third International Mathematics and Science Study) is an international comparative study of education in mathematics and science with the important question: *what can we learn from other countries?* It was conducted at grade 8 level in 1995 and again in 1999. The conceptual framework for this large scale curriculum study is based on the distinction of three curriculum levels (figure 1): the intended curriculum (what society at large prescribes students to learn, curriculum experts' opinions), the implemented curriculum (instruction at classroom level, teachers' opinions) and the attained curriculum (what is actually learnt by students).

To study the implementation process of the RME core curriculum at the level of junior secondary schools in the Netherlands, this framework was considered useful.

At the level of the attained curriculum, the TIMSS achievement test was carefully constructed in a process that is well-documented (Garden & Orpwood, 1996). Several experts in the field developed items testing for cognitive and procedural knowledge in a wide range of mathematics topics. With 40 different nations participating, a cross section of contents and levels had to be found that would be *equally unfair* for all.

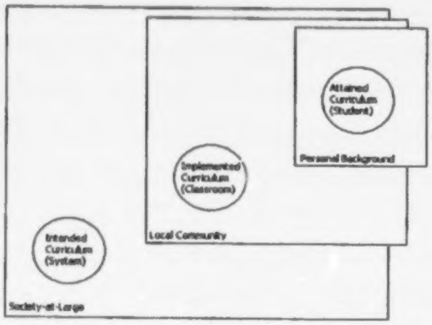

Figure 1. Framework for curriculum levels (Robitaille, 1993)

At the level of the intended curriculum, a Test Curriculum Matching Analysis was carried out. In each participating country curriculum experts were asked to review each test item and assess whether its content was covered by the intended curriculum for the majority of the target population. Results from this analysis are presented in table 1. For each nation participating in TIMSS-95 (testing grade 8), it shows what percentage of the TIMSS mathematics items were considered appropriate to the intended curriculum.

The Netherlands lingers at the bottom, with 71% of the TIMSS test being covered by the intended curriculum of the target population. The complement of 29% of the 157 TIMSS items was considered remote from the national curriculum. Kuiper, Bos & Plomp (1997 & 1999) already analysed this problem and their research (with a slightly different set of 150 TIMSS-95 mathematics items considered) displays a percentage of 69% of all test items that are reasonably well covered by the Dutch RME curriculum. Four years later in 1999, when curriculum experts again were asked to judge the TIMSS mathematics items, the percentage was 71% (out of 155 items), showing that no substantive change had occurred in the judgement (Bos & Vos, 2000). The minimal discrepancies (69% or 71%) can well be associated with the differences in the assessed sets of items.

It remains to be noted, that the approximately 70% portion of items that were considered to be in line with the national RME curriculum would have been much smaller if the experts also had considered the question format. With approximately 75% of all test items having a multiple choice format, there would remain very little to consider.

Table 1.
Nations and their percentage of test items from the TIMSS-95 international test matching the intended mathematics curriculum

Nation	% items addressing natl. curr. (n=157)	Nation	% items addressing natl. curr. (n=157)
Hungary	100	Singapore	90
United States	100	Ireland	89
Latvia (LSS)	99	Romania	88
Israel	98	France	86
Spain	98	Belgium (Fl)	86
Germany	96	Kuwait	86
Lithuania	96	Belgium (Fr)	85
Australia	95	Denmark	84
Japan	94	Switzerland	83
Slovak Rep	94	Iceland	83
Portugal	94	Colombia	82
Slovenia	93	England	81
Hong Kong	92	South Africa	80
Norway	92	Sweden	78
Korea	92	Russian Fed.	78
Czech Rep	92	Scotland	76
Iran, Isl. Rep	92	Cyprus	76
Canada	91	Bulgaria	74
Austria	90	Netherlands	71
New Zealand	90	Greece	46

Dutch students' performances on TIMSS-95 and TIMSS-99

When TIMSS-95 was carried out, serious doubts arose whether TIMSS would do justice to the Dutch target population. Additional research established that, although Dutch students were not prepared for the full set of TIMSS items by their curriculum, their abilities were well measured by TIMSS (Kuiper, Bos & Plomp, 1997, 1999). In other words: the approximately 70% of TIMSS items that matched with the curriculum gave them enough room to display their abilities. It was assumed that when learning mathematics with real-life contexts and integrated topics, students would still be able to display their abilities on isolated questions and the multiple-choice format was not to obstruct their performance.

Moreover, although it might not officially be intentional, somehow Dutch students were knowledgeable about the *remote* items and could attain reasonable scores on these items as well. It could be, that teachers would still follow the abandoned curriculum or mix forthcoming content of higher grades in their present teaching.

Another reason could be that students acquired their knowledge outside the mathematics classrooms, or that they just attempted the unknown tasks with an open mind.

In 1995 a sample of n=1921 students was tested, in 1999 a sample of n=2878 students was tested, according to the strict TIMSS sampling procedures (Beaton, Mullis, et.al., 1996; Kuiper, Bos & Plomp, 1997; Mullis, Martin, et.al., 2000; Bos & Vos, 2000). The tests from 1995 and 1999 were of comparable level, with half of the items being identical, and the other half being mostly *clones*. In the process of replacing items, only minor adjustments were made. For example item N19 in TIMSS-95 would read "*shade in 5/8 of the unit squares in the grid*", and its substitute in TIMSS-99 would read "*shade in 3/8 of the unit squares in the grid*".

Overall, Dutch grade 8 students performed well on both TIMSS mathematics achievement tests. In 1995 they scored an average percentage correct of 63% on the mathematics items, and in 1999 this was 65%. This small, though statistically insignificant improvement is confirmed in the international TIMSS-99 report, in which a different scale for measurement of country performances is used (Mullis, Martin, et.al., 2000). Both in TIMSS-95 and TIMSS-99 Dutch grade 8 students rank well above the international average. The new curriculum seems to have had a positive impact.

Do Dutch students show a better performance on the ± 70% portion of the TIMSS items that were considered to match their curriculum? In table 2 a trend for the average percentage correct on different sets of items is summarised. Looking at the data for 1995, the performances on the two complementary sets of items show no difference with the overall performance. On average 63% of Dutch students answered any item correctly, whether it matched the intended curriculum or not. The students performed just as proficient on the RME-matching items as on the set of items NOT covered by the curriculum.

Table 2.
Average percentage correct by Dutch students in TIMSS-95 and TIMSS-99, on subsets of items

	Average percentage correct	
	TIMSS-95 (n=1921)	**TIMSS-99 (n=2878)**
All TIMSS items (100%)	63	65
Non-RME-items (± 70%)	63	57
RME-items (±30%)	63	68

In 1999 there is a small (though not statistically significant) gap between achievements on items that match and do not match the curriculum. A larger percentage of students (68%) perform well on the test items that match the RME curriculum than on the items that are remote from it (57%). A reason for the slight discrepancy in the columns of table 2 could be that time was needed for the implementation of the new curriculum. 1993 was the year of introduction. Thus, in 1995, two years after the introduction of the RME curriculum, there was still a period of curriculum transition and the new curriculum had not yet established itself well.

Teachers might still incorporate topics from the abandoned curriculum. Four years later, in 1999, the implementation of the new curriculum was starting to show.

Dutch teachers' approval of the TIMSS test items

With students' achievement at the attained curriculum level, and the curriculum experts' judgement at the intended curriculum level, there is still an intermediate level in the conceptual framework to analyse: the implemented curriculum. The question is whether the grade 8 mathematics teachers had covered the content of the test items and whether students would have had an *opportunity to learn* this content. Would teachers also cover non-RME content?

The instrument for this research question consisted of a questionnaire, disseminated to the mathematics teachers of the 126 tested classes in which they were asked to judge the TIMSS test items by the following question: *if you were to set a test on all mathematics content which has been taught so far to the concerning class (tested in TIMSS), would you consider this item from its content to be suitable for this test?* It was explicitly stated that teachers were to ignore the format (multiple choice) and the difficulty level, and only indicate whether the content was taught.

The instrument of 1995 covered 16 mathematics test items out of 150. This small selection was based on the intended curriculum (curriculum experts had selected these 16 to match well with the RME curriculum) and proved to be too small to provide an analysis of teachers' coverage of the whole test. In 1999 the instrument was re-developed in a way that all 155 items were scrutinised. Yet, this number was considered too large to be included into one questionnaire, and the judgement of all items would become a tedious job for the teachers concerned. It could negatively affect their responses. Therefore, three mixed sets of items were created (of 52, 52 and 51 items), and each teacher would randomly receive one set. In this way all items would receive a judgement, although each item would only be seen by one-third of the teachers.

The response reached a satisfactory level of 89% (126 teachers were approached, 112 teachers responded). Further details on the methodology of this instrument can be found in Bos & Vos (2000). As no data from 1995 would be available to make a trend analysis, comparative data were created into two other dimensions:

- internationally, with mathematics teachers from Belgium (Flanders) judging the same mathematics items from TIMSS-99.
- cross-curriculary, with Dutch physics/chemistry teachers judging the 70 items from TIMSS-99 of their subject.

For the international comparison, Belgium (Flanders) was considered appropriate, because of similarities in economical, social and cultural aspects. Moreover their intended curriculum is very similar to the pre-RME-curriculum of the Netherlands and closely matches the TIMSS test. As can be seen in table 1, in 1995 the Belgian curriculum experts chose 86% of the TIMSS items to match their curriculum. In 1999 this percentage had increased to 98% (Mullis, Martin, et.al., 2000).

For the cross-curricular comparison, items from science were selected. The complete TIMSS science item set (n=135 items in 1995, n=143 items in 1999) covers the subject areas physics, chemistry, life science, earth science, environmental science and nature of sciences. In the Netherlands, we do not teach *integrated science* like in many other countries at grade 8 level, but instead there is the combined subject of physics/chemistry, the separate subject of biology, and the separate subject of geography. From the TIMSS test, we selected 70 items that could possibly be covered in the lessons of physics/chemistry. The selection consisted of items on physics (n=38), chemistry (n=19), together with items on environmental science (n=3) and the nature of science (n=10). The curriculum experts in this field had chosen two thirds of these 70 items to match with the intended curriculum, which is just slightly less than their counterparts did for mathematics (±70%, compare table 1).

In the analysis, for each item the percentage of approving teachers was calculated, by taking those who had indicated "yes" on the question whether they would include this particular item into a test covering all taught content. It was stipulated that if an item had a high teacher' approval, many students would have had an opportunity to learn its content.

In table 3 the percentage of approving teachers is divided into categories with ranges of 20%. Items in the first approval category (0-20%) have a very low teacher approval. Items in the last category (80-100%) have ample teacher approval. In three columns the percentages of items in the five approval categories is given, for the Belgian (Fl.) and Dutch mathematics teachers, and for the Dutch physics/chemistry teachers. To visualise the comparison, figure 2 displays the same data in a bar chart.

Table 3.
Approval rates of TIMSS-99 test items of Belgian (FL.) mathematics teachers, and Dutch mathematics and physics/chemistry teachers.

Percentage of approving teachers	**Belgium (Fl.)[1]**	**Netherlands**	
	% of math items (n=155)	**% of math items (n=155)**	**% of physics/chem. items (n=70)**
0-20	**0**	**1**	**3**
20-40	**1**	**3**	**10**
40-60	**4**	**6**	**37**
60-80	**24**	**24**	**33**
80-100	**71**	**66**	**17**

The general profile of answers by Belgian and Dutch mathematics teachers is fairly similar. Despite differences in the intended mathematics curriculum, there seems to be a tacit cross-border understanding in approval of the TIMSS test items. Both mathematics teacher groups display a high approval of the items, with the Belgian (Fl.) teachers displaying a slightly higher approval. According to the answers in both groups, more than two-thirds of the TIMSS mathematics test items had been taught by more than 80% of the teachers, and more than 90% of the items had been taught by more than 60% of the teachers. When looking at the 70% of items matching the Dutch RME curriculum, an average 87% of the mathematics teachers indicated that it was covered in their lessons (data not in table 3). The other 30% of non-RME items were covered by an average of 71% of the teachers. This could explain, why a majority of Dutch students proved proficient on the items that were not matching the intended curriculum: their teachers would still cover these topics.

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[1] We thank the Flemish TIMSS researchers, Prof dr. J. Van Damme and drs. A. Van den Broeck (K.U.Leuven), for making these data available (Departement Onderwijs Vlaanderen, 2000).

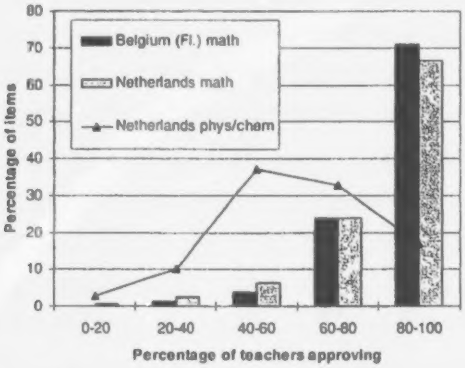

Figure 2.
Approval of TIMSS-99 test items by Belgian (Fl.) and Dutch mathematics teachers, compared to Dutch physics/chemistry teachers

Comparing the patterns of answers from the Dutch physics/chemistry teachers and the mathematics teachers proves difficult. Most physics/chemistry items receive an approval of half of the teachers, far less than the mathematics items. There are few physics/chemistry items with a high teacher' approval percentage, while in mathematics these outnumber all other items. To reach better understanding of this pattern, further cross-curricular research is needed, especially with regards to the physics/chemistry curriculum.

With the similar profiles of item approval by Belgian (Fl.) and Dutch mathematics teachers, it appears as if Dutch mathematics teachers still maintain characteristics in their instruction from the pre-RME era. Considering that their average years of teaching experience is 17 years (Bos & Vos, 2000), this means that the bulk of mathematics teachers in the Netherlands matured in their profession before 1993. It could mean that they still teach topics from the abandoned curriculum. As a consequence, it will take decades for the implemented curriculum to approach the new intended RME curriculum.

Conclusion

From the comparatively high performances of Dutch grade 8 students in the tests of the international comparative studies TIMSS-95 and TIMSS-99, it is obvious that the mathematics curriculum provides them with a solid foundation for doing mathematics. Yet, the implementation of the new RME-based curriculum at this level is still in progress, six years after its introduction in 1993. To analyse the process of implementation, a distinction in TIMSS items was made with the criterion of matching the intended RME curriculum or not. The performances of Dutch students on these complementary sets of items were calculated separately. Between 1995 and 1999, a slight improvement had taken place in the performance on the RME-matching items. Dutch students also did well on items that were not part of their intended curriculum, although their score decreased from 1995 to 1999.

To analyse the intermediate curriculum level of teacher instruction in the classrooms (implemented curriculum), an instrument was developed in which mathematics teachers were asked to judge all TIMSS mathematics items. A large majority of these teachers indicated that their students had an opportunity to learn about the content covered in the TIMSS test, whether the items matched the intended curriculum or not. Comparison with their Belgian (Fl.) colleagues displayed a fair agreement between the two groups. This could imply that the process of implementing the new RME curriculum is still proceeding slowly and Dutch teachers stay attached to the abandoned curriculum and that in Dutch mathematics classrooms a mix of two curricula is carried out. Further research into the implementation of the RME curriculum is advised.

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THE MICRO-DEVELOPMENT OF YOUNG CHILDREN'S PROBLEM SOLVING STRATEGIES WHEN TACKLING ADDITION TASKS

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The aim of this study was to determine the pathway of changes that occur in the problem solving strategies of 5-6 year old children when they are engaged in solving a specific form of addition task. Karmiloff-Smith's model of Representational Redescription (RR) suggests that higher conceptualisation and control of the employed strategy develops both before *and* after *the achievement of an efficient solution. This paper presents evidence that tends to support this hypothesis.*

INTRODUCTION

As the work of Kieren and Pirie (1994) and Lyndon and Pirie (1998), amongst others, demonstrates, mathematical understanding is always under construction. Yet very often in mathematics lessons, pupils who produce a correct answer or solution to a problem are not asked to justify or work further upon their solution. Correct answers are usually "ticked" while justifications, explanations and further work are often only asked for when errors occur. Justifying solutions is different from asking pupils to check their solutions. Indeed, Pawley and Cooper (1997) found that solution checking was not necessarily helpful for pupils, perhaps because it tends to increase cognitive load. Asking pupils to justify their solution is more akin to the "separate introduction of checking" which Pawley (1999) found led to significantly improved pupil performance on multiplicative tasks.

The aim of the research study from which the data presented in this paper comes is to try to understand the process that leads to this sort of enhanced performance. Our approach is to determine the pathways of change that occur in the *successful* problem solving strategies of young children as they tackle numerical tasks. This need for elaboration of a given solution *after success* has been addressed by some mathematics educators, notably within the theoretical framework of social constructivism. Research that has been carried out within this particular framework has used the classroom as a unit of analysis, and has shown the effectiveness of group-work and classroom discourse in promoting students' conceptual change and understanding (see, for example, Cobb *et al*, 1997). Group work on problems, discussion and argumentation on different ways of solving a task are certainly beneficial. However, it may be the case that not all students benefit to the same extent by group work and whole-class discussions. Reflecting and internally working upon their own problem solving approach, and knowledge that supports their own developed, efficient strategy can be another source of learning and conceptual change for children.

A comprehensive review of the literature (Voutsina 1999) indicates that Karmiloff-Smith's model of Representational Redescription (RR) (1984, 1992) offers the appropriate framework for this study. The model incorporates the idea of *success-based cognitive change*; that is, the possibility of conceptual change occuring *after* procedural success. It also suggests that higher conceptualisation and control of employed strategies develop both *before* and *after* the achievement of an efficient solution. The overall aim of the study is to determine the move 5-6 year old children make from procedural success to higher conceptualisation and understanding of the procedures they employ in the successful solution of an arithmetical task. This paper reports a typical example in order to illustrate the use of the RR model in arithmetic problem-solving, something that has not been done before.

THEORETICAL FRAMEWORK

The distinction between *procedural* knowledge (the ability to perform a task) and *conceptual* knowledge (the ability to understand the task) has been the subject of long debate in mathematics education (Skemp, 1971; Baroody & Ginsburg, 1986; Silver, 1987; Hiebert & Lefevre 1986; Bisanz & Lefevre, 1990). In the field of developmental psychology, Karmiloff-Smith (1992) has developed a theoretical model to account for development and learning based on the assumption that inflexible procedural behaviour rests on knowledge which is *implicit*; that is, knowledge which is *not* available as manipulable data. Karmiloff-Smith argues that after procedural success certain types of cognitive change may take place. In this process of change, implicit information embedded in an efficient problem-solving procedure progressively becomes more explicit, manipulable and flexible. In Karmiloff-Smith's developmental model, the issue of cognitive change is addressed in terms of knowledge explicitation that applies in a variety of domains, including mathematics.

Within the context of problem solving this idea of knowledge explicitation has only been studied in science, and with spatial, linguistic and notational tasks, but not any area of mathematics. In the framework of this study, children's developed strategies are seen as a product of the combination of different pieces and forms of arithmetical knowledge (conceptual, factual, procedural). Conceptual understanding, as it is built in the "micro-context" of an arithmetical task, is approached and studied on the basis of the idea of *knowledge redescription* (Karmiloff-Smith, 1992).

THE MODEL OF REPRESENTATIONAL REDESCRIPTION (RR)

Karmiloff-Smith argues that implicit information which is already stored in the mind, in a certain form of internal representations and is embedded in special-purpose procedures, is subject to an iterative process of redescription. The RR model is a *recurrent 3-phase model* with the following main characteristics:

Phase 1: "The procedural" phase

During this phase children's behaviour is considered to be "success-oriented". Separate units of behaviour are not brought in relation one to another. At the end point of this phase consistent successful performance is achieved, and this is what Karmiloff-Smith calls "behavioural mastery" (*ibid* p. 19).

Phase 2: The "meta-procedural" phase
In this phase, an overall organisation of the internal knowledge representations takes place. As a result, children generate "organisation-oriented" behaviour. They move beyond procedural success to a phase of internal representational organisation and the generation of a unified, single approach for all the parts of the problem.

Phase 3: The "conceptual" phase
During this phase the interaction between external data and internal representations is regulated and balanced as a result of the search for both internal and external control. Representations that sustain children's behaviour in the third phase are considered to be richer and more coherent, even though children's behaviour in this phase can seem identical to the behavioural output at phase 1.

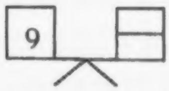
METHODOLOGY

The aim of the study is to describe and analyse changes in children's strategies on the basis of the premises and explanations that the RR model introduces. The study focuses on a number of cases. Changes in children's successful strategies are studied at a micro-developmental level. This means that the focus is on changes that occur within the context of a specific form of arithmetical task and within the boundaries of a sequence of limited-in number-sessions. For the data collection, the micro-developmental method is combined with the clinical method of interviewing. Children are interviewed individually while working upon specially designed tasks.

The "card" task asks children to find all the possible number bonds that result in a "target" number (for example, find all the possible number bonds to make 9, or 10 etc.). A pile of identical cards with incomplete number sentences, such as the one on the right, is at children's disposal.

□ + △ = 9

Children have to pick up one card at a time, put a number in the square and another one in the triangle in order to complete the number bond until there are no more possible ways to do so. The task is repeated with different "target" numbers. Each number bond that a child produces is considered as a step within the solution process. The hypothesis is that, in the context of this particular task, children initially approach each step of the task separately, employing different methods, and calling upon different pieces and types of knowledge. This kind of approach may well be successful. However, if appropriate motivation is given to children to keep working on the task, eventually, the different pieces of knowledge will be organised in a strategy applied consistently for every step, to the whole of the task. Similar-in goal tasks are presented to children to test the flexibility of the new strategy, and its transferability to similar goals. One such task is the "balance" task. On a piece of paper balances, such as the one shown below are drawn.



Children have to write one number on each of the blocks at the right side of the balance. The sum of these numbers must be equal to the number written on the block at the left side.

SAMPLE

Five children from a year-1 class of a South England infant school participated in the pilot study. Because the study focuses on children's evolving strategies after success and during a relatively limited number of sessions, children most competent in addition were selected to participate so that less time would be devoted to consider arithmetical misunderstandings and errors.

A CASE STUDY

An example of changes observed while working with a child is given below. Chris was 5 years 8 months old and participated in four sessions.

Key:
C: the experimenter
( ): movements, actions
[ ]: writing

Session 1

The target number was 7. Chris produced the following number combinations in the order shown in the inset below:

[6+1]
[5+2]
[4+3]

After writing down the first number, Chris counted on using his fingers to figure out the second. The interviewer asked Chris how he chose which number to write first. Chris replied:

Chr: Cause 6 it's just next to 7.
C: And why did you choose 5 after that?
Chr: Cause it's 1 more.
C: And how did you choose 4 afterwards?
Chr: It's 3 more.

The first number that Chris chose was the one that was closest to the target number. This choice allowed him to count less. It was an economical in counting method. Chris completed the task and produced the following number bonds:

[3+4]
[1+6]
[2+5]
[0+7]

To produce these number combinations Chris "swapped around" the previously produced number bonds, which is how he changed their addend order. After writing the last number bond Chris seemed to keep thinking. The interviewer asked:

C: So are these all?
Chr: There are more but I don't know them.

In this session Chris approached the task by focusing on the production of each number bond separately. The mixture of the two methods (economical in counting method and 'swapping') for the production of each number bond allowed Chris to be successful. Each one of these steps in the solution process was a separate unit of behaviour which was elaborated enough to allow success. However, Chris was not aware of his success. In terms of the RR model a first level of procedural success had been achieved.

Session 2

The target number was 8. Chris produced the following number combinations:

[7+1]
[0+8]
[8+0]
[1+7]
[2+6]
[3+5]

After writing down the last number bond Chris took some time to look at the completed number sentences and said:

Chr:	Then it gets higher and higher and higher…
C:	Which one is getting higher?
Chr:	(shows first numbers in the last three number bonds).
Chr:	It goes 7,…6, 5 (shows from the top to the bottom). Oh, 5, 6, 7 (shows from the bottom to the top). The lowest numbers go down there (shows 1, 2, 3 from the top to the bottom) and then like that (shows 5, 6, 7, from the bottom to the top)…like a zigzag. 4 and 4! **[4 + 4]**
C:	How did you think of that now?
Chr:	I don't know.

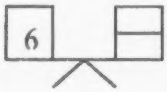
Chris noticed a pattern: a regularity of the numbers in the number bonds he had produced up to that point. He produced a new number bond following this pattern. However, this seemed to be a discovery which was not explicit enough yet to allow the formulation of verbal explanations.

Session 3

The target number was 6. Chris completed the task producing the following number bonds:

[0+**6**] [6+0] [**5+1**] [1+5] [2+**4**] [4+**2**] [3+**3**]

For the production of these number combination Chris applied his initial methods. At the end of the task however, he uttered the numbers in order, to check if he had finished. The numbers that he was showing while uttering them in order appear in bold. This was the first time that Chris realised the need to put the numbers in order so that he could check. However, it was clear that he had not yet fully grasped the rationale behind this strategy because he checked the numbers by moving from one column to the other. This would not allow one to know if all the possible numbers had been used either as first or second addends and thus whether all the possible number bonds had been produced. However, at the next run with the same task Chris employed the idea of "ordering" not only to check but to solve the task. This is considered as an important change and indication of Chris' movement away from the initial mixture of isolated though successful methods to an organisation-oriented "meta-phase" marked by the discovery of a new strategy. He applied this strategy consistently for all the steps in the task. Furthermore, he transferred the strategy to the balance task:



Chr:

5	4	3	2	1	0	6
1	2	3	4	5	6	0

Chr: No more (he said right away after finishing).
C: How do you know?
Chr: 1, 2, 3, 4, 5, 6, 0 (shows second numbers from the top).
That's all the different that you can make.

Chris produced these combinations in a few minutes by simply putting the numbers in the upper and lower boxes in descending and ascending order, correspondingly. Chris did not only transfer the "ordering" strategy to the balance-task but most importantly, this was the first time that he appeared to be certain of the completion of the task.

Session 4

In this session there were indications that Chris' approach to the task had been subjected to a process that made it more explicit. Chris applied the "ordering" strategy in the card-task for bigger target numbers such as 19 and 25. He justified the use of the strategy by saying: "It's easier… I know when I finish". Moreover, a violation of constraints imposed by Chris' initial theory of "economy in counting" was observed: Initially, in order to count less, Chris always used as first addend in the first number bond produced, the number which was closest to the "target" number. As a result, in the first number bonds the first addend was always bigger than the second. In the last session the first addend of the first number bonds produced was the smaller number. For example, when the target number was 19, Chris started the solution process producing these number bonds:

[0+19]
[1+18]
[2+17]

This was in opposition with Chris' initial theory of economy and probably happened because he realised that the use of the new strategy allowed him to avoid counting and any type of calculation all together. It allowed him to be successful and even more economic in effort.

DISCUSSION

Initially, Chris approached the task by applying a mixture of methods. The combination of these methods allowed Chris to complete the task successfully. However, he was not aware of his success since he kept looking for more number bonds even after all the possible number combinations had been generated. In the initial runs with the task, Chris focused his attention on each step of the solving procedure separately. He did not appear to have a whole view of the task that would allow him to relate one step with the other. Each one of Chris' steps in the procedure can be viewed as a separate unit of behaviour elaborated enough for successfully achieving its goal. Separate units of behaviour, in the sense that there was no conceptual or causal connection established among them, were integrated into a unified solving approach. This approach was followed consistently for some time. It can be considered that at this phase, and in terms of the RR model, a first level of *procedural* success had been reached. A significant instance followed: Chris noticed a regularity in the numbers of certain number combinations and followed it to produce the next number bond. He found out that the new number sentence was arithmetically correct, *and* maintained the regularity. Chris completed the task by applying a new strategy: the "ordering" strategy. An important change had occurred. Chris moved away from the initial mixture of isolated, though successful, methods and discovered a new strategy which he applied consistently for all the steps in the task. Within Chris' *meta-procedural* approach to the task, previously isolated steps in the solution procedure were now connected into a simplified strategy. Gradually, Chris proceeded to the consistent application of the strategy in every single run. He started relying on the previous step in order to proceed to the next one. Chris treated what was previously a sequence of isolated problems as a single one. A unified strategy was generated and applied to the whole of the task. He realised the efficiency of this strategy and also the economy in its use. He verbally justified the choice to use the new strategy on the basis of this realisation. Chris had shifted from his initial success-oriented approach to a new one that allowed him to organise the different steps of the task into a consistent whole. This new approach allowed him to keep track of the number bonds that he produced and thus allowed him to develop awareness of the completion of the task and of his success. These changes in strategy employed by Chris indicate problem solving behaviour that has gone over to a meta-procedural level. Chris was now at an organisation-oriented stage where a new, simplified procedure was generated which allowed him to acquire a better control over the components of the task.

The examination of this case supports the idea that the construction of mathematical understanding is an ongoing process, as Kieren and Pirie (1994) and Lyndon and Pirie (1998) argue. However, the interpretation of the phenomena observed under the light of the RR model showed that children's continuous search for better control and understanding of the problem situations in which they are involved is not merely restricted to, nor merely initiated by situations of difficulty, failure, or insufficient previous understanding. Stable states of success may also initiate and nurture children's movement towards a higher conceptualisation of the problem situations and their developed strategies.

CONCLUDING COMMENTS

Overall, the data reported in this paper is an example of changes that occurred while working with a particular child in the context of a broader research project. At this stage, the data shown suggest that children introduce qualitative changes and modifications to their successful strategies. These changes indicate the passage from initial success-oriented behaviour to an organisation-oriented phase in the RR framework during which the problem solver acquired a better control over the features of the task. If this evidence is replicated in the wider study, then one theoretical implication is that the form and type of changes that the RR model accounts for, seem to pertain to the changes observed in children's strategies within the setting of arithmetical problem solving situation. A practical implication of this for teaching is that if the process of Representational Redescription constitutes another way of constructing knowledge, then it is worthwhile giving children the time and space they need to work upon the knowledge that supports their own successes.

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Algebraic understanding and the importance of operation sense

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This paper examines young children's ability to generalise from their early experiences in arithmetic. A semi-structured interview was conducted with 87 children who had just completed their first three years of formal schooling. The purpose of this interview was to ascertain their understanding of 'turn arounds'. The results of the interviews indicated that many children are experiencing difficulties in reaching correct generalisations from their classroom experiences. These difficulties seem to be related to incorrect 'sense making', misleading teaching materials, and interference of new learning.

Introduction

With the recognition that students continue to experience many difficulties with algebraic concepts (Third International Mathematics and Science Study, 1998), the focus has moved to introducing algebraic ideas in the elementary grades. Kaput (1999) claims that we need to begin algebra early with an emphasis on sense making and understanding. Algebraic understanding evolves from viewing algebra as a study of structures abstracted from computation and relations, and as a study of functions (Kaput, 1999). Both these themes are believed to be appropriate for young children (NCTM Standards, 2000). As beginning algebra students progress from arithmetic thinking to algebraic thinking, they need to consider the numerical relations of a situation, discuss them explicitly in simple everyday language, and eventually learn to represent them with letters (Herscovics & Linchevski, 1994). This transition involves a move from knowledge required to solve arithmetic equations (operating on or with numbers) to knowledge required to solve algebraic equations (operating on or with the unknown or variable). It is believed that young children should be involved in making generalizations, using symbols to represent mathematical ideas, and in representing and solving problems (Carpenter & Levi 1999). This paper investigates young children's ability to generalise from their early experiences in arithmetic.

Early algebraic understanding

Two aspects are considered to be crucial in the transition from arithmetic to algebra. These are first, the use of letters to represent numbers and second, explicit awareness of the mathematical method that is being symbolised by the use of both numbers and letters (Kieran, 1992). This involves a shift from purely numerical solutions to a consideration of method and process. Yet many students experience difficulties in achieving this transition (Boulton-Lewis, Cooper, Atweh, Pillay & Wilss, 1998). Kieran and Chalouh (1992) suggest a reason for this is that most students are not given the opportunity to make explicit connections between arithmetic and algebra

(Kieran, 1992). It seems that knowledge of mathematical structure is essential for successful transition (Boulton-Lewis, Cooper, Atweh, Pillay, & Wills, 2000).

An understanding of algebraic structure is typically derived from knowledge of the structure of arithmetic. In this instance, knowledge of mathematical structure is knowledge about the sets of mathematical objects, relationship between the objects and properties of these objects (Morris, 1999). It is about relationships between quantities (e.g, equivalence and inequality), properties of quantitative relationships (e.g., transitivity and equality), properties of operations (e.g., associativity and commutativity), and relationships between the operations (e.g., distributivity). In a beginning algebra course it is implicitly assumed that students are familiar with these concepts from their work with arithmetic. From repeated classroom experiences in arithmetic it is assumed that by inductive generalisation students arrive at an understanding of the structure of arithmetic. Thus, knowledge of structure is considered to be at a meta-level, derived from experiences in arithmetic. How do classroom experiences impact on students' ability to derive structure?

Previous research has documented ways in which students' arithmetic experiences constitute obstacles for the learning of algebra. Most of this research has focussed on the differences between the two systems, for example, differing syntaxs (Lodholz, 1993), closure (Kieran, 1992), use of letters as shorthand (Booth, 1989), manipulations (Booth, 1989), and equality (Wagner & Parker, 1993). Recent research has begun to focus on the development of young children's algebraic thinking (Falkner, Levi, & Carpenter, 1999), with a focus on children's understanding of equality. This paper adds to this research by considering young children's understanding of the properties of the operations.

The specific aim of this paper was to investigate young children's ability to recognise the generality of the commutative property and to discuss this property in everyday language.

Method

Sample

The sample comprised of 87 children from four elementary schools in low to medium socio-economic areas. The children are all participants in a three year longitudinal study investigating early literacy and numeracy development. The average age of the sample was 8 years and 6 months and all had completed the first three years of formal education.

Interview

Five tasks were developed for the semi-structured interview. The two tasks reported on in this paper probe students' understanding of the generality of the commutative property, which is commonly referred to in our elementary curriculum as "turn arounds". Given the age of the sample and the number of years they had been studying mathematics, it was decided to limit the questions to addition and subtraction situations. By this stage of their schooling all children had completed their formal introduction to the concepts of addition and subtraction and could add and subtract numbers involving tens and ones. Task 1 involves probing children's understanding of 'turn arounds' for addition and Task 2 focuses on gauging whether children saw subtraction 'turn arounds' as being different from addition 'turn arounds'. For each task students were presented with 2 cards (see Figures 1 and 2).

2 + 3 = 3 + 2

31 + 16 = 16 + 31

Figure 1 Cards used for Task 1

After the completion of task 1 children were given the two cards presented in Figure 2 and were asked the same sequence of questions (see Figure 3) as for Task 1.

2 - 3 = 3 – 2

31 - 16 = 16 - 31

Figure 2 Cards used for Task 2

The script for the interview was as follows:

Script for Task 1 and task 2

Ask
What can you tell me about the number sentences on these cards?
Are they true or not true?

(If the child says that they are true) **Explain why they are true.**	(If the child says that they are not true) **Explain why they are not true.**
Can you give me some more examples?	**Can you give me some examples that are true?**
Describe the pattern in your own word	

Figure 3 Script for the interview

The interviews were audio-taped and the scripts transcribed for analysis.

Results

An examination of the transcripts indicated that the responses to the two tasks fell into five broad categories. The next section describes each of the categories and includes a typical response for each.

Category 1 (Correct generalisation)

For task 1, the child stated that the two cards for addition were true, gave some more examples and clearly explained in their own words how to form such examples. For task 2, the child stated that the two cards for subtraction were false (not true) and gave a valid example of why they were false. A typical response was as follows:

Task 1

What can you tell me about these number sentences.

They are turnarounds

You can really only do turnarounds with addition.

Can you give me some more examples

5 4

+4 +5

Are these statements true

Yes

Can you explain to me what a turn around is?

A Turnaround is when the like say that 5 plus 9 then the turnaround is 9 plus 5 - you just turn the numbers around

Task 2

What about these two?

They are turnarounds.

Are they true?

Not really. *Why*

No can't take 3 from 2 or 31 from 16

Category 2 (False generalisation)

The child stated that all four cards for the two tasks were true, gave some more similar examples and clearly explained in their own words how to form such examples. A typical response was as follows:

Task 1

They mean that it is a turn around.

Are they true or false

True

Can you write some more turn arounds for me

16 + 13 13 + 16

Explain

It's a turn around because if you had 31 and add 16 I could turn it around so that it is 16 plus 31

Task 2

What about these two?

They are both turn arounds.

Are they true or false? True

Can you give me some other examples of turn arounds?

Wrote 18-6
6-18

How would you explain that to someone else?

Now it is take away and if I had 18 over there and 6 over there I would turn it around so that I would have my 6 over there and my 18 over there.

Could you give me a subtraction example?

Subtraction means you take away. Subtraction is like a give away - If I had 12 bricks over there 8 bricks over there I could have 8 over there and 12 here so it is a turn around

Category 3 (Interference of question's format)

For this category the child stated that all four cards for the two tasks were false. A very common reason for this stance was that there should only be one number after the equal sign. A typical response was as follows:

Task 1

Not true

Because the = is meant to go last and the plus first

Because = is in the middle and there is another plus after the equal and it doesn't equal 3 + 2 it is meant to equal one number. You can have 2+3=5

After the equals how many numbers do you think there should be? one

Should be 31+ 16 = should be 47 - you have to take away the plus sign

Task 2

Not true

Because 31-16= 15 and they have a take away sign there and the number 31 at the end of the sum so it should be 31-16=15. It meant to be 2 take away 3 is one but they have 2 take away 3 equals 3 take away 2. But if it was three take away two it would equal one.

Category 4 (Interference of new learning)

The category represents responses where children seem to be experiencing some confusion due to over generalisations of new learning. For example, two children could now take 3 from 2 as they had recently been introduced to 'trading'. Another two posited responses as follows:

Task 1

True

Because you can add them up

What do you mean - can you write another one that is true

A plus or a take away

(wrote) 32+17=

The biggest number is at the top

What about this one here (16+32)

That's not true - you have to add more

Has the first number always have to be the biggest Yes

Is 2 plus 3 true No

Task 2

16 take away 31 and 2 take away 3 are not true because 2 can't take away 3

Because it is too small it is below 3

What about if I wrote down the plus one

$16+4 =$

$2 + 28 =$

Can I do the first one - YES

Can I do the second one - NO

Because 2 is too small.

Category 5 (unable to respond to the task)

Each response was coded according to the five categories. Table 1 summarises the frequency of responses in each category.

Table 1 Frequency of response for each category

Category	Category description	Frequency of response
1	Correct generalisation	23
2	False generalisation	24
3	Interference of the questions' format	31
4	Interference from 'new mathematics'	5
5	No response	4

The majority of children failed to reach the correct generalisations.

Discussion and conclusion

The results of this study present three differing themes. First young children certainly seem capable of reaching generalisations, even if these generalisations are mathematically incorrect. Most are engaged in sense making and understanding (Kaput, 1999). For example, Mitchell until recently could not make 'sense ' of 2-3. Children in his class have recently been introduced to the notion of trading. He stated *I can now do these. 2 take away 3 you can't do so you cross that one out (2) and put the 12 up there and then 12 take 3 is 9.* While this is not mathematically correct he felt very satisfied with his response. Young children are also capable of expressing their generalities in simple everyday language, a necessary step for progressing from arithmetic to algebraic thinking (Herscovics & Linchevski, 1994) and in most instances can offer further examples.

Second the types of classroom experiences that children are engaging in seem to interfere with them reaching valid generalisations. In this instance, teaching materials appear to be acting as cognitive obstacles to abstracting the underlying mathematical structure of arithmetic. A number of instances of these obstacles were presented in the data. The format of the cards themselves caused difficulties for some children. Samantha stated *I've never seen them written like this before. I write 31 take 16 = (and mimed the vertical algorithm). I don't know if they are true or false*, indicating that she had never seen expressions written in a horizontal format. The position of the equal sign also caused difficulties (see the response for category 3). Children's understanding of "=" also caused difficulties. Many of the responses in category 3 stated that *2+3 doesn't equal 3* and offered 2+3=5+2=7 as how the first card for Task 1 should be written. This confirms Anenz-Ludhow and Walgamuth's (1998) claim that many children in elementary grades generally think that the equal sign means that they should carry out the calculation that proceeds it and the number following the equal sign is the answer to the calculation. Again, this misunderstanding seems to be caused by teaching materials that predominantly involve getting answers to problems. In another section of the interview many children stated that *'equal means altogether, the answer'*.

Third children's misunderstandings also seem to be based on intuitive assumptions about applying new ideas to other familiar situations and interference from new learning in mathematics. Most of the children in the sample had had some classroom experiences with 'turn arounds' but, as indicated by the responses, these experiences seemed to be limited to addition situations. When presented with the subtraction task man simply applied their new understanding to this situation. Some tried to make sense of this intuitive assumption and changed their response during the interview. For example, Lucas stated *2-3=3-2 is true. 31-16=16-31 is true as well. I am thinking that they may not be true. Because 2 is not higher than 3 and 3 is higher than the 2.* But 36% were convinced that their intuitive assumption was correct. As indicated by the response to category 4, new learning seemed to also cause difficulties for some children. The introduction of subtraction interfered with their understanding of addition and the introduction of trading interfered with their understanding of subtraction.

From this research it seems from their classroom experiences with addition and subtraction young children have already developed misunderstandings with regard to the commutative property. When developing curriculum materials for the early years we must take into account that young children are engaging in sense making. We need to ensure that the ideas and materials we are presenting at this level help children abstract the structure of arithmetic rather than act as cognitive obstacles to future learning. From this study it seems that the misunderstandings these children are experiencing are based on pragmatic reasoning about new notions, the effects of misleading teaching materials and classroom experiences, and interference from new learning in mathematics.

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A MATTER OF PERSPECTIVE: VIEWS OF COLLABORATIVE WORK IN DATA HANDLING

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Abstract: *This paper reports on a study of students' collaborative group work in a grade 5/6 classroom using an open-ended task from the chance and data part of the mathematics curriculum. It considers (a) students' beliefs about collaborative group work compared to their actions; (b) observations of student knowledge, learning, and outcomes during collaboration compared to understanding displayed in individual interviews after work was completed; (c) students' accounts of events that took place in their groups compared with what was recorded on videotape; and (d) students' perceptions of the task and their beliefs about the mathematics curriculum.*

Introduction

This study is one of a series examining students' collaborative work using an open-ended data handling activity. Earlier studies identified factors associated with small group collaboration in a near-classroom situation, examined the use of an open-ended mathematical task in that environment, and documented help asked for and provided during the collaborative sessions (Chick & Watson, 1998; Watson & Chick, 2000, in press). In the grade 5/6 class used for this research, the teacher claimed to employ collaborative work in mathematical problem solving. She also said that the students knew the expectations, which included that everyone in the group understood how the result was obtained. For the current study, students were video-taped working in groups, and were also interviewed after the collaborative activity about their beliefs and understanding. These two sources of data provided an opportunity to observe and compare students' actions, beliefs, and potential contradictions in behaviour. This allowed the investigation of four key themes.

Views on collaboration. With the use of collaborative activities in the mathematics classroom becoming more common, it is recognised that students need to appreciate the benefits, be willing to work with others, and (sometimes) be taught necessary skills. It is also acknowledged that whereas some students see the benefits of collaboration (e.g., Watson & Chick, in press), others would prefer to work alone (e.g., Barnes, 1998). Ross and Cousins (1994) observed that students' intentions to seek and give help were not necessarily associated with their behaviours when participating in group work.

Did students learn what it appeared they did? Decisions about what students have learned in classroom settings may be made based on output produced (e.g., Chick & Watson, 1998), test scores (e.g., Webb, Troper, & Fall, 1995), comments recorded on videotape (Cobb, 1999), questions asked or answered (e.g., Watson & Chick, 2000), or post class interviews (e.g., Clarke, 1998; Frid, 1994). In this study transcribed videotapes were used to examine understanding during collaboration. This was compared to the understanding exhibited in individual follow-up interviews.

Recollection of events. During individual interviews students were not specifically quizzed about what happened in their groups, however they recounted what had occurred when asked to explain their group's poster and when asked if there was anything else that their group might have considered. Of interest were discrepancies in the accounts given by students and the actual events as recorded on videotape.

Is this maths? Students' negative views of mathematics and lack of confidence in the subject are well documented (e.g., Leder, Pearn, Brew, & Bishop, 1997). The introduction of chance and data to the curriculum had the potential to assist in softening the perceived profile of mathematics through the use of concrete materials, the inclusion of social applications, and the need for students to make personal judgements in decision-making. Of interest is what effect this has had on students' ideas of what mathematics is and its usefulness in everyday life.

Research Questions

In light of the above considerations, four questions were addressed. (i) Were students' beliefs about the value of collaboration as expressed during individual interviews consistent with their behaviour observed on the videotape? (ii) Were the observations of student knowledge, learning, and task outcomes consistent with that displayed in later individual interviews? (iii) Were students' accounts of events that occurred during their group work consistent with what was observed on videotape? (iv) To what extent did students question the mathematical nature of the task set?

Methodology

Procedure. Twenty-seven students in a combined grade 5/6 classroom in a suburban Australian primary (elementary) school took part in three 45-minute collaborative problem solving sessions involving an open-ended task from the chance and data part of the mathematics curriculum. The teacher assigned the students to groups of three, all with mixed gender. Seven groups had a mix of grades. Although the class had participated in group work on other occasions, the groups assigned for this activity were different from previous groups. The groups of three students were distributed around the classroom, half of the class at a time, with a video camera trained on each group. Except for a few initial instances of showing off for the cameras, the students ignored their presence. The task set for the students was to study a set of 16 data cards that contained the following information on each of 16 imaginary students: name, age, eye colour, favourite activity, weight, and number of fast food meals eaten per week. The students were asked to prepare a group poster displaying what they had learned about the 16 students. More details of the task are provided in Watson and Callingham (1997). After the three sessions, each group showed and explained its poster to the rest of the class, the two teacher/researchers who conducted the sessions (one of whom was the second author), and the classroom teacher. A week later the first author, who had not been previously involved with the class, interviewed all 27 students individually with a sequence of questions related to what occurred during the collaborative sessions, what the students understood of the task, and their views of working in groups. All students were happy to talk about the activity and none appeared to show any nervousness.

Analysis. The research team that analysed the data included, at various times, the authors, the second teacher/researcher present in the classroom, the transcriber of the videotapes, and a research assistant. The analysis of the data from interviews and videotape of the group sessions was similar to that advocated by Clarke (1998), adapted to meet the aims of the research questions. His *complementary accounts methodology* combines various sources, includes the reflective voice of the student, and employs a multifaceted analysis by a team of researchers. He asserts that an individual's learning process is embedded in a complex social context, and is "an integration of not just the obvious social events that might be recorded on a videotape, but also the individual's construal of those events, the memories invoked, and the constructions that arise as a consequence" (p. 100). In Clarke's approach students viewed classroom videotape during their interviews. This was not done here because it was felt important to make comparisons of the students' memories and beliefs about what had happened in the collaborative setting.

Data. There were 15 boys and 12 girls in the class; 5 boys and 4 girls were in grade 5. The data that were collected for each student included: gender of student, grade of student, assigned group, statements made about the activity and mathematics itself, understanding exhibited during group work and during the interview, descriptions of events occurring during group work that differed from that observed on videotape, description of collaborative or non-collaborative actions observed on videotape, and stated beliefs on collaboration during the interview. Where subjective decisions had to be made, these were decided by three or four members of the research team.

Results

Contradictions of belief and action on collaboration. Students were classified according to (a) whether their behaviours during group sessions were collaborative (or non-disruptive) or non-collaborative, and (b) whether the beliefs expressed during the individual interview supported collaborative work or not. Eleven students displayed predominantly collaborative behaviour, whereas 15 were classified as non-collaborative. In interviews 12 students said that, in general, collaboration is good, 12 had a negative view of collaboration, and 3 expressed both views with justifications (see Table 1). Some students expressed mixed views: they usually liked working in groups but not in this one, or they did not mind this group but did not always like working in groups. Of the three who strongly expressed both views, one girl showed both types of behaviour. She gave the boy in the group a dig with her elbow and hid the group's work from him so he could not collaborate, but she also gave him a great deal of help drawing graphs. In her interview she said she liked working in this group but did not always enjoy group work. This student was not classified in Table 1.

Table 1. Association of expressed belief in collaboration with observed behaviour during group work (n = 26[1]).

		Observation of Individual's Group Work Behaviour					
		Collaborative			**Non-collaborative**		
Belief expressed in individual interview	**Positive**	Male	Female	Total	Male[2]	Female	Total
		4	4	8	3	3	6
	Negative	Male	Female	Total	Male[2]	Female	Total
		1	2	3	9	2	11

[1] One girl displayed both positive and negative beliefs, and collaborative and non-collaborative behaviour. She is not included in the table.

[2] Two boys expressed both positive and negative views of group work, with justification.

Two boys, judged to display non-collaborative behaviour, also expressed mixed beliefs in relation to group work. One, who hoarded cards and flicked pencils and rubber bands, said that although "sometimes some of the people in the group were a bit silly, a bit uncooperative," the group worked quite well as "I got to know my partners better" and "[I'm] glad I worked with the group." The other boy was not disruptive but was often non-collaborative, ignoring questions from the other boy in his group and making all group decisions himself. In the interview he said "Yeah, we cooperated ... and I learned ... I guess with a group you just have to cooperate," but he also liked working alone because he could follow his own ideas. This boy was judged by the research team to be one of the brightest students in the class.

Four students who were judged to behave in a collaborative or non-disruptive fashion and who said that they liked to work in groups, were, in fact, observed to be ignored or abused by other group members. The other four students in the "collaborative-positive" cell of Table 1 were from the two most cooperative groups. The three students who generally behaved collaboratively but had negative views of group work were observed to be quiet students. In the case of one girl, one of the boys in her group was particularly disruptive. The third group member, a collaborative boy, noted that he would rather have worked on his own, as "sometimes when you needed the card another person had it, you couldn't get the card when you needed it."

Of students showing non-collaborative behaviour, 4 expressed positive views and 9 negative views, with, as noted above, 2 expressing both. Overall the groups from which these students came were judged among the least collaborative. There was a high representation of boys in the group displaying non-collaborative behaviour and having negative views of group work. Overall, about two-thirds of the class showed consistent behaviour and beliefs. No effect for students' grade level was noted.

Discrepancies in understanding. When the responses from the individual interviews were compared with the understanding displayed during group work sessions and the poster presentation, 16 of the students appeared consistent in their understanding of the task and what they had done. However, explanations from 11 students appeared quite different from those originally observed. Of these students, 6 (3 boys, 3 girls) had difficulty explaining what they had done in the group work sessions. One helped the others in her group construct their graphs during the group sessions but had trouble explaining her own graph during the interview. The other 5 students (of the 11) appeared to verbalise better understanding than they showed during group sessions. Three were girls, of whom one missed the final session and so did not contribute to her group's poster. One could state hypotheses about relationships among variables of which she appeared unaware during the group sessions, and one discussed a larger number of variables (for example differences between boys and girls) than during the group work where she appeared to consider only one, "hobbies". The two boys appeared to concentrate on the interview questions in contrast to their classroom behaviour, and provided better explanations than they had produced earlier.

Discrepancies in fact. In recounting events that occurred during group sessions, five students' descriptions differed markedly from events as recorded on videotape. One boy who hypothesised about the relationships among the variables claimed he did not graph any of these. He did graph at least one but due to the domineering attitude of the older girl in his group, he threw it away. He also claimed that all three had contributed to the group's graph, when in fact he and the other boy were only allowed to do "colouring in". Another boy explained why his graphs, rather than those of the other boy in the group, were used on the poster by saying a teacher/researcher told the group not to put "wrong data" on it; hence the other boy's graph was not used. Nothing said by the teacher/researcher on the videotape could be construed this way. The three girls with contradictory accounts of what happened were all dominant individuals in their groups. One said she wanted to graph eye colour, whereas during the group sessions she repeatedly stated that she did not like the idea at all. The second stated that her group had produced another graph that did not get on the final poster. Although a different graph was suggested by another group member, this was not even started. The third said her group discussed which graph to produce when in fact there was no discussion, she just demanded that her idea be carried out.

Is this maths? During group work 4 of the 27 students actively questioned whether the activity had anything to do with mathematics. Three boys and a girl asked variations on the theme: "What does this have to do with maths?" Another girl asked the second author in the course of the group work, "How can you *like* maths?" These five students' responses reflect some of the stereotypical views still present in the classroom. Students were not purposely questioned on these beliefs during their individual interviews, as the ideas were not part of the original research brief and because of the possibility of influencing responses to other interview questions.

Discussion

This class was not selected because it contained highly able students trained in a specific regime of collaborative behaviour. It was selected to observe what happens in a typical classroom in a typical school where collaboration is at least superficially known to be part of the teacher's program. The class was judged by its teacher to be "average", but she was confident students understood how to work collaboratively. The research team agreed with the teacher about the students' ability level and considered that the cognitive outcomes did not reach the level of other grade 6 children using the same task (e.g., Watson & Callingham, 1997). This study provides information for those who wonder how collaboration works in the "real world".

With some students indicating that sometimes they liked group work but not this time, or that they liked this group but not all group work, it may be that the specific make up of the group determines its success. It is interesting to note the differing experiences of the eight students who behaved collaboratively and expressed positive views about group work. Half were from groups judged to be the most disruptive and half were from the least disruptive groups. For the four students who had experienced abuse or been ignored, the contrast between views and experienced behaviour was not discussed in the interview. It is possible that they had experienced other more positive collaborative environments that compensated for this unpleasant experience or were reflecting a view they thought that the teacher wanted them to express. For the other four students, whose beliefs matched their behaviour, it might be expected that since their beliefs and behaviours coincided this contributed to relative harmony in their two groups. Whether these attitudes and actions were engendered by previous events in the classroom or the influence of the teacher is impossible to determine.

Of the 56% of students judged to have behaved in a non-collaborative fashion, a third were supportive of group work. Two of the girls in this category were among three dominant females who dictated terms within their groups. The research team suspected that some comments from students supporting collaboration—such as "It was easier to work together because you worked as a team"—closely resembled the opinions of their teacher. It is, however, impossible to determine if the students were speaking from genuine belief or from a "politically correct" position they had learned in the classroom. Of the two-thirds who were consistent in showing non-collaborative behaviour and negative beliefs most (9 of 11) were boys, including two whose views were equivocal. It is interesting that all members of one group were in this cell of Table 1, including the only two girls. This group comprised two grade 6 girls and a grade 5 boy and it appeared to the researchers that the grade and gender mix of this group, with a dominant but mathematically inept girl, was particularly unproductive.

The difference in understanding displayed at individual interviews from that observed during the group work sessions, particularly for the six students unable to explain the work done previously, is disturbing. Several possibilities exist to explain the apparent decline. Although all students appeared completely at ease during the individual interview with the first author, it may be that some were confused by the presence of the camera and a new researcher. It is also possible that in the intervening week students had forgotten what they had done. All appeared to remember the sequence of events but details may have been lost. Alternatively it is possible that students picked up comments and language from other members of their groups, which they utilised during group sessions and quickly forgot. The research team felt confident in the attribution of understanding during these sessions, certainly in several cases where the students appeared able to help others in their groups. The five students whose understanding of relationships appeared better at the time of the individual interview may have gleaned further information from the poster presentations made after the group sessions but before individual interviews, and further digested comments made by other group members. The fact that 41% of students displayed discrepancies in two apparently reliable settings, points to the need to collect information on student understanding from several sources.

The five students who gave discrepant accounts of events that occurred during group work were all in the non-collaborative category in terms of observed behaviour on the videotape. Two of the girls' beliefs about group work contrasted with their behaviour and one boy was equivocal about group work. Although it is impossible to be certain, it may be that these students wanted to make their own positions look more positive by describing events differently from the way they actually occurred.

The observation of student comments on mathematics and its relationship to the task was a serendipitous outcome of comments captured on videotape. The comments may reflect the fact that these children had had no graphing or other data handling activities in this class. They point to a need to include data handling activities at all levels and to connect them to the goals of the mathematics curriculum.

Conclusion

Overall the results of this small study of students' behaviour, in what was claimed by the teacher to be a collaborative environment, were disappointing to the research team. Critics of this study would undoubtedly say that the students had not been properly trained in the techniques of cooperative group. Given the social conditions and time available in many schools, however, it is fair to ask what is realistic in terms of training time for cooperative group work. Good, Mulryan, and McCaslin (1992)—who considered the role of teachers in preparing students, classroom management, age of students, and the role of explanations—concluded, "It is doubtful that there is a common shared experience of small group learning ... Experiences ... are better thought of as probabilistic than as predictable" (p.190).

If collaborative group work is an aim for teachers, then it would appear that in "typical" classrooms such as the one studied, more preliminary work is required to create the proper expectations in students. It is also necessary to consider carefully the assignment of students to mixed-gender and mixed-aged groups. The teacher of the class in this study believed her students would work cooperatively in the groups she assigned, yet the groups appeared to display many shortcomings.

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The Teacher's and Students' Important Roles in Sustaining and Enabling Classroom Mathematical Practices: A Case for Realistic Mathematics Education

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Abstract

This paper is a preliminary report of a second-grade classroom teacher experiment. The primary aim of this paper is to address the important role the classroom teacher and the students play in sustaining the classroom mathematical practices. To advance our ideas, we use excerpts from one discussion to illustrate the teacher's keen ability to consider aspects of the children's self-generated models. By capitalizing on such instances, the teacher not only provides possible learning opportunities for individual students but also makes it possible for the researchers to anticipate the emergence of subsequent more sophisticated practices.

In this paper, we provide a preliminary report of a second-grade classroom teaching experiment that was conducted during the 2000-01 academic year. The overall aim of this project was to support the children's flexible, mental reasoning with two-digit quantities. In particular, we explored the possibility of accounting for the children's learning as they moved from using informal to more formal ways of interpreting problems and recording their thinking. As a secondary goal, we hoped to refine an instructional sequence, Aunt Mary's Candy, to support the children's arithmetical reasoning. Our reasons for doing so were quite pragmatic. Because the classroom teacher was also using a newly adopted curriculum (that was compatible with the instructional sequence), the teacher needed to determine how she could infuse the sequence with the regular curriculum without threatening the integrity of either.

In our discussion here, we elaborate the significant role the classroom teacher and students can play in sustaining and the classroom mathematical practices. The teacher, for her part, must recognize important aspects of the students' interpretations and associated representations as they work individually, with partners or when they engage in whole class discussions. As she does so, she must employ strategies to make these interpretations explicit for the students. More generally, she must develop ways to infuse the children's idiosyncratic methods with the collective ways of interpreting and communicating mathematical ideas. At the same time, she must be flexible enough to advance ideas that support the emergence of new mathematical practices. As such, the teacher is faced with enormous challenges as she supports her students' mathematical learning. The students, for their part, must make sense of their activity and develop ways to notate and communicated their ideas to others. Before we address these issues, however, first we couch our discussion in theoretical assumptions and methodological issues that framed our efforts. We then provide examples from one classroom discussion to illustrate how the teacher and students can sustain and enable the classroom mathematical practices.

Theoretical Considerations

Instructional Design Theory of Realistic Mathematics Education. With regard to Realistic Mathematics Education [RME], Gravemeijer (1999) makes a strong case for the role curriculum might play in supporting children's formal mathematical reasoning. In particular, he elaborates several heuristics for developing and implementing instructional activities. One such heuristic is the role students' models can play in supporting their mathematical learning. These models are thought to emerge from the students' own understanding of a problem situation that is couched in a rich context. As students engage in problem solving within this context, they develop informal ways of representing their interpretations and solution methods. After some time, they develop more sophisticated ways to model their interpretations. Eventually their models take on a life of their own and can be used to reason mathematically without referring to the original problem context, that is, these models are transformed into more formal ways of reasoning.

In our view, mathematical meanings are socially accomplished by reflective individuals (Lave & Wenger, 1991; Cobb & Bowers, 2000). These meanings are negotiable as individuals reflect on their own and others' actions. That is, these meanings become social objects as individuals mutually orient themselves to one another's activity. One the other hand, these meanings have their roots in individual students' sense-making as they participate in mathematical activity that is valued by the teacher, the school district and the community at large. As such, we assume that the individuals' sense-making activity and the social situations in which they engage are inseparable and situated in various communities of practice. Although we acknowledge that social situations contribute to what constitutes knowing and doing mathematics in various communities, because our interest is that of mathematics learning in classrooms, our primary focus here is to elaborate the socially-situated nature of mathematics learning within this particular classroom microculture. (e.g., Cobb & Bowers; Lave & Wenger).

By coordinating RME with this situative view, the models that the students manipulate, adapt, or transform, although idiosyncratic per se, contribute to sustaining and enabling the taken-as-shared mathematical practices that emerge during whole class discussions.

Aunt Mary's Instructional Sequence. As one of the goals of the project, in collaboration with the classroom teacher, Ms. Wilson, we developed a series of activities to support the students' flexible manipulation of two-digit quantities for addition and subtraction situations. We adapted tasks from previous classroom teaching experiments conducted by the Purdue Problem Centered Mathematics Curriculum Project (Cobb, Yackel, Wheatley, Wood, NcNeal, Preston, & Merkel, 1992) and the Mathematizing, Modeling and Communicating in Reform Classrooms Project (Cobb, Yackel & Gravemeijer, 1995). Using some of these previously developed materials, we collaborated with Ms. Wilson to design an ongoing context about candy that her Aunt Mary made and distributed at various community functions, or gave to family members and friends. As the students worked in this context, they first created physical collections (with multilink cubes) to represent packages (collections of ten) and pieces of candy (collections of ones) and later drew pictures to reason about addition and subtraction situations involving Aunt Mary's candy. Making physical and pictorial collections allowed the students to engage in informal problem solving situations. These experiences, in turn, eventually allowed them to develop ways of notating and symbolizing that fit with more formal ways of notating and interpreting addition and subtraction situations (cf. Gravemeijer, 1999).

The activities Ms. Wilson introduced were designed so that the students could develop their own notational methods to explain their thinking about collections of tens and ones. Yet, certain ways of speaking about and notating how they broke apart and recombined collections of tens and ones became commonly used by many of the students. Typically, students drew rectangles to represent packages of ten candies and small circles to represent individual, loose candy. When solving subtraction situations, they usually crossed off loose pieces and subtracted additional pieces to take away the amount they needed to subtract. Whereas there was some variation in how they marked the packages, their notational methods and accompanying interpretations appeared to be understandable to the other students.

Methodology

Data Collection and Analysis. Data collection techniques included videotaping the daily lessons, making field notes, and collecting samples of the students' independent and small-group work. We gathered additional information as we talked to children each day and observed their mathematical activity. In addition to these data collection techniques, we conducted clinical interviews with each child at the beginning and end of the project to document their progress.

To analyze the data, we first perused our field notes to identify key lessons. After watching videotape recordings of several daily lessons, the videotape recording of one representative lesson was transcribed so that we could conduct further analysis. We also analyzed the children's written work samples to characterize their notational methods. Analysis of the children's written work was then triangulated with the whole-class discussion to reconstruct to the events of the lesson.

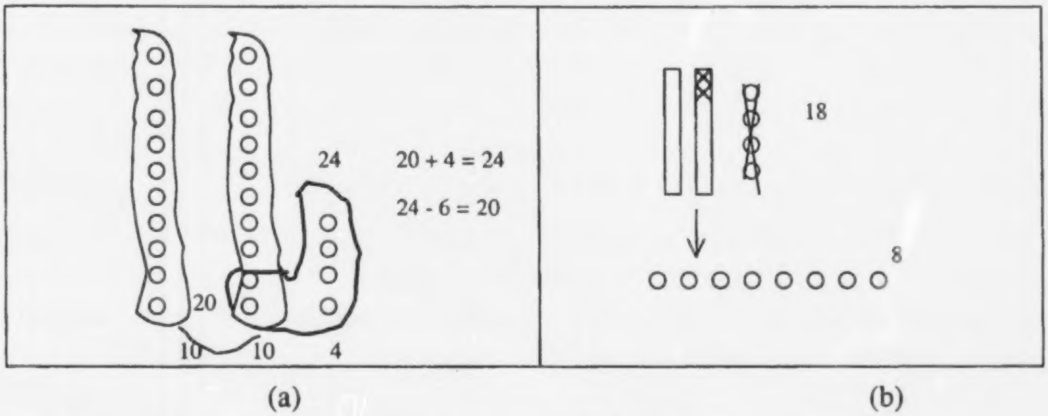
Classroom Episodes

The example we use here was taken from a lesson that occurred several weeks into the instructional sequence. The example is part of a whole class discussion that occurred after the children worked on several addition and subtraction tasks. After the children had completed several of the problems, Ms. Wilson asked them to join her in the front of the room to share their ideas. To begin the discussion, Ms. Wilson asked two children to come to the blackboard at the same time and show how they solved the following problem:

Aunt Mary has ▯▯○○○○ pieces of candy on the counter. Uncle Johnny eats

○○ pieces of candy. Show how much candy she has now.
○○
○○

Each child then explained his or her drawings on the board. We enter the discussion as Alice, the first child, explained how she solved the problem. She had drawn two packages showing each of the pieces of candies and four loose pieces (See Figure 1a). She then labeled each package with the numeral ten, the four loose pieces, and circled the six candies that she needed to take away. Interestingly, she also wrote two number sentences (20 + 4 = 24, 24 - 4 = 20). As her sentences indicated, she did not include the two loose pieces that she took away as she recorded her answer of 20. We enter the discussion as Ms. Wilson asked Alice to explain her drawing:



(a) (b)

Figure 1. (a) Alice's and (b) Luke's written records for 24 - 6 = ?

Teacher:	Alice would you tell us about your drawing?
Alice:	Yeah. Well first of all, I knew that if I had a package of ten (points to one of her packages) well um, what I did, I writed out and separated them and circled the ten and put a ten at the bottom. And then I made another package of ten…over here and then I circled that and put a ten and I added this ten with this ten and that made it twenty. And then I knew that I had to add the four [pieces] on (points to the four loose pieces) so I put the four over here and I got twenty-four.

We infer from Alice's explanation that telling about her drawing meant to describe what she had drawn. As the discussion ensued, Ms. Wilson asked Alice about her picture. In doing so, Ms. Wilson prompted Alice to provide a rationale for her drawing.

Teacher:	And what you are showing is what Aunt Mary has on the counter right now. Right?

Alice: (Nods yes.)

Teacher: Can I ask you a question before you go on to explain?

Alice: (Nods yes.)

Teacher: You said that you laid them out as pieces that and you circled them so that we would know that they were packages. What made you lay them out in pieces like that?

Alice: Because I knew if I did it the other way, that it would be harder. [It would be harder] to draw it to make the package torn apart

Teacher: Okay, so you did it so that you could see the packages when you wanted to tare them apart. Okay. So you have your twenty-four sitting out there, so now what are you going to do?

Ms. Wilson's question to Alice about why she showed loose pieces in each package is very important. By asking this question, "What made you lay them out in pieces like that?," Ms. Wilson made it possible for Alice to explain why she needed to show all the individual pieces. In essence, Ms. Wilson implicitly communicated to Alice and the other students that her drawing was valued. By making all the pieces visible, Alice could easily manipulate the pictorial collections to reason sensibly. When we consider the fact that Alice predominantly used counting strategies during her interview session and had difficulty solving tasks involving numbers over 20, her rationale seems quite fitting. Further, by asking this question, Ms. Wilson made it possible for the other students to understand why Alice chose to show the packages this way. The students had an opportunity to consider the particular details of Alice's drawing.

Following this exchange, Alice explained how her number sentences fit with her pictures but did not notice that her answer would be 18, not 20. As the discussion continued, Ms. Wilson prompted Alice to explain her answer:

Teacher Okay and then you circled some other pieces there. Um, what did you circle those pieces for?

Alice: Oh because, um, remember I said that Uncle Johnny ate six [pieces]

Teacher: Okay. And show where those six are that you [circled].

Alice: Well, I knew I had four, and then I knew that I had to use another package. So that I can get six because there is no other, um, there's no other way to get to six. So I took the four and put a line, and then I circled the four, I mean two, and then I counted them and figured out four, (points to the two pieces in the package and counts) and then five, and then six.

At this juncture, Alice moved beyond merely describing her drawing to giving a reason for why she circled the six loose pieces. We suspect that Ms. Wilson's question about the circled pieces was particularly significant in helping Alice explain her thinking. Initially, Alice may not have realized that she needed to use her picture to explain her thinking. However, at this point in the exchange, she explained in some detail her thinking using the drawing.

Following this exchange, Ms. Wilson asked Alice to determine how many pieces of candy Aunt Mary had left. After counting the remaining candy, Alice changed her answer to 18. She then changed her number sentence 24 - 6 = 18.

As the discussion continued, Ms. Wilson redescribed Alice's picture to the class. As she did so, she communicated to the students that Alice had made an important contribution to the class. We share part of her comment to illustrate how Ms. Wilson capitalized on aspects of Alice's drawing:

Teacher: This is a really neat way of looking at it. [This is] something I hadn't really thought about, Alice. But it really makes a lot of sense to me. She knew that Uncle Johnny ate more loose pieces than she had loose pieces sitting on the counter. So what she decided to do is to go ahead and show the pieces loose, and then she could work with them real easily. But she wanted to make sure we knew that she still had two packages. So she just drew the candies as if Aunt Mary had not yet quite packaged them up yet. She [Aunt Mary] had them in rows of what she was going to package. So you can see the loose pieces (points to the two packages Alice has drawn)...

Ms. Wilson's comment here addressed to two important points. The first point relates to representing Aunt Mary's packages as loose pieces. Alice had a reason for drawing the loose pieces in each package. It was easier for her to work with the packages if she drew all the pieces. Also, by making the loose pieces, her pictures were understandable to the rest of the class. Second, we note that Ms. Wilson and Alice, together, made it possible for certain mathematical ideas to emerge during the discussion. For Ms. Wilson's part, she aligned Alice's reasoning with how the class thought about and represented Aunt Mary's candy. Alice also contributed to this process when she provided a rationale for making her packages as loose pieces and explained her solution process.

After redescribing Alice's drawing, the second child, Luke, came to the blackboard and explained his drawing (see Figure 1b):

Luke: I crossed off the two pieces to make six and so, um, so that that would then be eight little pieces. And then those x's are for showing that Uncle Johnny had already had eaten them. And then I did that [drew the 8 pieces] so that you could see that there were eight more pieces (points to the eight loose pieces he has drawn on the blackboard)...And, and that's how I got the number eighteen (points to the numeral 18).

Luke, interestingly, used his drawing to explain how he solved the problem. In response to Luke's explanation, Ms. Wilson again underscored the significance of his as well as Alice's drawings and associated interpretations:

Teacher: What Luke did was a really, really neat thing. Because what he did was he went ahead and drew what Aunt Mary had on the counter, okay? And kind of like Alice, he imagined this [package] (points to the package that Luke has used to cross off two pieces) being broken up into pieces I think. And he said (points to the fours loose pieces that are crossed out) "I have four pieces that I can take away for what Uncle Johnny eats plus I need two more from this package" (points to the two pieces crossed out from one of the packages). And then he shows us (makes a circle with her hand around the eight loose pieces left) the eight pieces what he has left when he takes them away.

As Ms. Wilson redescribed Luke's picture, she specifically indicated how his picture was different from Alice's. Luke did not show the individual pieces contained in each package. As she indicated, he "imagined" that he could break apart a package to subtract the two additional pieces. By making this distinction explicit, Ms. Wilson communicated that she particularly valued how he reasoned with his drawing. By redescribing his explanation, Ms. Wilson also implicitly communicated that Luke had offered what constituted an acceptable explanation. Liam, for his part, contributed to this process by explaining his thinking using the pictorial collections.

Final Remarks

As we revisit our classroom example, we note that this discussion constrained and enabled the classroom mathematical practices. On the surface of it, the children's contributions were not mathematically different–both children offered very similar methods for solving the task. However, as we consider the quality of the children's explanations, their drawings point to different ways they participated in these practices. Whereas both children subtracted the loose pieces by taking away two pieces from one of the packages, how they made their pictorial collections signified very different ways of reasoning with the packages. Alice needed to count individual pieces, whereas Luke appeared to act with the packages as abstract collections (Cobb & Wheatley, 1988). The challenge then for the classroom teacher was to align the students' idiosyncratic interpretations with the emerging practices. As in our example, by juxtaposing these two children's explanations on the blackboard, Ms. Wilson could highlight the different interpretations the children might give as they decomposed and recomposed pictorial collections of tens and ones.

This example also points to the possible learning opportunities that arise during classroom discussions. Children like Alice might curtail how they made packages as pictorial collections. As a consequence of participating in this discussion, Alice had the opportunity to understand how other children reasoned with the packages. As such, she may develop interpretations that fit with how Luke acted with the pictorial collections. Luke could act with the collections without having to remake the individual pieces that composed each package. Further we clarify a collective conceptual shift that the class may make as they begin to act with their drawings in ways that are similar to how Luke solved the task. They may move from counting to making pictorial collections to reason with pictorial collections of abstract tens and ones.

These situations could also be learning opportunities for students like Luke. As they reflect on aspects of their drawings, they may curtail their methods by no longer needing to mark off individual pieces from a package. They may begin to reason mentally about the packages and use numerals to show how they partition collections of ten solve subtraction situations.

Finally, we note that the classroom mathematical practices are socially accomplished as the teacher and the students participate in these discussions. Whereas they contribute in different ways, it is clear that their contributions are equally important in advancing the mathematical practices. This is particularly the case in classrooms where children's self-generated models are valued and capitalized on. Such accounts as the one we have reported here remind us of the necessary role students' models play in learning mathematics with understanding.

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CHARTING ARGUMENTATION SPACE IN CONCEPTUAL LOCALES

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We describe a procedure for developing pedagogical knowledge about the potential argumentation space of groups of children in a conceptual locale. The method used involved the collection of small groups of children who had made significantly different responses to diagnostic test items, and the recording and analysis of their subsequent researcher-managed arguments in discussion. An example of the method is presented which shows how groups of 11-year old children developed arguments about the ordering of decimals, in response to a classic diagnostic item involving the ordering of 185, 73.5, 73.32, 57, 73.64. The analyses of these discussions led to a chart of the key elements of argument that arose, as well as general strategies for managing such discussions that were productive. These are considered as devices for helping teachers to plan argumentation in their classrooms.

Introduction

The identification and characterisation of the way children understand the mathematics presented to them in the curriculum has long been the focus of research in the psychology of mathematics education and continues to be a source of empirical and theoretical investigation. While much of this work was, and is still, conducted within a 'misconceptions' or 'alternative frameworks' paradigm, there is continuing development of work on children's mathematical thinking which elaborates on contextual, social and socio-cultural factors, and on the significance of inquiry discourse. (See for instance, Kirschner & Whitson, 1998, Forman & van Oers, 1998, Cobb & Bauersfeld, 1995 and Cobb et al, 2000.) In this study we are interested in how children may reveal and develop their understanding through collaborative argument in group discussion. We are particularly interested in discovering and describing productive lines of argument in relation to particular content conceptual locales, which may help teachers to develop productive discussions in their regular classrooms.

It has long been 'known' that children's errors and misconceptions can be the starting point for effective *diagnostically-designed mathematics teaching*. The key mathematical work on this in the UK was done in the 1980s by the ESRC Diagnostic Teaching Project (Bell et al, 1983), in which cognitive conflict was seen as the route to developing understanding. Argument in discussion between conflicting positions is seen as one important source of such conflict. The TIMSS video study reported that Japanese mathematics teaching typically makes use of a diagnostic approach: teachers are prepared with notes on a variety of *likely responses* to a key lead question, with guidance as to the thinking these responses indicate, and constructive teaching suggested related to each (Schmidt et al, 1996). This was related to the success of Japanese children's mathematical learning and particularly their problem solving capabilities. Dialogic methods involve the characteristics of conversation and the rigours of reason and persuasion: *sustained* talk and listening, statements of understanding or thinking-in-progress, the use and consideration of evidence, cognitive conflict and the making of new connections (Andrews et al, 1993; Costello et al, 1995; Inagaki, Hatano & Morita, 1998; Ryan and Williams, 2000).

Charting argumentation space: a methodology

There can be no genuine discussion or argument without a 'problematic', ie. an unresolved or not trivially-resolvable problem. This induces some purpose and some tension that sustains a discussion. The problematic for a particular group of children was established through prior testing which provided a range of student responses and methods of solution. The children were set the task of persuading each other by clear explanation and reasonable argument of the answer. The giving of clarifications, reasons, justifications and informal 'proof' was the rationale for the discussion. We use the term 'argumentation space' to describe the collection of relevant arguments likely to be used productively in children's arguments about a particular problematic. In this paper we show how we are beginning to chart argumentation spaces in ways which may help teachers to plan classroom discussions to develop productive arguments. In addition we outline the main strategies which we found supported productive argument in group discussions.

In this study a primary school cohort of 74 'year 6' (i.e. 11-year old) children was *screened* with a test that was designed to reveal common errors that had already been identified as relevant to their mathematics curriculum and level (Ryan & Williams, 2000). Essentially this involved identifying the most important common errors on tests for which we had collected a National sample of data (N = 1759) covering the entire mathematics curriculum for Key Stage 2 (end of UK primary school). From these errors, which had been coded and entered into a Rasch analysis (Ryan, Doig & Williams, 1998), we identified the most interesting errors based on the criteria that they should be: (a) common enough to reward a teacher's attention, (b) relevant to a significant locale of the curriculum being taught at the given age level in focus and (c) significant in terms of the literature on the psychology of learning.

The result was a diagnostic pencil and paper test of some 30 items lasting about 30 to 40 minutes and (later) a 20 minute mental test. The test items were drawn from the whole primary school curriculum. By way of an example, we will cite the case of an item called 'Ordering', which asked children to sort the numbers 185, 73.5, 73.32, 57, 73.64 from smallest to largest. In the National sample we found two common errors as expected: 57, 73.5, 73.32, 73.64, 185 ('decimal point ignored') and 57, 73.32, 73.64, 73.5, 185 ('longest is smallest'). These errors had been identified in the APU study of the early 1980s (Assessment of Performance Unit, 1982). The 'decimal point ignored' error is believed to have an important bearing on the development of children's number concept, and is typical of children's over-generalisation of whole number conceptions to the wider field of rational numbers. From each of the three year 6 classes, we selected 4 children for each discussion group on the basis that they had provided a range of responses on the test items.

There were 9 groups (36 children). The children from each group were from the same class and knew each other well, though were not necessarily from the same friendship group. Their teachers advised us on the likely successful dynamics for each group. They were mixed groups of boys and girls. Each group was withdrawn for discussion and videotaped in sessions lasting from between 30 to 50 minutes. Most children were involved in two sessions of taping. They recalled their test item response (an interval of a few days only) and were invited to present an argument for their response to the group. We, as researchers, adopted the teacher's role in discussion: generally four students at a time. All discussions were transcribed and analysed.

The analysis of argument follows Toulmin's scheme in general (developed by Cobb & Bauersfeld, 1995 and Cobb et al, 2000; Krummheuer, 1997, and others). Propositions relevant to the issue are 'backed' by arguments that are then subject to testing. In general children find it unnecessary to argue propositions which are believed to be shared, (i.e. taken-as-shared) so any particular discourse reflects the presumed shared points of departure, including the rules of argument in such situations. In this, the Researcher as a quasi-teacher, assumes the authority and seeks to ensure reasonableness, the need for the inquiry to persuade by good thinking and argument, and so on (Costello et al, 1995). An important role in productive argument may be played by tools in practice, which may provoke the formulation of connections between components of mathematical knowledge, new constructions and hence productive backing. The number line has been shown to play a significant role in many such contexts, and does so in the following example.

Results

Here we present part of one transcript for an argument about the ordering of decimals. Some commentary and analytical categories used are shown in bold to the right:

Kim: OK. I put 57 there – …. Then I put 73.5, … Then I put 73 point thirty-two, then I put 73.64 point sixty-four, then I put 185. **Everyday language**

Natalie: Well, I got 57 at the beginning too. And then I got 73.5. Then I got 73 point three- two Then I got 73 point six- four. Then I got 185. **Mathematical language**

RES: Could you explain why you put 73.5 before 73.32 (three, two)? **Focus**

Natalie: Because 73.32 (three, two) has got two digits after the decimal point and 73.5 has only got one. **Backing: separating decimal as wholes**

Elise: I'm not so sure, because 73.5 is basically 73 and a half. 73.64 (six, four) is, I'm not sure if it would be over a half or under…Actually I think the same as Kim… because, like Natalie said, there are two digits there, and two digits there, and only one digit there. **Intro fraction referent: conflict, backing**

RES: What do you think Richard? Richard: Same as Elise. **Check alternatives**

RES: The same… If I had a number line… Are you used to seeing a number line? (*children nod*). And I had 72. 72 would be back there. 73 would be there. 74 would be there. Where would you put 73.5? Do you want to do that Richard? **Intro tool: number line**

Richard: (*puts 73.5 half way between 73 and 74*) **Number line product**

RES: Can anybody put any other numbers in between 73 and 74? **Check alternatives. Press**

Kim: Yeah (*puts 73.64 above 73.5*) **Number line product**

RES: Why have you put in bigger than 73.5? **Check backing**

Kim: Because it's over a half. **Backing:fraction equivalent**

RES: Any other numbers you could put on that number line?
Do you want to have a go Natalie? **Press**

Natalie: 73 point two-five **Number line produces new argument!**

RES: 73.25 (?), where would that go?...
Could you tell us why you put 73.25 *just* there? **Focus on 0.25**

Natalie: It's a quarter of the number. **backing: fraction equivalent**

RES: Do you agree with that? (*children nod.*) So, it's gone... why has it gone exactly there? Is that because it is halfway towards a half? **Check backing**

Natalie: Yeah.

RES: Could you put a number on that number line Richard? **Develop number line**

Richard: Erm, 73.45...(*places it between 73.25 and 73.5... places 73.75 between 73.5 and 74*).

RES: 73.75, right? That's ...? **More 2-place decs**

Richard: Three-quarters. **Backing: fraction equivalence**

RES: So you put that halfway between 73 and a half, and 74...Where do you think 73.32 should go Kim: Before 73.5 RES: Why?

Kim: Because 73.5 is a half and 73.32 (?) is just after a quarter. **Resolution of referents**

RES: Could you say *why* it's just after a quarter? **Check backing**

Kim: Because a quarter is 73.25 (?) and 73.32 is bigger than 73.25 (*All agree*) I now think 73.32 is there, and 73.5 is there. **Kim sees change of mind**

RES: You all want to change your minds now?
Now why did we go wrong in the first place? **Seeks reflection**

Kim: Because we saw them as two-digit numbers, and we thought that the two-digit numbers were more than a one-digit number **Making new explicit**

Elise: I would say that 73.25 is a quarter, and it's less than 73.5 because that's a half, and 73.32 is just over a quarter, so it would be just under 73.5 **Fraction-decimal explicit**

Extracting the most productive and essential elements of this and other arguments about 'decimal point ignored' allow us to make a summary chart (Fig 1 below). This summarises the lines of argument we found that we think teachers will find useful in preparing for a particular discussion about 'ordering'.

Note the arguments advanced are 'backed' in Toulmin's sense (Toulmin, 1969) by the introduction of three key tools and references, without the introduction of these by the teacher/researcher or by a child, the arguments may critically follow different lines. These were:

- the placing of known decimals and fractions on a number line (e.g. 0.5 at ½, 0.25 at ¼).
- the equivalence or cancelling of fractions (e.g. 50% = 50/100 = 5/10 = ½)
- the equivalence of metric measures (e.g. 500mm = 50cm = 0.5m).

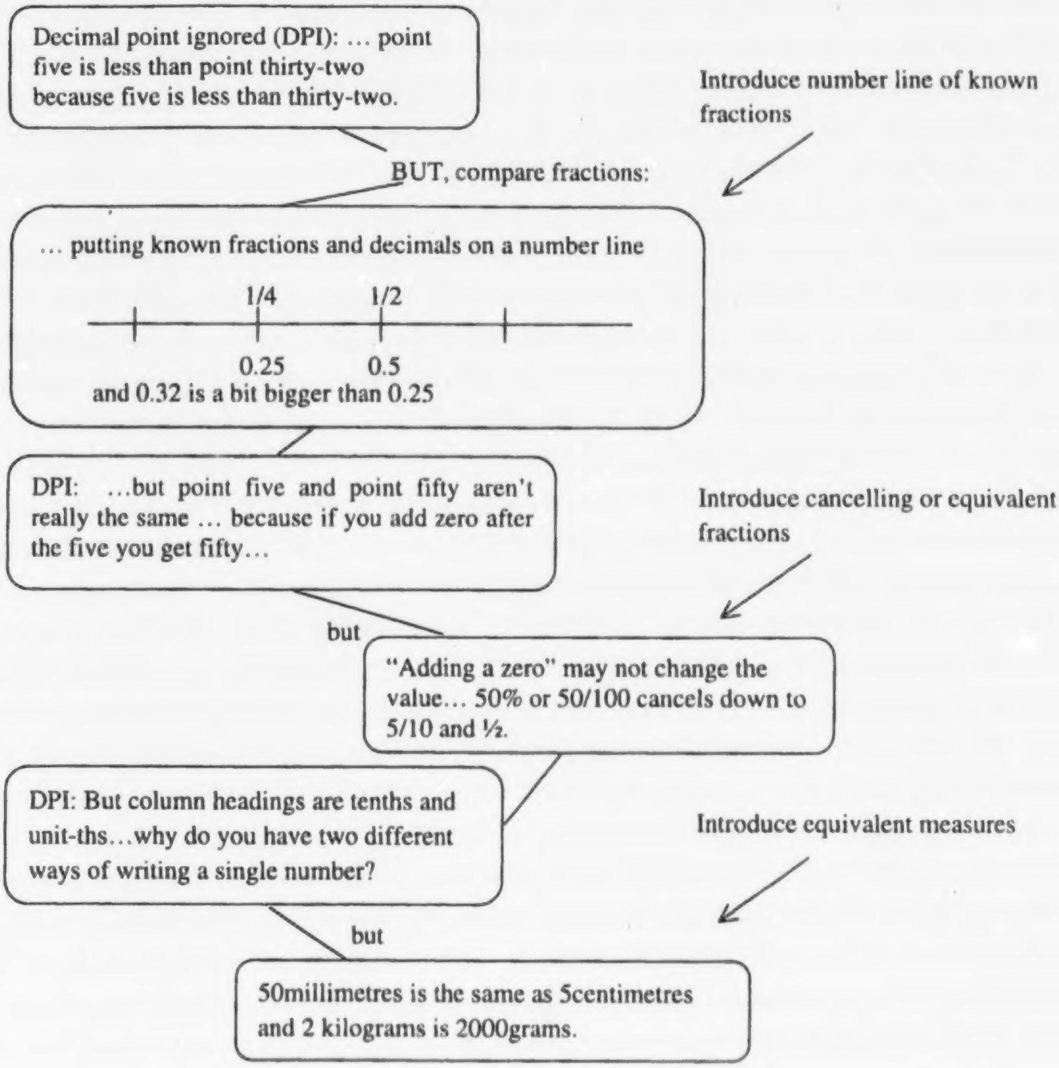
Decimal point ignored (DPI): … point five is less than point thirty-two because five is less than thirty-two.
Introduce number line of known fractions
BUT, compare fractions:
… putting known fractions and decimals on a number line
1/4
1/2
0.25
0.5
and 0.32 is a bit bigger than 0.25
DPI: …but point five and point fifty aren't really the same … because if you add zero after the five you get fifty…
Introduce cancelling or equivalent fractions
but
"Adding a zero" may not change the value… 50% or 50/100 cancels down to 5/10 and ½.
DPI: But column headings are tenths and unit-ths…why do you have two different ways of writing a single number?
Introduce equivalent measures
but
50millimetres is the same as 5centimetres and 2 kilograms is 2000grams.

Fig 1: A chart of argumentation space for 'Decimal Point Ignored'

The difference between numbers with one decimal place and two decimal places is particularly critical here, as was evident in this episode when 0.25 was placed at ¼ on the number line. In another discussion the critical difference was manifest in argument about the value of 0.5 and 0.50, which were considered by some children to represent different numbers, even with different places on the number line. One child actually suggested that 'nought point five' should perhaps be written as 0.05 when ordered with 'other' two decimal place numbers. This discourse manifests a conceptual world of whole numbers extended to decimal numbers which consists of pairs of numbers separated by a point, (i.e. *x* point *y*) and that the number after the point is, or should be, a fixed length of string digits. This is the root of DPI errors and their backing arguments.

In the next stage of the analysis we sought to categorise the researchers' inputs to the discourse and influence on the discussions in general. Being aware of the children who made the common errors in advance, the researcher manifestly sought to ensure that the arguments for the error were clearly voiced, as well as to ensure that potentially productive tools and referents were introduced at some point: hence the significance of the particular conceptual locale to pedagogical content knowledge.

However, we examined general teaching strategies that seemed generally productive across problem contexts and conceptual locales. In eliciting and sustaining argument, we include the eliciting of variety of 'answers' and arguments, asking children to listen and sometimes paraphrase others' views, seeking further clarification of arguments, (sometimes helping to formulate and encourage a minority point of view), seeking alternatives and dissent, and seeking reasons and 'backing'. These are all strategies that forestall closure and encourage productive conflict. In the final post-resolution stage of discussion, strategies that encouraged reflection included asking children whether and why they had changed their mind, what the argument or misconception had been, and how they would summarise what they had learnt.

Conclusions and discussion

We have shown that it is possible to use dialogue generated in research to chart an argumentation space which describes the children's arguments in response to provocative diagnostic items in a conceptual locale. The concept of an argumentation space located around a diagnostic item is designed to be helpful in supporting teachers' pedagogical content knowledge. We interpret these spaces as providing potential classroom discourses structuring potential zones of proximal development of individuals within a class. The dialogues teachers might generate in replication of the research setting might thereby provide opportunities for individuals to learn by testing their responses against those of their peers, and being given an opportunity to evaluate and shift their position accordingly.

We are currently investigating and evaluating how helpful these 'charts' can be to teachers in practice, and whether the resulting dialogues will be successful in helping children learn. The next step in the project involves a study with teachers delivering, marking and interpreting the diagnostic test and observation of their subsequent teaching through discussions based on these argumentation spaces. We will present some evaluation of its effectiveness in helping teachers to develop their practice at the PME presentation.

We believe that the approach adopted here has certain conceptual strengths and weaknesses. Clearly, desirable pedagogical content knowledge with respect to a conceptual field or even locale cannot be altogether encapsulated in one chart. In fact, even the literature in diagnostic teaching – a conflict-based method – is far from solely based on conflicting students in discussion: the use of particular tools and representations in particular have a most significant role. Furthermore there is a danger in peer discussion, often cited by teachers and in the literature, that students will be persuaded by the weakest of arguments.

However, the strength of developing this methodology for teachers as a tool in their practice is that it has certain general features as well as the particularities in the locally structured chart for a locale. Thus, we would hope that this approach as a teaching method will help teachers to improve their practice very generally. We are optimistic that the method will encourage at least some teachers to see themselves as teacher-researchers, and that they will wish to extend these or begin constructing their own charts for locales we have not yet explored.

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Learning to Teach Mathematics Differently: Reflection Matters[1]

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The nature of mathematics teaching, influenced by constructivist and social theories, is better understood as the result of investigations of accomplished teachers, but questions still remain as to how teachers' learn this complex form of teaching. This research reports an investigation of the interplay between beginning elementary teachers' reflective thinking and changes made in their mathematics teaching. The findings reveal teachers who change in teaching progressively increase in depth of reflection on their teaching and in appropriately interpreting their students' intentions and mathematical thinking. Reflections of teachers with little change in mathematics teaching consist of descriptions, evaluations, and rationalizations of events; moreover, they are unable to view circumstances from the students' perspective.

It is widely accepted that changes in perspectives on students' learning and the development of knowledge in mathematics reflect trends in cognition and the nature of knowledge influenced by post-modern philosophies. Together these trends have significantly changed the view of mathematics teaching to a more complex pedagogy that represents a major shift from the long-standing emphasis on single computational procedures (MSEB, 1989). Although, the nature of teaching, influenced by constructivist and social theories, is better understood as the result of investigations of accomplished teachers (e.g., Jaworski, 1994; Schoenfeld, 1998; Wood, Nelson & Warfield, in press), questions still remain about how teachers' learn this complex form of teaching.

A significant body of research exists which indicates teachers have learned as evidenced by changes in their practice, beliefs and knowledge. But the question still remains, how do we know if teachers *are* learning? To answer this question it is necessary to investigate the *process of reflection* which is thought to be central in teachers' learning (Schön, 1987). The purpose of this research is to examine how teachers used both their reflective thinking as a process in pedagogical reasoning and their classroom practice to transform their mathematics teaching in ways advocated (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000).

OVERVIEW

Several studies have shown that teachers develop the pedagogy necessary to create reform-oriented mathematics classes as they reflect on events that occur in their classrooms (Fennema, et al., 1994; Cobb, Wood, & Yackel, 1990; Simon & Schifter,

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1991). Fundamental to the thinking about teacher learning is the contention that "attempts to influence teachers knowledge and beliefs will not be at their most effective unless they draw on teachers' first-hand experiences of interacting with their students during mathematics instruction" (Cobb, Wood, & Yackel, 1990, pp. 141-142). Although these studies identify changes in teaching and teachers' beliefs, other studies have shown that while some teachers develop complex forms of teaching, others make little change (e.g., Vacc & Bright, 1998). Furthermore, among teachers who make changes, some teachers continue to develop in their teaching while others do not (Franke, et al., 1998). This raises questions about the commonly accepted notion that elementary classrooms are a source of opportunities for teachers' learning.

Along with classrooms as sites for teachers' learning, reflection is thought to be an essential process in teachers' learning and central in their capacity for pedagogical reasoning (Shulman, 1987). Subsequently, reflective thinking is viewed as central to much of the thinking in mathematics education about teacher learning. Studies, such as that of Mewborn (1999), have investigated what teachers' found problematic in classroom situations as a means to examine their reflective thinking. In her study, Mewborn found that teachers' reflective thought followed Dewey's (1933) five phases; moreover, to be a reflective thinker required teachers to hold relativistic beliefs. Although this research contributes to an understanding *about* the reflective thinking of teachers, it is still not well understood how teachers use reflective thinking to make sense of their work and how reflection influences the changes they make in their teaching.

Thus, the purpose of this research is to investigate how teachers use reflective thinking in their pedagogical reasoning and how their thinking relates to changes in teaching. It is contended that substantial change in teaching occurs as teachers reflect on their lessons, examining how classroom events that occurred compare with their intentions, and making alterations based on their reflections. In this study, the process of reflection was examined by investigating what teachers' notice when observing their classrooms, the interpretation they make of events, and the changes they propose in their practice. These findings about the process of reflection were then compared with analysis of the teachers' videotaped classrooms with regard to their development of socially interactive learning environments and the mathematics that occurred during the lessons.

THEORETICAL ORIENTATION

Cognitive and social context theoretical perspectives guide the research on the processes of teachers' learning. Both of these perspectives hold the view that the human mind is generative, creative, proactive, and reflective and that humans interpret and give meaning to events and things in their lives. Social environment provides opportunities for learning and differences in social context are thought to affect the nature of what is learned. Therefore, the social context teachers create for student learning affects their own opportunities for learning within the classroom. Thus, from a social interactive perspective the teachers' creation of classroom social structures important to students' participation in the discourse of inquiry is of central interest.

From a cognitive perspective the construct *reflection* drawn from the work of Dewey (1933) is used to describe the process by which teachers give meaning to their own and students actions and is used to examine teachers' learning. Reflection, in this study, is defined as the distancing of one's self from the object of reflection. Borrowing from Dewey's (1933) notion of that reflective thinking; consists of "a state of doubt, hesitation of perplexity in which thinking originates," and "an act of searching, hunting or inquiring to find material that will resolve the doubt" and "dispose of the perplexity" (p.14). The early work of Shulman (1987) on pedagogical reasoning and the recent research of Mewborn (1999) on the characteristics of reflection are used to further delineate the process of teacher reflection. Using both cognitive and social interactive perspectives allow for an explanation of teachers' learning that considers the interplay between teachers' reflection as a central process in pedagogical reasoning and their the development of classroom learning environments.

The characteristics of teaching follow conceptualizations of Jaworski (1994) that describes a teaching triad that consists of three dimensions, management of learning, sensitivity to students, and mathematical challenge. In addition, the interactive dimensions of participation and questioning from Wood (Wood & Turner-Vorbeck, in press) further characterize teaching.

DATA SOURCE, METHODOLOGY AND ANALYSIS

Approach to Teacher Development

The study reported in this paper is part of a larger 2-year research and development project in which the research goal was to investigate how elementary teachers' develop their classroom teaching in accordance with reform schemes.[2] The development aspect of the project consisted of creating an approach to inservice teacher education for elementary teachers that utilized three central themes that incorporated tenets of constructivism, social constructivism and sociology. Taken together, these themes reflected a stance toward working with teachers that placed importance on individual development of teaching in conjunction with the formation of public or common knowledge of teaching through the generation of a community of professional practitioners. In order to promote teachers' learning and yet attempt to develop an approach to development that would be less labor intensive for teacher educators, certain aspects of technology were used as support (cf. Wood, 1999 for further detail about the approach).

For the purpose of promoting teacher personal reflection, a component was created in the professional development approach that required teachers' to reflect on their teaching activity in conjunction with their students' mathematical thinking during mathematics lessons. Teachers made video recordings of their lessons and then later used these videotapes to examine the events that occurred during their lessons. However, in order to engage in investigation of their instruction, teachers needed support in developing their skills in making observations and reflecting on videotaped classroom events. As a means of support, a 3-step procedure was created, following Jaworski (1988), for responding to the tapes by writing in "reflective journals." These steps were to: write their expectations for the lesson prior to teaching; make detailed records of the discourse during class discussion; and, compare and contrast the records of the events described with their expectations. Following this, they were to write a "plan of action" to carry out in the classroom based on the results of their reflections.

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[2] The project is "Recreating Teaching Mathematics in the Elementary School" funded by the National Science Foundation.

Teachers and Data Source

Seven beginning (2 years of previous teaching experience) elementary teachers participated in the 2-year research and development project that involved an intensive investigation of their learning to teach. The seven teachers taught in the same school district in first through fifth grades (6-11 year-olds).

The reflective journals and the class videotapes served as the primary data sources for the research on individual teacher reflection and learning to teach. Each teacher made written comments following the format described above for the purpose of recording their reflections before, during, and after watching their classroom videotapes. Secondary data sources were each teacher's written journal responses to specific questions asked by researchers during the group working sessions and e-mail exchanges between the teachers and research team.

The primary source of data for analysis of classroom teaching consisted of videotape recordings of mathematics lessons recorded by each teacher twice per month. Each lesson videotaped was viewed and logged by the research staff as a detailed record to be used in the analysis.[3]

Methodology and Analysis

The methodology and analysis followed a qualitative research paradigm and procedures similar to those of Glaser and Strauss (1969) and Strauss & Corbin (1990) in which categories were developed from the data, examined for confirming and disconfirming evidence and revised. The specific methodology and analysis for examining reflection and classroom practice are described next.

Processes of Reflection

For the teachers' journal entries a written running record of each entry was compiled by one member of the research team of each teacher's: a) expectations, b) description of events, and c) reflections on the lesson. The three entries were analyzed and refined, analyzed again, and a running record for each teacher written by the same member of the research team. The teachers' responses were categorized as description, reflection or rationalization. The section of the journal that consisted of reflections on the lessons was additionally categorized in terms of depth of reflection. These categories revealed not only the nature of what aspects of students' thinking teachers' reflected on but also the quality of their reflection. The categories for quality of reflection consisted of *thought on action, thought on thought,* to *thought on thoughts of children.* The teacher journal responses to researcher-asked questions and the content of the e-mail messages provided supporting evidence for the running records. Each member of the research team individually analyzed each teacher's written journal entries and then met to discuss their analyses with the other members of the research group. Differences in interpretations were discussed and resolved by looking within the data for confirming or disconfirming evidence.

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[3] Research team members were Luiza de Souza, Caroline Van Tuinen, Michael Palm, Janet Warfield, and myself.

Classroom Practice

The classroom videotapes and logs were used to investigate each teacher's practice. The videotape logs for each of the classroom lessons were coded by the research team following an extension of the coding scheme developed and used in previous studies of teaching (Wood & Turner-Vorbeck, in press). The research team individually coded each lesson log for one teacher at a time and then met to discuss their analysis. Differences in interpretation were discussed and resolved by looking within the logs for confirming or disconfirming evidence. Certain videotaped lessons collected at the beginning and end of each year served as baseline data. The baseline videotapes were used to gauge changes in teaching over the duration of the project. Following a discussion of the coding of the baseline tapes, each member wrote a summary for each teacher of the interaction patterns, discourse, teacher questions and mathematical topics discussed.

Integration of Teacher Reflection and Classroom Practice

The remaining coded logs, along with the baseline lessons, were matched with journal entries in order to integrate teacher reflections on their classroom practice with video recordings of the lesson. The analysis of the data consisted of comparing and contrasting journal entries of teacher reflection with the log of the lesson being reflected up on. In addition, subsequent lessons were analyzed in order to match instances of teacher insight from reflection and proposed plans for change with video recorded observations of teaching.

RESULTS AND DISCUSSION

Teacher Reflection

The findings from the analysis of the data on teacher thinking and instances of reflection revealed differences among the teachers in terms of the content and depth of their reflection on children's mathematical thinking in the context of their class. Additionally, teachers differed in the intensity or quality of their reflection. That is, from the beginning one group of teachers on reflection saw aspects of their classroom interaction as problematic and gave thought to how their role as teacher influenced the situation (e.g., "*I need to listen more & not lead children to what I want to hear. I need* *to just let them explain their thinking."*) Later, these same teachers' reflections became 'thoughts about their thoughts' (e.g., *". . .solve the problem by subtracting at the very beginning so I knew she knew how to solve the problem."*) And still later, these teachers' thoughts were about what children's thinking or intentions might be in the situation (e.g., *"The other children were busy coloring to find the answer I felt that they thought they were going to find the answer quickly (or faster) than it took them!"*) Over the 2-year period, these teachers appeared to deepen in their understanding of children's mathematical thinking and their view of the problem, solution, or both.

Another group of teachers' responses consisted of summaries of events (e.g., *"all students were on task"*) or evaluations of students' behavior (e.g., *"the students are all doing a much better job of sharing responsibilities."*) When these teachers' did reflect, these primarily consisted of rationalizations (e.g., *"I feel we never have a good discussion when the camera is rolling"* and *"I feel the problem lies before in earlier grades, parental involvement, and many students just being lazy."* Moreover, they often attributed to children's action reasons that were inherent in the child (e.g., *"I had a hard time with the lower math students. They failed to build on other, earlier problems.")* Thus, they seemed not to think about how their children might be thinking about or making sense of a problem or solution; they were seemingly unable to view the situation from the child's perspective. Additionally, they seldom considered how their role, as teacher, might influence the situation.

Classroom Practice

Analysis of the teachers' classroom practice also revealed differences in the growth of their teaching. The analysis of the first baseline classroom videotape revealed that all of the teachers initially taught mathematics conventionally. That is, the focus was on computational procedures, interaction and discourse that consisted of the typical IER pattern described by Mehan (1979) and others. The teachers who substantively changed in their teaching created classroom environments similar to those advocated by reform in mathematics education (NCTM, 1989; 2000). These classes involved teachers and children in discussion that was characterized as *inquiry and argument* (c.f., Wood & Turner-Vorbeck, in press). The teachers that made little change in their teaching continued their conventional interaction characterized by teachers' 'test questions', students' answers, followed by teacher evaluation.

SUMMARY AND IMPLICATIONS

Taken together, the findings revealed two distinct groups of teachers existed that differed in the nature of their reflection and in their development of the pedagogical skills necessary to create interactive learning environment and the mathematics advocated in the reform agenda. Teachers who made changes in their mathematics classrooms progressively increased in the depth of their reflection. They also differed in their interpretation of their students' intentions and thinking—what Jaworski (1994) refers to as "student sensitivity." Conversely, the reflections of teachers who made little change in their teaching consisted of quite different content. Their responses consisted of descriptions, evaluations, and rationalizations of the events that occurred and they were seemingly unable to view the situation from the child's perspective.

Reflection has long been considered as an essential process in teachers' learning and central in their capacity for pedagogical reasoning, both of which are necessary if teachers are to develop and continue to generate more sophisticated forms of pedagogy. These findings provide further insights into how teachers' do or do not learn to teach differently and the role of reflection in the process that are important for mathematics teacher education.

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EXPLORING MULTIPLICATIVE AND ADDITIVE STRUCTURE OF ARITHMETIC SEQUENCES

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Arithmetic sequence is used in this study as a means to explore preservice elementary school teachers' connections between additive and multiplicative structures as well as several concepts related to introductory number theory. Vergnaud's theory of conceptual fields is used and refined to analyze students' attempts to test membership of given numbers and to generate elements that are members of a given infinite arithmetic sequence. Our results indicate that participants made a strong distinction between two types of arithmetic sequences, sequences of multiples (e.g. 7, 14, 21, 28, ...) and sequences of "non-multiples," (e.g. 8, 15, 22, 29, ...). Students were more successful in recognizing the underlying structure of elements in sequences of multiples, whereas for sequences of non-multiples students often preferred algebraic computations and were mostly unaware of the invariant structure linking the two types.

An arithmetic sequence is a sequence of numbers with a common difference. The topic of arithmetic sequence, along with other sequences, is usually introduced in high school and the standard approach utilizes algebraic representation and manipulation. Despite being a part of a high-school, rather than elementary school, curriculum the topic of arithmetic sequence is frequently approached in mathematics courses for preservice elementary school teachers. This is mainly because arithmetic sequences surface in the discussions of pattern recognition and understanding relations, generalization, and problem posing techniques (Brown and Walter, 1990; Principles and Standards for School Mathematics, 2000). However, little research has been done on students' learning of this topic. In this study we explore preservice elementary school teachers' understanding of the structure underlying arithmetic sequences.

THEORETICAL FRAMEWORK

Vergnaud's Theory of Conceptual Fields

In several publications over the last two decades Vergnaud developed, proposed and elaborated on the theory of conceptual fields (Vergnaud, 1988, 1994, 1996, 1997). The development of the theory of conceptual fields was motivated by the need to establish connections between explicit mathematical concepts, relations and theorems and between students', at times implicit, dynamic conceptions and competencies related to these mathematical concepts, relations and theorems. The following are among the established terms of reference:

- *Conceptual field* (1996, p. 225) is a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situation.
- A *scheme* (1996, p. 222; 1997, p. 12) is the invariant organization of behavior for a certain class of situations.
- A *theorem-in-action* is a proposition that is held to be true by the individual subject for a certain range of the situation variables (1996, p. 225).

Cognitive Development in a Conceptual Field - Theory Extension

Vergnaud claims that the theory of conceptual field is "a theory of representation and cognitive development" (1996, p. 220). Problem situations serve as triggers in generating and promoting cognitive development. When students are faced with a new situation, "they use the knowledge which has been shaped by their experience with simpler and more familiar situation and try to adapt it to this new situation" (Vergnaud, 1988, p. 141). This description is similar to Piagetian accommodation and assimilation. We elaborate further on the mechanism for the development of a particular scheme.

Theorems-in-action are identifiers of students' knowledge, as they describe mathematical relationships, either correct or incorrect, that are taken into account by students when they choose a path to solve a problem. Vergnaud suggests that "theorems-in-action have the potential to be the links among situations in the conceptual fields" (1988, p. 145). We add that theorems-in-action may also serve as separators, rather than links, and recognizing either could serve as a stepping-stone to learning.

Scheme development may occur in two ways: A student may recognize differences in two seemingly similar classes of situations. As an outcome, different theorems-in-action will be invoked, and, as a result, different routes will be taken by a student in dealing with each of the two classes of situations. A student may also recognize invariant structure in two classes of situations that were formerly perceived as "different". This may lead to an adaptation of two previously used theorems-in-action into one more general theorem-in-action that is applicable for both classes of situations. Furthermore, identifying invariant structure in situations may serve as a bridge that takes a student from one conceptual field to another.

METHODOLOGY

Participants

Participants in this study were preservice elementary school teachers enrolled in a core mathematics course. Students' work with arithmetic sequences in this course included pattern recognition, generating sequences given the first element and the difference, and developing and implementing formulas for calculating the *n*-th element as well as the sum of the first *n* elements of the sequence. Students also worked with elementary number theory topics, including divisibility, factors and multiples, and the division algorithm . Twenty out of 64 students enrolled in the course volunteered to participate in a clinical interview, which was the main source of the data collection.

Situations

Situation is a key feature in Vergnaud's definition of a scheme and a conceptual field. In this research we take a broad interpretation of situations, and classify as situations not only contextualized word-problems, but also mathematical problems and questions that are "abstract" or "decontextualized"; that is, not rooted in "real world" context. The following interview questions represent the core of the situations that were presented to participants.

1. Describing and exemplifying. Give several examples of arithmetic sequences. Can you think of an example that is different from others?

2. Testing membership.

Consider the following sequence of numbers

(a) 2, 5, 8, 11, 14, ...

Is it arithmetic? Is the number 360 (or 440) an element in this sequence (assuming it is infinite)? Why? Is there another way to verify this?

(b) The same question with respect to sequence 3, 6, 9, 12, ...

(c) The same question with respect to sequence 17, 34, 51, ... and number 204

(d) The same question with respect to sequence 8, 15, 22, 29, ... and number 704.

3. Generating examples of members. Can you think of a "large" number that is an element in sequence 2, 5, 8, ... (If necessary, "large" was described as a 3- or 4-digit number). Can you think of a large number that is definitely not an element in this sequence?

The same questions were posed with respect to sequences listed in 2(b), (c) and (d) above.

Objectives

We explore students' attempts to deal with the situations, specifically aiming to:

(1) identify and describe strategies (rules of action) used as participants encounter problem situations related to arithmetic sequences,

(2) analyze students' strategies and uncover underlying theorems-in-action,

(3) suggest a path for a development of individual's scheme within the context of presented situations, and

(4) test empirically the (above) extension of the theory.

STUDENTS' SCHEMES: RESULTS AND ANALYSIS

Listing the elements in a given arithmetic sequence by adding the common difference will eventually generate "large" elements and determine whether a given number is the element in the sequence. In light of Vergnaud's theory the common difference can be seen as an invariant identifying a class of situations. For participants in this study listing the elements was not the preferred choice, however, this strategy was mentioned by 8 participants as a verifying strategy or as a default for not being able to generate a better strategy.

Using the formula $a_n = a_1 + (n-1)d$ was a popular choice of strategy. The formula is applied in routine questions to find the n-th element when the first element and the common difference are known. Furthermore, it can be used to calculate any one of the four variables when the other three are known. Application of formulas was the exclusive strategy suggested by only one of the participants. However, 17 participants used formulas for situations similar to 2a and 2d, whereas they applied considerations of form and pattern for sequences similar to 2b and 2c. This fact, taken together with the significant amount of prompting and invitation to think of "another way" during the interview, suggests that participants preferred formulas when the pattern in the sequence was not obvious to them; that is, when they weren't aware of the multiplicative invariants in the structure of the elements. Therefore, participants invoked a scheme previously established to deal with arithmetic sequence related questions, the scheme of plugging numbers into the formula.

Multiples and non-multiples. It became apparent from participants' responses to question 1 (request to provide examples of several arithmetic sequences) that there is a class of arithmetic sequences preferred by students. Each participant provided between 4 and 8 examples of arithmetic sequences. Most of these examples were sequences of multiples of a small natural number, such as 3, 6, 9, 12 or 5, 10, 15, 20, ... with a possible exception of a sequence of odd numbers.

When the interviewer explicitly asked for "something different", the usual reaction was to provide sequences of multiples of "large" numbers, such as 50, 100, 150, 200, ... etc. or list multiples in a descending order. Though participants readily accepted other sequences, such as 2, 5, 8, 11, ..., as "arithmetic", they were not a part of their immediate repertoire of examples.

Realizing that some arithmetic sequences are "sequences of multiples" provides a tool for testing membership or generating "large" elements without relying on formulas. However, 8 participants overgeneralized the observation that every element in an arithmetic sequence is a multiple of the common difference *d* to hold for any arithmetic sequence. This tendency is exemplified in the following excerpt from the interview with Leah.

Interviewer: Would you please consider the following sequence: 8, 15, 22, 29, so far it's an arithmetic sequence, how would you continue?
Leah: 36, 43?
Interviewer: Okay. And how about the number 704? Is it an element in this sequence?
Leah: I'm going to check and see if 7 is a factor of 704, (pause) no ...
Interviewer: No for what?
Leah: Um, 704 is not going to be in this sequence because 7 is not a factor of 704.
Interviewer: Okay. How about 700?
Leah: Yeah, um, 7 is a factor of 700, so I think it's going to be in the sequence. 7 x 100 is 700.

Leah claims that the number 700 is an element in a sequence 8, 15, 22, ... because 7 is a factor of 700. In such cases the student's theorem-in-action was challenged by the interviewer by pointing out contradictory evidence. As a result of these types of challenges some participants refined their scheme by limiting it to certain kind of situations. Once the difference between the two classes – multiples and non-multiples – was realized, it became apparent that the same scheme cannot be used to accommodate both. As a result the scheme of considering multiples of *d* was restricted to sequences of multiples only.

In the following excerpt Sally considers the sequence 8, 15, 22, ... and the number 704.

Interviewer: So 704 is not divisible by 7, none of these elements in this sequence you believe will be divisible by 7, so can you draw conclusions from what you have now?
Sally: It's, it's um very possibly in this set.
Interviewer: Um hm. What, what will convince you?

Sally: (laugh) Well just because it's not divisible by 7, doesn't mean it's in the set, right?
Interviewer: Can you give me an example of a number that you know for sure that is not in this arithmetic sequence?
Sally: Um hm, um 700. . .
Interviewer: Another one. . .
Sally: Um, 77.
Interviewer: Okay. And how about 78?
Sally: It may be in the set, but it's not divisible by 7. . .
Interviewer: (laugh) So 77 you're sure is not, 78 you're not sure.
Sally: Right.
Interviewer: 79?
Sally: Could be. . .
Interviewer: Could be. 80?
Sally: Could be. . .

Sally is confident that multiples of 7 are not elements in the given sequence, but she believes that any number that is not a multiple if 7 "could be" in the sequence. At this stage students are able to differentiate and note that previously generated theorems-in-action are not fruitful in a new situation. However, they have not yet revised their theorems-in-action to generate rules-of-action for the new class of situation. Whereas a number's property of "being a multiple" gives a clear indication of its belonging to a sequence of multiples and non-belonging to a sequence of non-multiples, the property of "being a non-multiple" identifies that a number doesn't belong to a sequence of multiples, but gives no explicit hint with respect to the number's membership in a given sequence of non-multiples.

It was a common observation that sequences of multiples have two identifying features (invariants), multiples of *d* and a common difference, whereas sequences of non-multiples have only one identifying feature of common difference. Identifying a class of situations as "multiples" often left participants without appropriate tools to deal with non-multiples. In 17 cases the participants took advantage of multiples in considering situations 2(b) and 2(c), but regressed to the use of formulas for 2(a) and 2(d).

A further development was to realize that any arithmetic sequence of whole numbers can be considered as a translation along the number line of a corresponding sequence of multiples. Therefore, the next important step in developing individual's scheme is to recognize the invariant multiplicative structure of elements in an arithmetic sequence of non-multiples, often referred to by participants as "multiples adjusted".

Interviewer: Let's take one more. 8, 15, 22, 29 another sequence.
Lily: Okay, so this is a difference of 7. . .
Interviewer: How about the number 704?
Lily: (pause) 704, no (pause) this is, these numbers are plus 1 of multiples of 7, multiples of 7 plus 1. . .

Consideration of multiples and "adjustment" where necessary clearly equips Lily with a powerful scheme. Such an "adjustment" is expressed by Megan in a more mathematical way as she considers division with remainder.

Interviewer: And what about 704?
Megan: No, because that's got a remainder of (pause) 4, not 1. . . it needs to have a remainder of 1.
Interviewer: So can you please describe for me your general strategy? How would you decide whether a number I give you does belong to this sequence or doesn't belong to it?
Megan: Um, if it's divisible by 7, with the remainder of 1 then it does belong to the set.

Six participants eventually succeeded in suggesting some adjustment of either multiples or numbers divisible by *d*, and 2 explicitly mentioned the common reminder in division by *d* thereby identifying the multiplicative structure in a sequence of non-multiples.

Towards a Unified Scheme. Although the invariant of "multiples" in the sequences of multiples and the invariant of "multiples adjusted" in the sequences of "non-multiples" are analogous, the learner may still see multiples and non-multiples as two separate classes of situations having a different invariant structure. Therefore different schemes are invoked in dealing with these situations. The identification of a multiplicative invariant within non-multiples is essential for the development of a unified scheme. Identifying similarities, other than lexical, between the two classes, can be a next step in scheme development. It is a further sophistication to consider multiples with adjustment (that can be zero) or common remainder (that can be zero) in division by *d* as the invariant that unifies both classes of situations and allows an individual to invoke the same scheme for any arithmetic sequence. Recognizing invariant structure in two previously-treated-as-different multiplicative invariants supports the development of a unified more mature scheme. However, the interview situations provided little opportunity for participants to extend their scheme in a way that it could accommodate both classes.

CONCLUSION

Greer (1992) proposed that the analysis of the relationship between the conceptual fields of additive and multiplicative structures is a long-term objective on the agenda for further research in mathematics education. Our research is a step in this direction. Furthermore, this study contributes to prior research on preservice elementary school teachers understanding of elementary number theory (Zazkis & Campbell, 1996), specifically the concepts of multiples, divisibility and division with remainder. In addition, analysis of students' development of a specific problem related scheme leads to a more profound understanding of scheme development in general and in such provides an extension of Vergnaud's theory of conceptual fields. Furthermore, our research points at a possible direction for a pedagogical approach to the topic, an approach that capitalizes on the common structure of elements in an arithmetic sequence.

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OPEN-ENDED TASKS: THE DILEMMA OF OPENNESS OR AMBIGUITY?

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Open-ended tasks have been shown to offer considerable potential for learning in mathematics. They have substantial benefits for learning and assessment. However, when using open-ended tasks, questions need to be posed as to how open can they be. In making tasks "open" there is some chance that the task can become ambiguous. This paper reports on a small study of students' responses to open-ended tasks where some tasks were clear in their goals whereas others were more open, and hence open to greater interpretation by the students.

Tirosh (1999) notes that questions and questioning have a significant role in reform in mathematics education since new movements in mathematics education compel teachers and teacher educators to move away from the traditional forms of teacher-directed and closed questioning techniques that have dominated mathematics classrooms in the past. One tool that has been recognized as having significant value is that of open-ended questions or tasks. The work of researchers and teachers across a range of nations has been particularly valuable in identifying the features of open-ended tasks and how these are of benefit to teaching and learning (Becker & Shimada, 1997; Boaler, 1998; Chappell & Thompson, 1999; Sullivan, Warren, & White, 1999). While such research has been particularly useful in demonstrating the value of open-ended approaches to teaching mathematics, it has been limited in its analysis of the questions being posed. This paper explores the use of open-ended tasks across three classrooms, noting the ways in which language influences the interpretations of the given tasks. It is proposed that the tasks are useful in facilitating greater responses from the students, and hence, making assessment more authentic. However, it is noted that there is some need for concern when tasks become so open as to become ambiguous.

Open-ended Tasks

Sullivan et al (1999) define open-ended tasks as having more than one possible response (goal) and where there are multiple pathways for resolution (activity). They propose that such openness in activity and goal "fosters some of the more important aspects of learning mathematics, specifically, investigating, creating, problematizing, mathematizing, communicating and thinking" (p. 250), which they see as being substantially different from the more restrictive processes associated with recall and rote learning. Closed tasks have typically and predominantly been used in mathematics classrooms and examinations and can be seen as having one answer, for example, *"What is the area of a rectangular piece of paper with the dimensions of 4 cm and 6 cm?"* Clearly, there is only one correct answer, 24 cm^2. While there may be a number of ways in which the answer can be calculated, such as repeated addition (6+6+6+6= 24), using a diagram marking the rectangle into small squares, or multiplying 4 by 6, in most cases there is a preferred activity for resolution. In contrast, an open-ended task is one that allows for a range of "correct" responses and a range of ways of achieving those responses. An open-ended task would be one such as: *"If the area of a rectangular piece of paper is 24 cm^2, what might be its dimensions?"* This type of question offers greater scope for teachers in assessing students' understanding of area, and also allows students' greater scope in demonstrating what they know about area.

Chappell and Thompson (1999) note that the advantages of using open-ended tasks in mathematics are that they encourage students to move beyond the skills-based approaches typical of mathematics classrooms. They demand that they students think more deeply about the concepts; and that they make connections between concepts. This point is reinforced by the work of Boaler (1998) who found that the students who used open-ended approaches in mathematics developed deeper and more connected forms of knowing than their peers in traditional classrooms. She argues that the learning in the open-ended classrooms was more like the learning that occurs in the world beyond school and hence there is greater transfer from school to beyond school.

Sullivan et al (1999) raise the issue of content specificity of open-ended tasks as this allows the mathematics to be made transparent to the students so enabling their learning to be more directed. An example these authors cite is "*A number is rounded off to 5.6. What might the number be?*" (Sullivan et al., 1999, p. 250). In these tasks, the content is specific to mathematics and students are able to answer it according to the current levels of understanding—some may use two decimal places whereas others use three or more decimal places. The rationale behind open-ended tasks is that they are to be seen as open enough for interpretation so that students answer them in ways that they understand.

The Project

This paper reports on the analysis of a number of tasks given to students in the upper primary school in Queensland, Australia. A set of five open-ended tasks was given to 115 students, but only two of the tasks will be discussed in this paper. The tasks were all open-ended and were designed to cover a range of areas of the mathematics curriculum. Students were given the tasks during a mathematics lesson, and conversation was allowed. Two multi-age classrooms participated in the study, with each classroom having approximately 60 students and two teachers. The students were in the final two years of primary schooling and aged 11–13. The students were given the tasks to complete and later a number of interviews were conducted with selected children in order to gain some appreciation of their thought processes when undertaking the tasks. These students were selected on the basis of the responses that they had offered in the tasks.

Task 1: Data Handling Task

The first task to be discussed involved the interpretation of statistics. The context was a real one to the students whose school was close to a major road that lead to an island where there was only a very small bridge to cover the main thoroughfare of traffic. There had been substantial reports on the need for a bridge or alternative thoroughfare to cater for the increasing traffic.

Task: At the Chevron Island Bridge, the average number of people per car is 2.5. Draw what this might look like if there are 16 cars on the bridge.

The task was open in its goal and method of resolution. It allowed the teachers to access students' understanding of what the statistics meant in the world beyond school mathematics. Frequently, the mean is something that is calculated: *"Forty people in 16 cars, what would the mean be?"* However, this task asked what this might look like when posed in the context beyond school. Similarly, it assessed whether or not students understood what a mean of 2.5 meant, and how this is manifested in the everyday. It also allowed for further information to be posed by students that would not otherwise be possible through a closed question. This type of task provided a rich source of information about students' understanding of measures of central tendency, interpretation of data, and application of data.

The responses to this task fell into a number of categories. Many of the students were able to work through the task so that they could provide a range of cars that would have different number of people in them and where the average number of people was 2.5 per car (see Fig 1a).

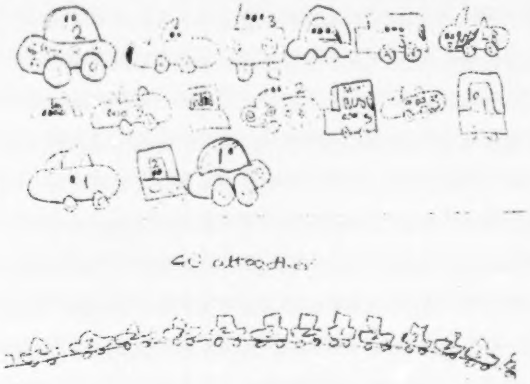

Figure 1a and 1b: *Responses offered by students*

Other responses indicated that students realised that there needed to be a total of 40 people in the 16 cars (see Fig 1b). In noting explicitly the need for a total of 40 people, students used a range of strategies, including those listed below, to arrive at 40 people.

Others took a more systematic approach and had alternate cars with two and three people in each (see Fig 2). While this produced a mean of 2.5, it did not show the depth of understanding evident in the responses above. Students appear to have calculated a simple method of means of 2.5 using a pairing strategy of 2 and 3 and then applied this to the 16 cars.

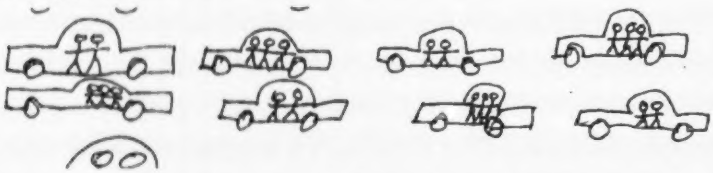

Figure 2: *[extract of] Systematic representation of 2:3 pairings*

Another strategy used involved the translation of "half". Some students (see Fig 3a) showed three people per car with one person being smaller than the others. This would suggest that these students interpreted the 0.5 to mean a "small" or "half" person, as was confirmed through the interviews. One student commented that a child was "half an adult so there are two adults and one child in each car." Others showed a similar representation, but with a code to show that there were two adults per car and one child whereby the one child represented "half a person" or "0.5 people".

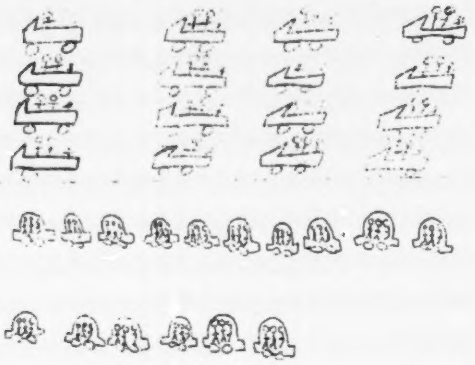

Figure 3a and 3b: *Cars showing representations of "half"*

Others (see Fig 3b) showed "half a person" by only drawing half a body so that there was literally a "half person" in each car. This representation indicates a literal translation of the data and hence would suggest that there is a need for further work to be undertaken with these students.

As can be seen from these responses, the open-ended task allowed for considerable diversity in responses and a range of representations. This offers potential for effective diagnosis of students' understandings. While the goal was restricted in some sense (in that there needed to be 16 cars on the bridge), the ways in which people were represented in these cars was open. The responses offered by the students varied in both the goal (by having different amounts of people in each car) and the activity through which they solved the task. Students had varying degrees of success with the task but all were able to produce some documentation of their understanding of the task and the concepts involved. This allowed for the teacher to make a range of judgements about students' levels of understanding.

Task 2: Estimation and Rounding Task

The role of language and openness need to be considered with open-ended tasks. The use of language in open-ended tasks may make them "open" to interpretation as well as open mathematically. Consider the following task:

Task: My dog weighs about 20 kilograms. How much could she weigh?

This task was designed to assess the estimation and rounding skills of students so that the words of "about" and "could" were central to the notion of estimation and rounding. However, the responses of the students indicated a number of possible interpretations of the task. Many of the responses were difficult to categorize since there would need to be some unsubstantiated interpretation of the results, so only those that clearly fell into a category are considered in the calculations.

Around half of the students interpreted the question to be one of estimation of weights. There was some degree of variability in responses with 38% of the students offering responses around the 20kg measure, as was the expected response. These responses centred on what would normally be considered those typical of a mathematical context. These students offered responses that were either a single weight such as *18.5kg*, or a range of weights such as *18-22kg;* or *between 22.5 and 19.8*, while others constructed a list of weights such as *17, 18 , 19, 20, 21, 22.* Others (16%) offered responses that were weights around 30–40kg that may be considered too high for estimation purposes but may indicate some conceptualization of estimation. Some of the students interviewed thought that this weight was close to 20kg while others thought that this is what their dog (at home) weighed. Hence, it is difficult to classify the answers as being correct or otherwise without further information as to the rationale behind the responses offered by these students.

What was interesting was the cluster of responses whereby students interpreted the task to be futures-orientated, where the task was translated as meaning "If my dog weighs 20 kg now, what might she weigh in the future?" Typical of this group of responses are the following answers:

"When she's bigger, she'll get to 25kg."
"A puppy could be a weight of 20kg but when it's older it could weigh 40 to 45 kg."
"If I had a Husky that weighed 20 kg as a puppy, it might weigh 60–70kg."
"My puppy is half grown. He weighs about 20kgs. When he is fully grown, he will way [sic] about 70kgs."

These responses indicate that the language of the task created a different interpretation from that intended. When asked about the response he made to this task, one boy offered the following comment. "In our group one of the girls who is good at maths said that we really needed to estimate that if a puppy would weigh 20 kgs, what might is be when it was fully grown." This comment indicates two important issues, the first being that of interpretation of the task, and second, that the role of group work also impacts on the translation of the task. While it is not possible to say how many students followed group dynamics in their responses, it is possible that in some groups dominant personalities were able to sway particular students in proposing answers of a particular type.

The use of "My dog" may cause students to personalize the task. Some students interpreted the question to be highly contextual and related to their own dog so that little or no mathematizing was undertaken. This group of consisted of 17% of the student responses. Typically the responses were written in a way that suggested that the students were interpreting the question as if it were about their own dogs, real or imagined. Responses in this category were typically:

"My dog is an obese German Shepherd who weighs 60kg."
"My dog Wally weighs about 20kg. It is very old. The vet said it should weigh 40kgs at least."
"I think my dog is a bit fat. She could weigh about 15-17kgs."

Included in this group of responses was a cluster of responses that seemed to estimate what a dog might weigh and tried to think about which dogs might weigh particular weights around 20 kg such as:
"A cocker spaniel might weigh 30 kgs".
"A Labrador would weigh about 40-50kgs."

There seems to be some transfer from the context of the question to the context beyond the mathematics classroom as if the question being posed asks the students to consider what types of dogs may weigh round 20kgs. While they could not accurately guess a dog that would be about 20 kgs, their reasoning is exemplified by the comment made by one girl—*"I tried to think about what dog might weigh about 20 kgs. A foxy [fox terrier] is only little so I think they would not even be 10kgs. Dogs like Rottweilers and Great Danes are really big and would weight lots. I could not think of a dog that would be about 20 kgs. A Labrador would be more than 20 kgs but it was the closest that I could guess that would be near to 20kgs."*

With this question, there is some sense that the question could have been worded so that it would be less ambiguous. As Sullivan et al (1999) argued, the content specific nature of open-ended tasks can provide support for learners. This task may have been ambiguous and hence open to too much interpretation when the intention of the task was to assess/access students' understanding of estimation and rounding of mass. The task might have been better worded if it were *"If a vet has rounded a dog's weight to 20 kgs, what might the dog have weighed?"* However, such wording raises a dilemma as to the openness of the task. The use of signifiers such as "rounded" can serve as a key word and as such may provide the cue for what the students need to do to undertake the task. Where a key word approach has been used, it is difficult to ascertain whether or not students have understood what the task is asking or where there has been a reliance on key words. As a strategy, using key words to solve word problems has been found to be successful. Schoenfeld (1992), reporting on a text book series widely used in the USA, found that most word problems (approx. 90%) in the series could be solved using superficial key words rather than requiring any mathematical understanding or interpretation. Using tighter or more explicit wording may have resulted in more students being able to answer the question "correctly", it raises the issue as to the purpose of questioning in mathematics classrooms.

One of the advantages of the open-ended tasks is that they allow students to see the application of tasks to contexts beyond the classroom. While tighter wording of the problem may have meant that students were then able to provide estimates of the pre-rounded weight, it raises issues as to whether or not students understand the transfer between contexts. As Boaler (1998) argues "It seemed that the act of using mathematical procedures within authentic activities allowed the students to view the procedures as tools that they could use and adapt. The understandings and perceptions that resulted from these experiences seemed to lead to increased competence in transfer situations" (p. 59). In this instance, the wording of this question was akin to what would be heard in many beyond-school contexts, such as a veterinarian's surgery where the weight of the dog determines the dosage of a medicine. Following Boaler's contention, it would suggest that such openness may be useful for its links, and hence transfer, to the non-school contexts.

Openness Versus Ambiguity

While the first example highlights the value of open-ended tasks for assessment, some questions regarding the value of the second task as an assessment item need to be raised. In contrast, the second task could be seen as being poorly worded due to the ambiguity of the task. As noted earlier, however, this ambiguity can also be of value insofar as creating an openness to the task. By tightening the wording of the task in order to reduce ambiguity, it is possible that the task becomes too prescriptive due to key words defining what is to be undertaken—in this case, the use of rounding would reduce ambiguity insofar as the goals of the task but would also provide a cue as to what needed to be done. Durkin and Shire (1991) have shown that ambiguity is a part of mathematics education. Words such as rational, odd, base and so on have particular meanings in mathematics that are very different from their in non-mathematical contexts. Similarly, homophones such as pi and pie; two and too; or whole and hole also produce ambiguities for students. Walkerdine (1982) has argued that students often identify a particular word as being key in a sentence or task and as a consequence select the wrong discourse in which to locate and respond to the task. For example, in the second task, the students have interpreted the word "could" to mean a futures perspective and have responded in this sense, rather than as a rounding context as intended by the teacher. They have identified a futures discourse and responded correctly in this context.

As has been recognised within the mathematics education community, there are particular social and cultural norms that work with mathematics classrooms that students must become conversant with. Part of such competency is recognizing the unspoken rules of interactions with mathematics and what are seen to be valid and legitimate forms of responding. Students need to become familiar with what are socially legitimate forms of knowledge within the classroom and what are not. Ambiguity in wording, for example, can confuse students in what are socially acceptable responses. In the case cited here, the potential for inclusion of futures-perspectives can be a legitimate part of the mathematics classrooms, so the responses can be seen as appropriate. However, many other tasks need to be considered carefully as the ambiguous wording may cause students to offer inappropriate responses as a consequence of misinterpretation of the words and their relevant contexts. For example, the use of "odd" numbers can result in students perceiving such numbers as being "strange" due to the ambiguity of the term between the mathematics and non-mathematics contexts. Students must make the transition from one context to another. Indeed, many of the errors made by students can be seen to be linguistically related rather than mathematical.

While the value of open-ended tasks has been shown in this and other studies, there is a need for support for students when beginning open-ended approaches to teaching. In the case cited here, these students had little or no experience with open-ended tasks. Their responses to the first task suggest that they are able to deal with the tasks, but as the second task indicates, some explicit teaching maybe of value in contexts where there is some ambiguity in the task. In this case, the ambiguity can be a feature of the openness of the task but also a hindrance, particularly if there is some assessment associated with the tasks. Coming to know mathematically and pedagogically, means coming to understand the expectations of the social and cultural norms that are embedded in such tasks. In the second task, where the ambiguity may be seen as a valuable characteristic, it may be of value to make this ambiguity an explicit teaching feature so that students come to know the unspoken rules or norms of the mathematics classroom.

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